Title: PSI 2017/2018 - Cosmology - Lecture 14

Date: Apr 26, 2018 10:15 AM

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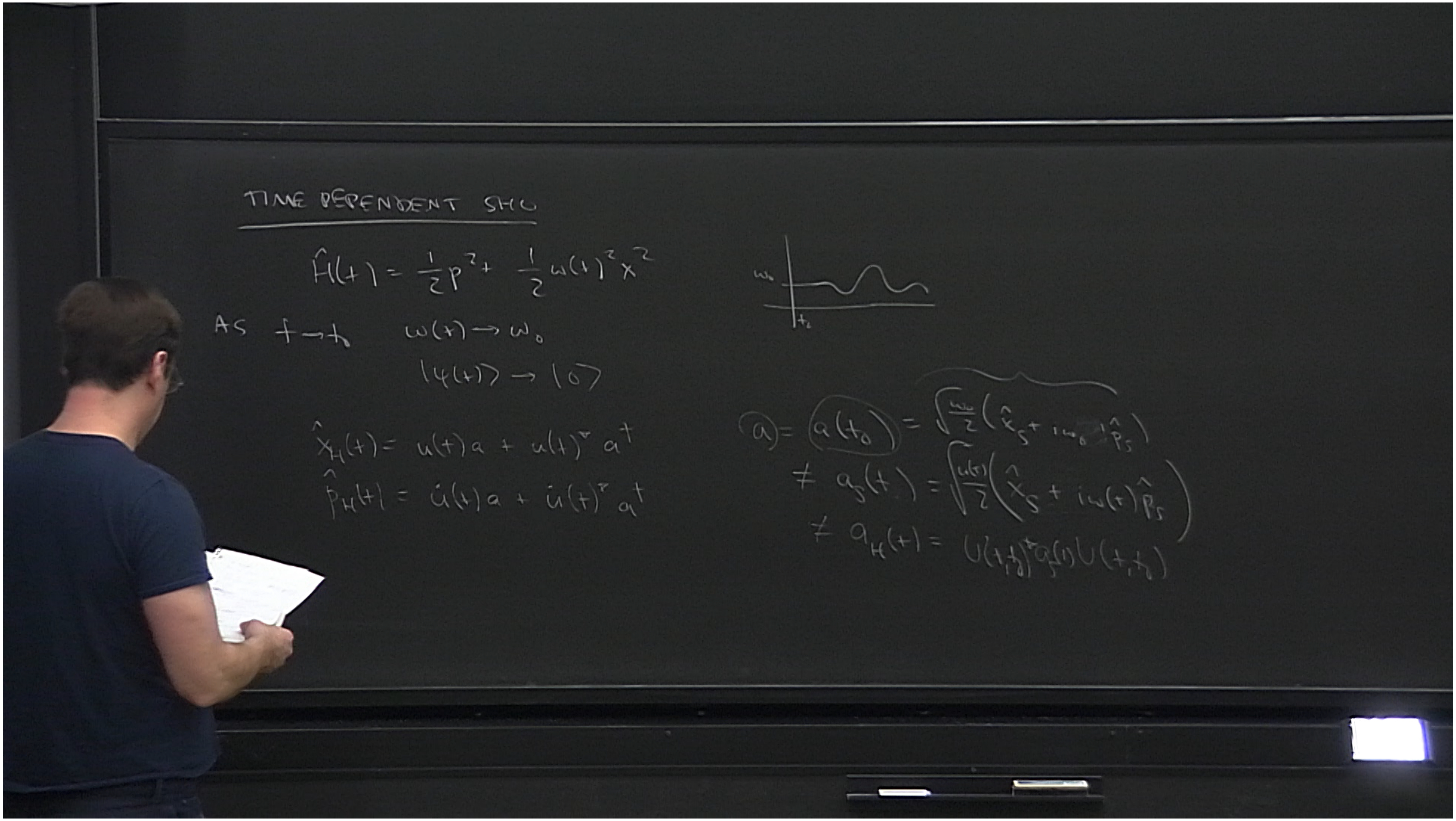
Abstract:


TIME DEPENDENT SHO
$\hat{H}(t) = \frac{1}{2}p^2 + \frac{1}{2}\omega(t)^2 x^2$
AS $t \to t_0$ $\omega(t) \to \omega_0$
$|\psi(t)\rangle \to |0\rangle$
$\hat{x}_H(t) = u(t)a + u(t)^* a^\dagger$
$\hat{p}_H(t) = \dot{u}(t)a + \dot{u}(t)^* a^\dagger$
ω_0
t_0
$a = a(t_0) = \sqrt{\frac{\omega_0}{2}}(\hat{x}_S + i\omega_0 \hat{p}_S)$
$\neq a_S(t) = \sqrt{\frac{\omega(t)}{2}}(\hat{x}_S + i\omega(t)\hat{p}_S)$
$\neq a_H(t) = U(t,t_0)^* a_S(t) U(t,t_0)$

TIME DEPENDENT SHO

$$\hat{H}(t) = \frac{1}{2}p^2 + \frac{1}{2}\omega(t)^2 x^2$$

AS $t \to t_0$ $\quad \omega(t) \to \omega_0$

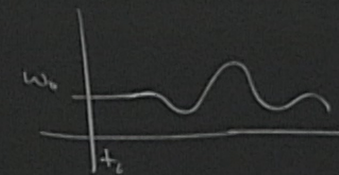
$$|\psi(t)\rangle \to |0\rangle$$

$$\hat{x}_H(t) = u(t)\,a + u(t)^* a^\dagger$$

$$\hat{p}_H(t) = \dot{u}(t)\,a + \dot{u}(t)^* a^\dagger$$

WHERE $\ddot{u}(t) = -\omega(t)^2 u(t)$

$$u(t) \to \frac{1}{(2\omega_0)^{1/2}} e^{-i\omega_0 t}$$



$$a = a(t_0) = \sqrt{\frac{\omega_0}{2}}\left(\hat{x}_S + i \ldots\right.$$

$$\neq a_S(t) = \sqrt{\frac{\omega(t)}{2}}\left(\hat{x} \ldots\right.$$

$$\neq a_H(t) = \ldots$$

$$\langle x^2(t)\rangle = \langle \psi(t) | \hat{x}_S^2 | \psi(t) \rangle$$

$$= \langle 0 | \hat{x}_H(t)^2 | 0 \rangle$$

$$= \langle 0 | \left(u(t)\, a + u^*(t)\, \cancel{a^\dagger} \right) \left(u(t)\, \cancel{a} + u^*(t)\, a^\dagger \right) | 0 \rangle$$

$$= u(t)\, u(t)^* \underbrace{\langle 0 | a a^\dagger | 0 \rangle}_{=1}$$

$$= |u(t)|^2$$

TIME DEPENDENT SHO

$$\hat H(t) = \frac{1}{2}p^2 + \frac{1}{2}\omega(t)^2 x^2$$

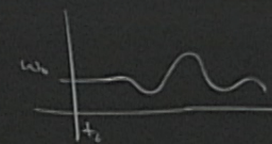
AS $t \to t_0$ $\quad \omega(t) \to \omega_0$

$$|\psi(t)\rangle \to |0\rangle$$

$$\hat x_H(t) = u(t)\, a + u(t)^* a^\dagger$$
$$\hat p_H(t) = \dot u(t)\, a + \dot u(t)^* a^\dagger$$

WHERE $\ddot u(t) = -\omega(t)^2 u(t)$

$$u(t) \to \frac{1}{(2\omega_0)^{1/2}} e^{-i\omega_0 t}$$



$$a = a(t_0) = \sqrt{\frac{\omega_0}{2}}\left(\hat x_S + i\omega_0^{-1}\hat p_S\right)$$
$$\neq a_S(t) = \sqrt{\frac{\omega(t)}{2}}\left(\hat x_S + i\omega(t)\hat p_S\right)$$
$$\neq a_H(t) = U(t,t_0)^\dagger a_S(t)\, U(t,t_0)$$

$\langle x^2(t)\rangle = \langle$

$=$

$=$

$=$

$=$

$$\langle x^2(t)\rangle = \langle \psi(t)|\, \hat{x}_S^2 \,|\psi(t)\rangle$$

$$= \langle 0|\, \hat{x}_H(t)^2 \,|0\rangle$$

$$= \langle 0|\, \big(u(t)a + u^*(t)a^\dagger\big)\big(u(t)a + u^*(t)a^\dagger\big) \,|0\rangle \qquad [a, a^\dagger] = 1$$

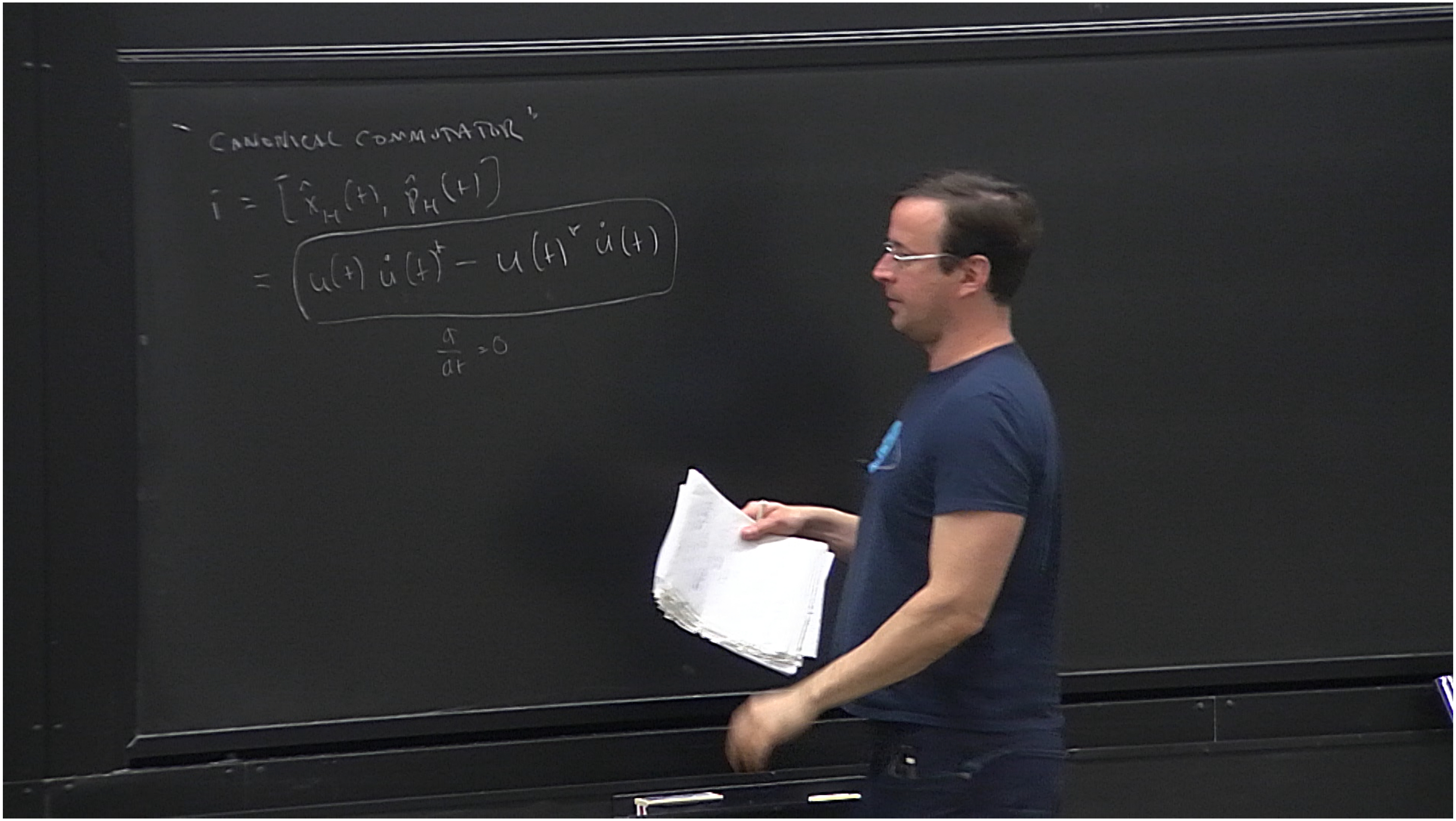
$$= u(t)\,u(t)^* \underbrace{\langle 0|\, a a^\dagger \,|0\rangle}_{=1}$$

$$= |u(t)|^2$$

- CANONICAL COMMUTATOR:

$$i = [\hat{x}_H(t), \hat{p}_H(t)]$$

$$= u(t)\,\dot{u}(t)^* - u(t)^*\,\dot{u}(t)$$


"CANONICAL COMMUTATOR"
$i = [\hat{x}_H(t), \hat{p}_H(t)]$
$= u(t)\,\dot{u}(t)^* - u(t)^*\,\dot{u}(t)$
$\frac{d}{dt} = 0$

- CANONICAL COMMUTATOR:

$$i = [\hat{x}_H(t), \hat{p}_H(t)]$$

$$i = u(t)\,\dot{u}(t)^* - u(t)^*\,\dot{u}(t)$$

$$\begin{pmatrix} \hat{x}_H(t) \\ \hat{p}_H(t) \end{pmatrix} = \begin{pmatrix} u(t) & u(t)^* \\ \dot{u}(t) & \dot{u}(t)^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

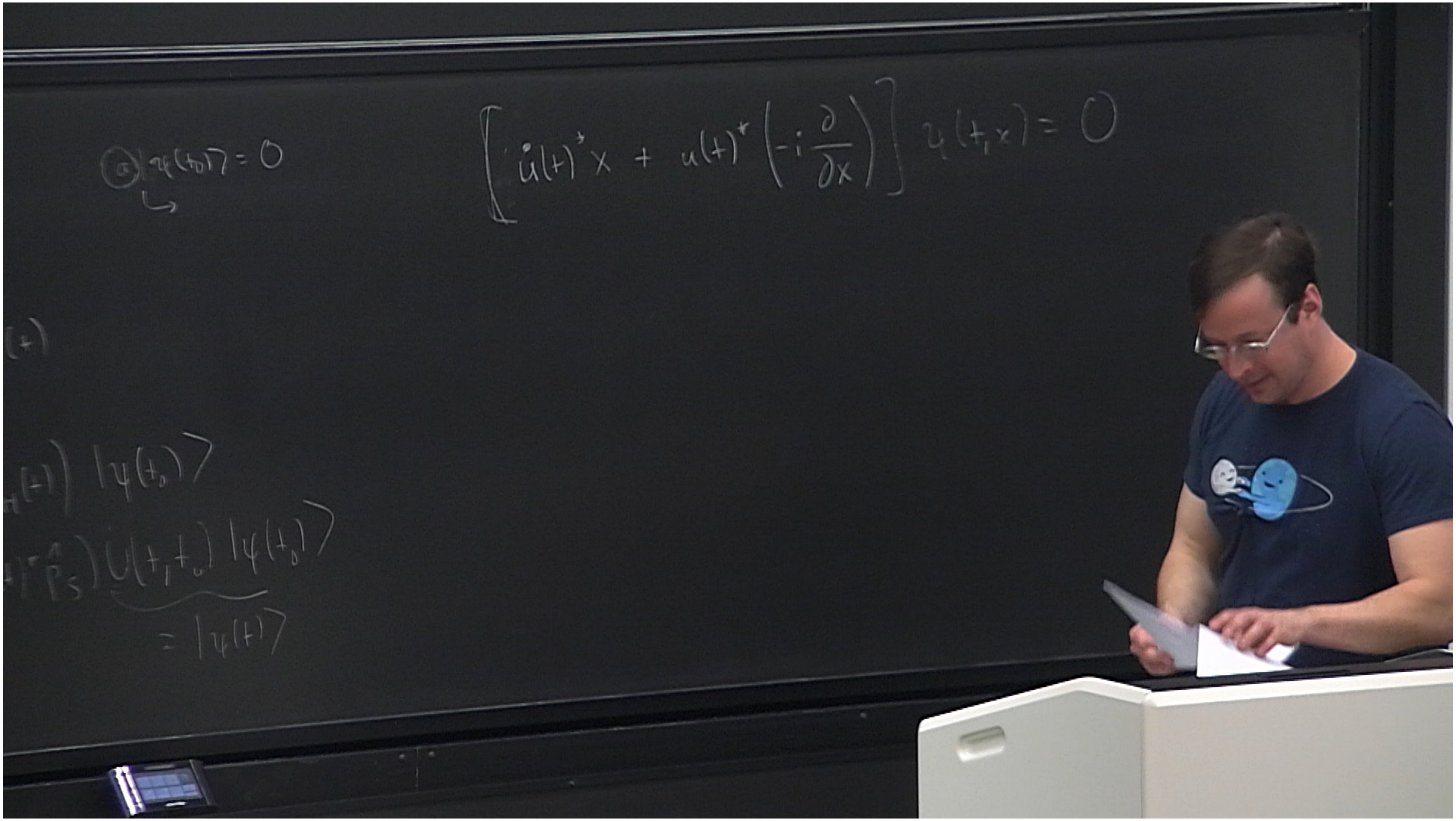
$$\begin{pmatrix} \hat{x}_H(t) \\ \hat{p}_H(t) \end{pmatrix} = \begin{pmatrix} u(t) & u(t)^* \\ \dot{u}(t) & \dot{u}(t)^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

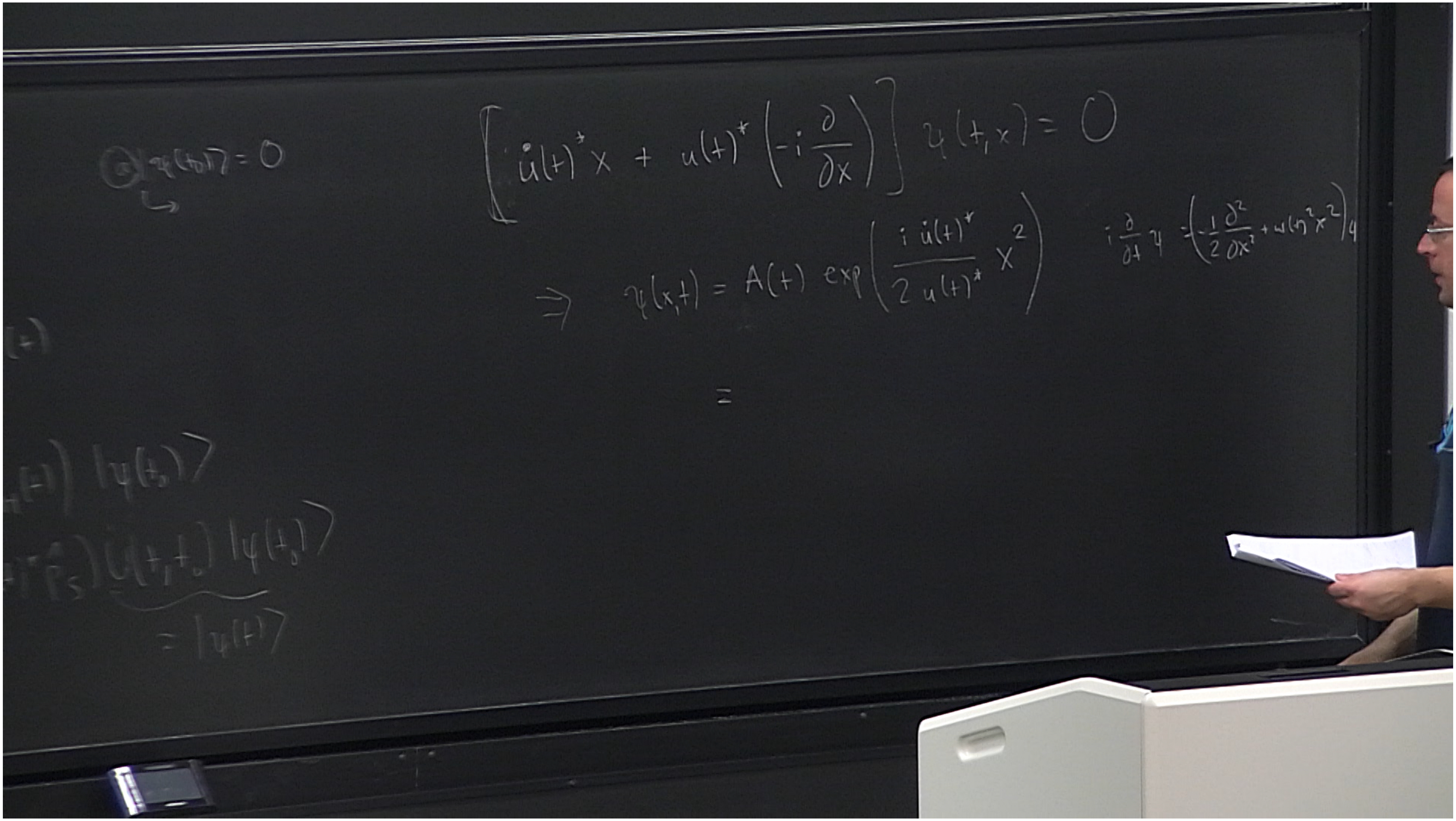
$$a|\psi(t_0)\rangle = 0$$

INVERT $\Longrightarrow$

$$a = \dot{u}(t)^* \hat{x}_H(t) + u(t)^* \hat{p}_H(t)$$

$$\begin{aligned} 0 &= a|\psi(t_0)\rangle \\ &= \left(\dot{u}(t)^* \hat{x}_H(t) + u(t)^* \hat{p}_H(t)\right)|\psi(t_0)\rangle \\ &= U(t,t_0)^\dagger \left(\dot{u}(t)^* \hat{x}_S + u(t)^* \hat{p}_S\right) \underbrace{U(t,t_0)|\psi(t_0)\rangle}_{=|\psi(t)\rangle} \end{aligned}$$


