

Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 23

Date: Apr 03, 2018 04:00 PM

URL: <http://pirsa.org/18040028>

Abstract:

QFT for Cosmology, Achim Kempf, Lecture 23

Note Title

Plan: Unruh effect \rightarrow Hawking effect

Unruh effect in 1+1 dimensions

The metric: In inertial, cartesian ds^2 : $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix}$

Consider an observer's trajectory $x^\mu(\tau)$
and use the observer's proper time τ as the parameter.

□ Velocity

$$\dot{x}^\mu(\tau) := \frac{dx^\mu(\tau)}{d\tau}$$



Proposition: $\eta_{\mu\nu} \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) = 1 \quad \forall \tau$



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Proposition: $\eta_{\mu\nu} \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) = 1 \quad \forall \tau$

Proof: At any point in time, τ , in rest frame: $\bar{x}^\mu(\tau) = (\tau, 0)$

$$\Rightarrow \dot{\bar{x}}^\mu(\tau) = (1, 0)$$

$$\Rightarrow \eta_{\mu\nu} \dot{\bar{x}}^\mu(\tau) \dot{\bar{x}}^\nu(\tau) = 1 \quad \text{which is a } \underline{\text{scalar}}$$

$$\Rightarrow \eta_{\mu\nu} \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) = 1 \quad \text{in all cd systems for all } \tau.$$

... *indeed* ...

Proposition: $\gamma_{\mu\nu} \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) = 1 \quad \forall \tau$

Proof: At any point in time, τ , in rest frame: $\bar{x}^\mu(\tau) = (\tau, 0)$

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$$\Rightarrow \gamma_{\mu\nu} \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) = 1 \quad \text{in all cd systems for all } \tau.$$

D Acceleration: $\ddot{x}^\mu(\tau) := \frac{d\dot{x}^\mu(\tau)}{d\tau}$

Proposition: $\ddot{x}_\mu(\tau) \dot{x}^\mu(\tau) = 0 \quad \forall \tau$

Proof. $\nabla_\mu \cdot d(\vec{v} \times \vec{r}) = \vec{v} \nabla \cdot \vec{r} - \vec{r} \nabla \cdot \vec{v}$ "proper acceleration"

Special case of uniform acceleration: $a(\tau) = a \quad \forall \tau$

Proposition: A trajectory of uniform acceleration a is given by:

$$x^\mu(\tau) = (t(\tau), x(\tau)) = \left(\frac{1}{a} \sinh(a\tau), \frac{1}{a} \cosh(a\tau) \right)$$

Proof: $\dot{x}_\mu(\tau) = (\cosh(a\tau), \sinh(a\tau))$

is obeying $\dot{x}_\mu \dot{x}^\mu = 1 \quad \checkmark$

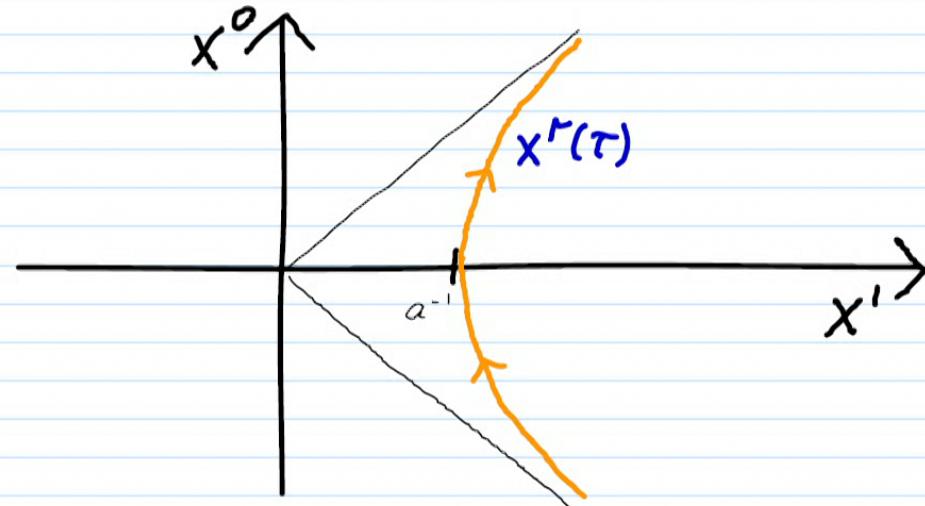
$\Rightarrow \tau$ really is the proper time

and, crucially:

$$\ddot{x}_\mu(\tau) = (a \sinh(a\tau), a \cosh(a\tau)) \text{ obeying } \ddot{x}_\mu \ddot{x}^\mu = -a^2 \quad \checkmark$$

This trajectory also obeys:

$$X_p(\tau)X^k(\tau) = X^0(\tau)^2 - X^1(\tau)^2 = -\frac{1}{a^2}$$



i.e., it is a hyperbola of deceleration followed by acceleration.

