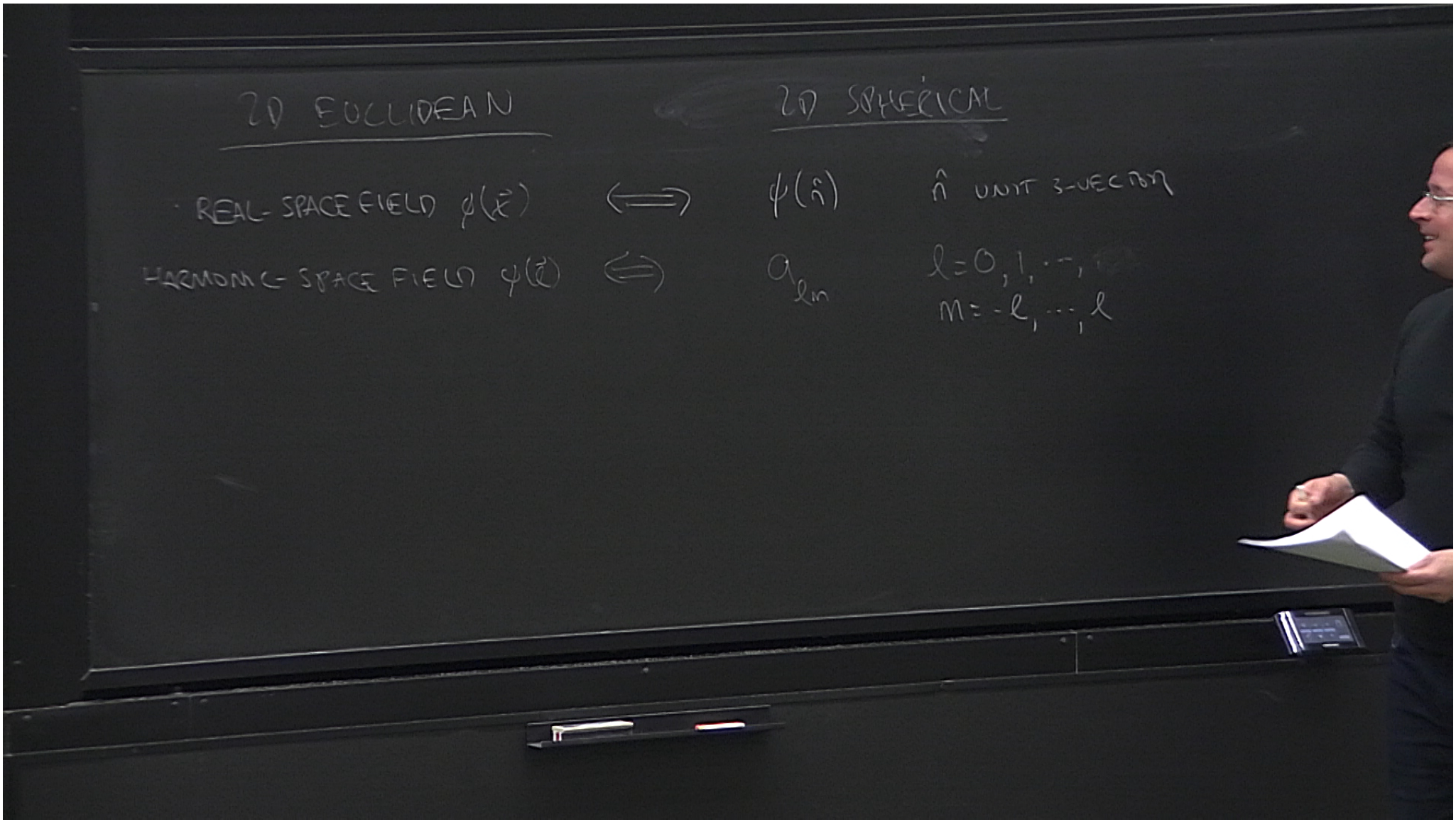
Title: PSI 2017/2018 - Cosmology - Lecture 7

Date: Apr 17, 2018 10:15 AM

URL: http://pirsa.org/18040019

Abstract: First look at CMB: surface of last scattering, Planck spectrum, multipole expansion, dark matter evidence


2D EUCLIDEAN
2D SPHERICAL
REAL-SPACE FIELD $\phi(\vec{x})$ $\Longleftrightarrow$ $\psi(\hat{n})$ $\hat{n}$ UNIT 3-VECTOR
HARMONIC-SPACE FIELD $\psi(\vec{k})$ $\Longleftrightarrow$ $a_{\ell m}$ $\ell = 0, 1, \cdots,$
$m = -\ell, \cdots, \ell$

2D EUCLIDEAN		2D SPHERICAL	
REAL-SPACE FIELD $\phi(\vec{x})$	$\Longleftrightarrow$	$\psi(\hat{n})$	$\hat{n}$ UNIT 3-VECTOR
HARMONIC-SPACE FIELD $\psi(\vec{\ell})$	$\Longleftrightarrow$	$a_{\ell m}$	$\ell = 0, 1, \cdots,$ $m = -\ell, \cdots, \ell$

POWER SPECTRUM

$$\langle \tilde{\phi}(\ell)\, \tilde{\phi}(\ell')^* \rangle = P(\ell)\, (2\pi)^2 \delta(\ell' \ell') \Longleftrightarrow$$

2D EUCLIDEAN		2D SPHERICAL	
REAL-SPACE FIELD $\phi(\vec{x})$	$\Longleftrightarrow$	$\psi(\hat{n})$	$\hat{n}$ UNIT 3-VECTOR
HARMONIC-SPACE FIELD $\phi(\vec{k})$	$\Longleftrightarrow$	$a_{\ell m}$	$\ell = 0, 1, \cdots,$ $m = -\ell, \cdots, \ell$