$\left[\dot{u}(t)^* x + u(t)^* \left(-i \frac{\partial}{\partial x}\right)\right] \psi(t,x) = 0$
$a|\psi(t_0)\rangle = 0$
$= |\psi(t)\rangle$


$\left[i\,\dot{u}(t)^* x + u(t)^* \left(-i \frac{\partial}{\partial x}\right) \right] \psi(t,x) = 0$
$i \frac{\partial}{\partial t} \psi = \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + \omega(t)^2 x^2\right) \psi$
$\Rightarrow \psi(x,t) = A(t) \exp\left(\frac{i\,\dot{u}(t)^*}{2\,u(t)^*} x^2 \right)$
$=$
$= |\psi(t)\rangle$

$$\left[\dot{u}(t)^* x + u(t)^*\left(-i\frac{\partial}{\partial x}\right)\right]\psi(t,x) = 0$$

$$i\frac{\partial}{\partial t}\psi = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \omega(t)^2 x^2\right)\psi$$

$$\Rightarrow \quad \psi(x,t) = A(t)\,\exp\left(\frac{i\,\dot{u}(t)^*}{2\,u(t)^*}\,x^2\right)$$

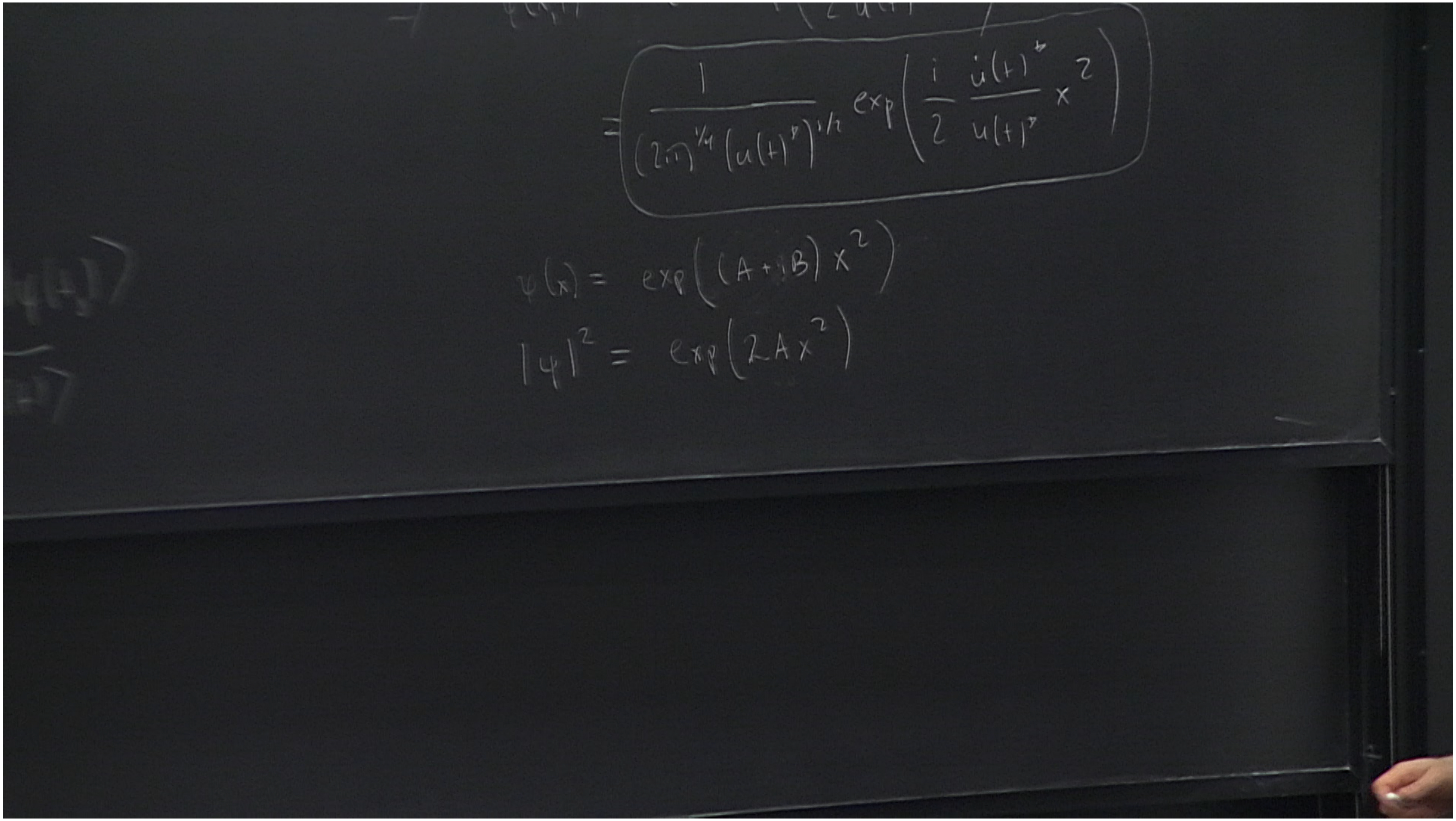
$$= \frac{1}{(2\pi)^{1/4}\left(u(t)^*\right)^{1/2}}\exp\left(\frac{i}{2}\,\frac{\dot{u}(t)^*}{u(t)^*}\,x^2\right)$$

$$\left[\dot{u}(t)^* x + u(t)^*\left(-i\frac{\partial}{\partial x}\right)\right]\psi(t,x) = 0$$

$$i\frac{\partial}{\partial t}\psi = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + \omega(t)^2 x^2\right)\psi$$

$$\Rightarrow \quad \psi(x,t) = A(t)\exp\left(\frac{i\,\dot{u}(t)^*}{2\,u(t)^*}x^2\right)$$

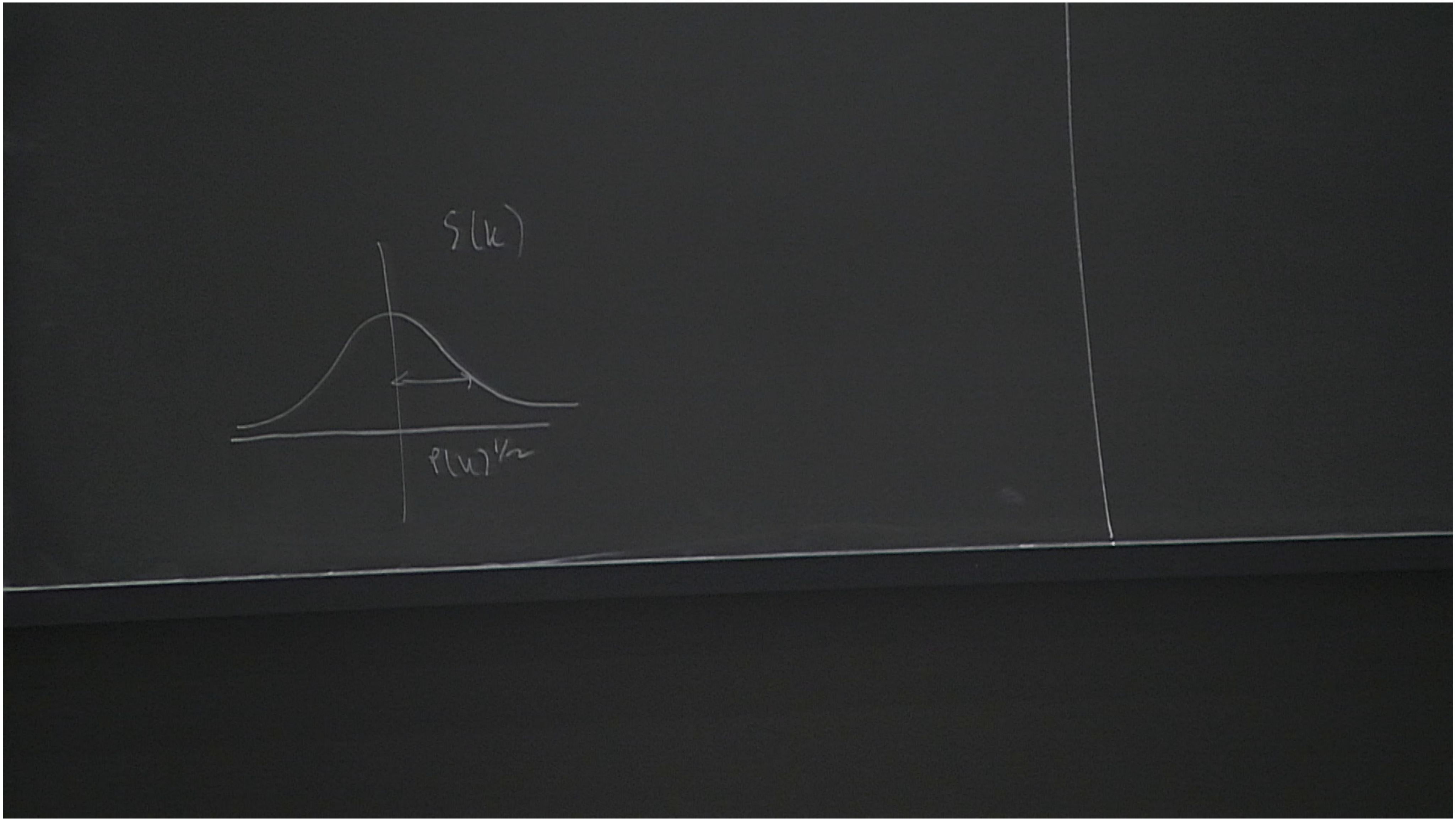
$$= \boxed{\frac{1}{(2\pi)^{1/4}(u(t)^*)^{1/2}}\exp\left(\frac{i}{2}\frac{\dot{u}(t)^*}{u(t)^*}x^2\right)}$$

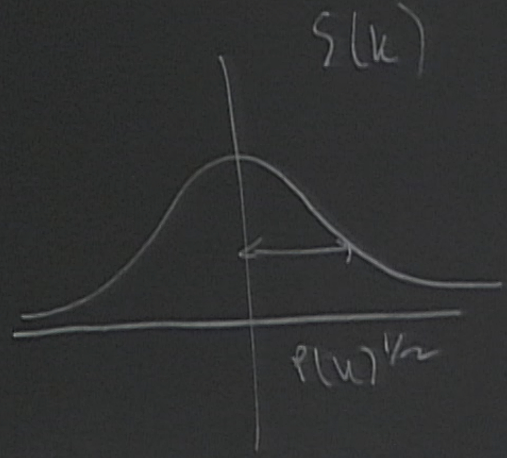

$= \dfrac{1}{(2\pi)^{1/4}(u(t)^*)^{1/2}} \exp\left(\dfrac{i}{2}\dfrac{\dot{u}(t)^*}{u(t)^*}x^2\right)$
$\psi(x) = \exp\left((A+iB)x^2\right)$
$|\psi|^2 = \exp\left(2Ax^2\right)$

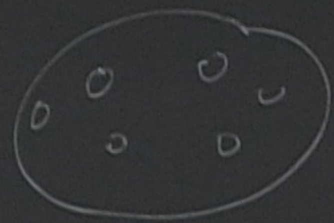
$$= \boxed{\frac{1}{(2\pi)^{1/4}\left(u(t)^*\right)^{1/2}} \exp\left(\frac{i}{2}\,\frac{\dot{u}(t)^*}{u(t)^*}\,x^2\right)}$$

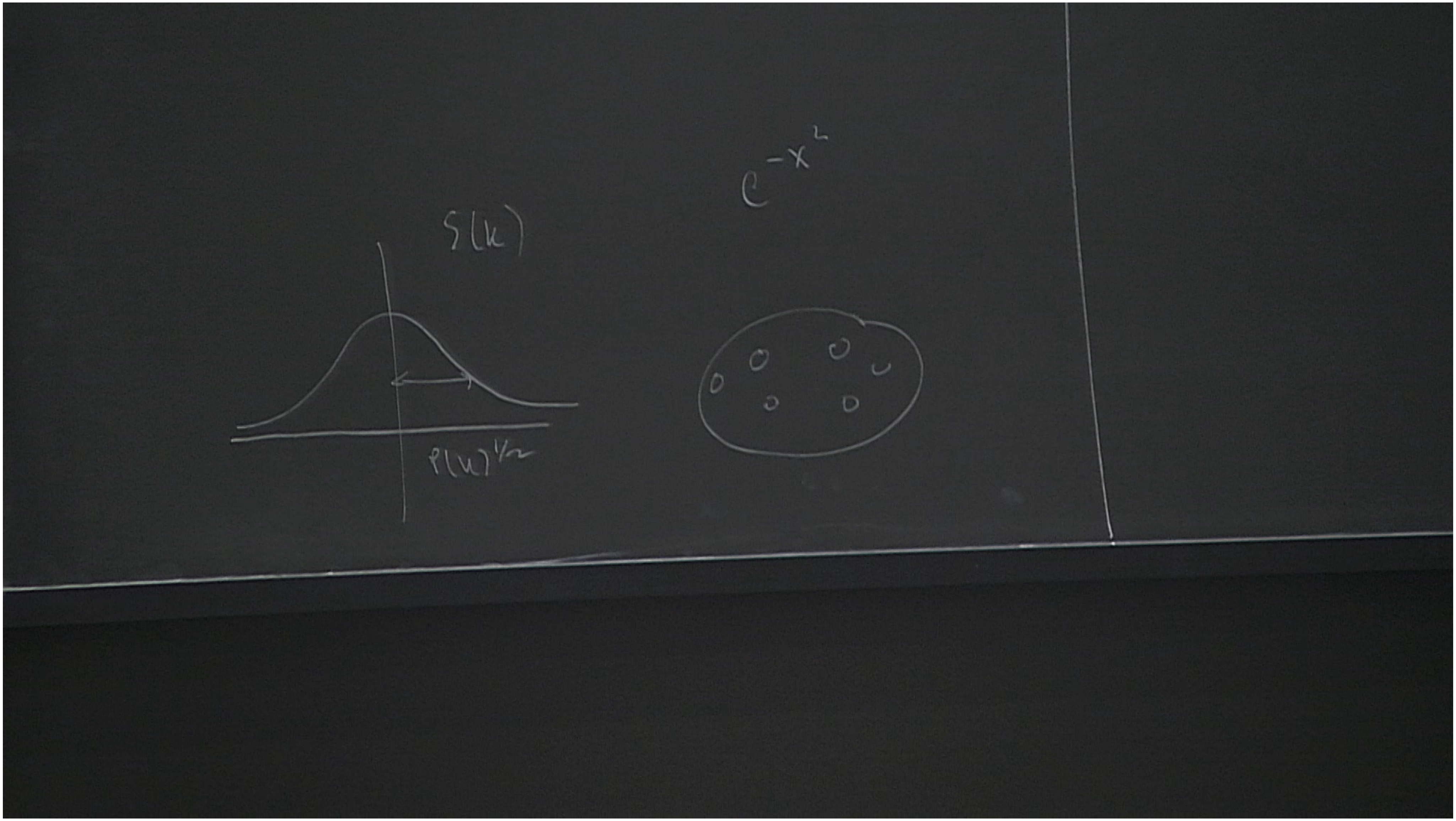
$$\psi(x) = \exp\left((A+iB)\,x^2\right)$$

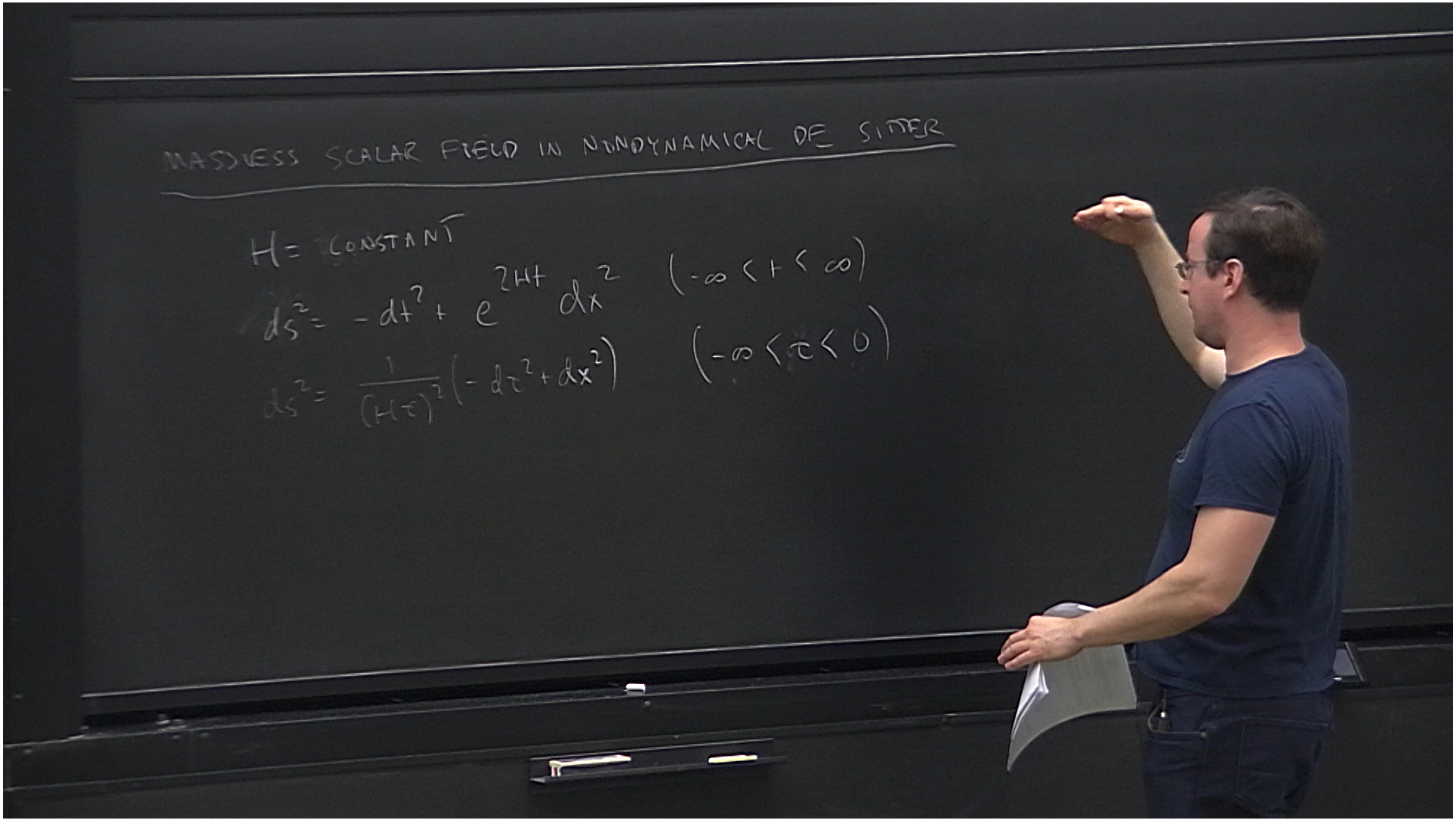
$$|\psi|^2 = \exp\left(2Ax^2\right) \qquad \text{GAUSSIAN!}$$


S(k)
ρ(k)^{1/2}


S(k)
ρ(k)^{1/2}




S(k)
e^{-x^2}
P(k)^{1/2}


MASSLESS SCALAR FIELD IN NONDYNAMICAL DE SITTER
H = CONSTANT
$ds^2 = -dt^2 + e^{2Ht} dx^2 \quad (-\infty < t < \infty)$
$ds^2 = \frac{1}{(H\tau)^2}(-d\tau^2 + dx^2) \quad (-\infty < \tau < 0)$

