Title: PSI 2017/2018 - Cosmology - Lecture 13

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Abstract:

$$S = \int d^4x \sqrt{-g}\left(\frac{M_{pl}^2}{2} R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right)$$

(b)

(a)

$$S = \frac{M_{pl}^2}{2}\int d^4x \sqrt{g^{(3)}}\left(N R^{(3)} + N^{-1} E_{ij} E^{ij} - N^{-1} E^2\right)$$

$$+ \int d^4x \sqrt{g^{(3)}}\left(\frac{1}{2} N^{-1}\left(\dot{\phi} - N^i \partial_i \phi\right)^2 - \frac{1}{2} N g^{(3)ij}(\partial_i \phi)(\partial_j \phi) - N V(\phi)\right)$$

$$E_{ij} = N K_{ij} = \frac{1}{2}\left(\dot{g}^{(3)}_{ij} - \nabla^{(3)}_i N_j - \nabla^{(3)}_j N_i\right)$$

$$E = g^{(3)ij} E_{ij}$$

(b) $$S = \frac{M_{pl}^2}{2}\int d^4x \sqrt{g^{(3)}}\left(N R^{(3)} + N^{-1}\left(-6H^2 - 4H\,\Delta E + (\Delta E)^{ij}(\Delta E_{ij}) - (\Delta E)^2\right)\right)$$

$$+ \int d^4x \sqrt{g^{(3)}}\left[\frac{1}{2}N^{-1}\dot{\phi}^2 - NV\right]$$

$$g_{ij} = a(t)^2\left(\delta_{ij} + h_{ij}\right)$$

$$\delta\phi = 0$$

$$\Delta E_{ij} = E_{ij} - H g^{(3)}_{ij}$$

$$= \frac{1}{2}a^2\dot{h}_{ij} - \frac{1}{2}\left(\nabla^{(3)}_i N_j + \nabla^{(3)}_j N_i\right)$$

$$\Delta E = g^{ij}\,\Delta E_{ij}$$

$$H^2 = \frac{\rho}{3M_{pl}^2} = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$$

$$\Rightarrow S = \frac{M_{pl}^2}{2}\int d^4x\sqrt{g^{(3)}}\left[N R^{(3)} + N^{-1}\left(-4H\Delta E + (DE)^{ij}(DE)_{ij} - (DE)^2\right]\right.$$

$$+ \int d^4x \sqrt{g^{(3)}}\left(-N^{-1} - N\right)V$$

$$N = 1 + \Psi$$

$$S = \frac{M_{pl}^2}{2}\int d^4x\,\sqrt{g^{(3)}}\left[\overbrace{(1+\psi)}^{\text{2ND}}R^{(3)} + (1-\psi)(-4H\,\Delta E) + \underbrace{(\Delta E)^{ij}(\Delta E)_{ij} - (\Delta E)^2}_{\text{2ND}}\right]$$

$$+ \int d^4x\,\sqrt{g^{(3)}}\left(-2-\underbrace{\psi^2}_{\text{2ND}}\right)V$$

$$\left.\begin{array}{l} \int d^4x\,\sqrt{g^{(3)}}\,\overbrace{R^{(3)}} \\ \int d^4x\,\sqrt{g^{(3)}}\,\psi R^{(3)} \end{array}\right\}\ \text{BOTH SECOND ORDER}$$

$$\delta R^{(3)} \neq 0 \qquad \delta\left[\int \sqrt{g^{(3)}}\,R^{(3)}\right] = \int \sqrt{g^{(3)}}\left(-\underbrace{G^{(3)ij}}_{R^{(3)ij} - \frac{1}{2}R^{(3)}\delta^{ij}}\right)(\delta h_{ij})$$

$$S \supset \frac{M_{pl}^2}{2}\int d^4x\,\sqrt{g^{(3)}}\,(-4H\,\Delta E) + \int d^4x\,\sqrt{g^{(3)}}\,(-2V) \leftarrow \text{FIRST ORDER TERMS}$$

$$I = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{g^{(3)}}\, H\, (DE)$$

$$= \frac{M_{Pl}^2}{2} \int d^4x \sqrt{g^{(3)}}\, H \left(\frac{1}{2} a^2 g^{ij} \dot{h}_{ij} - g^{ij} \nabla_i N_j \right)$$

$\delta\phi = 0$

$$H^2 = \frac{\rho}{3M_{pl}^2} = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$$

$$\Rightarrow \quad S = \frac{M_{pl}^2}{2}\int d^4x\sqrt{g^{(3)}}\left[NR^{(3)} + N^{-1}\left(-4H\Delta E + (DE)^{ij}(DE)_{ij} - (DE)^2\right)\right]$$

$$+ \int d^4x\sqrt{g^{(3)}}\left(-N^{-1} - N\right)V$$

$V(\phi) = \underbrace{V(\phi(t))}$

$$N = 1 + \Psi$$

$$I = \frac{M_{pl}^2}{2}\int d^4x\,\sqrt{g^{(3)}}\,H\,(DE)$$

$$= \frac{M_{pl}^2}{2}\int d^4x\,\underbrace{\sqrt{g^{(3)}}\,H}\left(\frac{1}{2}a^2 g^{ij}\dot{h}_{ij} - g^{ij}\cancel{\nabla_i N_j}\right)$$