POWER SPECTRUM

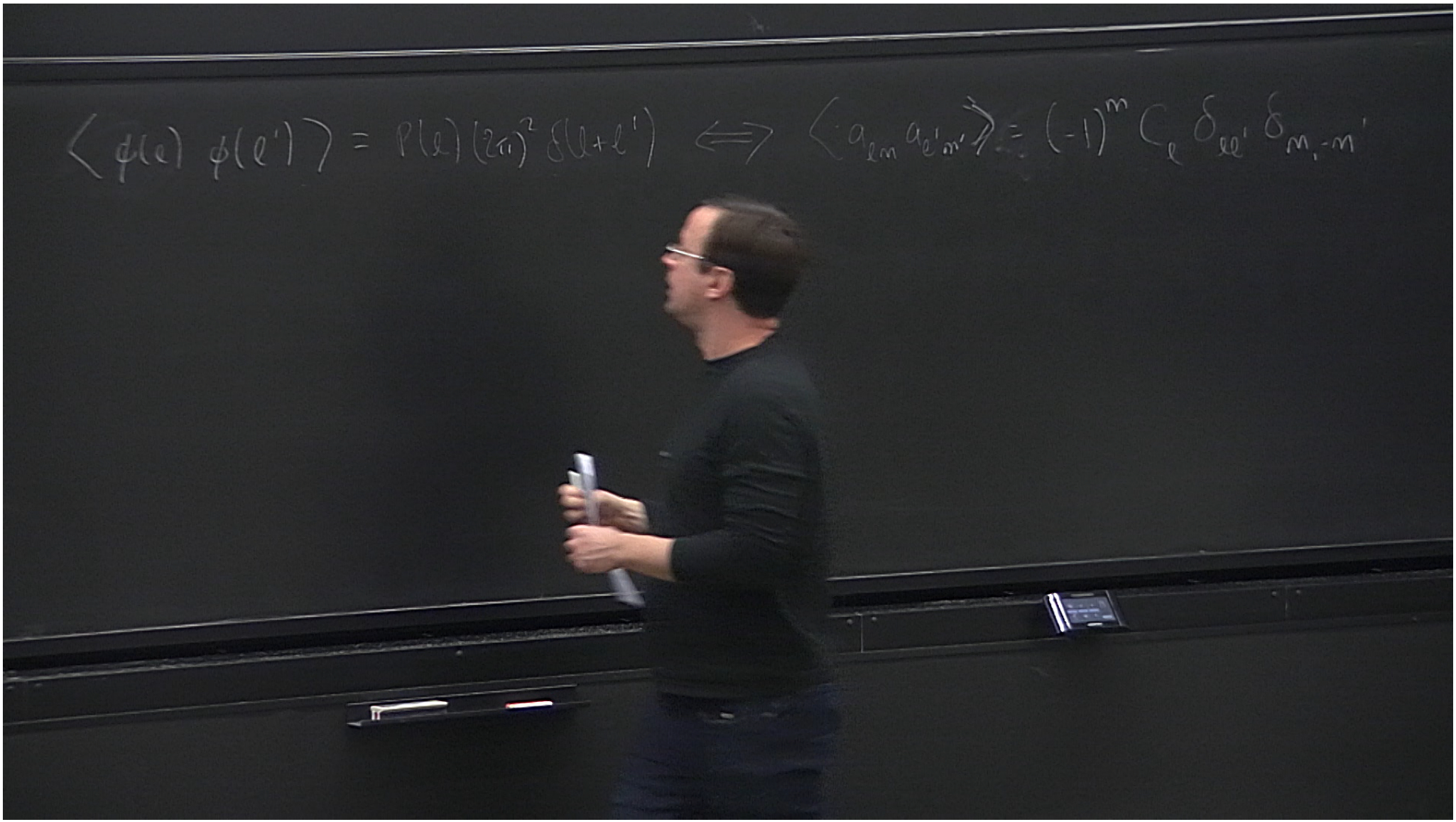
$$\langle \phi(\ell)\, \phi(\ell')^* \rangle = P(\ell)\, (2\pi)^2 \delta(\ell - \ell') \Longleftrightarrow$$

2D EUCLIDEAN		2D SPHERICAL	
REAL-SPACE FIELD $\phi(\vec{x})$	$\Longleftrightarrow$	$\psi(\hat{n})$	$\hat{n}$ UNIT 3-VECTOR
HARMONIC-SPACE FIELD $\varphi(\vec{\ell})$	$\Longleftrightarrow$	$a_{\ell m}$	$\ell = 0, 1, \cdots,$ $m = -\ell, \cdots, \ell$

POWER SPECTRUM

$$\langle \varphi(\ell)\, \varphi(\ell')^* \rangle = P(\ell)\, (2\pi)^3 \delta(\ell - \ell') \Longleftrightarrow \langle a_{\ell m}\, a^*_{\ell' m'} \rangle = C_\ell\, \delta_{\ell\ell'}\, \delta_{mm'}$$

REAL FIELD $\varphi(-\ell) = \varphi(\ell)^* \Longleftrightarrow a^*_{\ell m} = (-1)^m a_{\ell, -m}$


$\langle \phi(\ell)\, \phi(\ell') \rangle = P(\ell)\,(2\pi)^2\, \delta(\ell+\ell') \iff \langle a_{\ell m}\, a_{\ell' m'} \rangle = (-1)^m\, C_\ell\, \delta_{\ell\ell'}\, \delta_{m,-m'}$