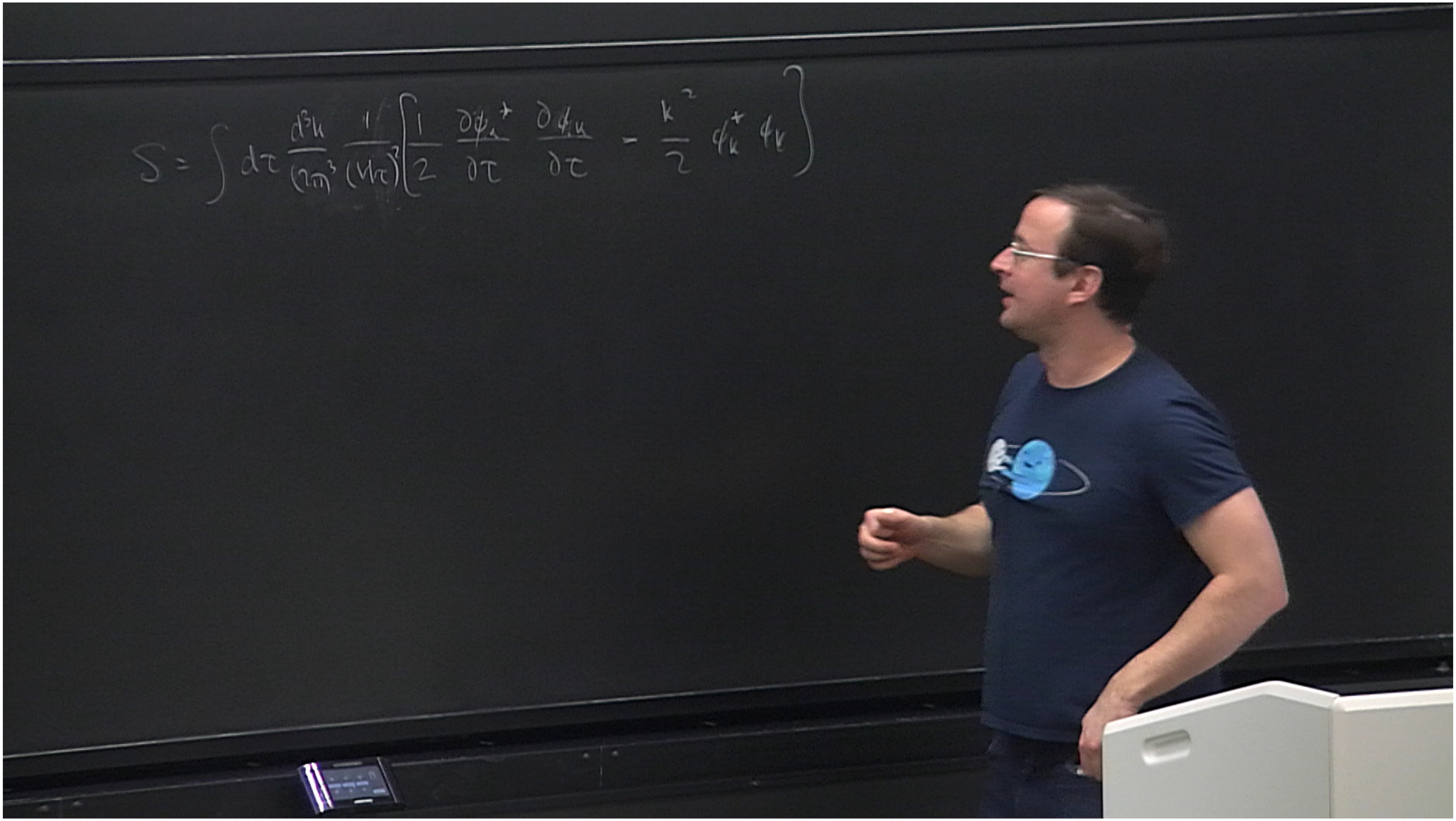
MASSLESS SCALAR FIELD IN NONDYNAMICAL DE SITTER

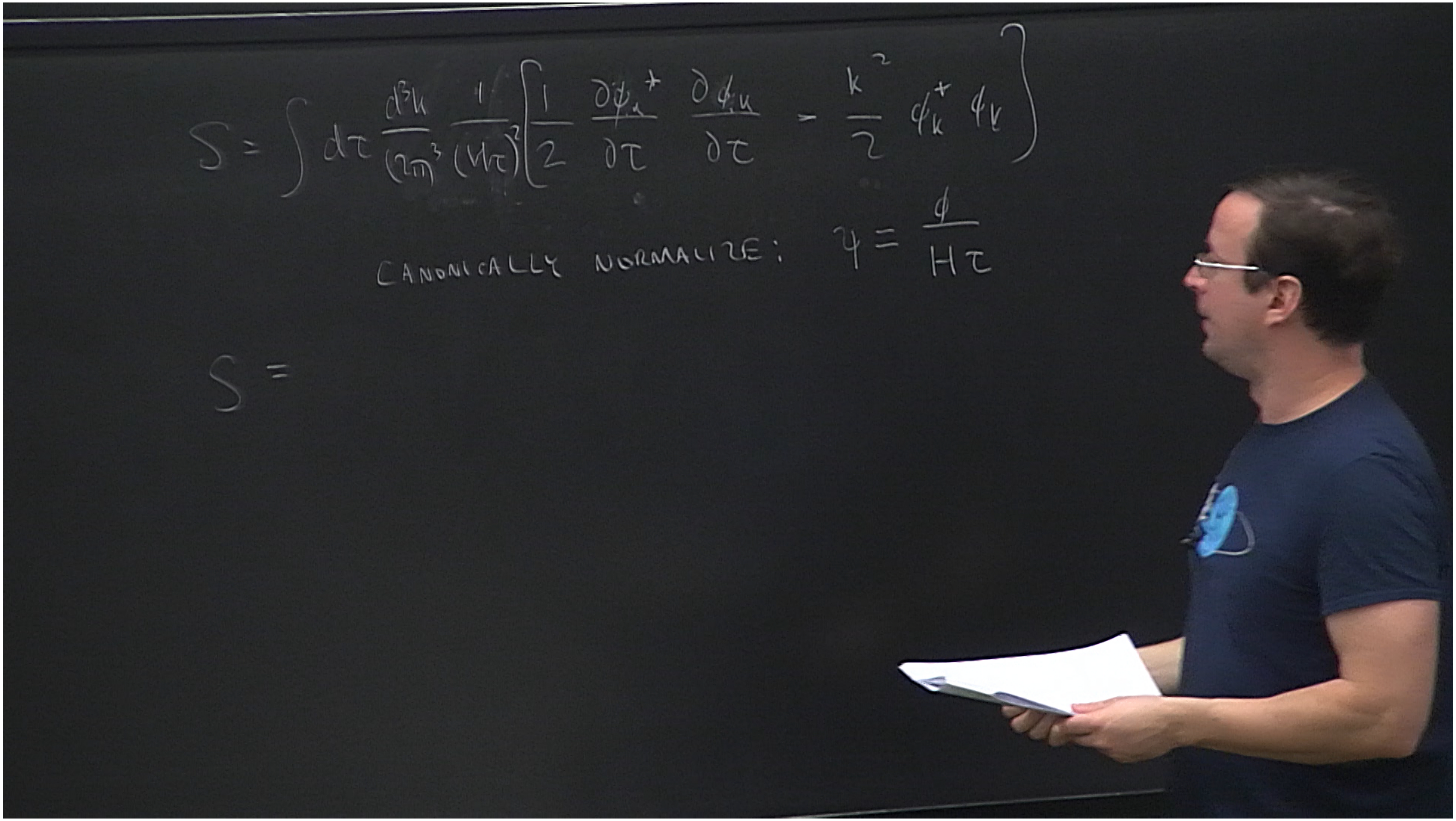
$H =$ CONSTANT

$$ds^2 = -dt^2 + e^{2Ht}\, dx^2 \qquad (-\infty < t < \infty)$$

$$ds^2 = \frac{1}{(H\tau)^2}\left(-d\tau^2 + dx^2\right) \qquad (-\infty < \tau < 0)$$

$$S = \int d^4x \sqrt{-g}\left(-\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi)\right)$$


$S = \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{(H\tau)^2} \left[\frac{1}{2} \frac{\partial \phi_k^*}{\partial \tau} \frac{\partial \phi_k}{\partial \tau} - \frac{k^2}{2} \phi_k^* \phi_k \right]$


$S = \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{(H\tau)^2} \left[\frac{1}{2} \frac{\partial \phi_k^*}{\partial \tau} \frac{\partial \phi_k}{\partial \tau} - \frac{k^2}{2} \phi_k^* \phi_k \right]$
CANONICALLY NORMALIZE: $\psi = \frac{\phi}{H\tau}$
$S =$

$$S = \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{(H\tau)^2} \left[\frac{1}{2} \frac{\partial \phi_k^*}{\partial \tau} \frac{\partial \phi_k}{\partial \tau} - \frac{k^2}{2} \phi_k^* \phi_k \right]$$

CANONICALLY NORMALIZE: $\psi = \frac{\phi}{H\tau}$

$$S = \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{(H\tau)^2} \left[\frac{1}{2} \left(H\tau \frac{\partial \psi_k^*}{\partial \tau} + H \psi_k^* \right) \left(H\tau \frac{\partial \psi_k}{\partial \tau} + H \psi_k \right) - \frac{k^2}{2} (H\tau)^2 \psi_k^* \psi_k \right]$$

$$= \int d\tau \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2} \left(\frac{\partial \psi_k}{\partial \tau} \right)^* \left(\frac{\partial \psi_k}{\partial \tau} \right) + \frac{1}{2\tau} \left(\frac{\partial \psi_k^*}{\partial \tau} \psi_k + \psi_k^* \frac{\partial \psi_k}{\partial \tau} \right) - \frac{1}{2} \left(k^2 - \frac{1}{\tau} \right) \psi_k^* \psi_k \right]$$

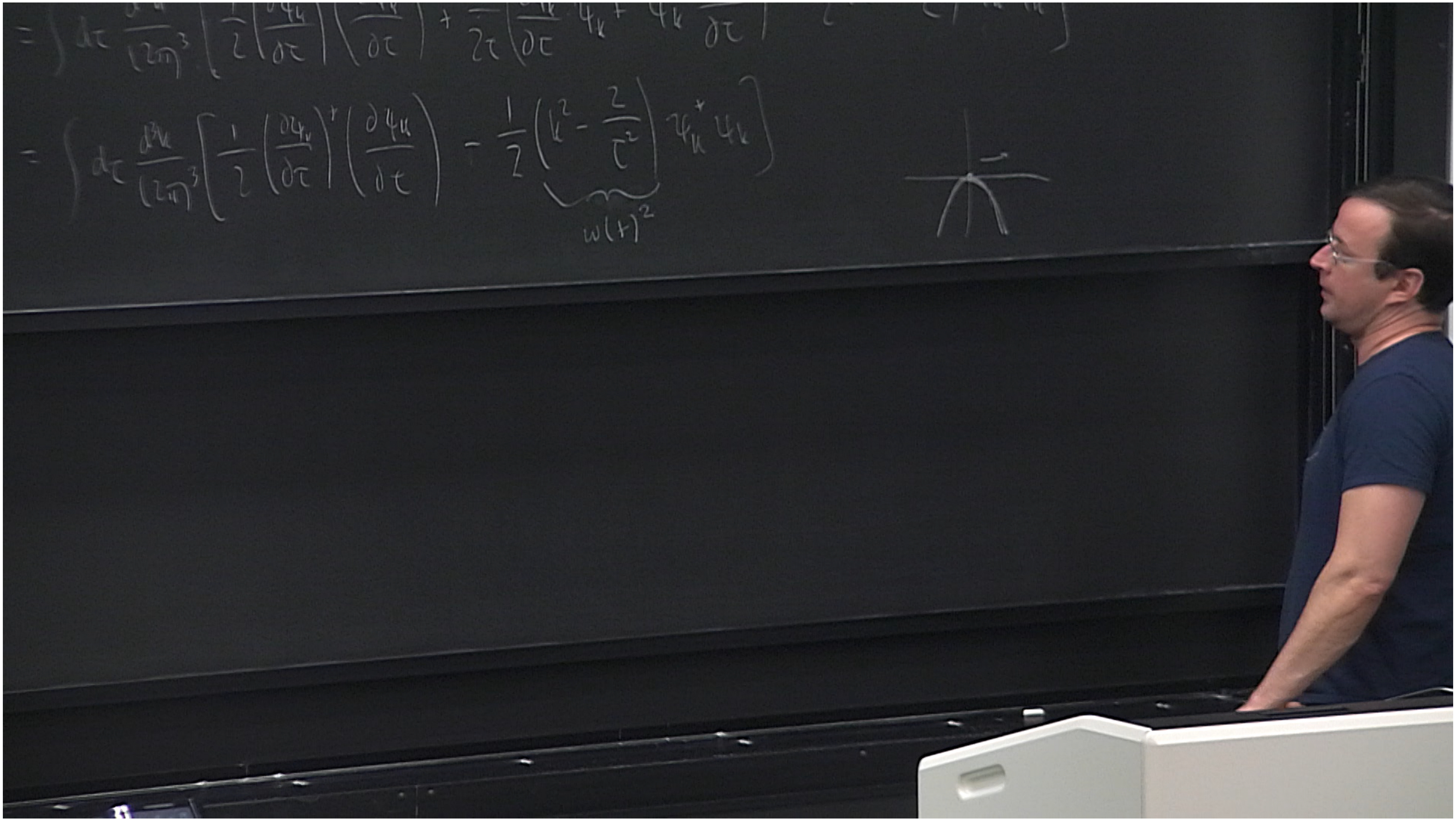
$$S = \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{(H\tau)^2}\left[\frac{1}{2} \frac{\partial \phi_k^*}{\partial \tau} \frac{\partial \phi_k}{\partial \tau} - \frac{k^2}{2} \phi_k^* \phi_k\right]$$

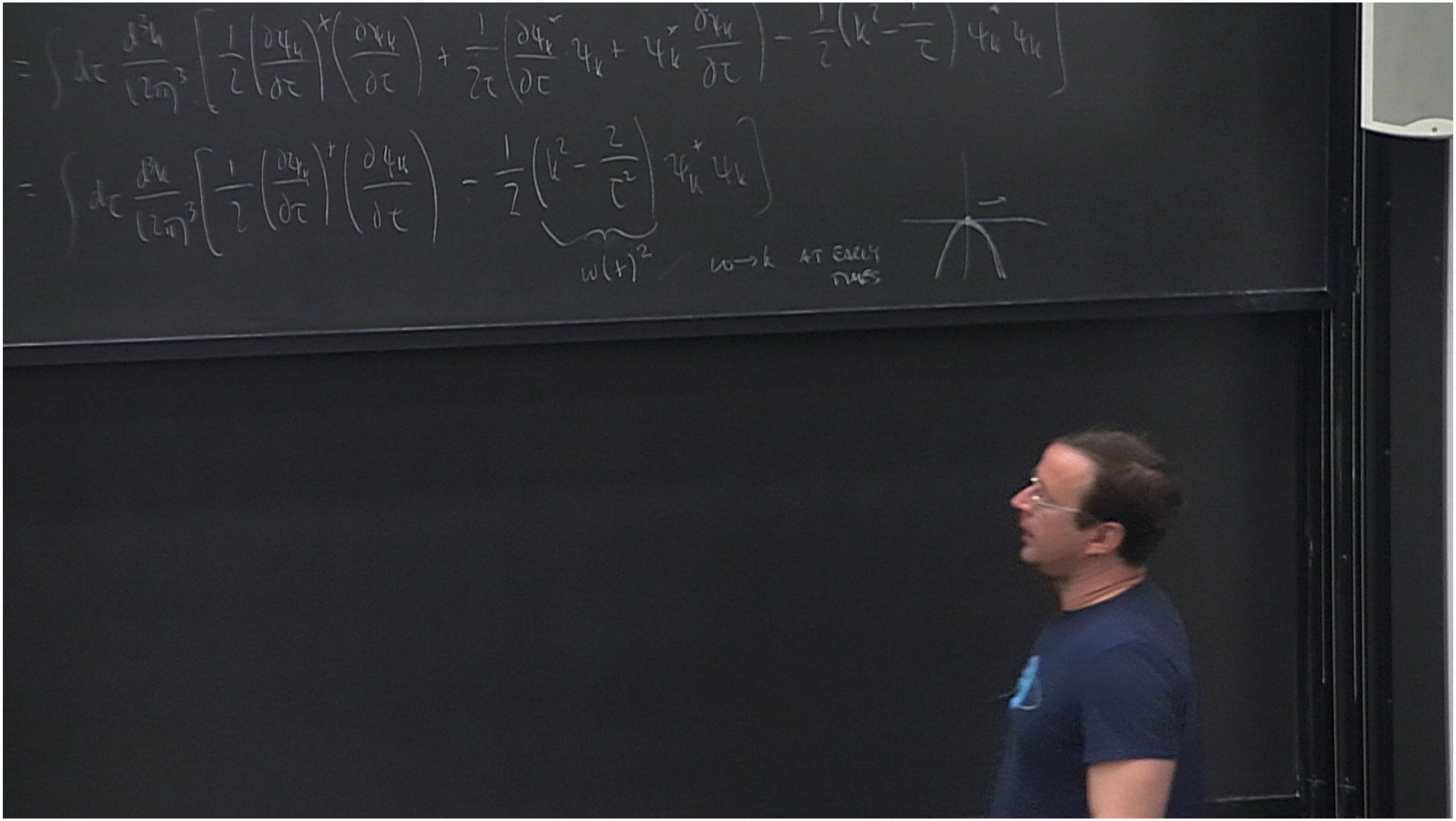
CANONICALLY NORMALIZE: $\psi = \dfrac{\phi}{H\tau}$

$$S = \int d\tau \frac{d^3k}{(2\pi)^3} \frac{1}{(H\tau)^2}\left[\frac{1}{2}\left(H\tau \frac{\partial \psi_k^*}{\partial \tau} + H\psi_k^*\right)\left(H\tau \frac{\partial \psi_k}{\partial \tau} + H\psi_k\right) - \frac{k^2}{2}(H\tau)^2 \psi_k^* \psi_k\right]$$

$$= \int d\tau \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\left(\frac{\partial \psi_k}{\partial \tau}\right)^*\left(\frac{\partial \psi_k}{\partial \tau}\right) + \frac{1}{2\tau}\left(\frac{\partial \psi_k^*}{\partial \tau}\psi_k + \psi_k^* \frac{\partial \psi_k}{\partial \tau}\right) - \frac{1}{2}\left(k^2 - \frac{1}{\tau}\right)\psi_k^* \psi_k\right]$$

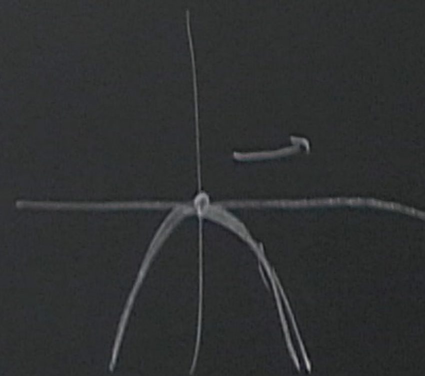
$$= \int d\tau \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\left(\frac{\partial \psi_k}{\partial \tau}\right)^*\left(\frac{\partial \psi_k}{\partial \tau}\right) - \frac{1}{2}\underbrace{\left(k^2 - \frac{2}{\tau^2}\right)}_{\omega(t)^2}\psi_k^* \psi_k\right]$$


$= \int d\tau \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2}\left(\frac{\partial\psi_k}{\partial\tau}\right)^*\left(\frac{\partial\psi_k}{\partial\tau}\right) - \frac{1}{2}\underbrace{\left(k^2-\frac{2}{\tau^2}\right)}_{\omega(t)^2}\psi_k^*\psi_k\right]$


$= \int d\tau \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2} \left(\frac{\partial \psi_k}{\partial \tau}\right)^* \left(\frac{\partial \psi_k}{\partial \tau}\right) + \frac{1}{2\tau} \left(\frac{\partial \psi_k^*}{\partial \tau} \psi_k + \psi_k^* \frac{\partial \psi_k}{\partial \tau}\right) - \frac{1}{2} \left(k^2 - \frac{1}{\tau}\right) \psi_k^* \psi_k \right]$
$= \int d\tau \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2} \left(\frac{\partial \psi_k}{\partial \tau}\right)^* \left(\frac{\partial \psi_k}{\partial \tau}\right) - \frac{1}{2} \underbrace{\left(k^2 - \frac{2}{\tau^2}\right)}_{\omega(t)^2} \psi_k^* \psi_k \right]$
$\omega \to k$ AT EARLY TIMES

$$\left.\frac{\psi_k}{\tau}\right) - \frac{1}{2}\underbrace{\left(k^2 - \frac{2}{\tau^2}\right)}_{\omega(t)^2}\psi_k^*\psi_k\Big]$$

$\omega \rightarrow k$ AT EARLY TIMES



STARTING IN VACUUM STATE

$$\hat{\psi}^H_k(\tau) = u_\psi(k,\tau)\, a_k + u_\psi(k,\tau)^* a^\dagger_{-k}$$

WHERE $\ddot{u}_\psi(k,\tau) = -\left(k^2 - \frac{2}{\tau^2}\right) u_\psi(k,\tau)$