$$= \frac{M_{pl}^2}{2}\int d^4x\,\sqrt{g^{(3)}}\,H\left(\frac{1}{2}g^{ij}\dot{g}_{ij} - 3H\right)$$

$$= \frac{M_{pl}^2}{2}\int d^4x\,H\left(\frac{d\sqrt{g^{(3)}}}{dt} + \sqrt{g^{(3)}}\,(-3H^2)\right)$$

$$= \frac{M_{pl}^2}{2}\int d^4x\,\sqrt{g^{(3)}}\left(-\dot{H} - 3H^2\right) = \boxed{-\frac{1}{2}\int d^4x\,\sqrt{g^{(3)}}\,V}$$

$$h_{ij} = a^{-2} g_{ij}$$

$$\dot{h}_{ij} = a^{-2}\left(\dot{g}_{ij} - 2H g_{ij}\right)$$

$$\frac{d}{dt}\left(\sqrt{g^{(3)}}\right) = \frac{1}{2}\sqrt{g^{(3)}}\left(g^{ij}\dot{g}_{ij}\right)$$

$$\dot{H} + 3H^2 = \frac{V}{M_{pl}^2}$$

$$h_{ij} = 2\zeta\delta_{ij} + \gamma_{ij} \qquad \delta^{ij}\gamma_{ij} = \partial_i\gamma_{ij} = 0$$

$$\cdots \quad (c) + (d)$$

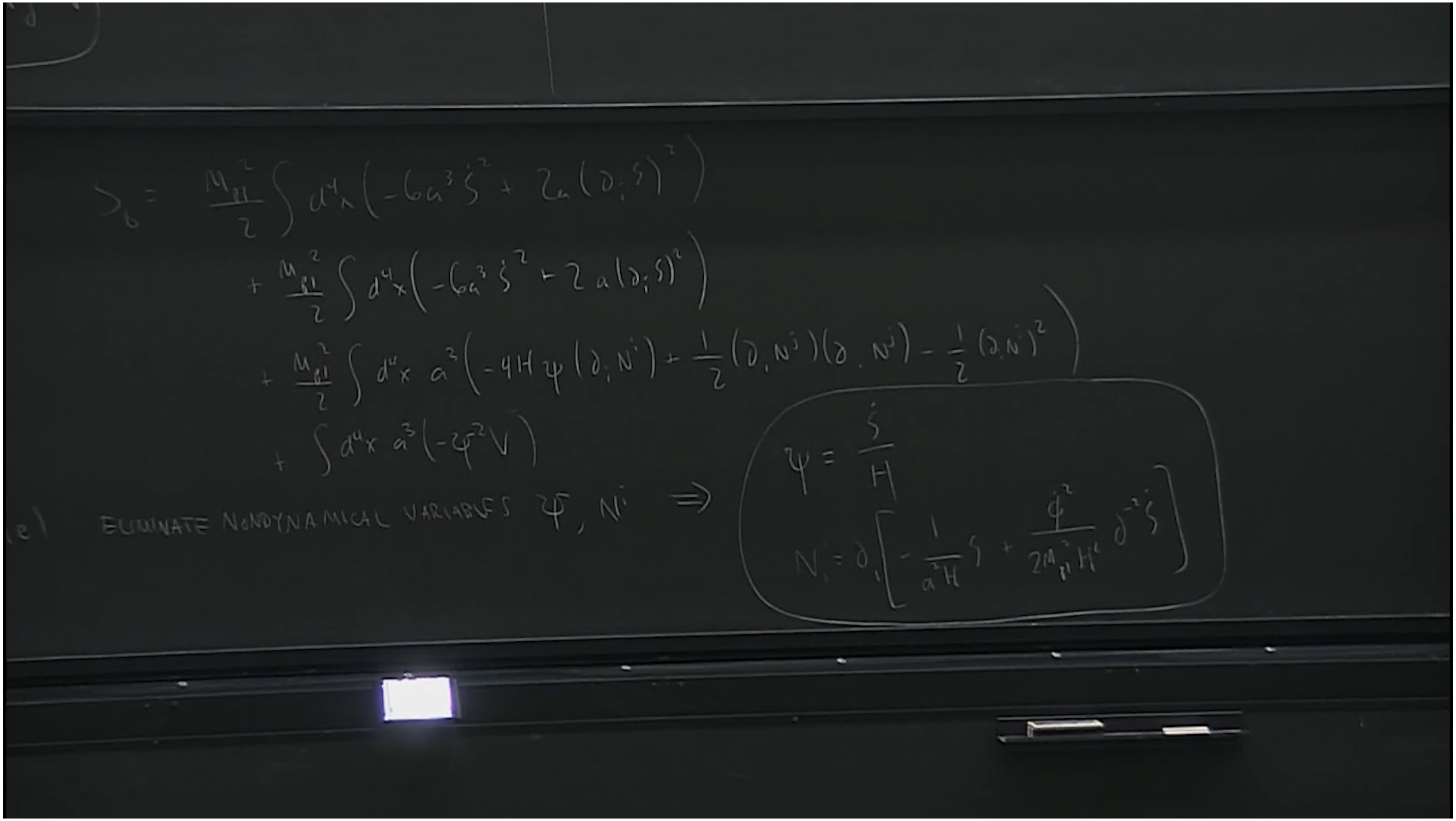
$$S = \underbrace{S_0} + \underbrace{S_2}$$