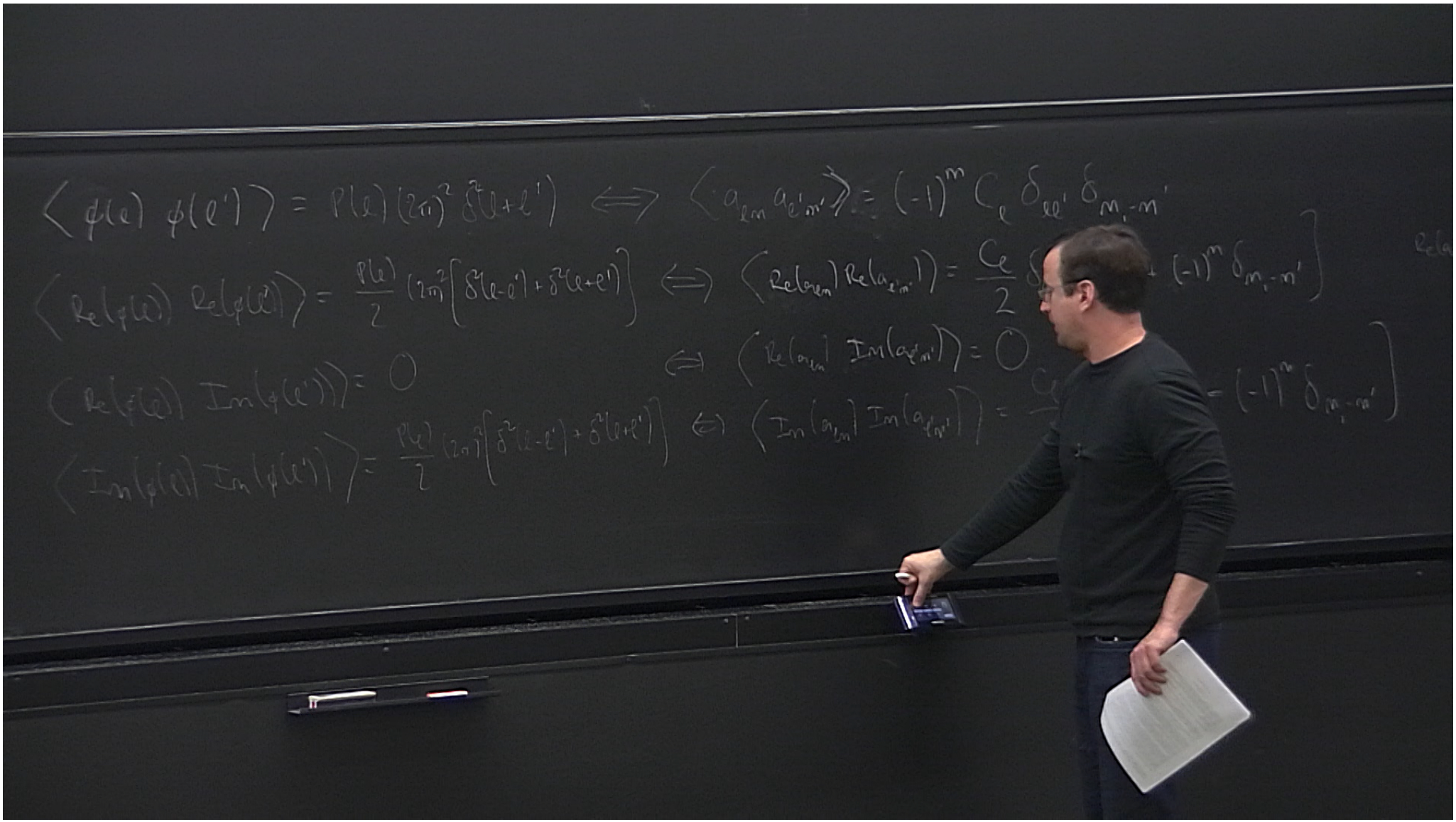
$$\langle \phi(\ell)\, \phi(\ell') \rangle = P(\ell)\,(2\pi)^2\, \delta^2(\ell+\ell') \iff \langle a_{\ell m}\, a_{\ell' m'} \rangle = (-1)^m\, C_\ell\, \delta_{\ell\ell'}\, \delta_{m,-m'}$$

$$\langle \mathrm{Re}[\phi(\ell)]\, \mathrm{Re}[\phi(\ell')] \rangle = \frac{P(\ell)}{2}\,(2\pi)\left[\delta^2(\ell-\ell') + \delta^2(\ell+\ell')\right] \iff \langle \mathrm{Re}[a_{\ell m}]\, \mathrm{Re}[a_{\ell' m'}] \rangle = \frac{C_\ell}{2}\, \delta_{\ell\ell'}\left[\delta_{mm'} + (-1)^m\, \delta_{m,-m'}\right]$$

$$(-1)^m C_\ell \, \delta_{\ell\ell'} \, \delta_{m,-m'}$$

$$= \frac{C_\ell}{2} \delta_{\ell\ell'} \left[\delta_{mm'} + (-1)^m \delta_{m,-m'} \right]$$

$$\mathrm{Re}[a_{\ell m}] = \frac{a_{\ell m} + a_{\ell m}^*}{2}$$


$\langle \phi(\ell)\, \phi(\ell') \rangle = P(\ell)(2\pi)^2 \delta^2(\ell+\ell') \iff \langle a_{\ell m} a_{\ell' m'} \rangle = (-1)^m C_\ell \delta_{\ell\ell'} \delta_{m,-m'}$
$\langle \mathrm{Re}(\phi(\ell))\, \mathrm{Re}(\phi(\ell')) \rangle = \frac{P(\ell)}{2}(2\pi)^2 \left[\delta^2(\ell-\ell') + \delta^2(\ell+\ell')\right] \iff \langle \mathrm{Re}(a_{\ell m})\, \mathrm{Re}(a_{\ell' m'}) \rangle = \frac{C_\ell}{2}$
$+ (-1)^m \delta_{m,-m'}$
$\langle \mathrm{Re}(\phi(\ell))\, \mathrm{Im}(\phi(\ell')) \rangle = 0 \iff \langle \mathrm{Re}(a_{\ell m})\, \mathrm{Im}(a_{\ell' m'}) \rangle = 0$
$\langle \mathrm{Im}(\phi(\ell))\, \mathrm{Im}(\phi(\ell')) \rangle = \frac{P(\ell)}{2}(2\pi)^2 \left[\delta^2(\ell-\ell') + \delta^2(\ell+\ell')\right] \iff \langle \mathrm{Im}(a_{\ell m})\, \mathrm{Im}(a_{\ell' m'}) \rangle =$
$= (-1)^m \delta_{m,-m'}$

$$\langle \tilde{\phi}(\ell)\, \tilde{\phi}(\ell') \rangle = P(\ell)\,(2\pi)^2 \delta(\ell-\ell') \iff \langle a_{\ell m}\, a^*_{\ell' m'} \rangle = C_\ell\, \delta_{\ell\ell'}\, \delta_{mm'}$$