AND $u_\psi(k,\tau) \to \frac{1}{(2k)^{1/2}} e^{-ik\tau}$ AS $\tau \to -\infty$

STARTING IN VACUUM STATE

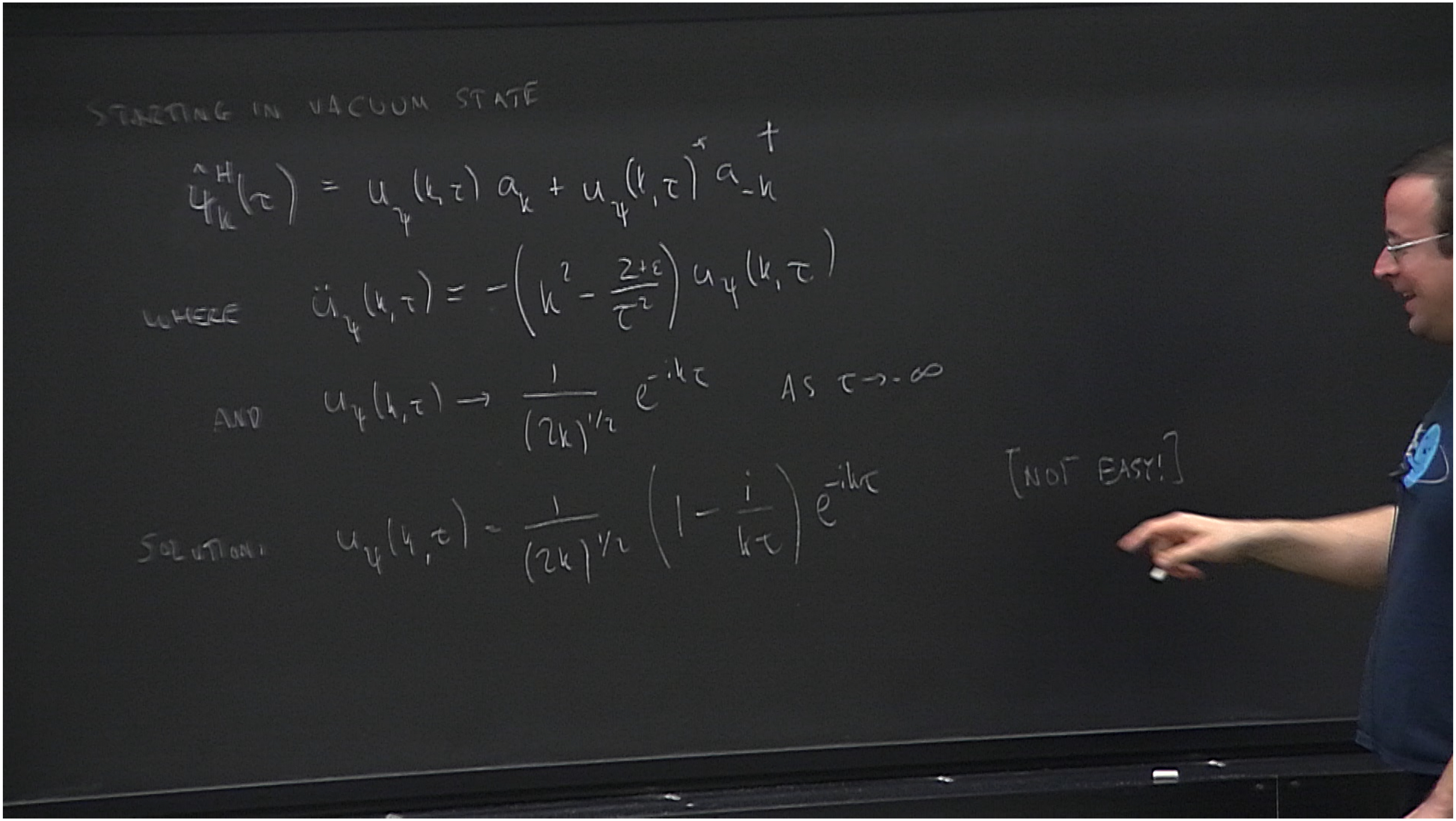
$$\hat{\psi}^{H}_{k}(\tau) = u_{\psi}(k,\tau)\, a_{k} + u_{\psi}(k,\tau)^{*}\, a^{\dagger}_{-k}$$

WHERE $$\ddot{u}_{\psi}(k,\tau) = -\left(k^{2} - \frac{2}{\tau^{2}}\right) u_{\psi}(k,\tau)$$

AND $$u_{\psi}(k,\tau) \to \frac{1}{(2k)^{1/2}}\, e^{-ik\tau} \quad \text{AS } \tau \to -\infty$$

[NOT EASY!]

SOLUTION: $$u_{\psi}(k,\tau) = \frac{1}{(2k)^{1/2}}\left(1 - \frac{i}{k\tau}\right) e^{-ik\tau}$$


STARTING IN VACUUM STATE
$\hat{\psi}^H_k(\tau) = u_\psi(k,\tau)\, a_k + u_\psi(k,\tau)^* a^\dagger_{-k}$
WHERE $\ddot{u}_\psi(k,\tau) = -\left(k^2 - \frac{2+\epsilon}{\tau^2}\right) u_\psi(k,\tau)$
AND $u_\psi(k,\tau) \to \frac{1}{(2k)^{1/2}} e^{-ik\tau}$ AS $\tau \to -\infty$
[NOT EASY!]
SOLUTION: $u_\psi(k,\tau) = \frac{1}{(2k)^{1/2}} \left(1 - \frac{i}{k\tau}\right) e^{-ik\tau}$

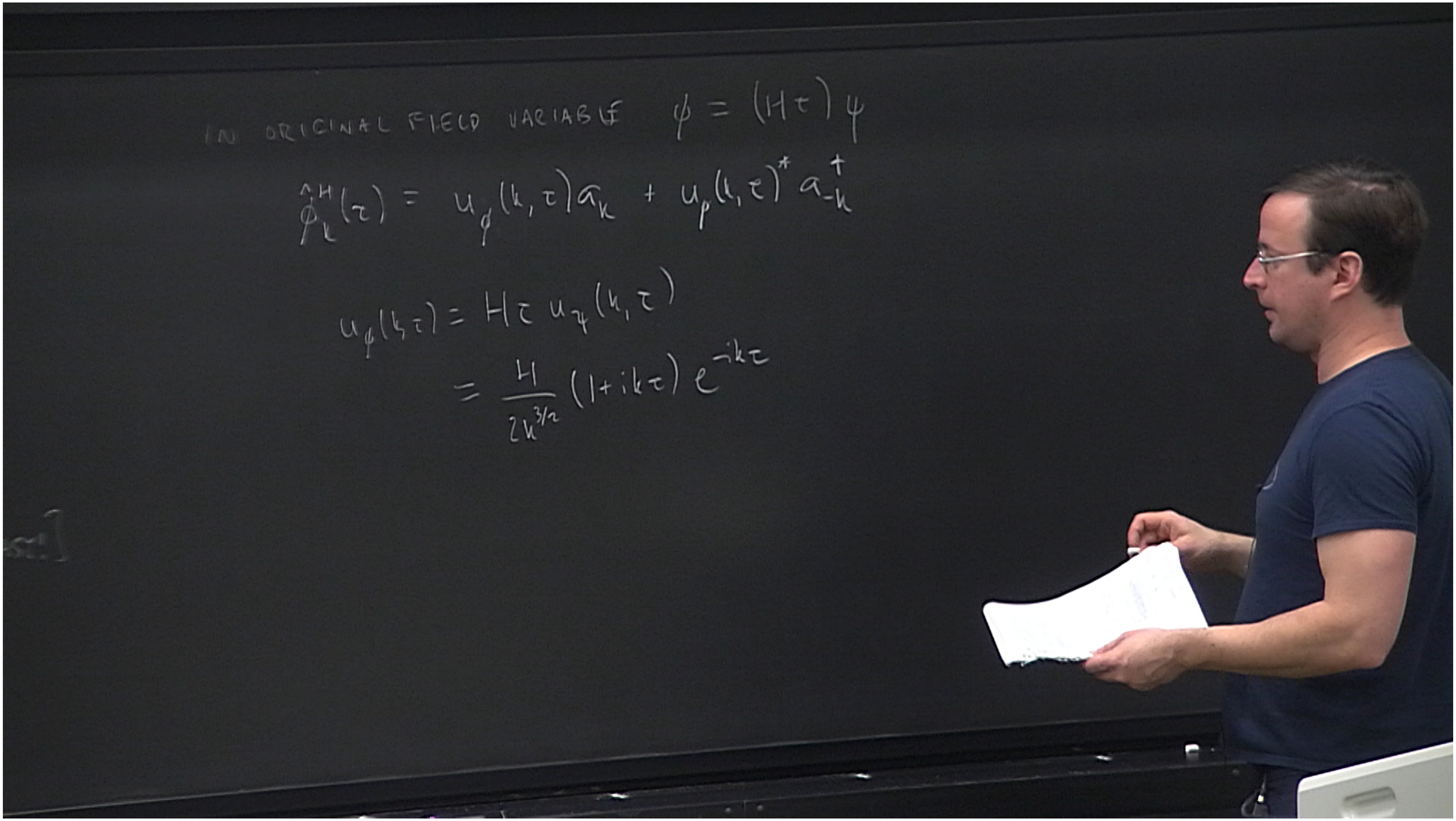
STARTING IN VACUUM STATE

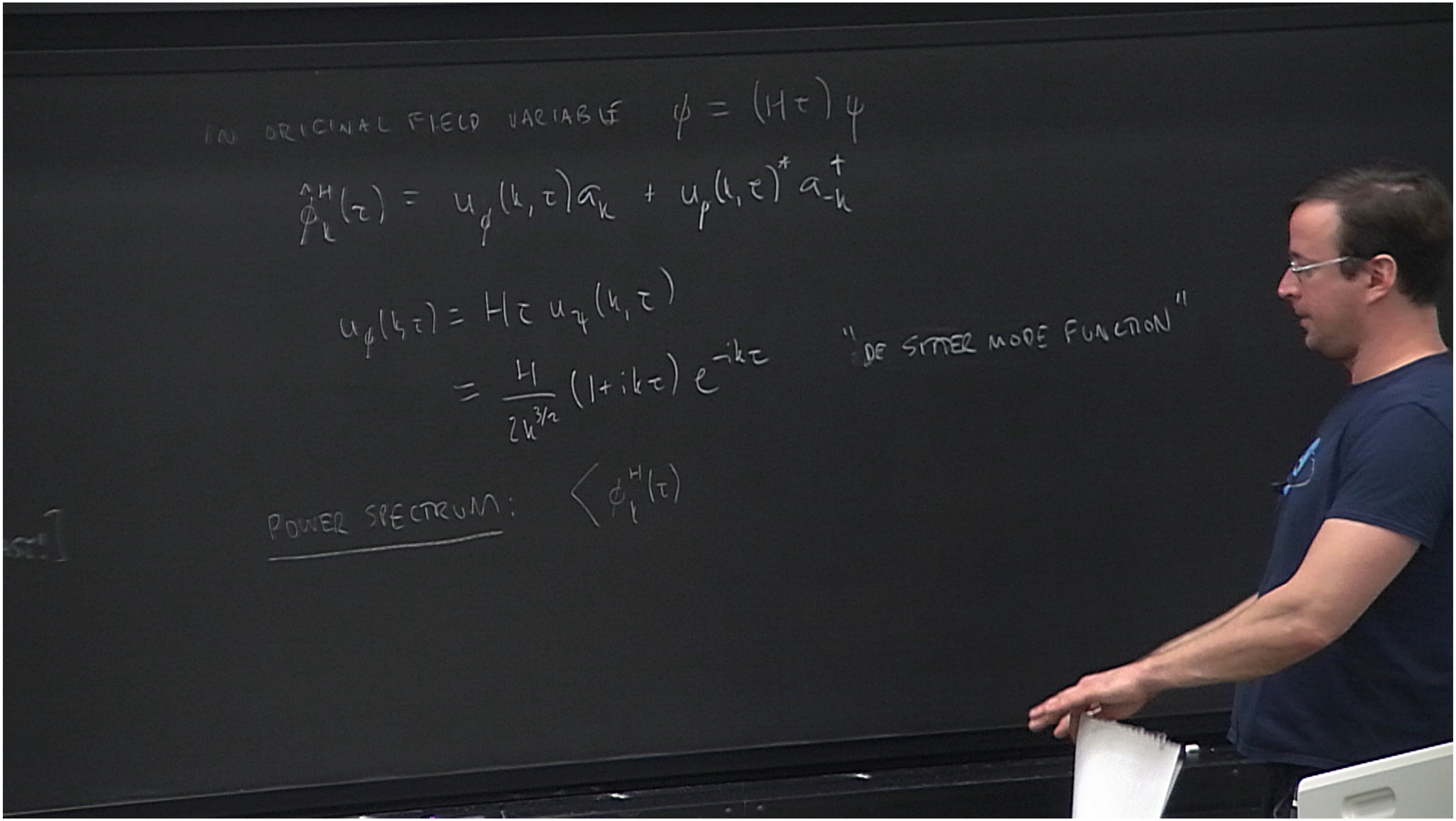
$$\hat{\psi}^H_k(\tau) = u_\psi(k,\tau)\, a_k + u_\psi(k,\tau)^*\, a^\dagger_{-k}$$

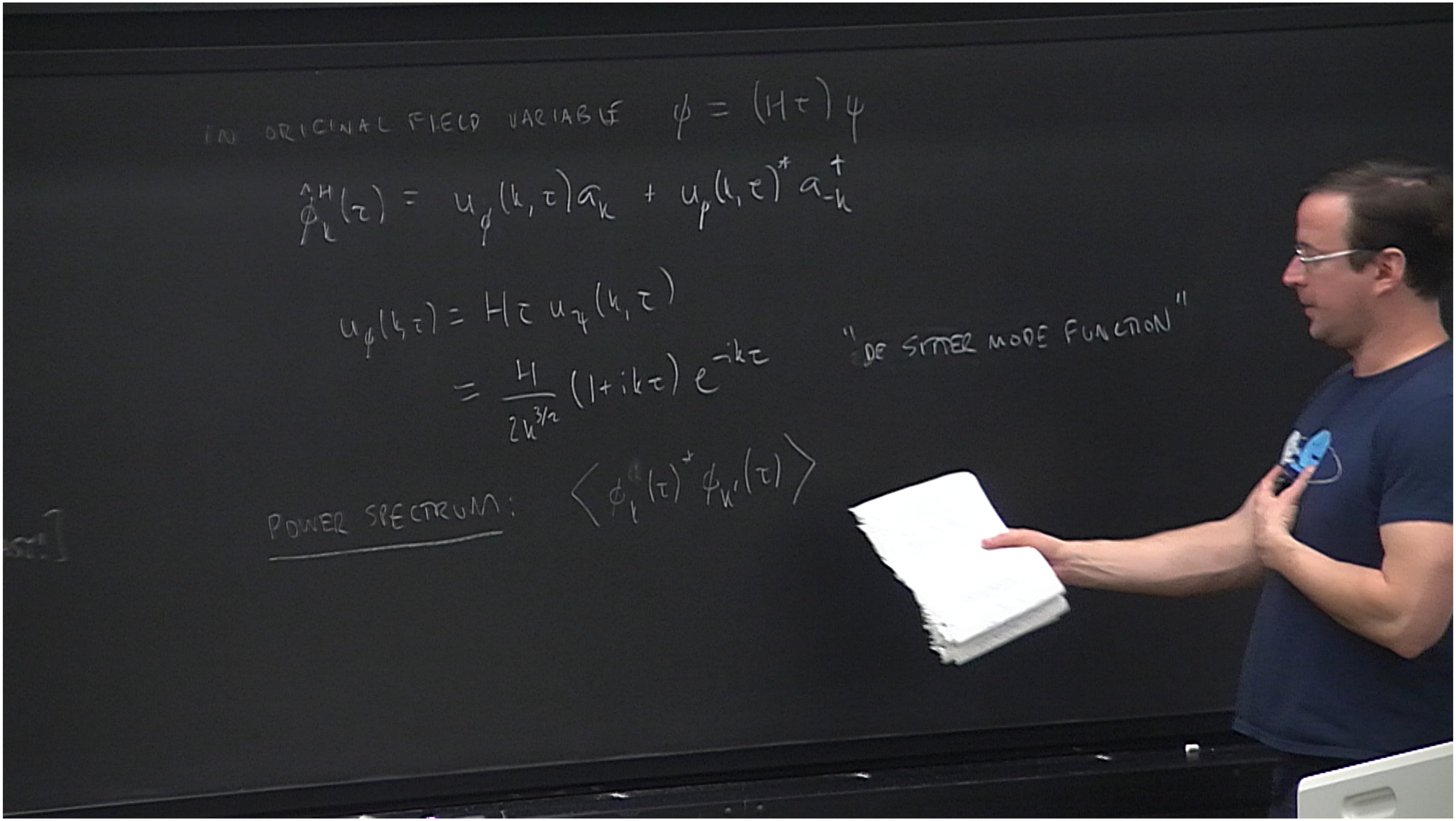
WHERE $\ddot{u}_\psi(k,\tau) = -\left(k^2 - \frac{2}{\tau^2}\right) u_\psi(k,\tau)$

AND $u_\psi(k,\tau) \to \frac{1}{(2k)^{1/2}} e^{-ik\tau}$ AS $\tau \to -\infty$

SOLUTION: $\boxed{u_\psi(k,\tau) = \frac{1}{(2k)^{1/2}}\left(1 - \frac{i}{k\tau}\right) e^{-ik\tau}}$ [NOT EASY!]


IN ORIGINAL FIELD VARIABLE $\phi = (H\tau)\psi$
$\hat{\phi}^H_k(\tau) = u_\phi(k,\tau)a_k + u_\phi(k,\tau)^* a^\dagger_{-k}$
$u_\phi(k,\tau) = H\tau\, u_\psi(k,\tau)$
$= \frac{H}{2k^{3/2}}(1+ik\tau)e^{-ik\tau}$


IN ORIGINAL FIELD VARIABLE $\phi = (H\tau)\psi$
$\hat{\phi}^H_k(\tau) = u_\phi(k,\tau)a_k + u_\phi(k,\tau)^* a^\dagger_{-k}$
$u_\phi(k,\tau) = H\tau\, u_\psi(k,\tau)$
$= \frac{H}{2k^{3/2}}(1+ik\tau)e^{-ik\tau}$
"DE SITTER MODE FUNCTION"
POWER SPECTRUM: $\langle \phi^H_k(\tau)$


IN ORIGINAL FIELD VARIABLE $\phi = (H\tau)\psi$
$\hat{\phi}^H_k(\tau) = u_\phi(k,\tau)a_k + u_\phi(k,\tau)^* a^\dagger_{-k}$
$u_\phi(k,\tau) = H\tau\, u_\psi(k,\tau)$
$= \frac{H}{2k^{3/2}}(1+ik\tau)e^{-ik\tau}$
"DE SITTER MODE FUNCTION"
POWER SPECTRUM: $\langle \phi^H_k(\tau)^* \phi_{k'}(\tau) \rangle$

IN ORIGINAL FIELD VARIABLE $\phi = (H\tau)\psi$

$$\hat{\phi}^H_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)^*\, a^\dagger_{-k}$$

$$u_\phi(k,\tau) = H\tau\, u_\psi(k,\tau)$$

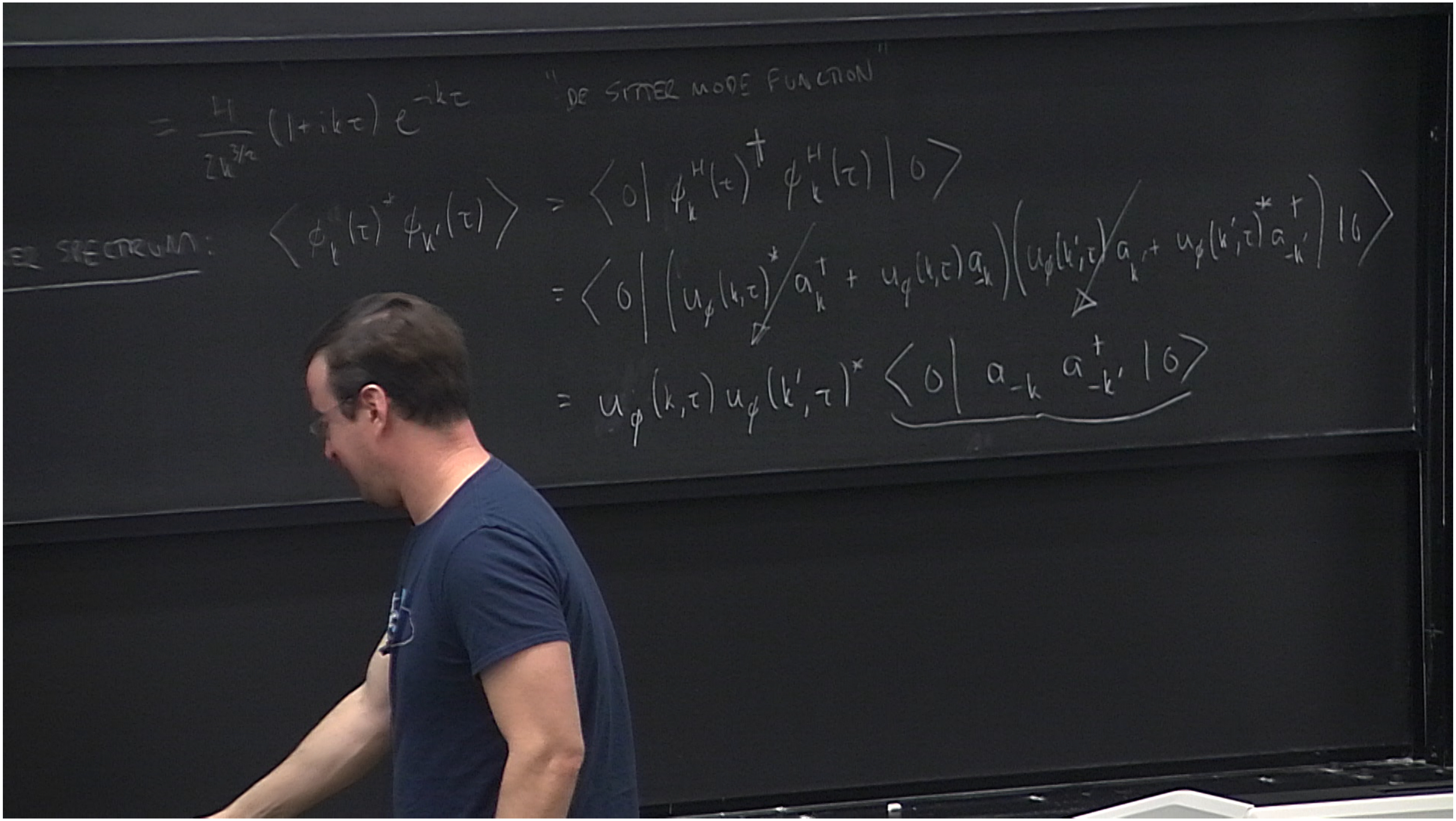
$$= \frac{H}{2k^{3/2}}(1+ik\tau)\, e^{-ik\tau}$$ "DE SITTER MODE FUNCTION"