Notice: Our uniformly accelerated traveler has horizons:

- can't influence events below the line $X^0 = -X^1$, i.e., with $X^0 + X^1 \leq 0$
 - can't be influenced by events above the line $X^0 = X^1$, i.e., with $X^0 - X^1 \geq 0$
- } (A)

Inertial light cone coordinate system:

The inertial cartesian coordinates are fine to describe particle motion.

For wave equations, often light cone coordinates have advantages. (Esp. in 1+1D):

$$\tilde{x}^\mu(x^0, x^1) := (u(x^0, x^1), v(x^0, x^1))$$

$$\begin{aligned} \text{with: } u(x^0, x^1) &:= x^0 - x^1 \\ v(x^0, x^1) &:= x^0 + x^1 \end{aligned} \quad \left. \right\} (B)$$

The metric: In inertial, cartesian coords x^μ : $g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix}$

$$\begin{array}{|c|} \hline \text{In inertial light cone coords } \tilde{x}^\mu: g_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0, 1/2 \\ 1/2, 0 \end{pmatrix} \\ \text{i.e.: } ds^2 = du dv \\ \hline \end{array}$$

Exercise: Check this, using $g_{\mu\nu}(\tilde{x}) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu}(x)$.

The trajectory above in inertial light cone coords:

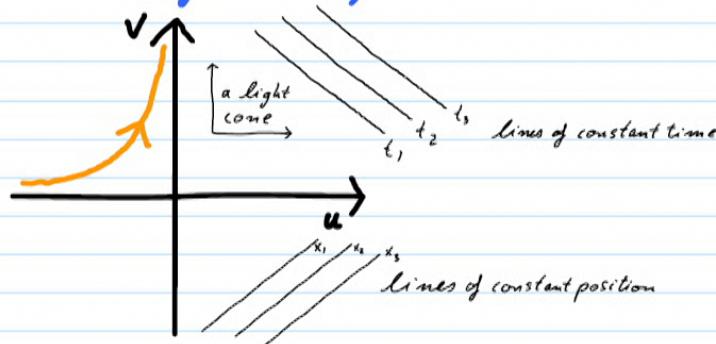
$$\tilde{x}(\tau) = (u(\tau), v(\tau))$$

$$\text{with } u(\tau) = t(\tau) - x(\tau) = -\frac{1}{a} e^{-a\tau}$$

$$v(\tau) = t(\tau) + x(\tau) = \frac{1}{a} e^{a\tau}$$

Notice: From (A) \wedge (B): the traveller

- can't influence events (u, v) with $v \leq 0$
- can't be influenced by events (u, v) with $u \geq 0$



A coordinate system that is comoving with the traveler

We want a coordinate system ξ^{μ} so that our traveler's trajectory is:

$$\xi^r(\tau) = (\tau, 0)$$

But this fixes the new cds only on the trajectory!

Q: How to continue the new cds to elsewhere?

A: We can require (in 1+1 dimensions) that the light cones are still at 45° , i.e., that

$$g_{\mu\nu}(\xi) = f(\xi) \begin{pmatrix} 1, 0 \\ 0, -1 \end{pmatrix},$$

$$\text{i.e. } ds^2 = f(\xi) (d\xi^0{}^2 - d\xi^1{}^2), \text{i.e., } \underbrace{ds^2 = 0}_{\substack{\text{condition for} \\ \text{light-like tangent}}} \Rightarrow d\xi' = \pm d\xi^0$$

Proposition:

Under the change of coordinates

$$\begin{aligned}x^0(\xi) &= \tilde{\alpha}' e^{a\xi'} \sinh(a\xi^0) \\x'(\xi) &= \tilde{\alpha}' e^{a\xi'} \cosh(a\xi^0)\end{aligned}\quad \left.\right\}^{(T)}$$

we have that the trajectory $\xi^r(\tau) = (\tau, 0)$

is indeed the trajectory of our traveler:

$$x^r(\tau) = (\tilde{\alpha}' \sinh(a\tau), \tilde{\alpha}' \cosh(a\tau))$$

And in addition: The Minkowski metric $g_{\mu\nu}(x) = g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reads in the ξ coordinates:

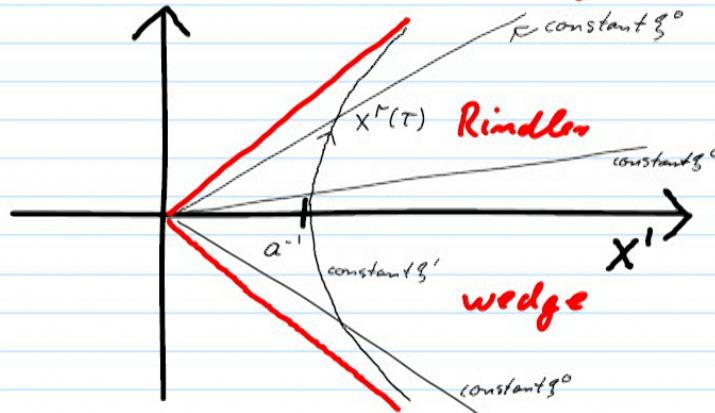
$$g_{\mu\nu}(\xi) = e^{2a\xi'} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

\Rightarrow In this coords,
light travels
still at 45° .

In (T), why did we map the new cds to the old: $\xi^r \rightarrow x^r$?

There is no inverse $x^r \rightarrow \xi^r$!

Why? Because all of $(\xi^0, \xi^1) \in \mathbb{R}^2$ maps only on to the Rindler wedge $x^r > |x^0|$



From (T):

For each ξ^1 , obtain a hyperbola within the Rindler wedge.

Together they cover exactly only the Rindler wedge.

We knew that the traveler has horizons.

His comoving cds ξ^r reaches only as far as to his horizons.

Accelerated light cone coordinates.

In $\tilde{\xi}^\mu$ cds, light still travels at 45° .

\Rightarrow It will be useful for wave equations to introduce accelerated light cone coordinates:

$$\tilde{\xi}^\mu(\xi) = (\tilde{\xi}^0(\xi), \tilde{\xi}'(\xi)) = (\bar{u}(\xi), \bar{v}(\xi))$$

$$\text{where: } \bar{u}(\xi) = \xi^0 - \xi'$$

$$\bar{v}(\xi) = \xi^0 + \xi'$$

In the cds $\tilde{\xi}^\mu$ we have:

$$g_{\mu\nu}(\tilde{\xi}) = e^{a(\bar{v}-\bar{u})} \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

$$\text{i.e.: } ds^2 = e^{a(\bar{v}-\bar{u})} d\bar{u} d\bar{v}$$

Remark: We can also directly map the accelerated light cone cds $\tilde{\xi} = (\bar{u}, \bar{v})$ into the inertial light cone coordinates $\tilde{x} = (u, v)$: (Exercise: show this)

Important later! →

$$u(\bar{u}, \bar{v}) = -\frac{1}{\alpha} e^{-\alpha \bar{u}}$$

$$v(\bar{u}, \bar{v}) = \frac{1}{\alpha} e^{\alpha \bar{v}}$$

Summary:

Coordinate system

$$x = (x^0, x')$$

Form of the metric

$$g_{\mu\nu}(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{x} = (u, v)$$

$$g_{\mu\nu}(\tilde{x}) = \begin{pmatrix} 0 & \frac{1}{\alpha} \\ \frac{1}{\alpha} & 0 \end{pmatrix}$$

These cds
cover only
the Rindler wedge

$$\left\{ \begin{array}{l} \xi = (\xi^0, \xi') \\ \bar{\xi} = (\bar{u}, \bar{v}) \end{array} \right.$$

$$g_{\mu\nu}(\xi) = e^{2\alpha \xi'} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$g_{\mu\nu}(\bar{\xi}) = e^{\alpha(\bar{v}-\bar{u})} \begin{pmatrix} 0 & \frac{1}{\alpha} \\ \frac{1}{\alpha} & 0 \end{pmatrix}$$

Observation: These metrics are pairwise conformally related:

$$g_{\mu\nu}(x) \rightarrow \bar{g}_{\mu\nu}(x) := \Omega^2(x) g_{\mu\nu}(x)$$

Proposition: In 2 dimensions, the K.G. action is invariant:

$$\begin{aligned} S_g[\phi] &= \frac{1}{2} \int_{\mathbb{R}^2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \sqrt{|g|} d^2x \\ &= S_{\bar{g}}[\phi] \end{aligned}$$