FIELD $\tilde{\phi}(-\ell) = \tilde{\phi}(\ell)^*$ $\iff$ $a^*_{\ell m} = (-1)^m a_{\ell,-m}$

$$\langle \phi(\ell)\, \phi(\ell') \rangle = P(\ell)\,(2\pi)^2 \delta(\ell+\ell') \iff \langle a_{\ell m}\, a_{\ell' m'} \rangle = (-1)^m C_\ell\, \delta_{\ell\ell'}\, \delta_{m,-m'}$$

$$\langle \mathrm{Re}(\phi(\ell))\, \mathrm{Re}(\phi(\ell')) \rangle = \frac{P(\ell)}{2}(2\pi)^2 \left[\delta^2(\ell-\ell') + \delta^2(\ell+\ell')\right] \iff \langle \mathrm{Re}(a_{\ell m})\, \mathrm{Re}(a_{\ell' m'}) \rangle = \frac{C_\ell}{2}\, \delta_{\ell\ell'} \left[\delta_{mm'} + (-1)^m \delta_{m,-m'}\right]$$

$$\langle \mathrm{Re}(\phi(\ell))\, \mathrm{Im}(\phi(\ell')) \rangle = 0 \iff \langle \mathrm{Re}(a_{\ell m})\, \mathrm{Im}(a_{\ell' m'}) \rangle = 0$$

$$\langle \mathrm{Im}(\phi(\ell))\, \mathrm{Im}(\phi(\ell')) \rangle = \frac{P(\ell)}{2}(2\pi)^2 \left[\delta^2(\ell-\ell') + \delta^2(\ell+\ell')\right] \iff \langle \mathrm{Im}(a_{\ell m})\, \mathrm{Im}(a_{\ell' m'}) \rangle = \frac{C_\ell}{2}\, \delta_{\ell\ell'} \left[\delta_{mm'} - (-1)^m \delta_{m,-m'}\right]$$

AT EACH MULTIPOLE ℓ, THERE ARE $(2\ell+1)$ REAL DOF'S

$$\{\mathrm{Re}(a_{\ell 0}), \mathrm{Re}(a_{\ell 1}), \mathrm{Im}(a_{\ell 1}), \ldots, \mathrm{Re}(a_{\ell \ell}), \mathrm{Im}(a_{\ell \ell})\}$$

$$a_{\ell,-m} = (-1)^m a^*_{\ell m}$$

WITH $(2\ell+1)\times(2\ell+1)$ COVARIANCE MATRIX

$$\begin{pmatrix} C_\ell & & & & & \\ & C_\ell/2 & 0 & & & \\ & 0 & C_\ell/2 & & & \\ & & & \ddots & & \\ & & & & C_\ell/2 & 0 \\ & & & & 0 & C_\ell/2 \end{pmatrix}$$

AT EACH MULTIPOLE ℓ, THERE ARE $(2\ell+1)$ REAL DOF'S

$$\left\{ \mathrm{Re}(a_{\ell 0}),\ \mathrm{Re}(a_{\ell 1}),\ \mathrm{Im}(a_{\ell 1}),\ \cdots,\ \mathrm{Re}(a_{\ell \ell}),\ \mathrm{Im}(a_{\ell \ell}) \right\}$$

$$a_{\ell,-m} = (-1)^m a_{\ell m}^*$$

WITH $(2\ell+1)$-BY-$(2\ell+1)$ COVARIANCE MATRIX

$$\begin{pmatrix} C_\ell & & & & & \\ & C_\ell/2 & 0 & & & \\ & 0 & C_\ell/2 & & & \\ & & & \ddots & & \\ & & & & C_\ell/2 & 0 \\ & & & & 0 & C_\ell/2 \end{pmatrix}$$