POWER SPECTRUM: $\langle \phi^H_k(\tau)^* \phi_{k'}(\tau) \rangle = \langle 0 | \phi^H_k(\tau)^* \phi^H_k(\tau) | 0 \rangle$

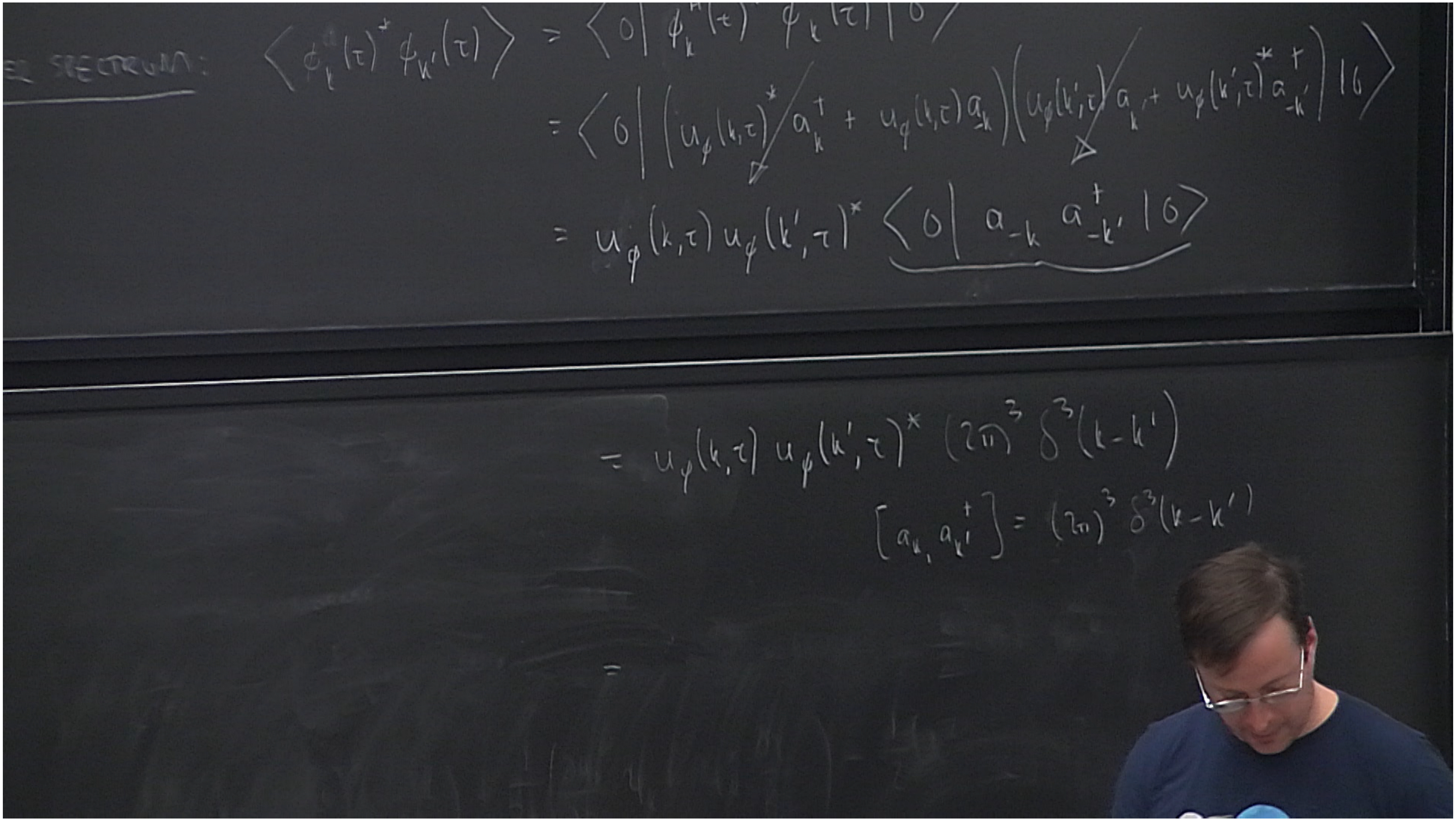
$$= \frac{H}{2k^{3/2}}(1+ik\tau)e^{-ik\tau}$$ "DE SITTER MODE FUNCTION"

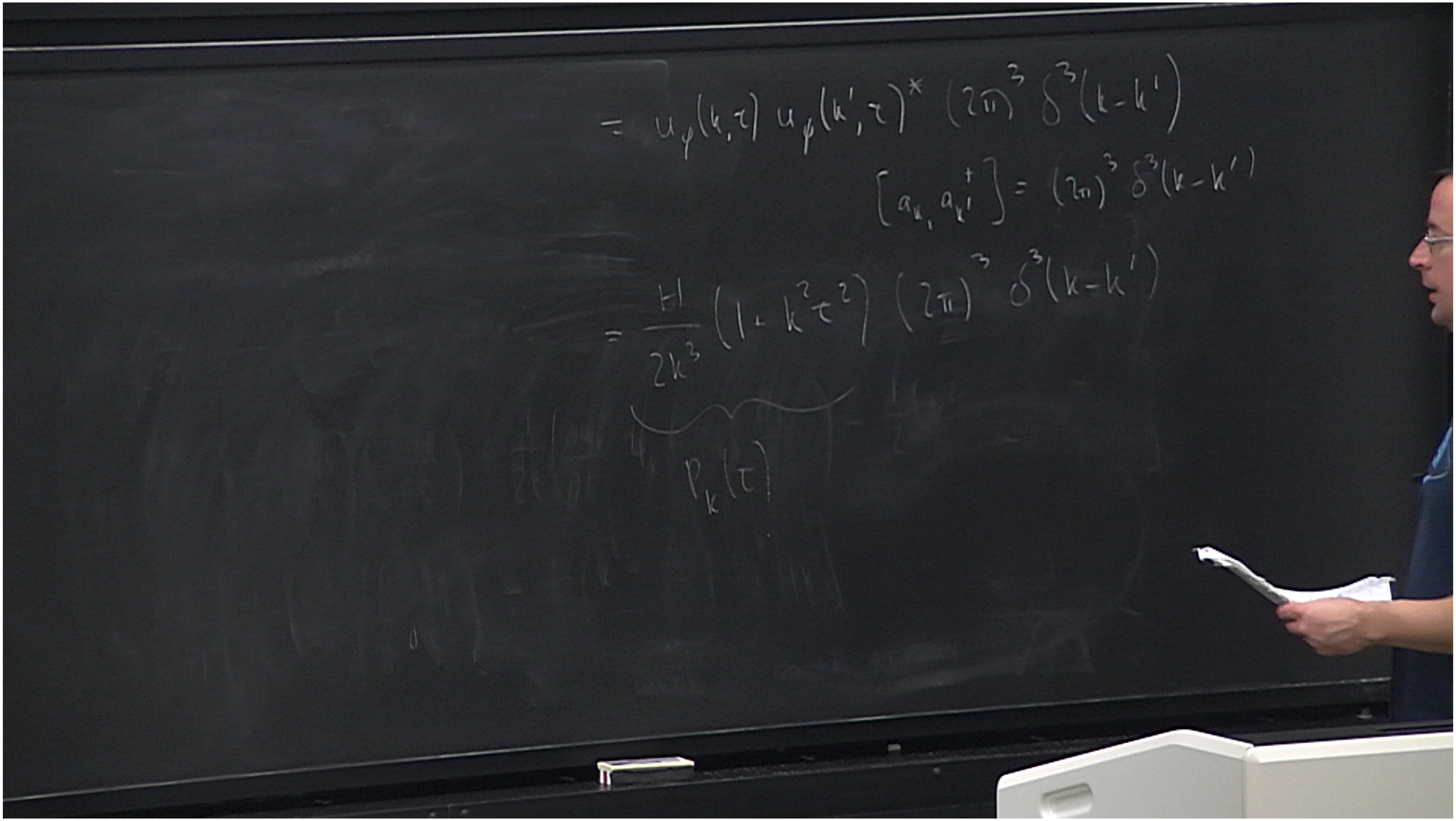
POWER SPECTRUM: $$\langle \phi_k(\tau)^* \phi_{k'}(\tau) \rangle = \langle 0 | \phi_k^H(\tau)^\dagger \phi_k^H(\tau) | 0 \rangle$$

$$= \langle 0 | \left(u_\phi(k,\tau)^* a_k^\dagger + u_\phi(k,\tau) a_{-k} \right) \left(u_\phi(k',\tau) a_{k'} + u_\phi(k',\tau) a_{-k'}^\dagger \right) | 0 \rangle$$



= H/(2k^{3/2}) (1+ikτ) e^{-ikτ}   "DE SITTER MODE FUNCTION"
ER SPECTRUM:
⟨φ_k(τ)* φ_k'(τ)⟩ = ⟨0| φ_k^H(τ)† φ_k^H(τ) |0⟩
= ⟨0| (u_φ(k,τ)* a_k† + u_φ(k,τ) a_-k)(u_φ(k',τ) a_k' + u_φ(k',τ)* a_-k'†) |0⟩
= u_φ(k,τ) u_φ(k',τ)* ⟨0| a_-k a_-k'† |0⟩


SPECTRUM:
⟨φ_k(τ)* φ_k'(τ)⟩ = ⟨0| φ_k(τ)* φ_k(τ) |0⟩
= ⟨0| (u_φ(k,τ)* a_k^† + u_φ(k,τ) a_-k)(u_φ(k',τ) a_k' + u_φ(k',τ)* a_-k'^†) |0⟩
= u_φ(k,τ) u_φ(k',τ)* ⟨0| a_-k a_-k'^† |0⟩
= u_φ(k,τ) u_φ(k',τ)* (2π)^3 δ^3(k-k')
[a_k, a_k'^†] = (2π)^3 δ^3(k-k')
=


$= u_\varphi(k,\tau)\, u_\varphi(k',\tau)^* \,(2\pi)^3 \delta^3(k-k')$
$[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^3(k-k')$
$= \underbrace{\frac{H}{2k^3}(1+k^2\tau^2)}_{P_k(\tau)} (2\pi)^3 \delta^3(k-k')$

$$= u_\varphi(k,\tau)\, u_\varphi(k',\tau)^* \,(2\pi)^3\, \delta^3(k-k')$$

$$[a_k, a_{k'}^\dagger] = (2\pi)^3\, \delta^3(k-k')$$

$$= \underbrace{\frac{H^2}{2k^3}\left(1+k^2\tau^2\right)}_{P_k(\tau)} (2\pi)^3\, \delta^3(k-k')$$

At $\tau = 0 \Rightarrow P_\varphi(k) = \dfrac{H^2}{2k^3}$

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^3(k-k')$$

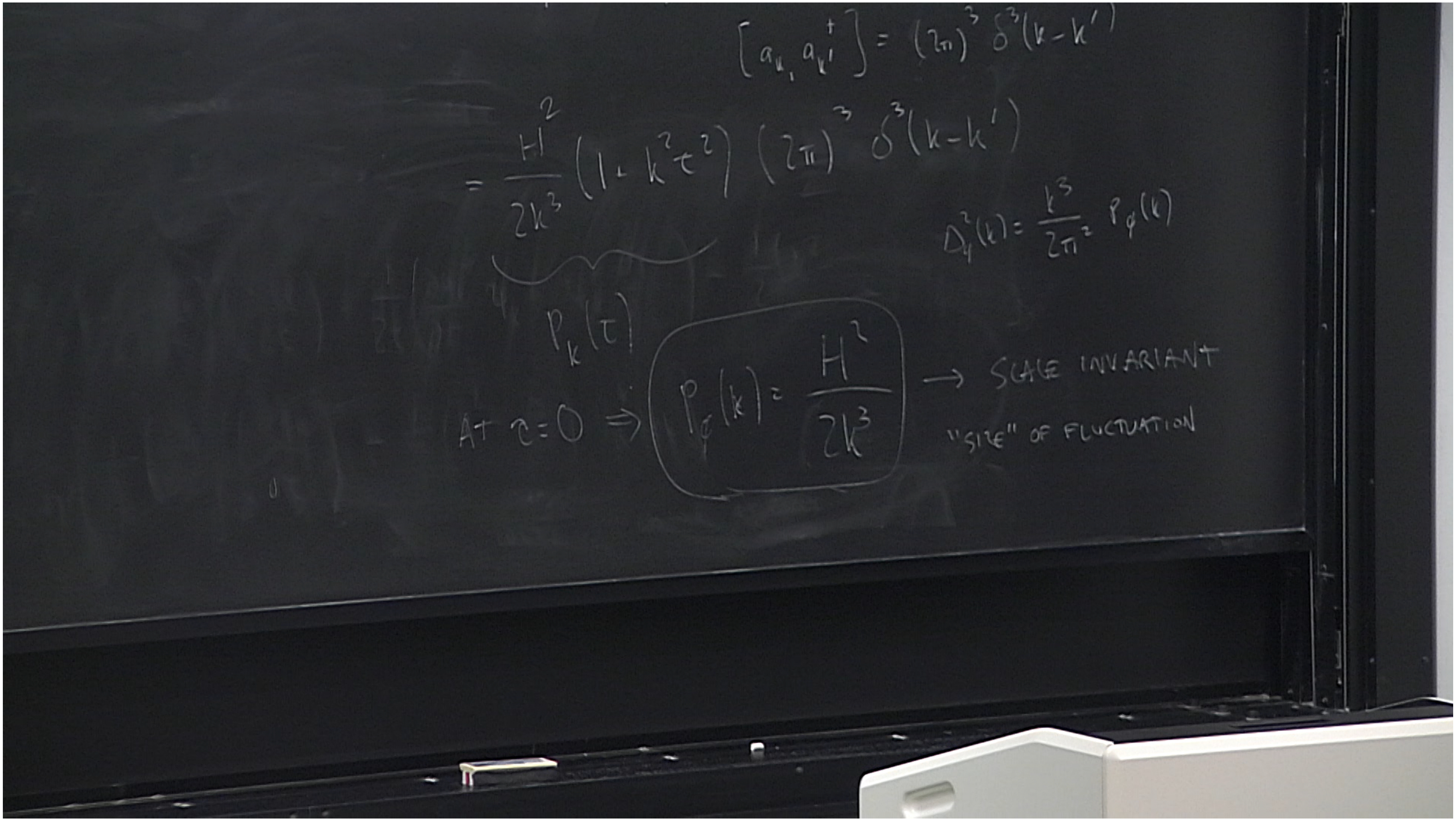
$$= \underbrace{\frac{H^2}{2k^3}(1+k^2\tau^2)}_{P_k(\tau)} (2\pi)^3 \delta^3(k-k')$$

At $\tau = 0 \Rightarrow P_\phi(k) = \dfrac{H^2}{2k^3} \longrightarrow$ SCALE INVARIANT

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^3(k-k')$$

$$= \underbrace{\frac{H^2}{2k^3}\left(1+k^2\tau^2\right)}_{P_k(\tau)} (2\pi)^3 \delta^3(k-k')$$

At $\tau = 0 \Rightarrow \boxed{P_\phi(k) = \frac{H^2}{2k^3}} \longrightarrow$ SCALE INVARIANT


$[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^3(k-k')$
$= \underbrace{\frac{H^2}{2k^3}(1+k^2\tau^2)}_{P_k(\tau)} (2\pi)^3 \delta^3(k-k')$
$\Delta_\phi^2(k) = \frac{k^3}{2\pi^2} P_\phi(k)$
At $\tau = 0 \Rightarrow \boxed{P_\phi(k) = \frac{H^2}{2k^3}}$ → SCALE INVARIANT
"SIZE" OF FLUCTUATION

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^3(k-k')$$

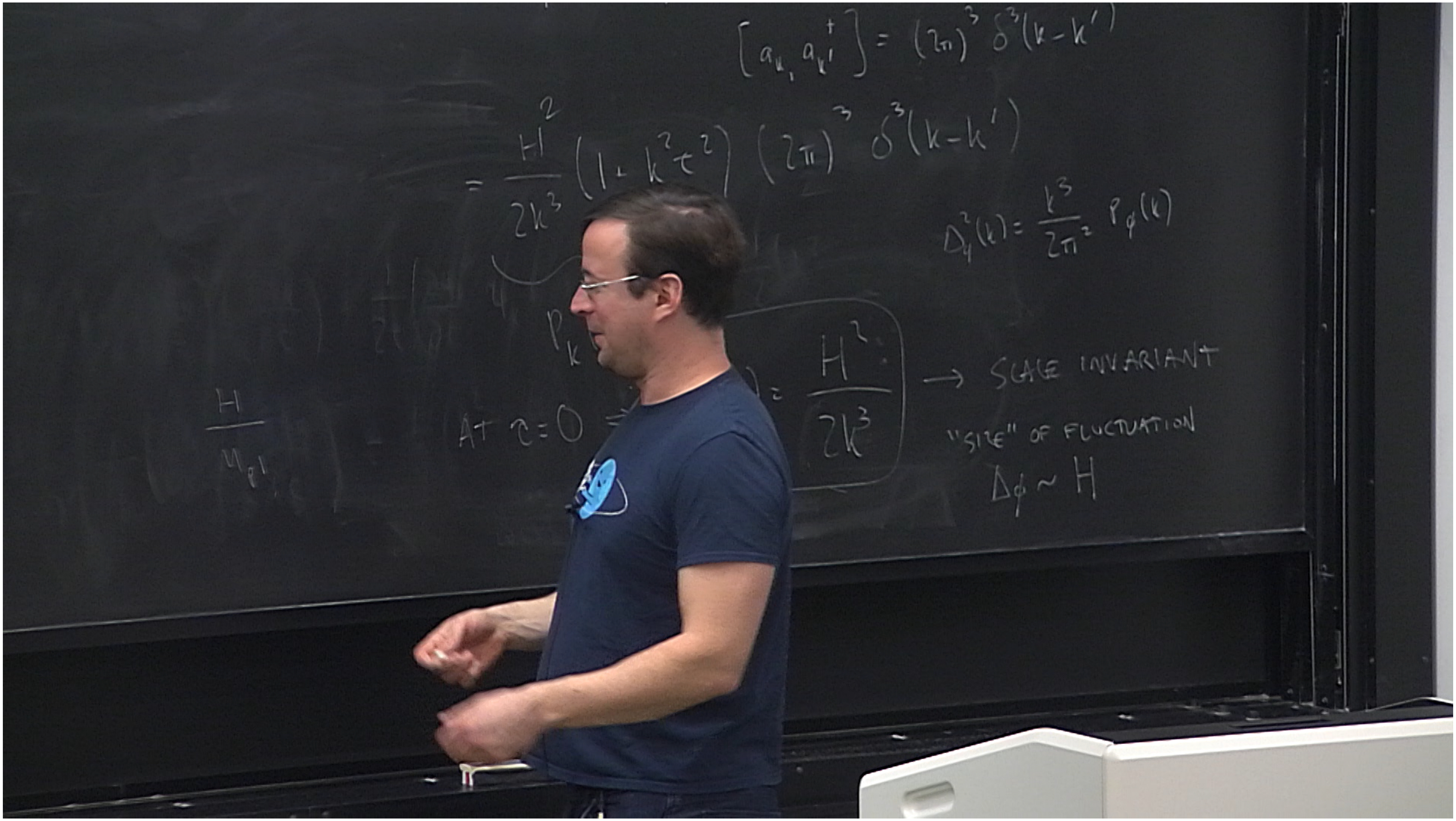
$$= \underbrace{\frac{H^2}{2k^3}\left(1+k^2\tau^2\right)}_{P_k(\tau)} (2\pi)^3 \delta^3(k-k')$$