Proof:

$$\text{We have } g^{\mu\nu}(x) \rightarrow \bar{g}^{\mu\nu}(x) = \Omega^{-2}(x) g^{\mu\nu}(x)$$

$$\text{and } \sqrt{|g|} \rightarrow \sqrt{|\bar{g}|} = \Omega^2(x) \sqrt{|g|} \text{ in 2 dimensions.}$$

✓

\Rightarrow The Klein Gordon action

$$\begin{aligned} S[\phi] &= \frac{1}{2} \int g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \sqrt{|g|} d^2x \quad \text{general cds} \\ &= \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x^\mu \phi)^2 - (\partial_\mu \phi)^2 dx^0 dx^1 \quad \text{inertial cartesian cds} \\ &= 2 \int_{\mathbb{R}^2} (\partial_u \phi)(\partial_v \phi) du dv \quad \text{inertial light cone cds} \end{aligned}$$

On Rindler Wedge: (easy to see because of conformal invariance)

$$\begin{aligned} S_{RW}[\phi] &= \frac{1}{2} \int_{\mathbb{R}^2} (\partial_{\tilde{x}}^\mu \phi)^2 - (\partial_{\tilde{x}}^\mu \phi)^2 d\tilde{x}^0 d\tilde{x}^1 \quad \text{accelerated cartesian cds} \\ &= 2 \int_{\mathbb{R}^2} (\partial_{\tilde{u}} \phi)(\partial_{\tilde{v}} \phi) du dv \quad \text{accelerated light cone cds} \end{aligned}$$

\downarrow because massive action is not conformal

Remark: A massive field would have a different equation motion in accelerated frames.

i.e.: accelerated observer can find out he's accelerating using masses.

The Klein Gordon equations:

In inertial light cone coordinates:

$$\frac{\delta S'}{\delta \phi} = \partial_u \frac{\delta S}{\delta \partial_u \phi} + \partial_v \frac{\delta S}{\delta \partial_v \phi} \quad \text{i.e. } \partial_u \partial_v \phi(u, v) = 0$$

Easily solved: $\phi(u, v) = A(u) + B(v)$, with A, B arbitrary functions.

For example: $\phi(u, v) = e^{-i\omega u} = e^{-i\omega(t-x)} = e^{i\omega(x^0 - x')}$

is a right-moving positive frequency solution.

The usual Minkowski space quantum field solution $\hat{\phi}(x^0, x')$ can be written this way:

$$\hat{\phi}(u, v) = \int_0^\infty \frac{dk}{T2\pi} \frac{1}{T2\omega} \left(\underbrace{e^{-i\omega u} a_k + e^{i\omega u} a_k^+}_{\text{right movers}} + \underbrace{e^{-i\omega v} a_{-k} + e^{i\omega v} a_{-k}^+}_{\text{left movers}} \right) \text{ and } \omega = |k|$$

The Klein Gordon equation:

In inertial light cone coordinates:

$$\frac{\delta S'}{\delta \phi} = \partial_u \frac{\delta S}{\delta \partial_u \phi} + \partial_v \frac{\delta S}{\delta \partial_v \phi} \quad \text{i.e. } \partial_u \partial_v \phi(u, v) = 0$$

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The Klein-Gordon equation in the accelerated frame:

In accelerated light cone coordinates: (covering only the Rindler wedge)

$$\frac{\delta S_{\text{Rw}}'}{\delta \phi} = \partial_{\bar{u}} \frac{\delta S_{\text{Rw}}}{\delta \partial_{\bar{u}} \phi} + \partial_{\bar{v}} \frac{\delta S_{\text{Rw}}'}{\delta \partial_{\bar{v}} \phi} \quad \text{i.e. } \partial_{\bar{u}} \partial_{\bar{v}} \phi(\bar{u}, \bar{v}) = 0$$

Easily solved: $\phi(\bar{u}, \bar{v}) = A(\bar{u}) + B(\bar{v})$, with A, B arbitrary functions.

For example: $\phi(\bar{u}, \bar{v}) = e^{-i\omega\bar{u}} = e^{-i\omega(\xi^0 - \xi')}$

is a right-moving positive frequency solution.