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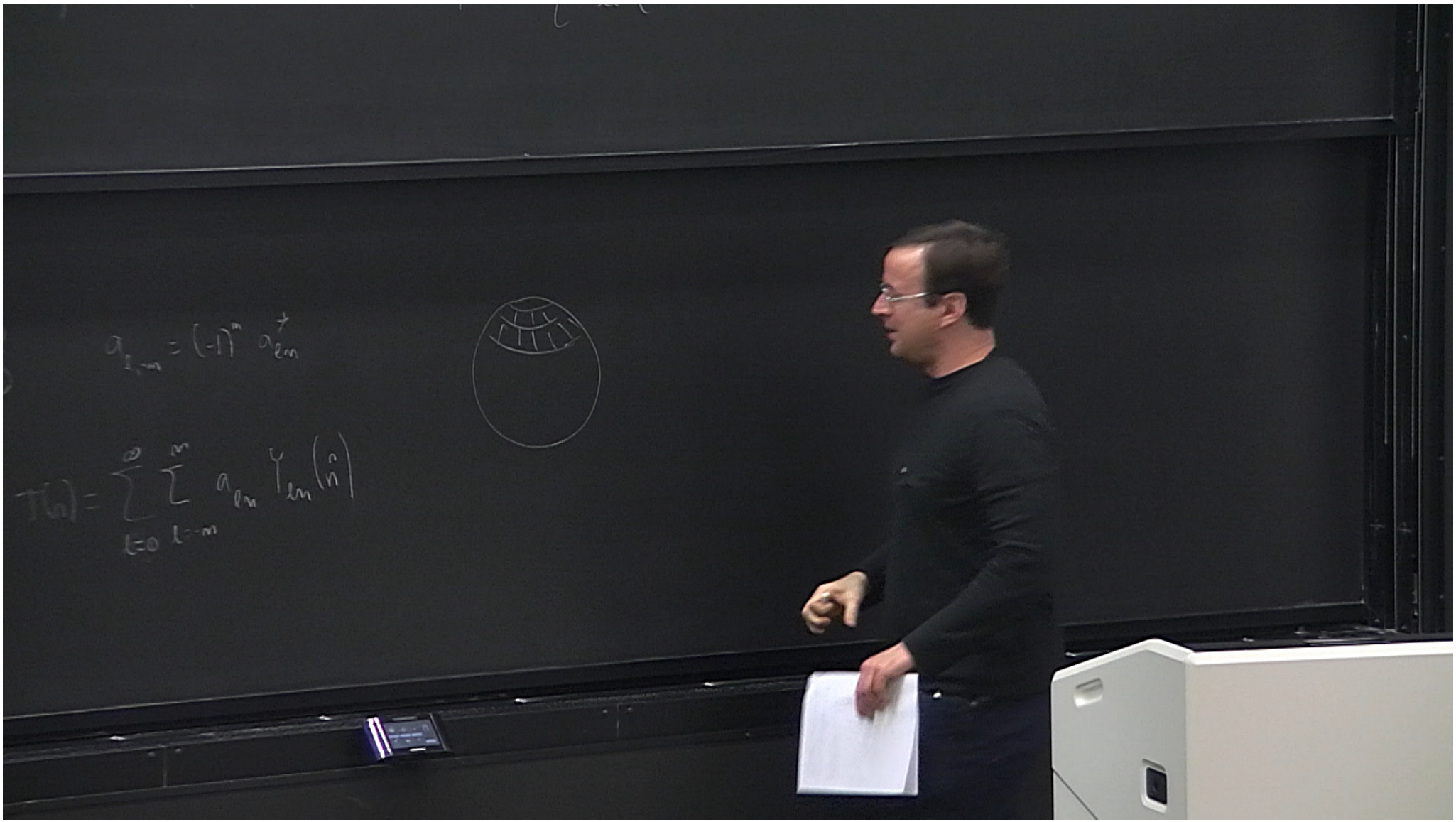
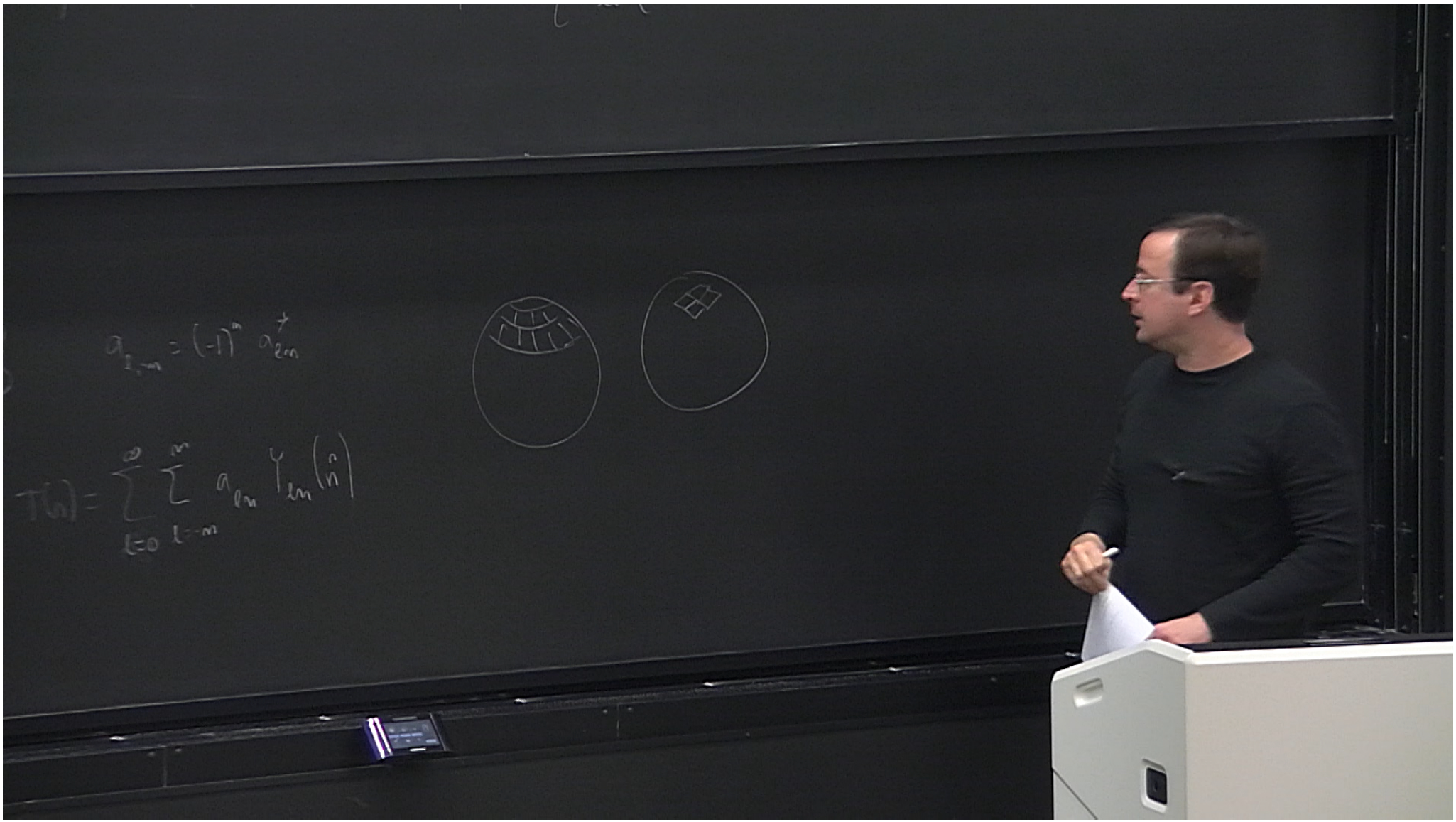
$$a_{\ell m} \longrightarrow T(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{\ell=-m}^{m} a_{\ell m} Y_{\ell m}(\hat{n})$$