$$S_2 = \frac{M_{pl}^2}{8}\int d^4x \left(a^3 \dot{\gamma}_{ij}\dot{\gamma}_{ij} - a(\partial_k\gamma_{ij})(\partial_k\gamma_{ij}) \right)$$

$$S_0 = \frac{M_{pl}^2}{2}\int d^4x \left(-6a^3 \ldots \right.$$

$$+ \frac{M_{pl}^2}{2}\int d^4x \left(-6a \ldots \right.$$

$$+ \frac{M_{pl}^2}{2}\int d^4x \; a^3 \left(\ldots \right.$$


$S_0 = \frac{M_{pl}^2}{2}\int d^4x\left(-6a^3\dot{\zeta}^2 + 2a(\partial_i \zeta)^2\right)$
$+ \frac{M_{pl}^2}{2}\int d^4x\left(-6a^3\dot{\zeta}^2 + 2a(\partial_i \zeta)^2\right)$
$+ \frac{M_{pl}^2}{2}\int d^4x\, a^3\left(-4H\psi(\partial_i N^i) + \frac{1}{2}(\partial_i N^j)(\partial_i N^j) - \frac{1}{2}(\partial_i N^i)^2\right)$
$+ \int d^4x\, a^3(-\psi^2 V)$
ELIMINATE NONDYNAMICAL VARIABLES ψ, N^i $\Rightarrow$
$\psi = \frac{\dot{\zeta}}{H}$
$N_i = \partial_i\left[-\frac{1}{a^2 H}\zeta + \frac{\dot{\phi}^2}{2M_{pl}^2 H^2}\partial^{-2}\dot{\zeta}\right]$

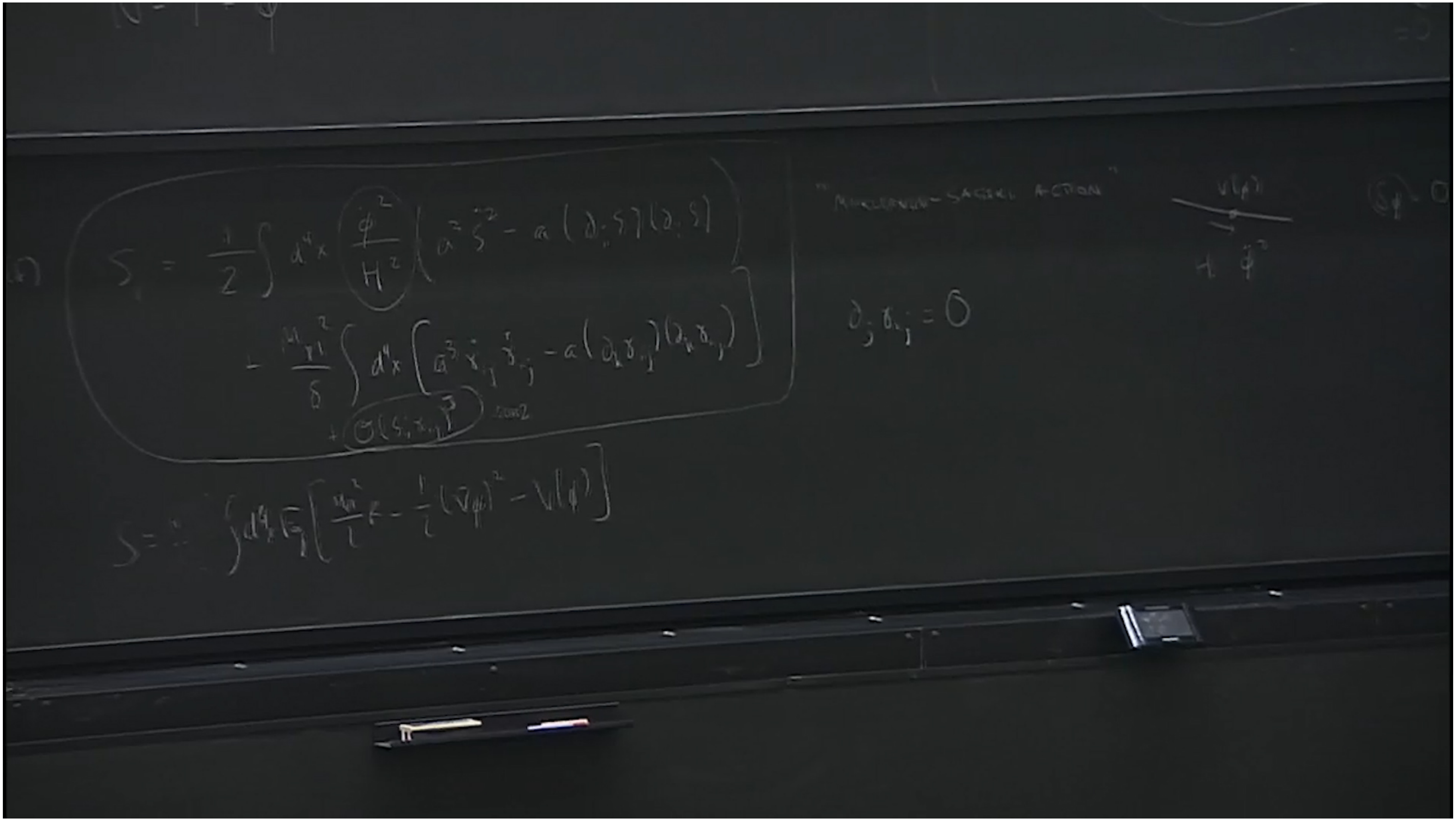
$$S = \frac{1}{2}\int d^4x \, \frac{\dot{\phi}^2}{H^2}\left(a^3 \dot{\zeta}^2 - a\,(\partial_i \zeta)(\partial_i \zeta)\right) + \frac{M_{pl}^2}{8}\int d^4x \left[a^3 \dot{\gamma}_{ij}\dot{\gamma}_{ij} - a\,(\partial_k \gamma_{ij})(\partial_k \gamma_{ij})\right]$$