In the accelerated frame, the quantum field in the Rindler wedge is:

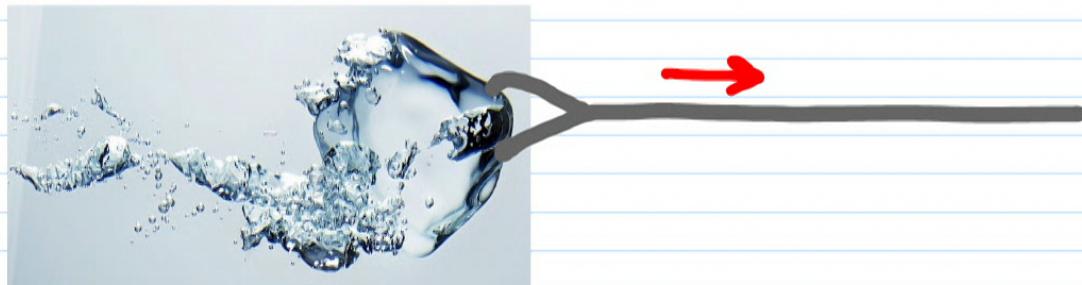
$$\hat{\phi}(\bar{u}, \bar{v}) = \int_0^\infty \frac{dk}{T2\pi} \frac{1}{T2\omega} \left(\underbrace{e^{-i\omega\bar{u}} b_k^+ + e^{i\omega\bar{u}} b_k^-}_{\text{right movers}} + \underbrace{e^{-i\omega\bar{v}} b_{-k}^+ + e^{i\omega\bar{v}} b_{-k}^-}_{\text{left movers}} \right) \text{ and } \omega = |k|$$

Notice: hermiticity conditions, K.G. eqn and CCRs obeyed.

For the inertial observer, the vacuum state obeys: $a_k |0_m\rangle = 0$

But for the accelerated observer, the vacuum state obeys: $b_k |0_R\rangle = 0$

We will assume that the state of the system is $|4_i\rangle = |0_m\rangle$.



Will acceleration melt ice?

We arrived at a typical situation:

$$\hat{\phi}(u, v) = \int_0^\infty \frac{dk}{12\pi} \frac{1}{12w} \left(e^{-iu\hat{a}_k} + e^{iu\hat{a}_k^+} + e^{-iv\hat{a}_{-k}} + e^{iv\hat{a}_{-k}^+} \right)$$

(A)

$$= \int_0^\infty \frac{dk}{12\pi} \frac{1}{12w} \left(e^{-i\bar{u}\hat{b}_k} + e^{i\bar{u}\hat{b}_k^+} + e^{-i\bar{v}\hat{b}_{-k}} + e^{i\bar{v}\hat{b}_{-k}^+} \right)$$

There must exist a Bogoliubov transformation linking the a_k, a_k^+ and b_k, b_k^+ !

Observation:

The left and right movers won't mix.

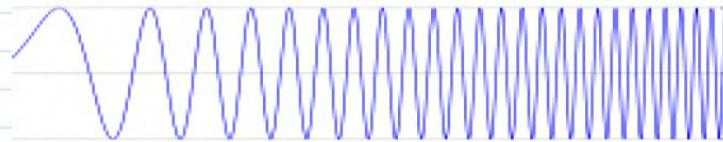
→ For simplicity we'll consider only the right movers.

Observation:

Among right movers all frequencies may mix :

$$b_{\Omega} = \int_0^{\infty} d\omega (\alpha_{\Omega\omega} \alpha_{\omega}^* - \beta_{\Omega\omega} \beta_{\omega}^*) \text{ with } \omega = k \quad (B)$$

Intuition: To the traveller, any monochromatic wave sounds like a chirp.



Exercise: Check that $[\alpha_k, \alpha_k^*] = \delta(k-k')$ and $[b_k, b_k^*] = \delta(k-k')$ imply:

$$\int_0^{\infty} d\omega (\alpha_{\Omega\omega} \alpha_{\Omega'\omega}^* - \beta_{\Omega\omega} \beta_{\Omega'\omega}^*) = \delta(\Omega - \Omega') \quad (C)$$

Calculation of α_{ω} and β_{ω} : (lengthy, for more details, see Mukhanov & Winitzki text.)

□ Substitute (B) into (A) and collect coefficients of α_{ω}

$$\Rightarrow \omega^{1/2} e^{-i\omega u} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}} \left(\alpha_{\omega} e^{-i\Omega' \bar{u}} - \beta_{\omega}^* e^{i\Omega' \bar{u}} \right)$$

□ Act with $\int_{-\infty}^\infty d\bar{u} e^{\pm i\Omega' \bar{u}}$ on the equation

and then use that $\int e^{i(\Omega - \Omega') \bar{u}} d\bar{u} = 2\pi \delta(\Omega - \Omega')$.