$a_{l,-m} = (-1)^m a^*_{lm}$
$T(n) = \sum_{l=0}^{\infty} \sum_{l=-m}^{m} a_{lm} Y_{lm}(\hat{n})$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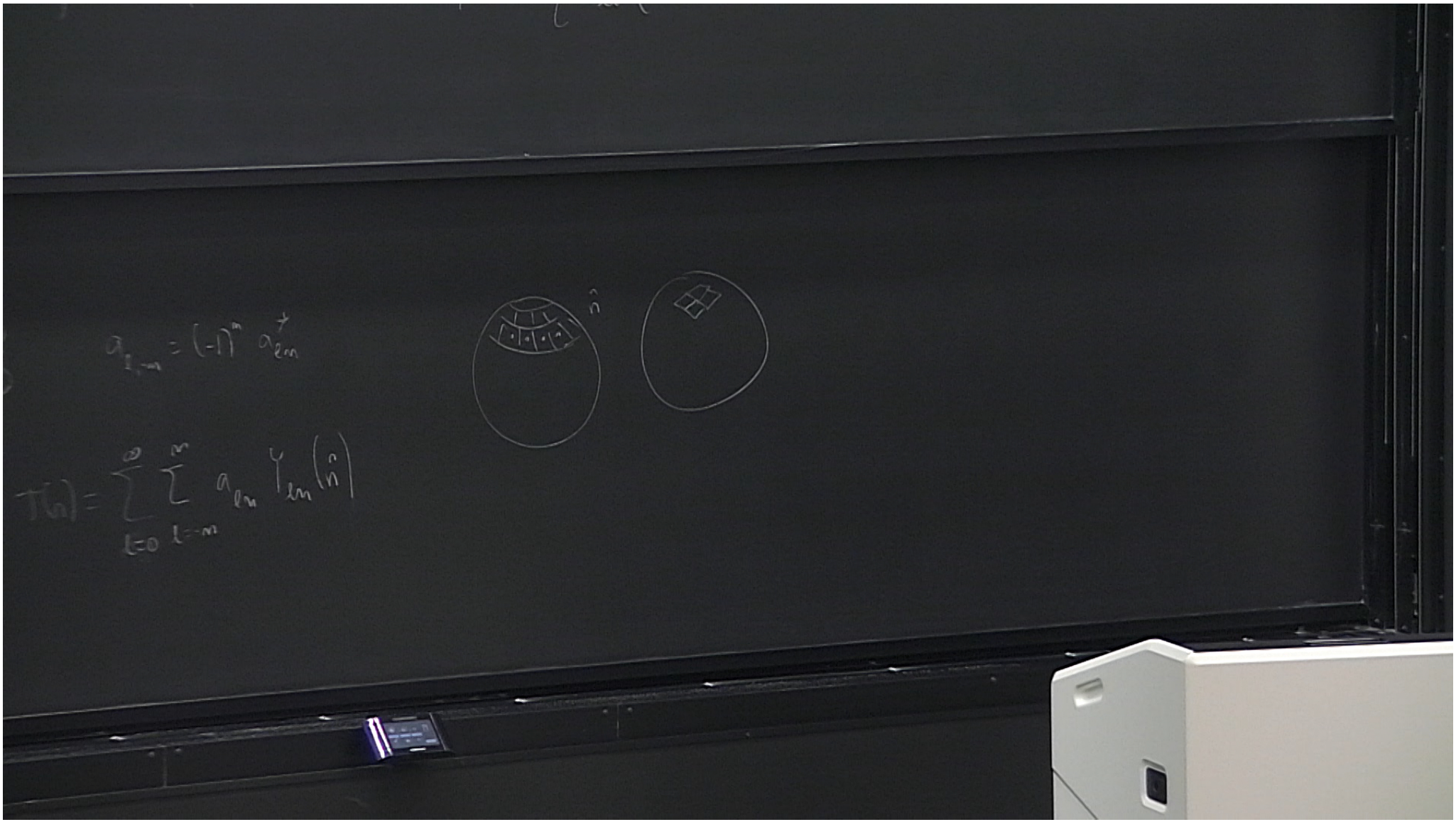
$a_{l,-m} = (-1)^m a^*_{lm}$
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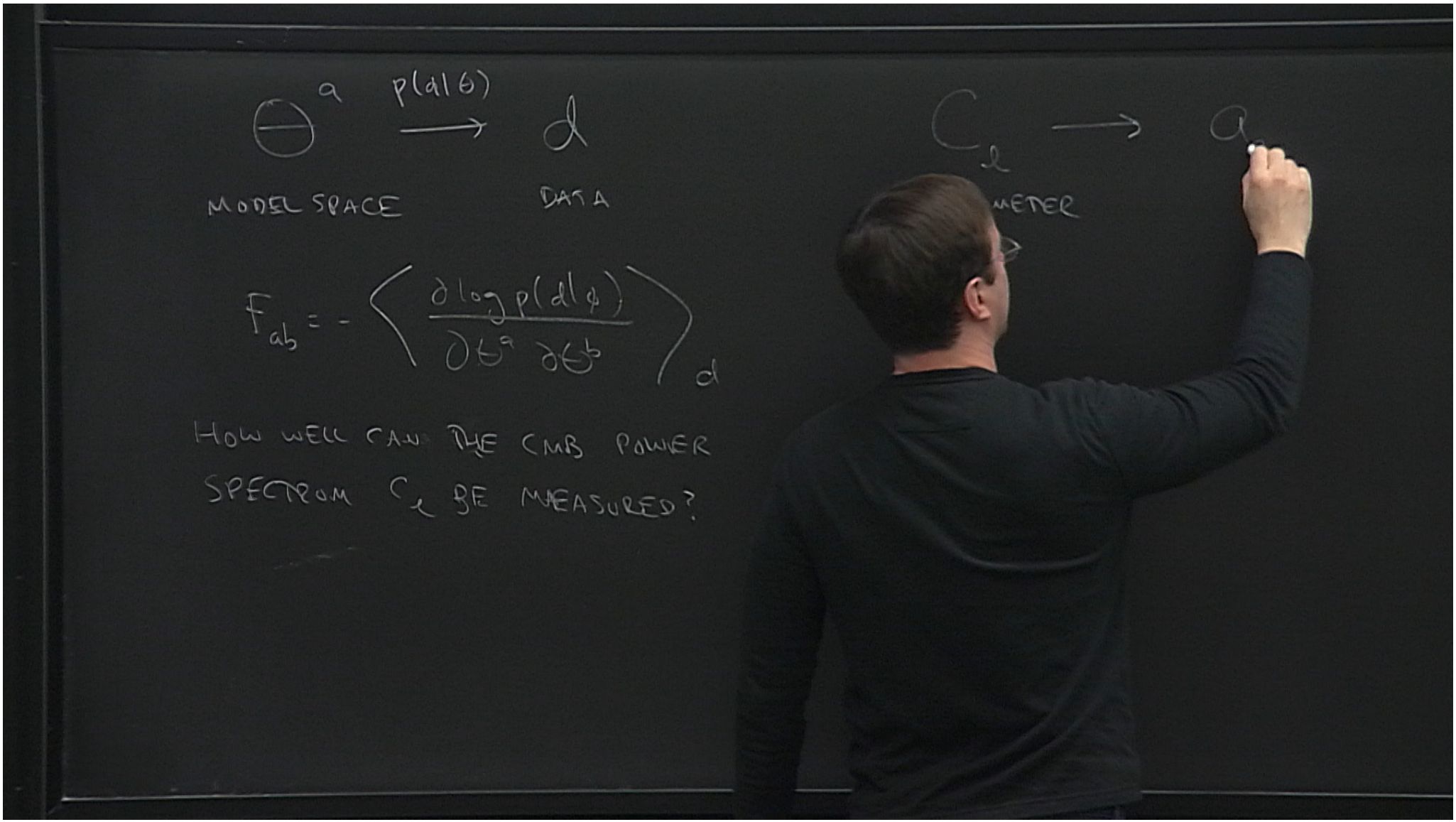
$\hat{n}$