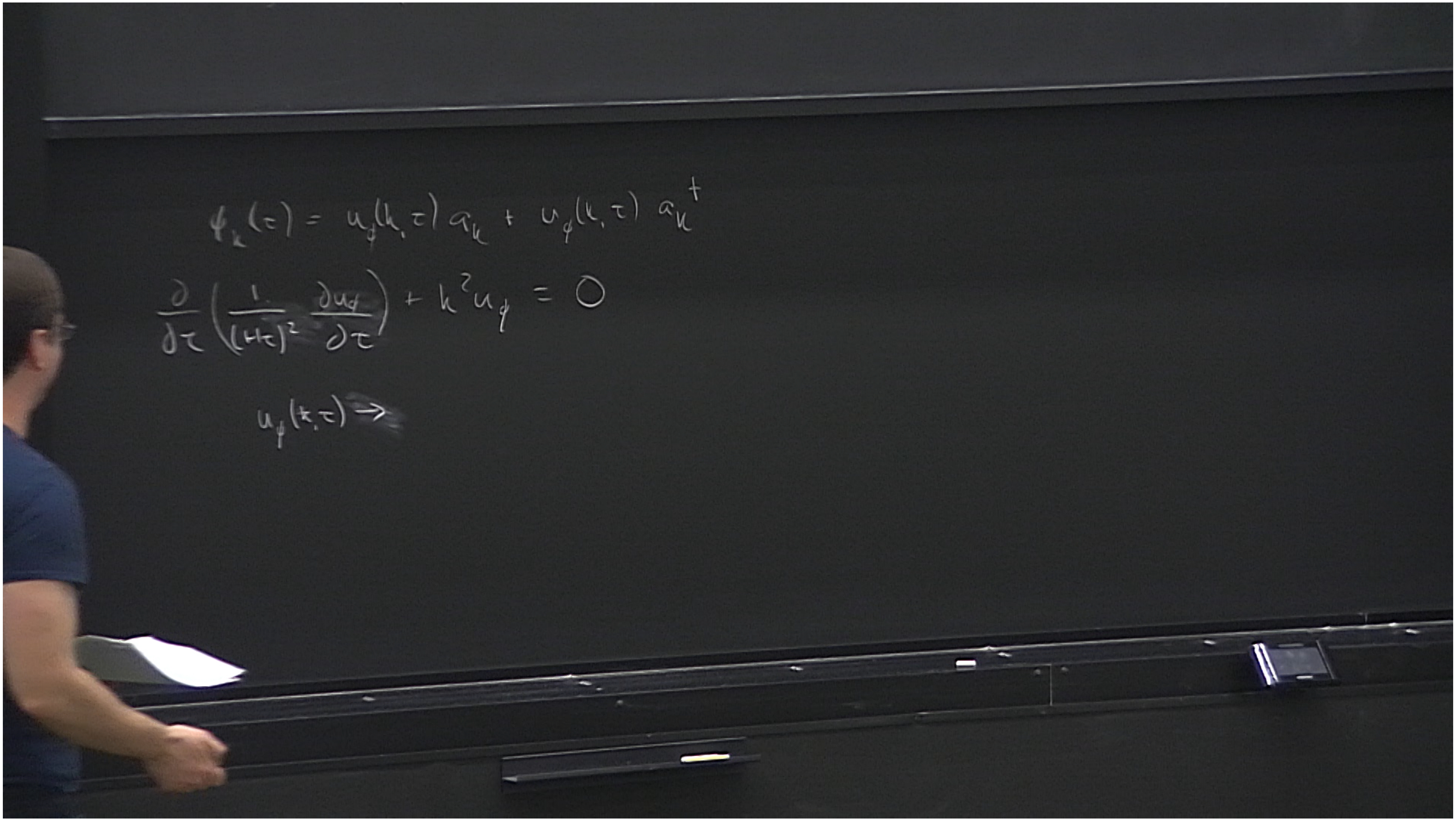
$$\Delta_\phi^2(k) = \frac{k^3}{2\pi^2} P_\phi(k)$$

At $\tau = 0 \Rightarrow \boxed{P_\phi(k) = \frac{H^2}{2k^3}}$ $\longrightarrow$ SCALE INVARIANT

"SIZE" OF FLUCTUATION

$$\Delta\phi \sim H$$


[a_k, a_k'^†] = (2π)^3 δ^3(k − k')
= H^2/2k^3 (1 + k^2τ^2) (2π)^3 δ^3(k − k')
Δ_φ^2(k) = k^3/2π^2 P_φ(k)
At τ = 0 ⇒
H^2/2k^3
→ SCALE INVARIANT
"SIZE" OF FLUCTUATION
Δφ ~ H
H/M_pl


$\phi_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)\, a_k^\dagger$
$\frac{\partial}{\partial\tau}\left(\frac{1}{(H\tau)^2}\frac{\partial u_\phi}{\partial\tau}\right) + k^2 u_\phi = 0$
$u_\phi(k,\tau) \rightarrow$

$$\phi_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)\, a_k^\dagger$$

$$\frac{\partial}{\partial\tau}\left(\frac{1}{(H\tau)^2}\frac{\partial u_\phi}{\partial\tau}\right) + k^2 u_\phi = 0$$

$$u_\phi(k,\tau) \to A(k)\, e^{-ik\tau} \quad \text{AS } \tau \to -\infty$$

$$\phi_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)\, a_k^\dagger$$

$$\frac{\partial}{\partial \tau}\left(\frac{1}{(H\tau)^2}\frac{\partial u_\phi}{\partial \tau}\right) + k^2 u_\phi = 0$$

$$u_\phi(k,\tau) \to \underbrace{A(k)}\, e^{-ik\tau} \quad \text{AS } \tau \to -\infty$$

$$i = [\phi, p_\phi] \qquad p_\phi = \frac{\delta S}{\delta \phi} = \frac{1}{(H\tau)^2}\frac{\partial \phi}{\partial \tau}$$

$$= \frac{1}{(H\tau)^2}\left(u\dot{u}^* - u^*\dot{u}\right)$$

$$\phi_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)\, a_k^\dagger$$

$$\frac{\partial}{\partial \tau}\left(\frac{1}{(H\tau)^2}\frac{\partial u_\phi}{\partial \tau}\right) + k^2 u_\phi = 0$$

$$u_\phi(k,\tau) \to \underbrace{A(k)}\, e^{-ik\tau} \qquad \text{AS } \tau \to -\infty$$

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$$= \frac{1}{(H\tau)^2}\left(u\dot{u}^* - u^*\dot{u}\right)$$

$$\Rightarrow |A|^2 = \frac{H^2}{2k^3}$$

$$\phi_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)\, a_k^\dagger$$

$$\frac{\partial}{\partial\tau}\left(\frac{1}{(H\tau)^2}\frac{\partial u_\phi}{\partial\tau}\right) + k^2 u_\phi = 0$$

$$u_\phi(k,\tau) \to \underbrace{A(k)}\, e^{-ik\tau} \qquad \text{AS } \tau \to -\infty$$

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$$= \frac{1}{(H\tau)^2}\left(u\dot{u}^* - u^*\dot{u}\right)$$

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$$\phi_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)\, a_k^\dagger$$

$$\frac{\partial}{\partial\tau}\left(\frac{1}{(H\tau)^2}\frac{\partial u_\phi}{\partial\tau}\right) + k^2 u_\phi = 0$$

$$u_\phi(k,\tau) \to A(k)\, e^{-ik\tau} \qquad \text{AS } \tau \to -\infty$$

$$i = [\phi, p_\phi] \qquad p_\phi = \frac{\delta S}{\delta\phi} = \frac{1}{(H\tau)^2}\frac{\partial\phi}{\partial\tau}$$

$$= \frac{1}{(H\tau)^2}\left(u\dot{u}^* - u^*\dot{u}\right)$$

$$\Rightarrow |A|^2 = \frac{H^2}{2k^3}$$

$$\phi_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)\, a_k^\dagger$$

$$S \to \lambda S$$

$$\frac{\partial}{\partial\tau}\left(\frac{1}{(H\tau)^2}\frac{\partial u_\phi}{\partial\tau}\right) + k^2 u_\phi = 0$$

$$u_\phi(k,\tau) \to A(k)\, e^{-ik\tau} \quad \text{AS } \tau \to -\infty$$

$$i = [\phi, p_\phi] \qquad p_\phi = \frac{\delta S}{\delta\dot\phi} = \frac{1}{(H\tau)^2}\frac{\partial\phi}{\partial\tau}$$

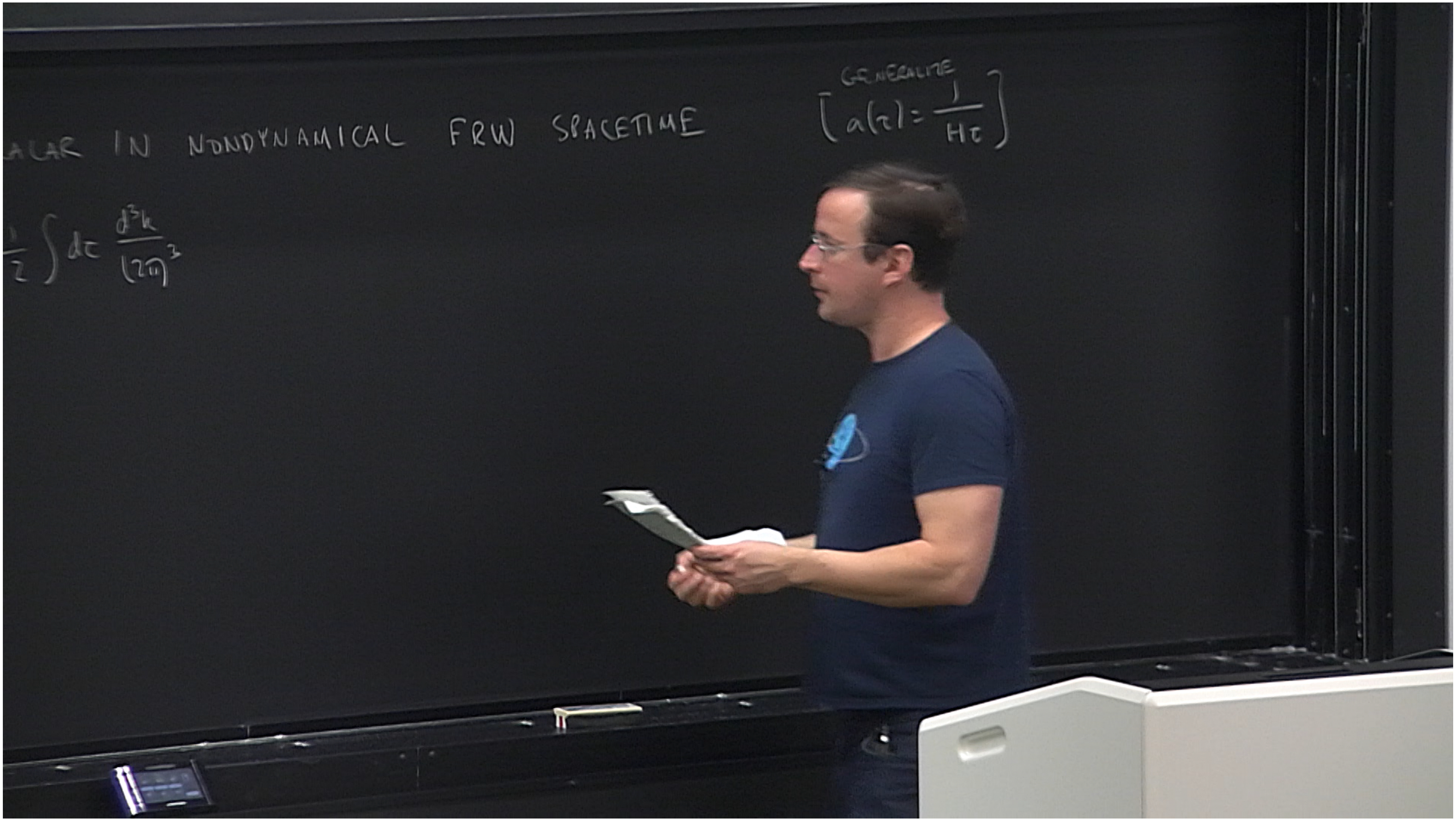
$$= \frac{1}{(H\tau)^2}\left(u\dot u^* - u^*\dot u\right)$$

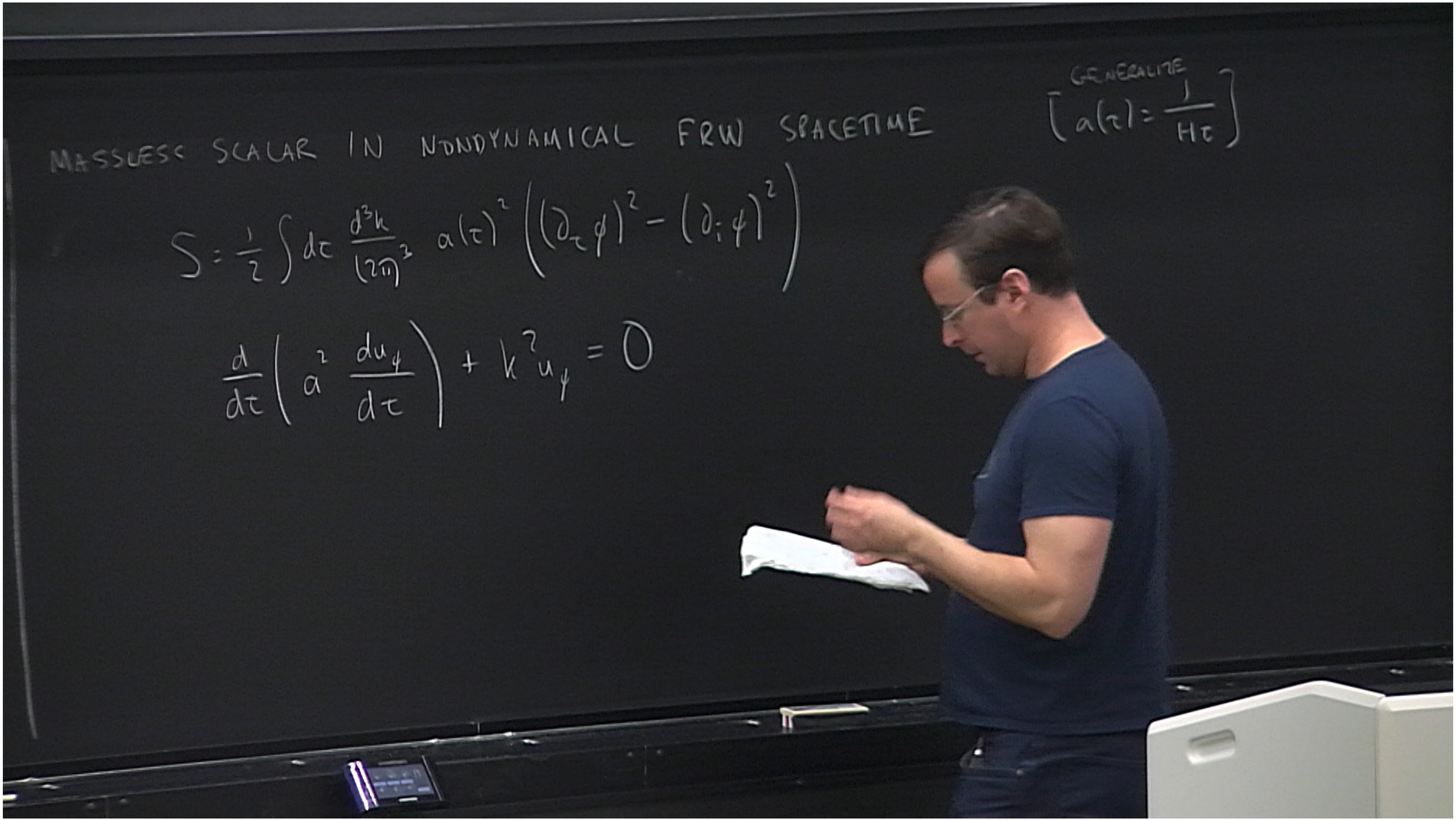
$$\Rightarrow |A|^2 = \frac{H^2}{2k^3}$$