Calculation of α_{ω} and β_{ω} : (lengthy, for more details, see Mukhanov & Winitzki text.)

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and then use that $\int_{-\infty}^\infty e^{i(\Omega - \Omega') \bar{u}} d\bar{u} = 2\pi \delta(\Omega - \Omega')$.

$$\Rightarrow \begin{cases} \alpha_{\Omega \omega} \\ \beta_{\Omega \omega} \end{cases} = \pm \frac{1}{2\pi} \sqrt{\frac{\Omega}{\omega}} \int_{-\infty}^\infty e^{\mp i\omega u + i\Omega \bar{u}} d\bar{u}$$

+ case:
- case:

□ Substitute (B) into (A) and collect coefficients of a_w

\Rightarrow

$$\bar{\omega}^{\frac{1}{2}} e^{-i\omega u} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}} \left(d_{\Omega' w} e^{-i\Omega' \bar{u}} - \beta_{\Omega' w}^* e^{i\Omega' \bar{u}} \right)$$

□ Act with $\int_{-\infty}^\infty d\bar{u} e^{\pm i\Omega' \bar{u}}$ on the equation

and then use that $\int_{-\infty}^\infty e^{i(\Omega - \Omega') \bar{u}} d\bar{u} = 2\pi \delta(\Omega - \Omega')$.

\Rightarrow

$$\begin{aligned} + \text{case: } & d_{\Omega' w} \\ - \text{case: } & \beta_{\Omega' w} \end{aligned} \} = \pm \frac{1}{2\pi} \sqrt{\frac{\Omega}{w}} \int_{-\infty}^\infty e^{\mp i\omega u + i\Omega' \bar{u}} d\bar{u}$$

Recall:

$$u(\bar{u}, \bar{v}) = -\frac{1}{a} e^{-a\bar{u}}$$

(encoding the chirping)

and, therefore: $\frac{du}{d\bar{u}} = e^{-a\bar{u}} \Rightarrow d\bar{u} = (-au)^{-1} du$

\Rightarrow

$$\left. \begin{array}{l} L_{\Omega w} \\ \beta_{\Omega w} \end{array} \right\} = \pm \frac{1}{2\pi} \sqrt{\frac{\Omega}{w}} \int_{-\infty}^{\infty} e^{\mp iwu + i\Omega\bar{u}} d\bar{u} = \pm \frac{1}{2\pi} \sqrt{\frac{\Omega}{w}} \int_{-\infty}^{\infty} e^{\mp iwu} (-au)^{-i\frac{\Omega}{a}-1} du$$

Now, using $\Gamma(r) = \int_0^\infty s^{r-1} e^{-s} ds$

$$\Rightarrow \left. \begin{array}{l} L_{\Omega w} \\ \beta_{\Omega w} \end{array} \right\} = \pm \frac{1}{2\pi a} \sqrt{\frac{\Omega}{w}} e^{\pm \frac{\pi\Omega}{2a}} e^{i\left(\frac{\Omega}{a} \ln \frac{w}{a}\right)} \Gamma\left(-i\frac{\Omega}{a}\right)$$

!

$$\Rightarrow \left. \begin{array}{l} \alpha_{\Omega\omega} \\ \beta_{\Omega\omega} \end{array} \right\} = \pm \frac{1}{2\pi a} \sqrt{\frac{\Omega}{\omega}} e^{\pm \frac{\pi\Omega}{2a}} e^{i\left(\frac{\Omega}{a} \ln \frac{\omega}{a}\right)} \Gamma\left(-\frac{i\Omega}{a}\right) !$$

Observation: $\Rightarrow |\beta_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}} |\alpha_{\Omega\omega}|^2 \quad (\text{D})$

So far acceleration $a \rightarrow 0$ we have $|\beta_{\Omega\omega}| \rightarrow 0$,
i.e. then no particles observed in travelers frame.

How many particles does an accelerated observer see if $a \neq 0$?

$$\langle 4_{\cdot\cdot} | \hat{N}_{\Omega} | 4_{\cdot\cdot} \rangle = \langle 0_m | \hat{N}_{\Omega} | 0_m \rangle$$

. . 1. 1 .