$\Theta^a \xrightarrow{p(d|\theta)} d$
MODEL SPACE
DATA
$C_\ell \longrightarrow a$
METER
$F_{ab} = -\left\langle \frac{\partial \log p(d|\theta)}{\partial \theta^a \partial \theta^b} \right\rangle_d$
HOW WELL CAN THE CMB POWER
SPECTRUM C_ℓ BE MEASURED?

$$\theta^a \xrightarrow{p(d|\theta)} d$$

MODEL SPACE — DATA

$$C_\ell \longrightarrow a_{\ell m}$$

1 PARAMETER — $(2\ell+1)$ NUMBERS

$$F_{ab} = -\left\langle \frac{\partial \log p(d|\theta)}{\partial \theta^a \, \partial \theta^b} \right\rangle_d$$

COMP. LIKELIHOOD $\quad p(C_\ell | a_{\ell m}) =$

HOW WELL CAN THE CMB POWER SPECTRUM C_ℓ BE MEASURED?

[CMB SKY IS PERFECTLY MEASURED]

$$\theta^a \xrightarrow{p(d|\theta)} d$$

MODEL SPACE — DATA

$$C_\ell \longrightarrow a_{\ell m}$$

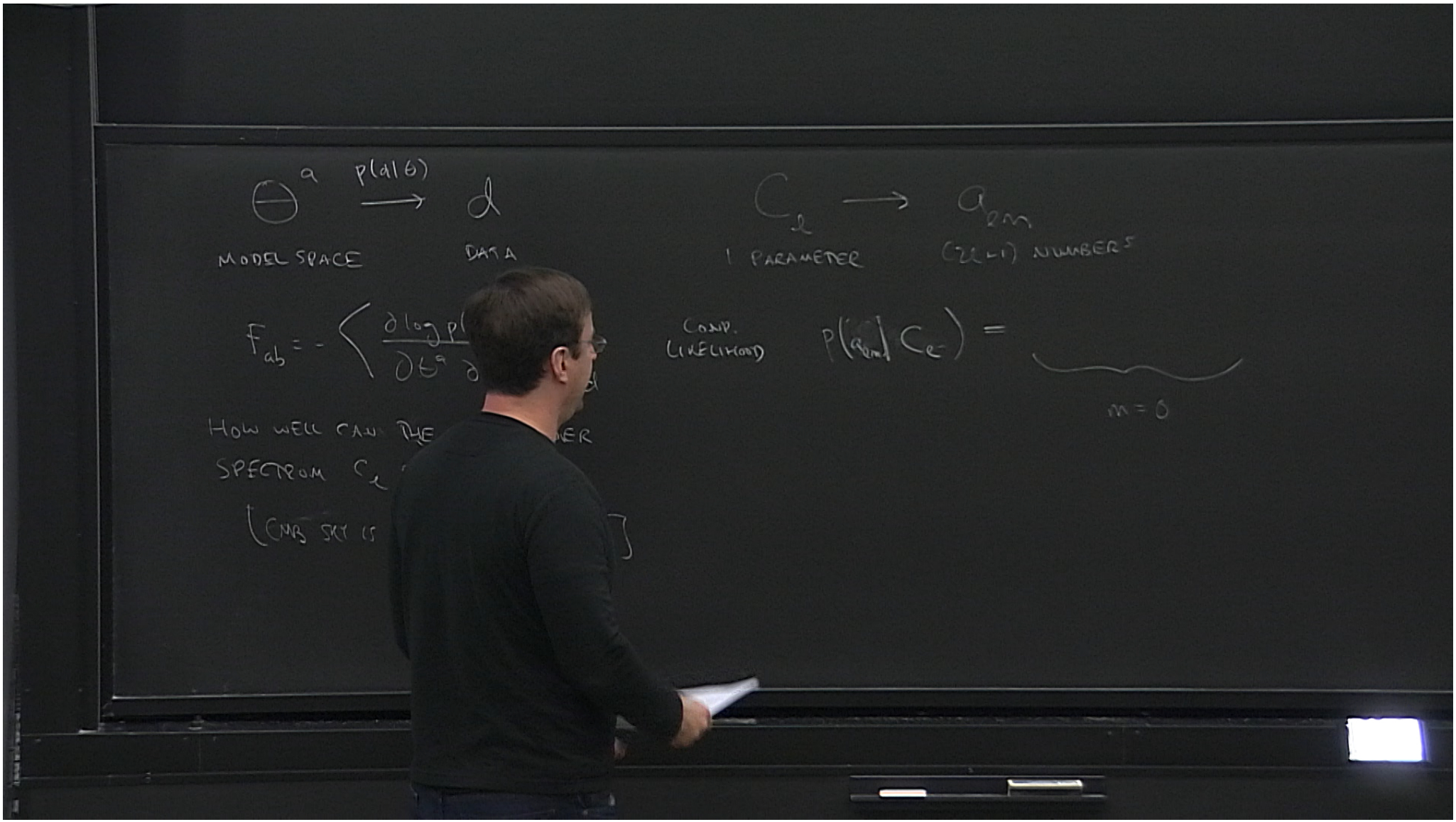
1 PARAMETER — $(2\ell+1)$ NUMBERS

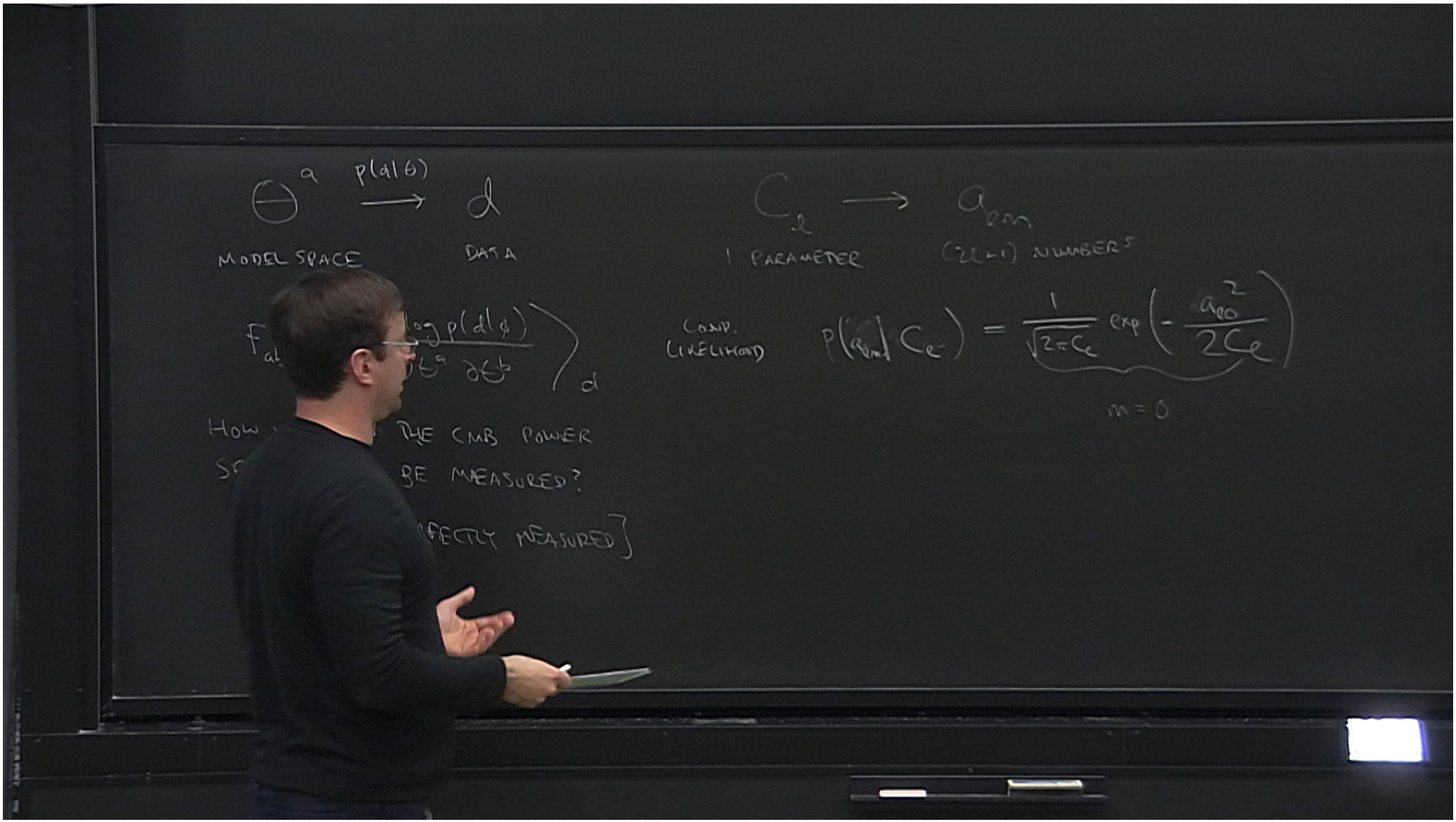
$$F_{ab} = -\left\langle \frac{\partial \log p(d|\theta)}{\partial \theta^a \, \partial \theta^b} \right\rangle_d$$

COND. LIKELIHOOD $\quad p(a_{\ell m} | C_\ell) =$

HOW WELL CAN THE CMB POWER SPECTRUM C_ℓ BE MEASURED?