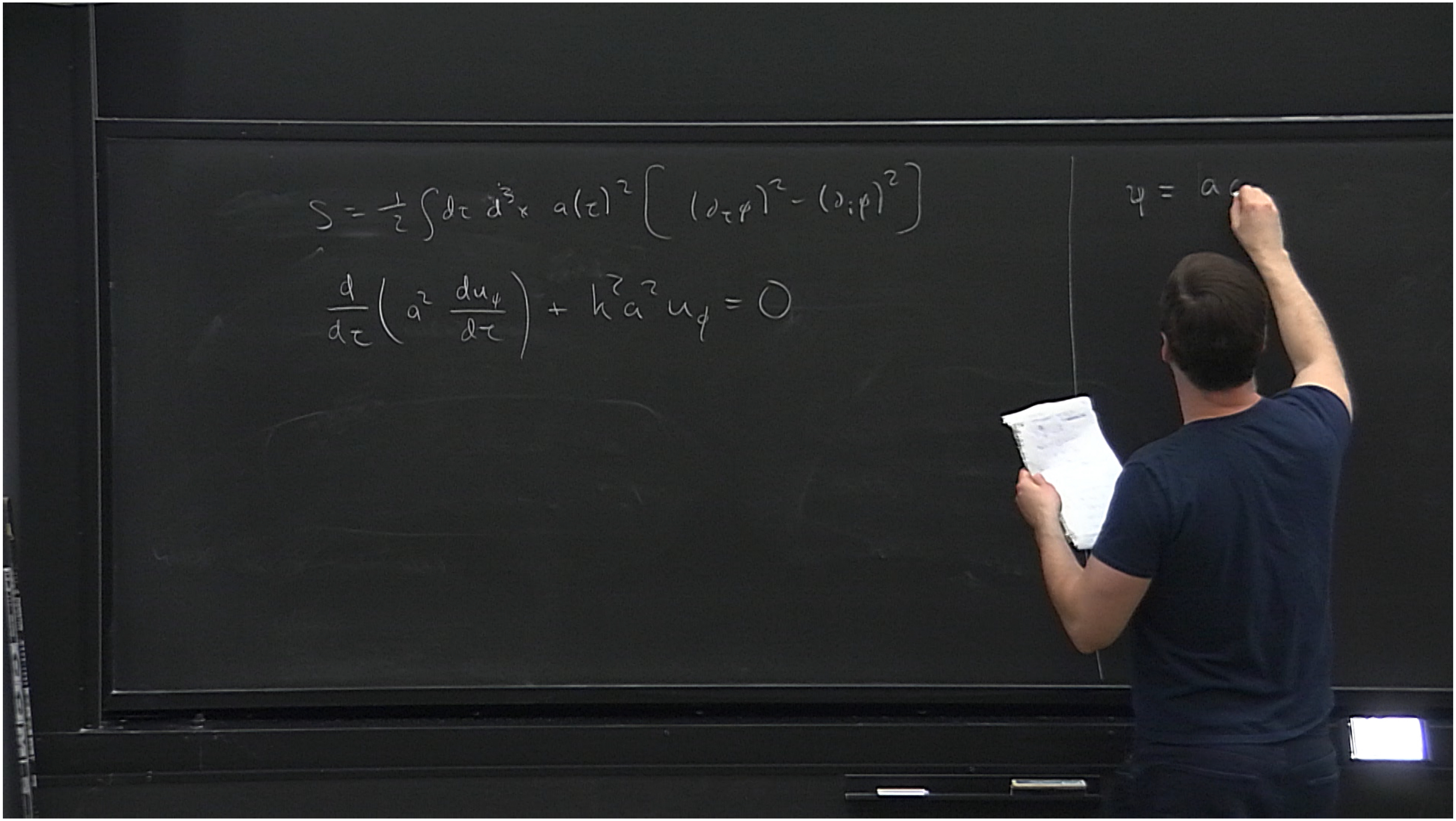
MASSLESS SCALAR IN NONDYNAMICAL FRW SPACETIME

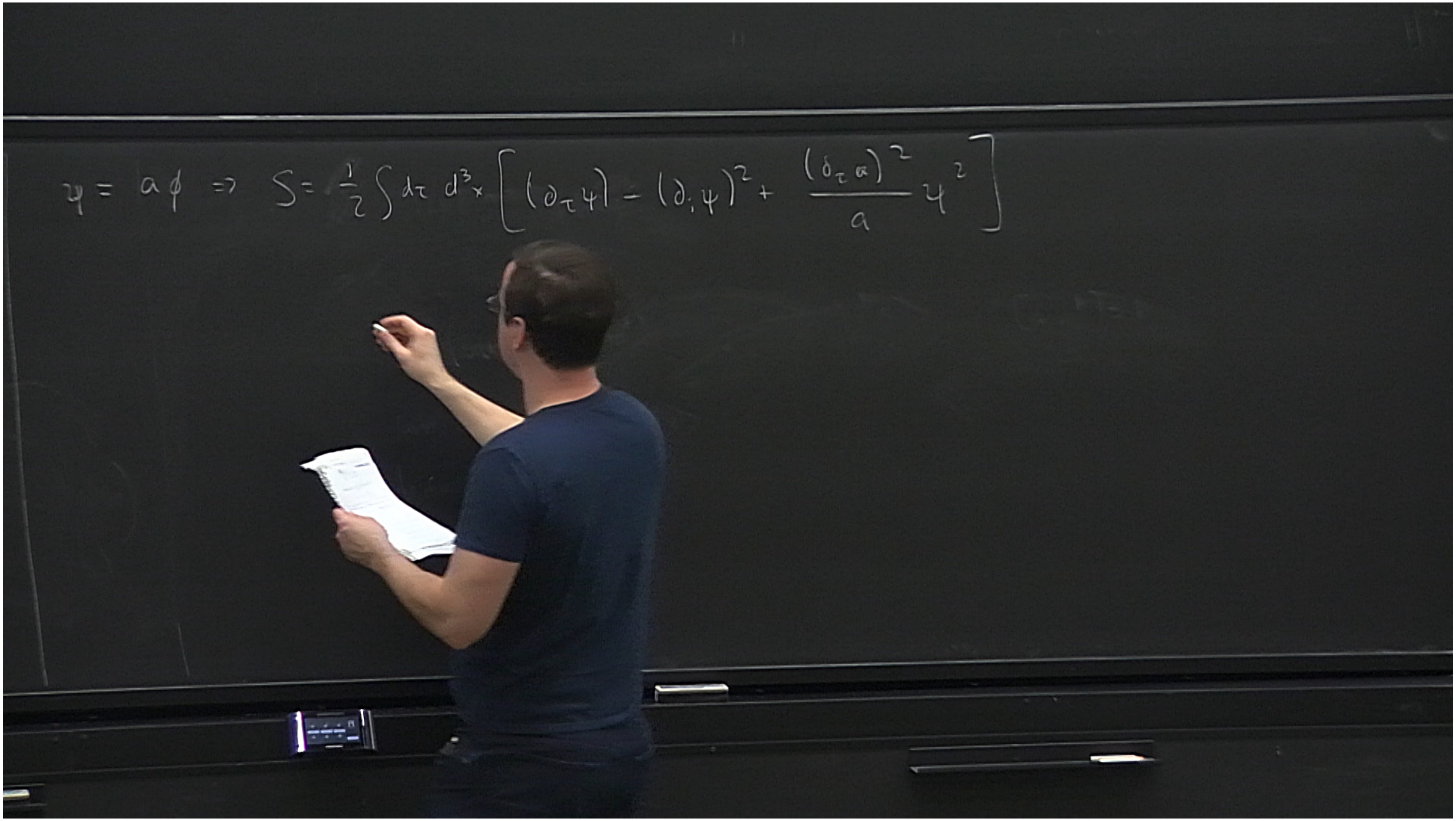
$[a(\tau) = -$

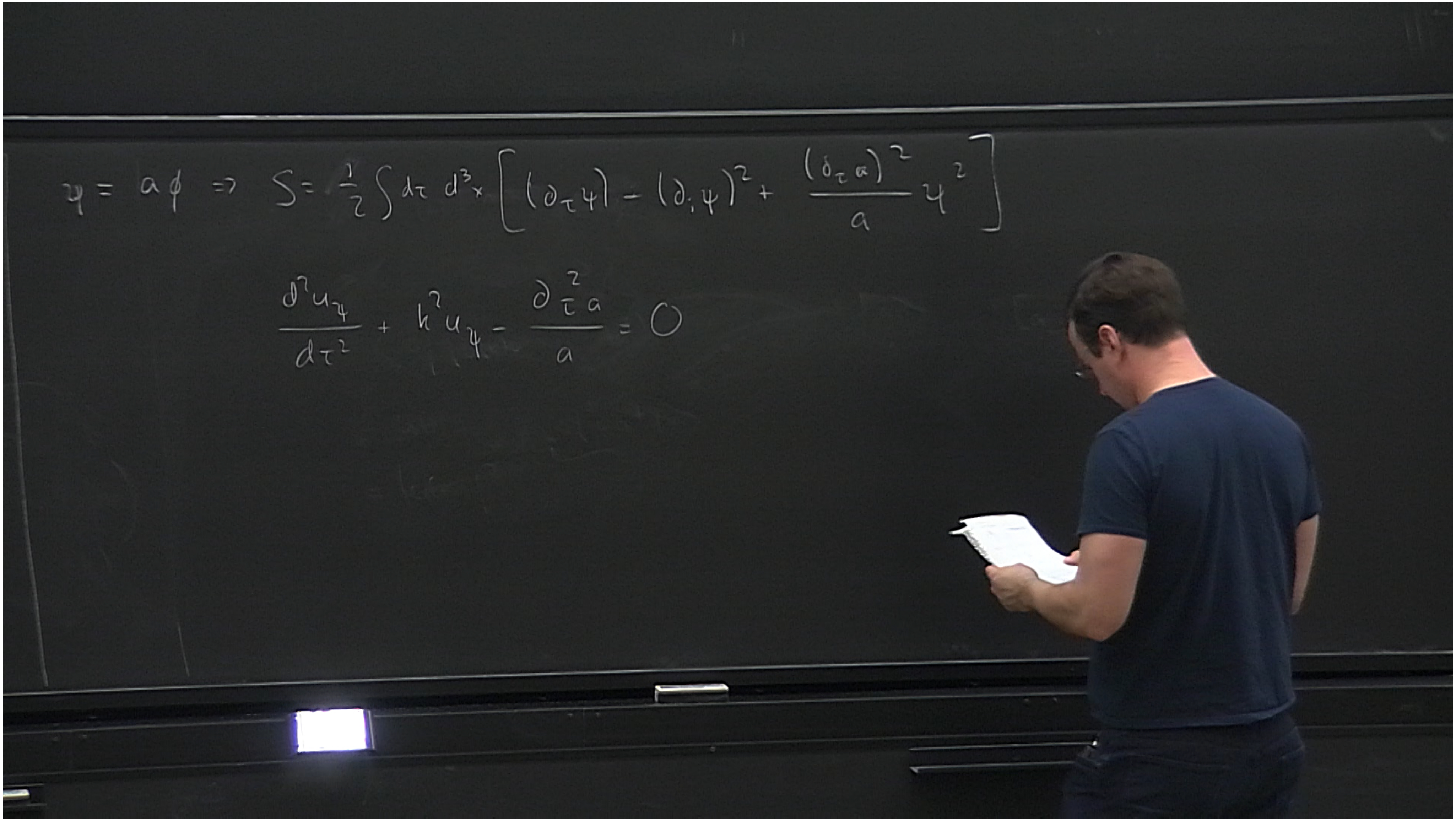
$$S = \frac{1}{2}\int d\tau \frac{d^3k}{(2\pi)^3}$$


ALAR IN NONDYNAMICAL FRW SPACETIME
GENERALIZE
$a(\tau) = \frac{1}{H\tau}$
$\frac{1}{2}\int d\tau \frac{d^3k}{(2\pi)^3}$


GENERALIZE
$\left[a(\tau) = \frac{1}{H\tau} \right]$
MASSLESS SCALAR IN NONDYNAMICAL FRW SPACETIME
$S = \frac{1}{2} \int d\tau \frac{d^3k}{(2\pi)^3} a(\tau)^2 \left((\partial_\tau \phi)^2 - (\partial_i \phi)^2 \right)$
$\frac{d}{d\tau}\left(a^2 \frac{du_\phi}{d\tau} \right) + k^2 u_\phi = 0$


$S = -\frac{1}{2}\int d\tau\, d^3x\; a(\tau)^2 \left[(\partial_\tau \varphi)^2 - (\partial_i \varphi)^2 \right]$
$\frac{d}{d\tau}\left(a^2 \frac{du_\varphi}{d\tau} \right) + k^2 a^2 u_\varphi = 0$
$\varphi = a$

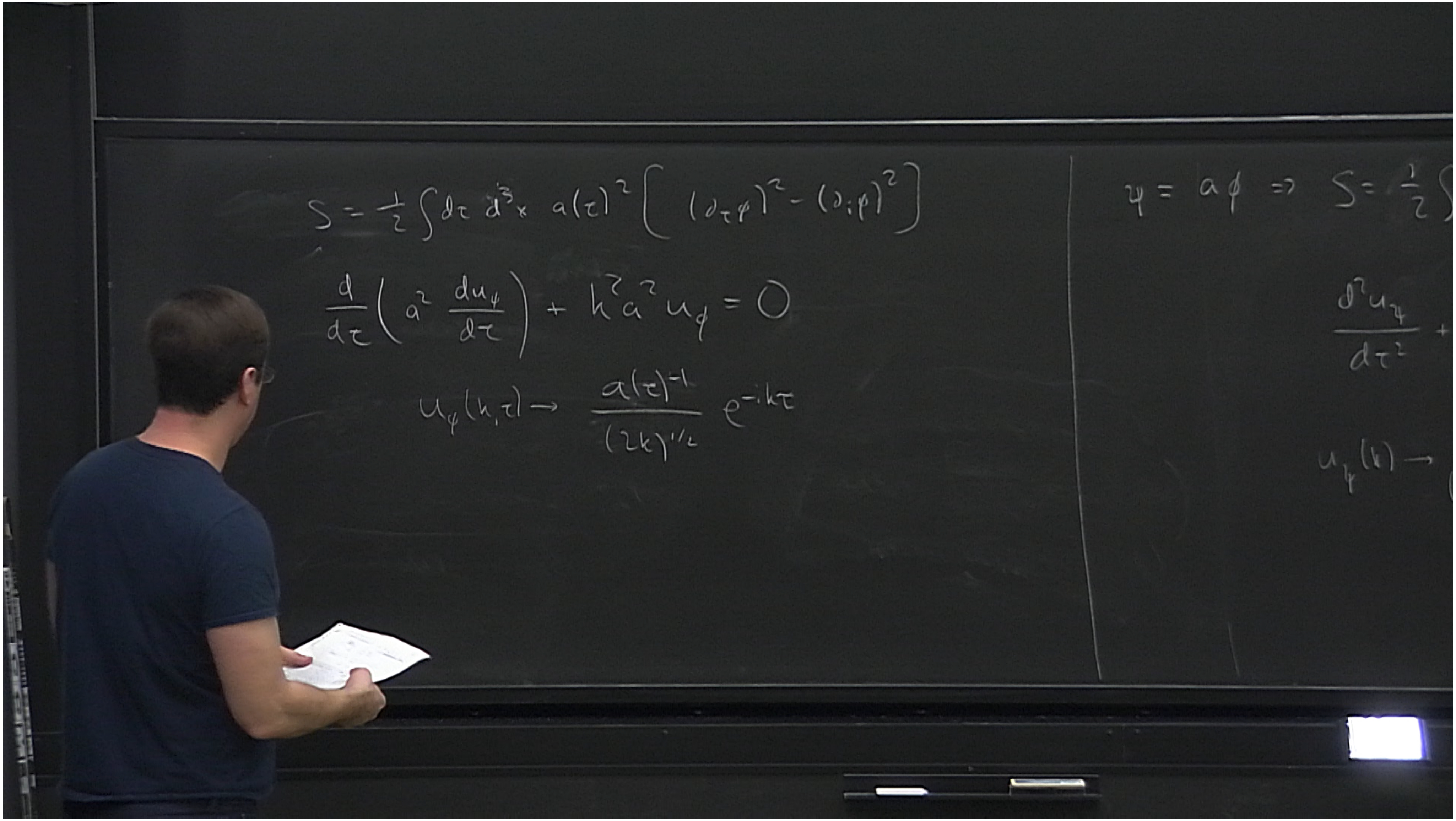

$\psi = a\phi \Rightarrow S = \frac{1}{2}\int d\tau\, d^3x \left[(\partial_\tau \psi) - (\partial_i \psi)^2 + \frac{(\partial_\tau a)^2}{a}\psi^2 \right]$

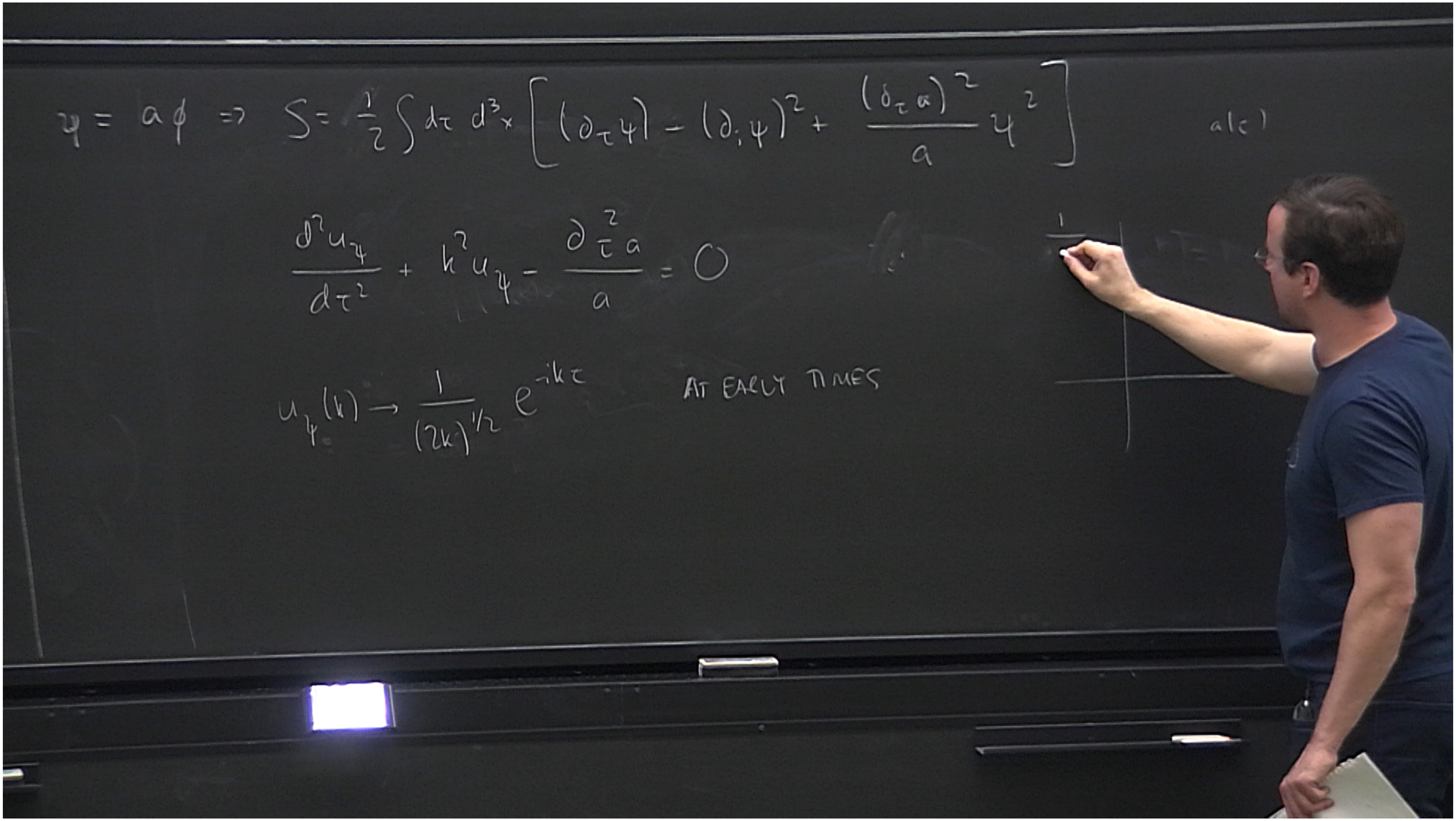

$\psi = a\phi \Rightarrow S = \frac{1}{2}\int d\tau\, d^3x \left[(\partial_\tau \psi) - (\partial_i \psi)^2 + \frac{(\partial_\tau a)^2}{a}\psi^2 \right]$
$\frac{d^2 u_\psi}{d\tau^2} + k^2 u_\psi - \frac{\partial_\tau^2 a}{a} = 0$

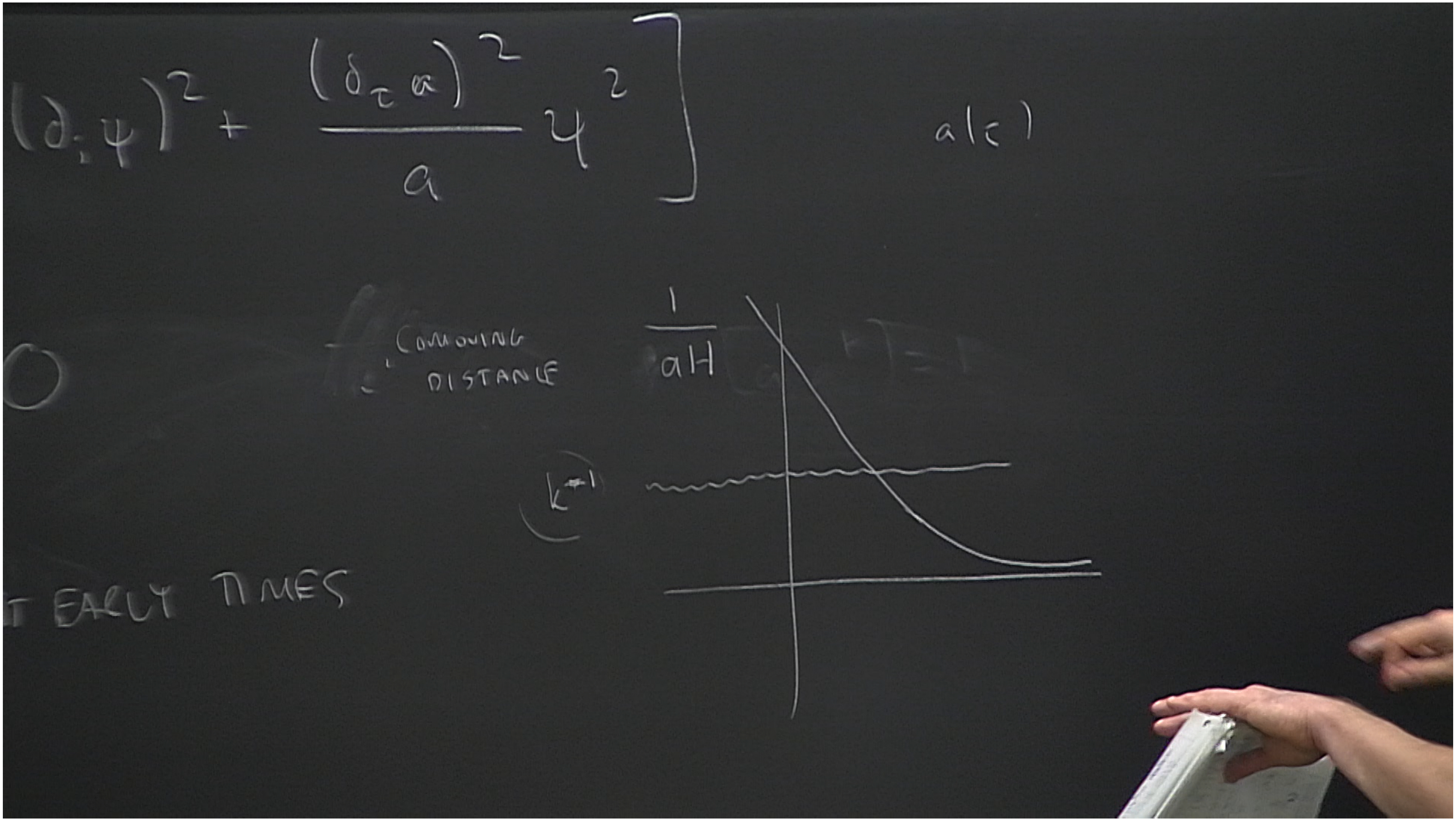
$$\psi = a\phi \Rightarrow S = \frac{1}{2}\int d\tau\, d^3x \left[(\partial_\tau \psi) - (\partial_i \psi)^2 + \frac{(\partial_\tau a)^2}{a}\psi^2 \right] \qquad a(\tau)$$

$$\frac{d^2 u_\psi}{d\tau^2} + k^2 u_\psi - \frac{\partial_\tau^2 a}{a} = 0$$

$$u_\psi(k) \to \frac{1}{(2k)^{1/2}} e^{-ik\tau}$$ AT EARLY TIMES

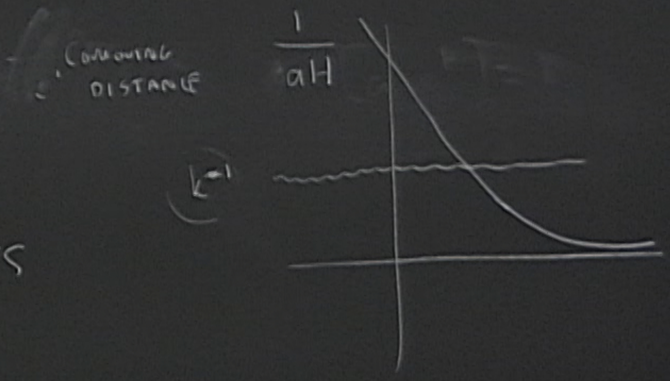

$S = -\frac{1}{2}\int d\tau\, d^3x\; a(\tau)^2 \left[(\partial_\tau \varphi)^2 - (\partial_i \varphi)^2 \right]$
$\frac{d}{d\tau}\left(a^2 \frac{du_\varphi}{d\tau} \right) + k^2 a^2 u_\varphi = 0$
$u_\varphi(k,\tau) \to \frac{a(\tau)^{-1}}{(2k)^{1/2}} e^{-ik\tau}$
$\psi = a\phi \Rightarrow S = \frac{1}{2}\int$
$\frac{d^2 u_\psi}{d\tau^2} +$
$u_\psi(k) \to$


$\psi = a\phi \Rightarrow S = \frac{1}{2}\int d\tau\, d^3x \left[(\partial_\tau \psi) - (\partial_i \psi)^2 + \frac{(\partial_\tau a)^2}{a}\psi^2 \right]$
$a(\tau)$
$\frac{d^2 u_\psi}{d\tau^2} + k^2 u_\psi - \frac{\partial_\tau^2 a}{a} = 0$
$u_\psi(k) \to \frac{1}{(2k)^{1/2}} e^{-ik\tau}$
AT EARLY TIMES