How many particles does an accelerated observer see if $a \neq 0$?

$$\langle \psi_{\omega} | \hat{N}_{\omega} | \psi_{\omega} \rangle = \langle 0_n | \hat{N}_{\omega} | 0_n \rangle$$

$$= \langle 0_n | \hat{b}_{\omega}^+ \hat{b}_{\omega}^- | 0_n \rangle$$

$$= \langle 0_n | \left(\int_0^{\infty} d\omega \alpha_{\omega}^* \hat{a}_{\omega}^+ - \beta_{\omega}^* \hat{a}_{\omega}^- d\omega \right) \left(\int_0^{\infty} d\omega' \alpha_{\omega'} \hat{a}_{\omega'}^+ - \beta_{\omega'} \hat{a}_{\omega'}^- d\omega' \right) | 0_n \rangle$$

$$= \int_0^{\infty} d\omega |\beta_{\omega}^*|^2$$

Using (C) and (D) \Rightarrow

∞

$$\beta_{\Omega\omega} \left. \right|_{-\infty}^{-2\pi/\omega}$$

$$\left. \right|_{-\infty}^{-2\pi/\omega}$$

Now, using $\Gamma(r) = \int_0^{\infty} s^{r-1} e^{-s} ds$

$$\Rightarrow \frac{L_{\Omega\omega}}{\beta_{\Omega\omega}} = \pm \frac{1}{2\pi a} \sqrt{\frac{\Omega}{\omega}} e^{\pm \frac{\pi\Omega}{2a}} e^{i\left(\frac{\Omega}{a} \ln \frac{\omega}{a}\right)} \Gamma\left(-\frac{i\Omega}{a}\right)$$

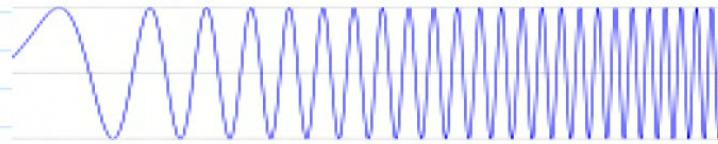
!

Observation: $\Rightarrow |\beta_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}} |L_{\Omega\omega}|^2 \quad (\text{D})$

So for acceleration $a \rightarrow 0$ we have $|\beta_{\Omega\omega}| \rightarrow 0$,
i.e. then no particles observed in traveler's frame.

How can we control the particle distribution?

Intuition: To the traveller, any monochromatic wave sounds like a chirp.



Exercise: Check that $[a_k, a_k^+] = \delta(k-k')$ and $[b_k, b_k^+] = \delta(k-k')$ imply:

$$\int_0^\infty d\omega (d_{\Omega\omega} d_{\Omega'\omega}^* - \beta_{\Omega\omega} \beta_{\Omega'\omega}^*) = \delta(\Omega - \Omega') \quad (C)$$

Calculation of $d_{\Omega\omega}$ and $\beta_{\Omega\omega}$: (lengthy, for more details, see

$$= \int_0^\infty d\omega |\beta_{\omega\Omega}|^2$$

Using (C) and (D) \Rightarrow

$$\langle q_i | \hat{N}_\Omega | q_i \rangle = \int_0^\infty d\omega |\beta_{\omega\Omega}|^2 = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1} \delta(\Omega - \omega) \quad \uparrow \text{Divergent}$$

Observation:

With infrared cut off through (accelerating) box of size V we have discrete k , discrete $\Omega(k)$ and $\delta(\Omega - \Omega')$ in (C) becomes $V \delta_{\Omega, \Omega'}$.

Then:

\Rightarrow Number density:

$$\bar{n}_\alpha := \frac{1}{V} \langle \hat{q}_\alpha | \hat{N}_\alpha | \hat{q}_\alpha \rangle = \frac{1}{e^{\frac{2\pi\alpha}{a}} - 1}$$

Compare: If a harmonic oscillator of energy levels $E_n = \Omega(n + \frac{1}{2})$ is exposed to a heat bath of temperature T , then its expected excitation number is

$$\bar{n} = \frac{1}{e^{\frac{\Omega}{kT}} - 1}$$

\Rightarrow The traveler's mode oscillators are excited as if exposed to a heat bath of the Unruh temperature:

$$T = \frac{a}{2\pi}$$

Observation:

- Could the quantum field be in the state $|0_R\rangle$?
- We'd expect that then inertial observers would see particles!
- But $|0_R\rangle$ is not a physically implementable state, even in principle! Why?
- $|0_R\rangle$ is a state with regions of diverging energy density!
Why? If $\hat{\phi}$ is in state $|0_R\rangle$ then, in accelerated cds, energy density is constant throughout that cds.
But this cds piles up at the horizons!

◻ $|O_R\rangle$ is a state with regions of diverging energy density!

Why? If $\hat{\phi}$ is in state $|O_R\rangle$ then, in accelerated cds,
energy density is constant throughout that cds.