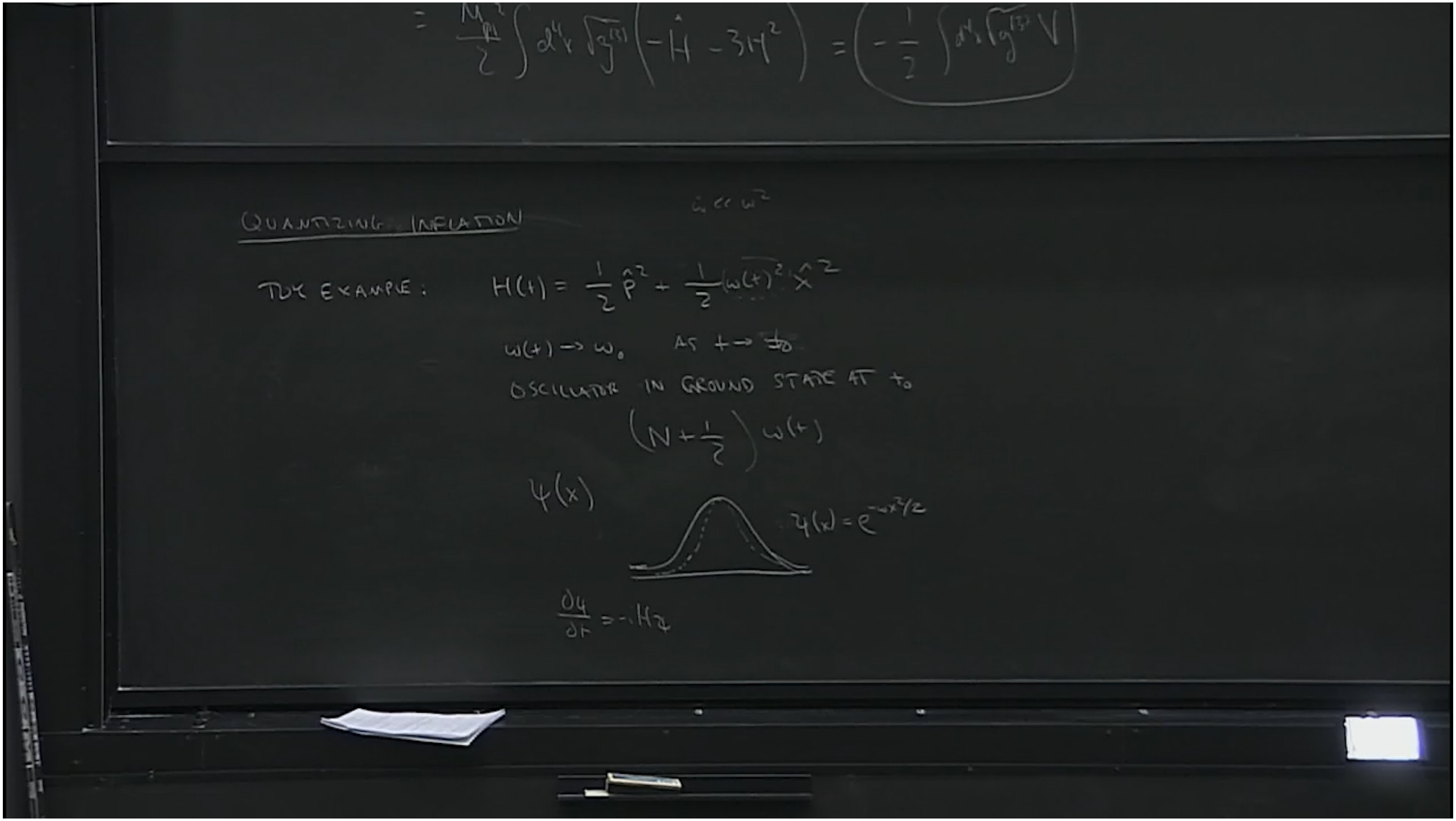
"MUKHANOV-SASAKI ACTION"