$(\partial_i \psi)^2 + \frac{(\partial_\tau a)^2}{a}\psi^2 \Big]$
$a(\tau)$
COMOVING DISTANCE
$\frac{1}{aH}$
k^{-1}
EARLY TIMES

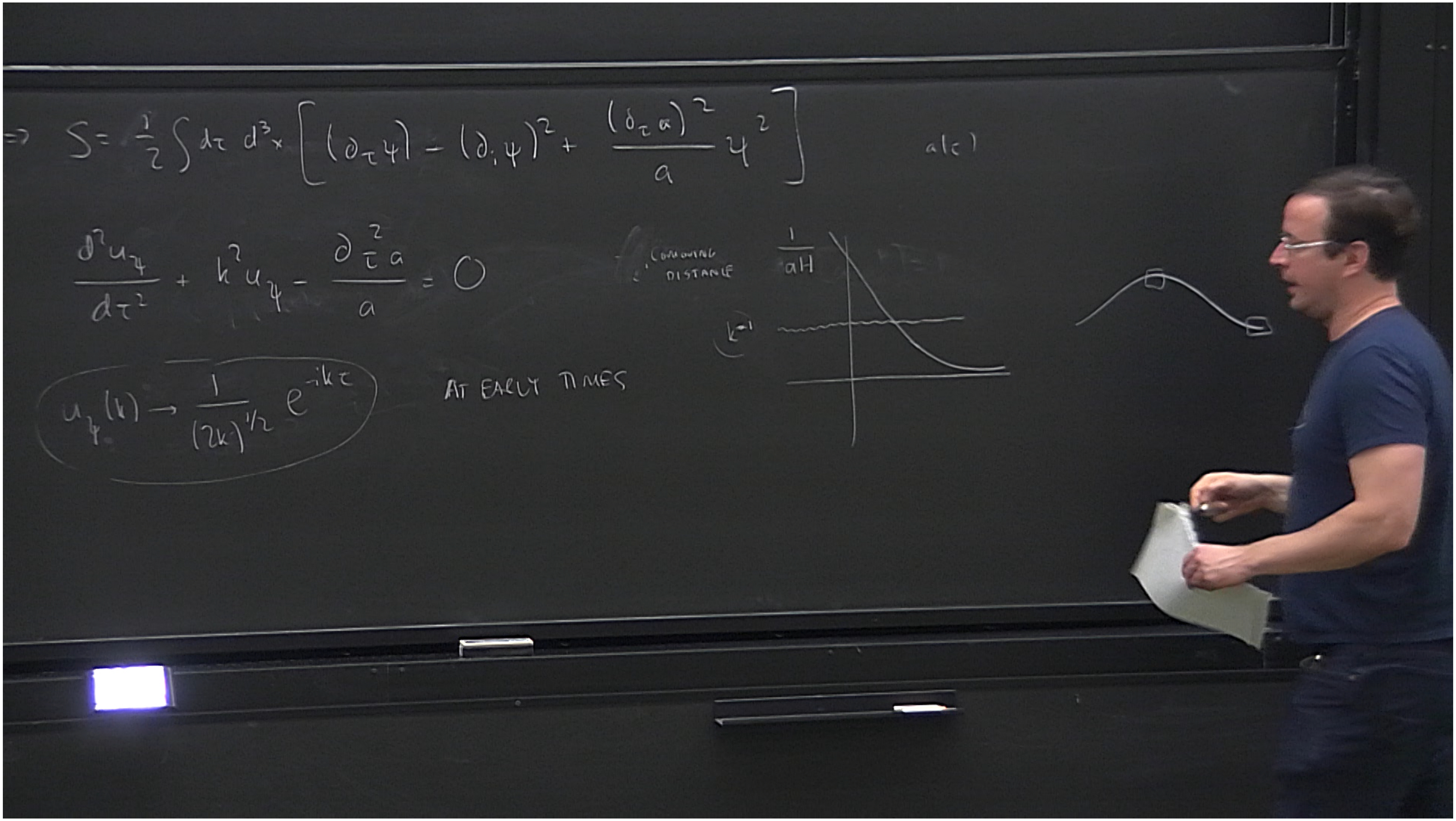
$$\psi = a\phi \Rightarrow S = \frac{1}{2}\int d\tau\, d^3x \left[(\partial_\tau \psi) - (\partial_i \psi)^2 + \frac{(\partial_\tau a)^2}{a} \psi^2 \right] \qquad a(\tau)$$

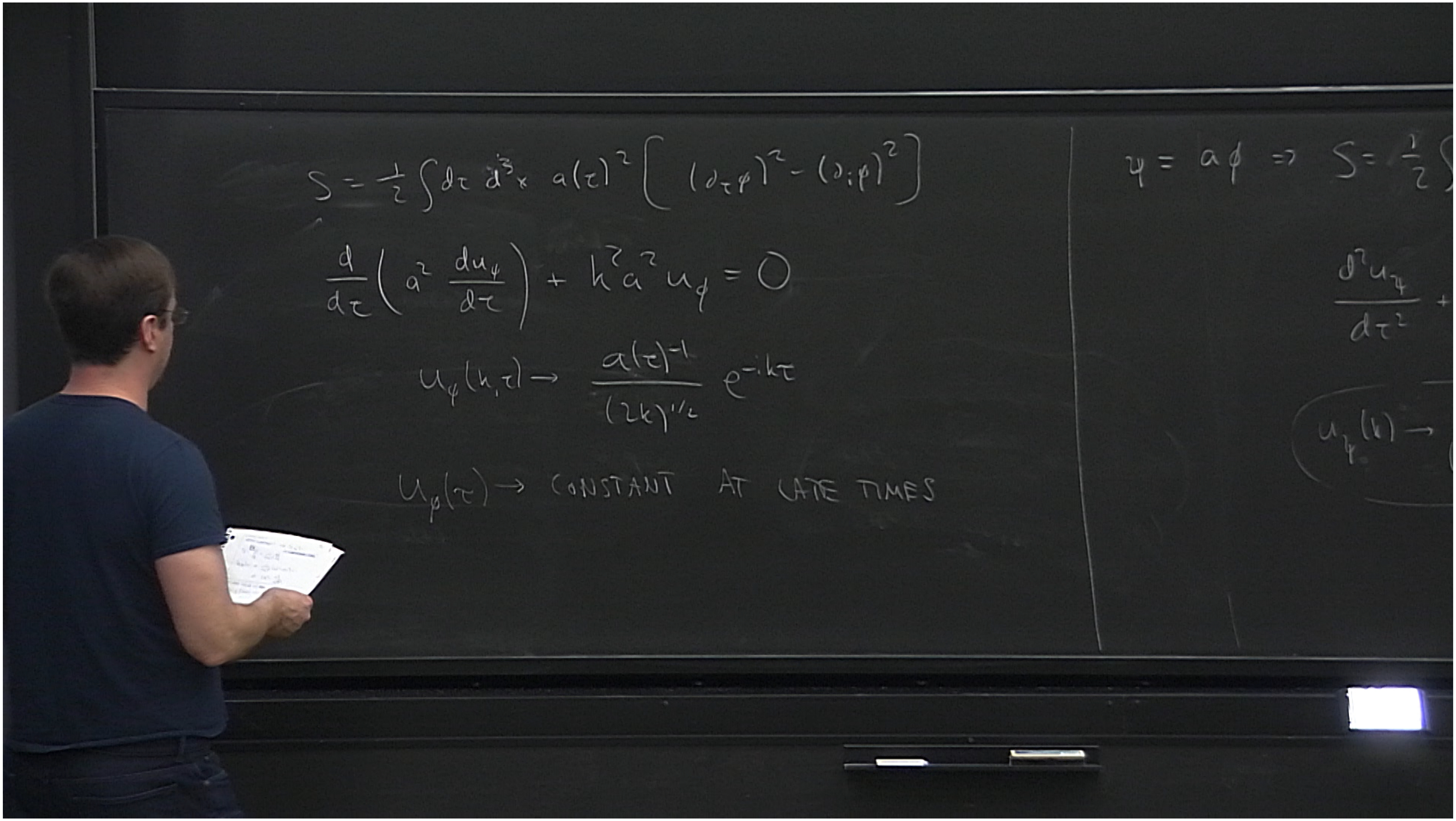
$$\frac{d^2 u_\psi}{d\tau^2} + k^2 u_\psi - \frac{\partial_\tau^2 a}{a} = 0$$

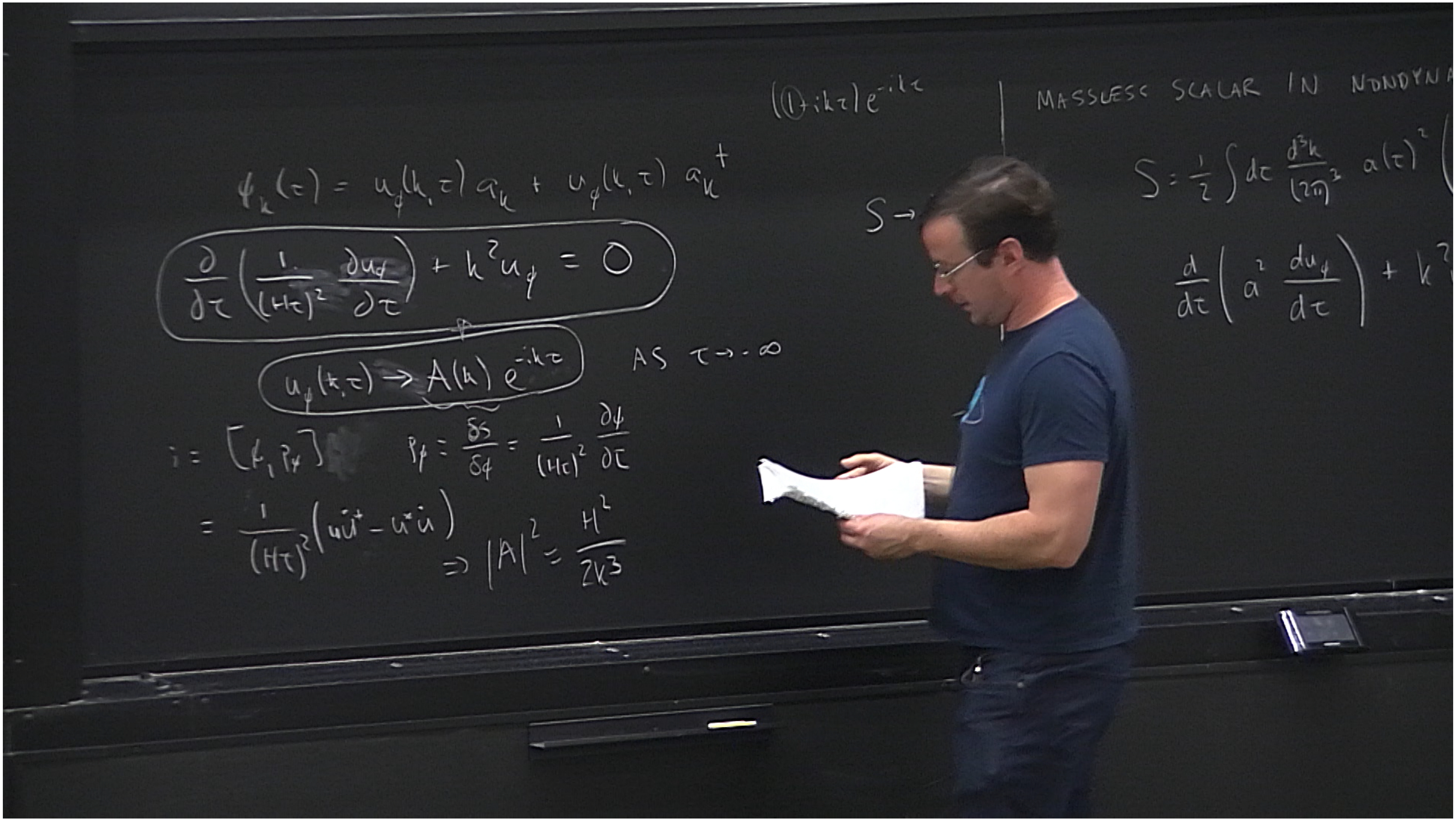


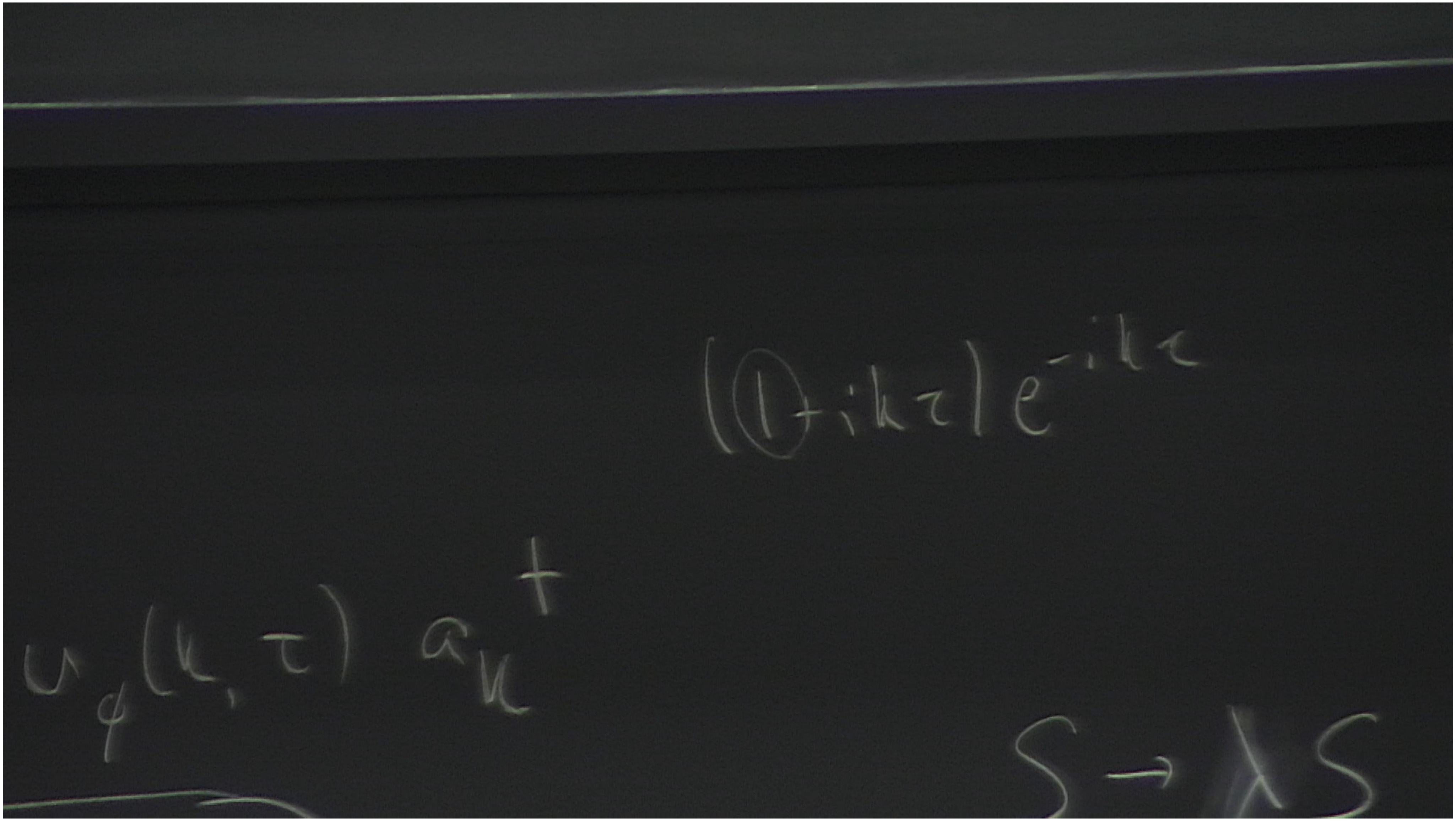
$$u_\psi(k) \rightarrow \frac{1}{(2k)^{1/2}} e^{-ik\tau}$$

AT EARLY TIMES


$\Rightarrow S = \frac{1}{2}\int d\tau\, d^3x \left[(\partial_\tau \psi) - (\partial_i \psi)^2 + \frac{(\partial_\tau a)^2}{a}\psi^2 \right]$
a(τ)
$\frac{d^2 u_\psi}{d\tau^2} + k^2 u_\psi - \frac{\partial_\tau^2 a}{a} = 0$
COMOVING DISTANCE
$\frac{1}{aH}$
k^{-1}
$u_\psi(k) \rightarrow \frac{1}{(2k)^{1/2}} e^{-ik\tau}$
AT EARLY TIMES


$S = -\frac{1}{2}\int d\tau\, d^3x\, a(\tau)^2 \left[(\partial_\tau \varphi)^2 - (\partial_i \varphi)^2 \right]$
$\frac{d}{d\tau}\left(a^2 \frac{du_\varphi}{d\tau} \right) + k^2 a^2 u_\varphi = 0$
$u_\varphi(k,\tau) \to \frac{a(\tau)^{-1}}{(2k)^{1/2}} e^{-ik\tau}$
$u_\varphi(\tau) \to$ CONSTANT AT LATE TIMES
$\psi = a\phi \Rightarrow S = \frac{1}{2}$
$\frac{d^2 u_\psi}{d\tau^2} +$
$u_\psi(k) \to$


MASSLESS SCALAR IN NONDYNA
$(1+ik\tau)e^{-ik\tau}$
$\phi_k(\tau) = u_\phi(k,\tau)\, a_k + u_\phi(k,\tau)\, a_k^\dagger$
$\frac{\partial}{\partial\tau}\left(\frac{1}{(H\tau)^2}\frac{\partial u_\phi}{\partial\tau}\right) + k^2 u_\phi = 0$
$u_\phi(k,\tau) \to A(k)\, e^{-ik\tau}$
AS $\tau \to -\infty$
$i = [\phi, p_\phi]$
$p_\phi = \frac{\delta S}{\delta\phi} = \frac{1}{(H\tau)^2}\frac{\partial\phi}{\partial\tau}$
$= \frac{1}{(H\tau)^2}(u\dot{u}^* - u^*\dot{u})$
$\Rightarrow |A|^2 = \frac{H^2}{2k^3}$
$S = \frac{1}{2}\int d\tau \frac{d^3k}{(2\pi)^3}\, a(\tau)^2$
$\frac{d}{d\tau}\left(a^2 \frac{du_\phi}{d\tau}\right) + k^2$


$(1 + ik\tau)e^{-ik\tau}$
$u_\phi(k,\tau)\, a_k^\dagger$
$S \to \lambda S$

$$S = \frac{1}{2}\int d\tau\, d^3x\; a(\tau)^2 \left[(\partial_\tau \varphi)^2 - (\partial_i \varphi)^2 \right]$$

$$\frac{d}{d\tau}\left(a^2 \frac{du_\varphi}{d\tau} \right) + k^2 a^2 u_\varphi = 0$$

$$u_\varphi(k,\tau) \to \frac{a(\tau)^{-1}}{(2k)^{1/2}} e^{-ik\tau}$$

$u_\varphi(\tau) \to$ CONSTANT AT LATE TIMES

$$|u_\varphi(k,\tau)|^2 \to \frac{H_*^2}{2k^3} + [\text{SLOW-ROLL CORRECTIONS}]$$

$$H_* = H(a)\Big|_{k = aH}$$

$\psi = a\phi \Rightarrow S = \frac{1}{2}\int$

$$\frac{d^2 u_\psi}{d\tau^2} +$$

$u_\psi(k) \to$

$$S = -\frac{1}{2}\int d\tau\, d^3x\; a(\tau)^2 \left[(\partial_\tau \varphi)^2 - (\partial_i \varphi)^2 \right]$$

$$\varphi = a\phi \Rightarrow$$

$$\frac{d}{d\tau}\left(a^2 \frac{du_\varphi}{d\tau} \right) + k^2 a^2 u_\varphi = 0$$

$$u_\varphi(k,\tau) \rightarrow \frac{a(\tau)^{-1}}{(2k)^{1/2}} e^{-ik\tau}$$

$u_\varphi(\tau) \rightarrow$ CONSTANT AT LATE TIMES

$$|u_\varphi(k,\tau)|^2 \rightarrow \frac{H_*^2}{2k^3} + [\text{SLOW-ROLL CORRECTIONS}]$$

$$H_* = H(a)\Big|_{k=aH}$$

$$S = -\frac{1}{2}\int d\tau\, d^3x\; a(\tau)^2\left[(\partial_\tau\varphi)^2 - (\partial_i\varphi)^2\right]$$

$$\psi = a\phi \Rightarrow$$

$$\frac{d}{d\tau}\left(a^2\frac{du_\varphi}{d\tau}\right) + k^2a^2u_\varphi = 0$$

$$u_\varphi(k,\tau) \to \frac{a(\tau)^{-1}}{(2k)^{1/2}}e^{-ik\tau}$$

$u_\varphi(\tau) \to$ CONSTANT AT LATE TIMES

$$|u_\varphi(k,\tau)|^2 \to \frac{H_*^2}{2k^3} + \text{[SLOW-ROLL CORRECTIONS]}$$

$$H_* = H(a)\Big|_{k=aH}$$