But this cds piles up at the horizons!

Recall: $T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - \gamma_{\mu\nu}(\partial_\lambda \phi)(\partial^\lambda \phi)$

\Rightarrow Need to study terms of the form $\langle O_R | (\partial \phi)^2 | O_R \rangle$.

Calculate: $\langle O_1 | (\partial_- \hat{\phi})^2 | O_- \rangle = \langle O_0 | (\underline{\partial} \hat{\phi})^2 | O_- \rangle$

Recall: $T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - g_{\mu\nu}(\partial_\lambda \phi)(\partial^\lambda \phi)$

\Rightarrow Need to study terms of the form $\langle O_R | (\partial \phi)^2 | O_R \rangle$.

Calculate: $\langle O_R | (\partial_u \hat{\phi})^2 | O_R \rangle = \underbrace{\langle O_R | \left(\frac{\partial \bar{u}}{\partial u} \right)^2 (\partial_u \hat{\phi})^2 | O_R \rangle}_{\text{enters } \langle O_R | T_{\mu\nu}(u, v) | O_R \rangle}$

calculation of
inertial observer!

Recall: $u(\bar{u}, \bar{v}) = -\frac{1}{a} e^{-a\bar{u}}$

$$\Rightarrow \frac{du}{d\bar{u}} = -a u \quad \Rightarrow$$

Calculate: $\langle \phi_R | (\partial_{\bar{u}} \hat{\phi})^2 | \phi_R \rangle = \langle \phi_R | \left(\frac{\partial \bar{u}}{\partial u} \right)^2 (\partial_{\bar{u}} \hat{\phi})^2 | \phi_R \rangle$

enters $\langle \phi_R | T_{\mu\nu}(u, v) | \phi_R \rangle$
 calculation of
 inertial observer!

Recall: $u(\bar{u}, \bar{v}) = -\frac{1}{a} e^{-a\bar{u}}$

$$\Rightarrow \frac{du}{d\bar{u}} = -a u \Rightarrow$$

$$= \frac{1}{(au)^2} \langle \phi_R | (\partial_{\bar{u}} \hat{\phi})^2 | \phi_R \rangle$$

$$\Rightarrow \frac{du}{da} = -au \Rightarrow$$

$$= \frac{1}{(au)^2} \langle 0_R | (\partial_u \hat{\phi})^2 | 0_R \rangle$$

same b/c calculated exact same way from (A)

$$= \frac{1}{(au)^2} \langle 0_m | (\partial_u \hat{\phi})^2 | 0_m \rangle$$

Finite after renormalization.

But: $u' \rightarrow \infty$ at the traveler's horizon!

\Rightarrow In states $|4\rangle = |0_R\rangle$, or $|4\rangle = b_m^+ |0_R\rangle$ etc,

$$= \frac{1}{(au)^2} \langle 0_R | (\partial_u \hat{\phi})^2 | 0_R \rangle$$

same b/c calculated exact same way from (A)

$$= \frac{1}{(au)^2} \langle 0_M | (\partial_u \hat{\phi})^2 | 0_M \rangle$$

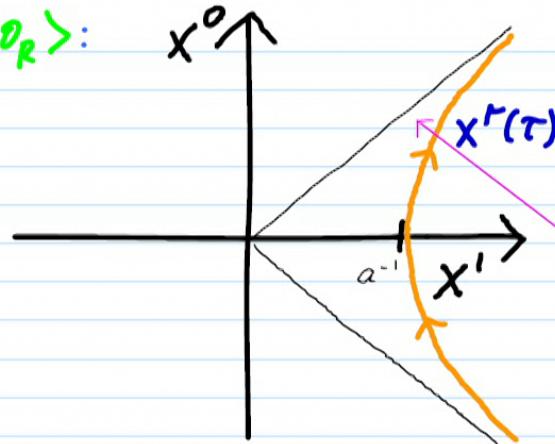
Final after renormalization.

But: $u^- \rightarrow \infty$ at the traveler's horizon!

\Rightarrow In states $|4\rangle = |0_R\rangle$, or $|4\rangle = b_{n_1}^+ |0_R\rangle$ etc,

$\langle 4 | T_{\mu\nu}(u, v) | 4 \rangle \rightarrow \infty$ as $u \rightarrow 0$ (future horizon)

If $\hat{\phi}$ is in state $|\phi_R\rangle$:



$$\langle \psi | T_{\mu\nu}(u, v) | \psi \rangle \rightarrow \infty$$