$V(\phi)$

$H \quad \dot{\phi}^2$

$$\partial_j \gamma_{ij} = 0$$


$S_2 = \frac{1}{2}\int d^4x \, \frac{\dot\phi^2}{H^2}\left(a^3 \dot\zeta^2 - a (\partial_i \zeta)(\partial_i \zeta) \right)$
$+ \frac{M_{pl}^2}{8}\int d^4x \left[a^3 \dot\gamma_{ij} \dot\gamma_{ij} - a (\partial_k \gamma_{ij})(\partial_k \gamma_{ij}) \right]$
$+ \mathcal{O}(\zeta, \gamma_{ij})^3$
ACTION
$\partial_j \gamma_{ij} = 0$
$V(\phi)$


$= \frac{M_{pl}^2}{2}\int d^4x \sqrt{g^{(3)}}\left(-\dot{H} - 3H^2\right) = \left(-\frac{1}{2}\int d^4x \sqrt{g^{(3)}}\, V\right)$
QUANTIZING INFLATION
$\dot{\omega} \ll \omega^2$
TOY EXAMPLE: $H(t) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega(t)^2\hat{x}^2$
$\omega(t) \to \omega_0$ AS $t \to t_0$
OSCILLATOR IN GROUND STATE AT t_0
$\left(N + \frac{1}{2}\right)\omega(t)$
$\psi(x)$
$\psi(x) = e^{-\omega x^2/2}$
$\frac{\partial\psi}{\partial t} = -iH\psi$

$$= \frac{M_{pl}^2}{2}\int d^4x\sqrt{g^{(3)}}\left(-\dot{H} - 3H^2\right) = \boxed{-\frac{1}{2}\int d^4x\sqrt{g^{(3)}}\,V}$$

QUANTIZING INFLATION

$\dot{\omega} \ll \omega^2$

TOY EXAMPLE: $H(t) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega(t)^2\hat{x}^2$

$\omega(t) \to \omega_0$ AS $t \to t_0$

OSCILLATOR IN GROUND STATE AT t_0

$$\left(N + \frac{1}{2}\right)\omega(t)$$

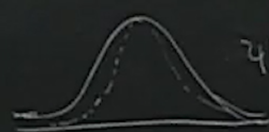
AT EARLY TIMES

$$\hat{x} = \frac{1}{\sqrt{2\omega_0}}(a + a^\dagger)$$

$$\hat{p} = -i\sqrt{\frac{\omega_0}{2}}(a - a^\dagger)$$

$$a|\psi(t_0)\rangle = 0$$

$\psi(x)$



$\psi(x) = e^{-\omega x^2/2}$

$$\frac{\partial\psi}{\partial t} = -iH\psi$$

SCHRODINGER PICTURE

$$\frac{d}{dt}|\psi(t)\rangle = -i H_S(t)\,|\psi(t)\rangle$$

$$|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle \qquad U(t,t_0) \neq e^{-iH(t-t_0)}$$

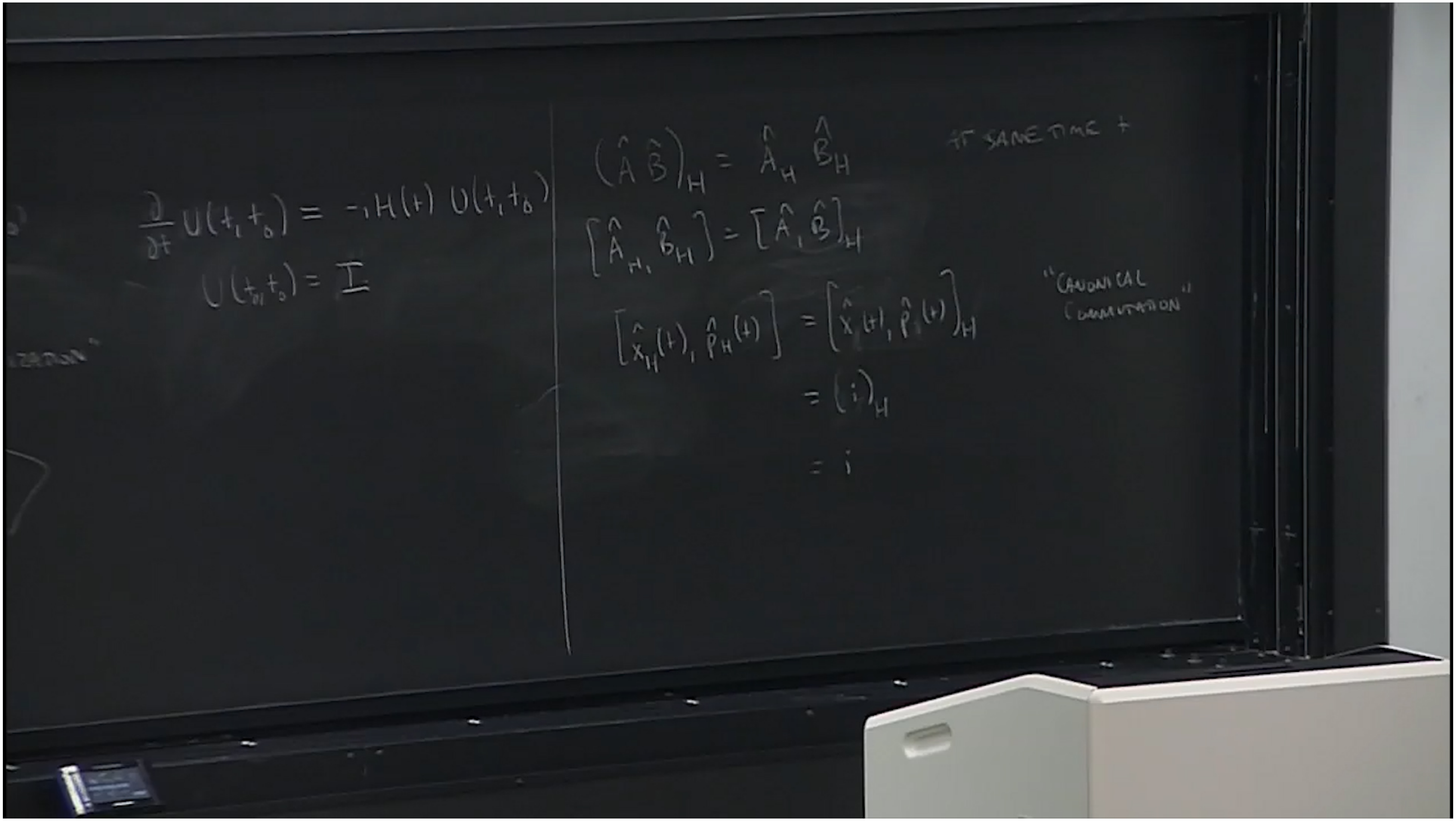
HEISENBERG PICTURE

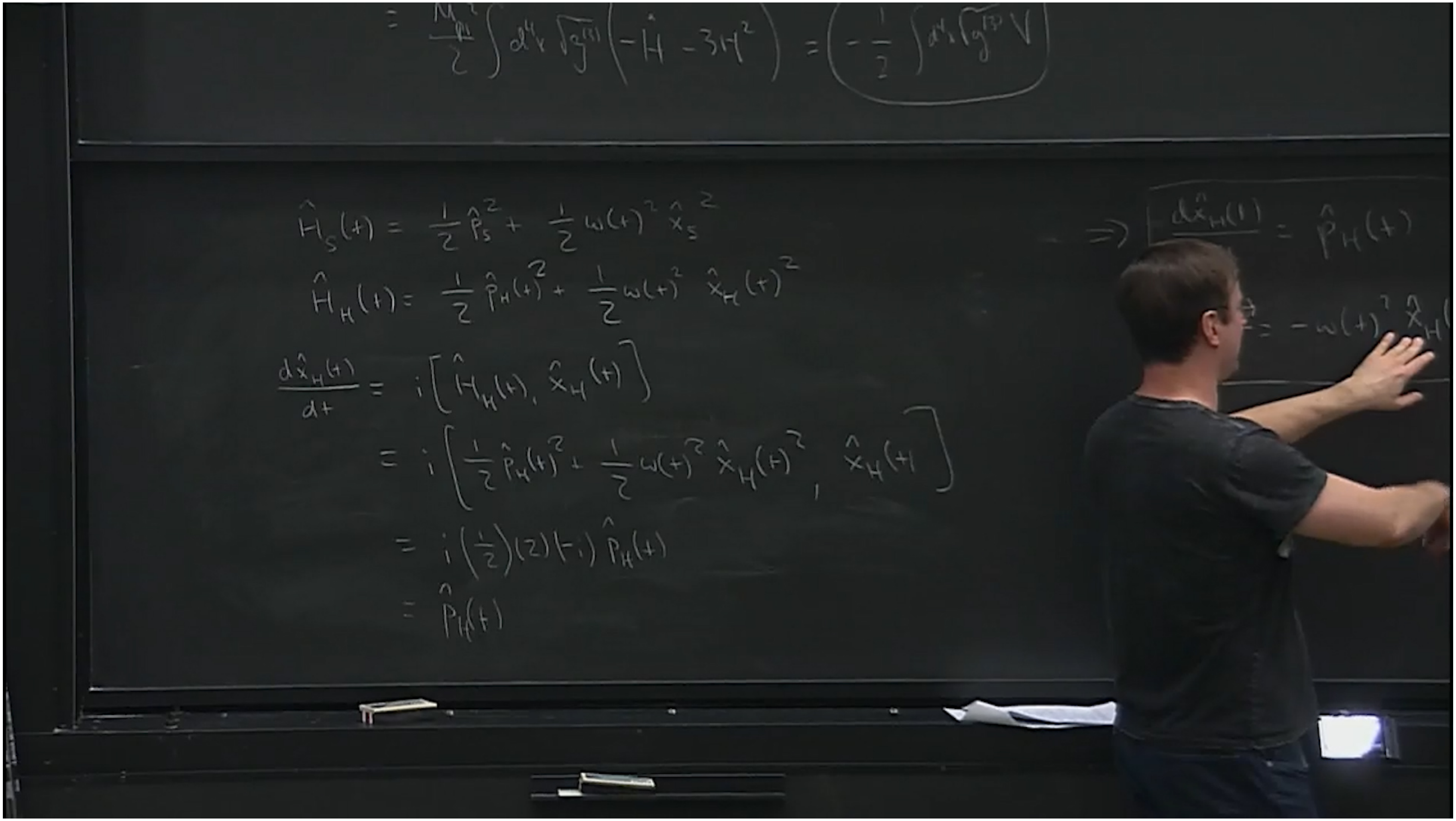
$$\mathcal{O}_H(t) \overset{\text{def}}{=} U(t,t_0)^\dagger\, \mathcal{O}_S(t)\, U(t,t_0)$$

$$\langle\psi(t)|\,\mathcal{O}_S(t)\,|\psi(t)\rangle = \langle\psi(t_0)|\,\mathcal{O}_H(t)\,|\psi(t_0)\rangle$$

$$\dot{\mathcal{O}}_H(t) = i\left[\hat{H}_H(t), \mathcal{O}_H(t)\right] + \left(\frac{\partial \mathcal{O}}{\partial t}\right)_H$$

↳ HEISENBERG-PICTURE HAMILTONIAN


$\frac{\partial}{\partial t} U(t, t_0) = -i H(t) U(t, t_0)$
$U(t_0, t_0) = I$
$(\hat{A}\hat{B})_H = \hat{A}_H \hat{B}_H$
AT SAME TIME t
$[\hat{A}_H, \hat{B}_H] = [\hat{A}, \hat{B}]_H$
$[\hat{x}_H(t), \hat{p}_H(t)] = [\hat{x}(t), \hat{p}(t)]_H$
$= (i)_H$
$= i$
"CANONICAL COMMUTATION"



$= \frac{M_{pl}^2}{2}\int d^4x\sqrt{g^{(3)}}\left(-\dot{H} - 3H^2\right) = \boxed{-\frac{1}{2}\int d^4x\sqrt{g^{(3)}}\,V}$
$\hat H_S(t) = \frac{1}{2}\hat p_S^2 + \frac{1}{2}\omega(t)^2\hat x_S^2$
$\hat H_H(t) = \frac{1}{2}\hat p_H(t)^2 + \frac{1}{2}\omega(t)^2\hat x_H(t)^2$
$\frac{d\hat x_H(t)}{dt} = i\left[\hat H_H(t), \hat x_H(t)\right]$
$= i\left[\frac{1}{2}\hat p_H(t)^2 + \frac{1}{2}\omega(t)^2\hat x_H(t)^2, \hat x_H(t)\right]$
$= i\left(\frac{1}{2}\right)(2)(-i)\hat p_H(t)$
$= \hat p_H(t)$
$\Rightarrow \frac{d\hat x_H(t)}{dt} = \hat p_H(t)$
$= -\omega(t)^2\hat x_H$

$$3!\gamma^2 \Big) = \boxed{-\frac{1}{2}\int d^4x \sqrt{g^{(3)}}\, V}$$

$$\hat{x}_H(t)^2$$

$$\Rightarrow \boxed{\begin{aligned} \frac{d\hat{x}_H(t)}{dt} &= \hat{p}_H(t) \\ \frac{d\hat{p}_H(t)}{dt} &= -\omega(t)^2\,\hat{x}_H(t) \end{aligned}}$$

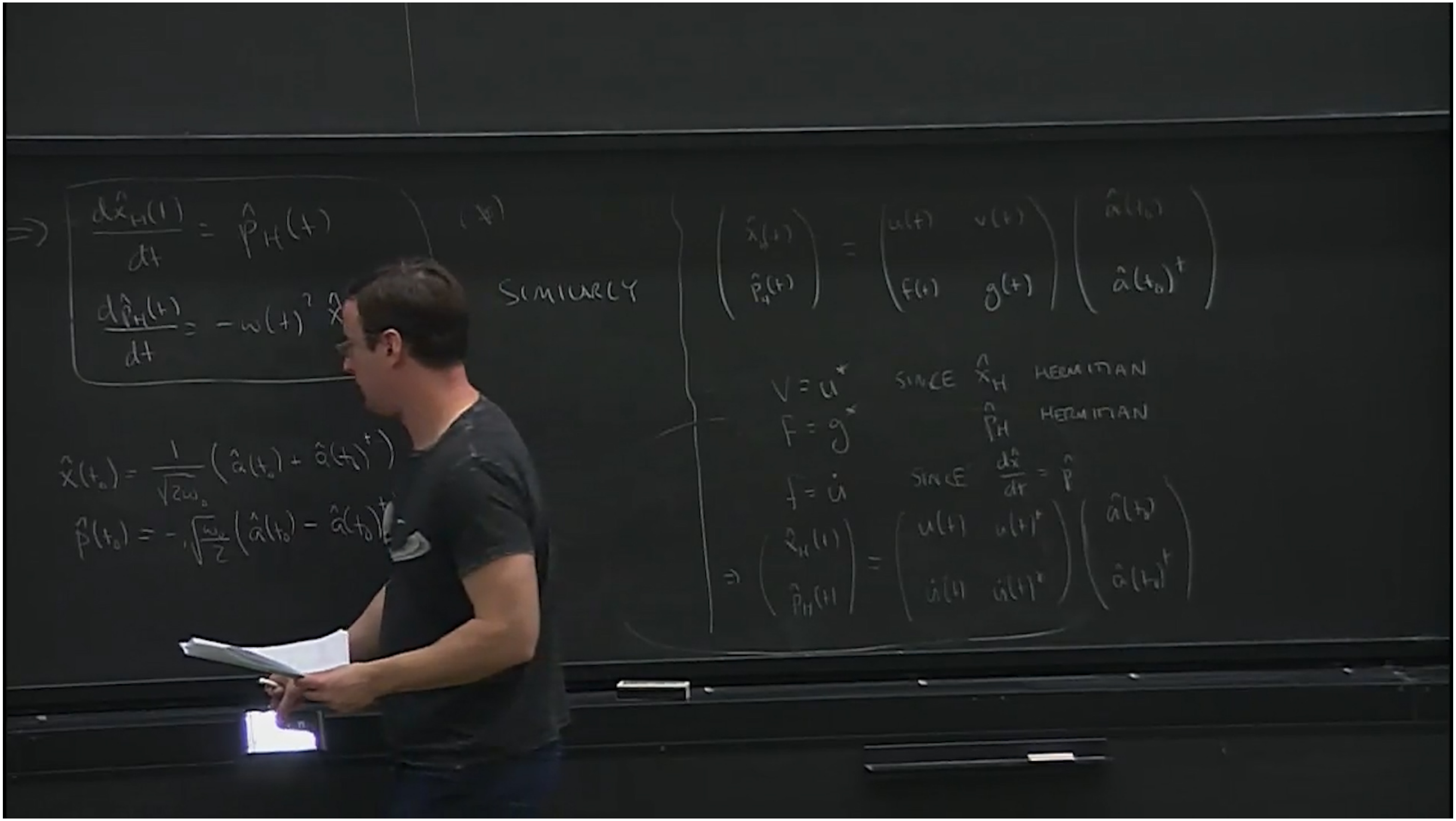
SIMILARLY

$$\begin{pmatrix} \hat{x}(t) \\ \hat{p}(t) \end{pmatrix} = \begin{pmatrix} u(t) & v(t) \\ f(t) & g(t) \end{pmatrix}$$

$$\hat{x}_H(t)^2,\ \hat{x}_H(t)\Big]$$

$$\hat{x}(t_0) = \frac{1}{\sqrt{2\omega_0}}\left(\hat{a}(t_0) + \hat{a}(t_0)^\dagger\right)$$

$$\hat{p}(t_0) = -i\sqrt{\frac{\omega_0}{2}}\left(\hat{a}(t_0) - \hat{a}(t_0)^\dagger\right)$$


$\Rightarrow \frac{d\hat{x}_H(t)}{dt} = \hat{p}_H(t)$
$\frac{d\hat{p}_H(t)}{dt} = -\omega(t)^2 \hat{x}$
(*)
SIMILARLY
$\begin{pmatrix} \hat{x}_H(t) \\ \hat{p}_H(t) \end{pmatrix} = \begin{pmatrix} u(t) & v(t) \\ f(t) & g(t) \end{pmatrix} \begin{pmatrix} \hat{a}(t_0) \\ \hat{a}(t_0)^\dagger \end{pmatrix}$
$v = u^*$ SINCE $\hat{x}_H$ HERMITIAN
$f = g^*$ $\hat{p}_H$ HERMITIAN
$f = \dot{u}$ SINCE $\frac{d\hat{x}}{dt} = \hat{p}$
$\Rightarrow \begin{pmatrix} \hat{x}_H(t) \\ \hat{p}_H(t) \end{pmatrix} = \begin{pmatrix} u(t) & u(t)^* \\ \dot{u}(t) & \dot{u}(t)^* \end{pmatrix} \begin{pmatrix} \hat{a}(t_0) \\ \hat{a}(t_0)^\dagger \end{pmatrix}$
$\hat{x}(t_0) = \frac{1}{\sqrt{2\omega_0}} \left(\hat{a}(t_0) + \hat{a}(t_0)^\dagger \right)$
$\hat{p}(t_0) = -i\sqrt{\frac{\omega_0}{2}} \left(\hat{a}(t_0) - \hat{a}(t_0)^\dagger \right)$

$\hat{p}_H(t)$ $\dot{u}(t)$ $\dot{u}(t)^*$ $a(t_0)$

EQ OF MOTION FOR $\hat{p}$ $\Rightarrow$ $\boxed{\ddot{u} = -\omega(t)^2 u}$ $x(t)$

$-2H g_j$

g_j

$\langle x^2 \rangle$

$\langle x^4 \rangle$

INITIAL CONDITIONS ARE:

$$u(t) \to \frac{1}{(2\omega_0)^{1/2}} e^{-i\omega_0 (t-t_0)}$$

$$\hat{x}_H(t) = u(t)\, a + u(t)^*\, a^\dagger$$

$$\hat{p}_H(t) = \dot{u}(t)\, a + \dot{u}(t)^*\, a^\dagger$$