[CMB SKY IS PERFECTLY MEASURED]


$\Theta^a \xrightarrow{p(d|\theta)} d$
MODEL SPACE
DATA
$C_\ell \longrightarrow a_{\ell m}$
1 PARAMETER
$(2\ell+1)$ NUMBERS
$F_{ab} = -\left\langle \frac{\partial \log p(}{\partial \theta^a \partial}\right.$
COND. LIKELIHOOD
$p(a_{\ell m} | C_\ell) =$
$m = 0$
HOW WELL CAN THE
SPECTRUM C_ℓ
(CMB SKY IS


$\Theta^a \xrightarrow{p(d|\theta)} d$
MODEL SPACE
DATA
$C_\ell \longrightarrow a_{\ell m}$
1 PARAMETER
$(2\ell+1)$ NUMBERS
COND. LIKELIHOOD
$p(a_{\ell m}|C_\ell) = \frac{1}{\sqrt{2\pi C_\ell}} \exp\left(-\frac{a_{\ell 0}^2}{2C_\ell}\right)$
$m=0$
THE CMB POWER
BE MEASURED?
FECTLY MEASURED

$C_\ell \longrightarrow a_{\ell m}$

PARAMETER (2ℓ+1) NUMBERS

$$p(a_{\ell m} | C_\ell) = \underbrace{\frac{1}{\sqrt{2\pi C_\ell}} \exp\left(-\frac{a_{\ell 0}^2}{2C_\ell}\right)}_{m=0} \prod_{m=1}^{\ell} \underbrace{\frac{1}{\sqrt{\pi C_\ell}} \exp\left(-\frac{\mathrm{Re}(a_{\ell m})^2}{C_\ell}\right)}_{\mathrm{Re}(a_{\ell m})} \underbrace{\frac{1}{\sqrt{\pi C_\ell}} \exp\left(-\frac{\mathrm{Im}(a_{\ell m})^2}{C_\ell}\right)}_{\mathrm{Im}(a_{\ell m})}$$

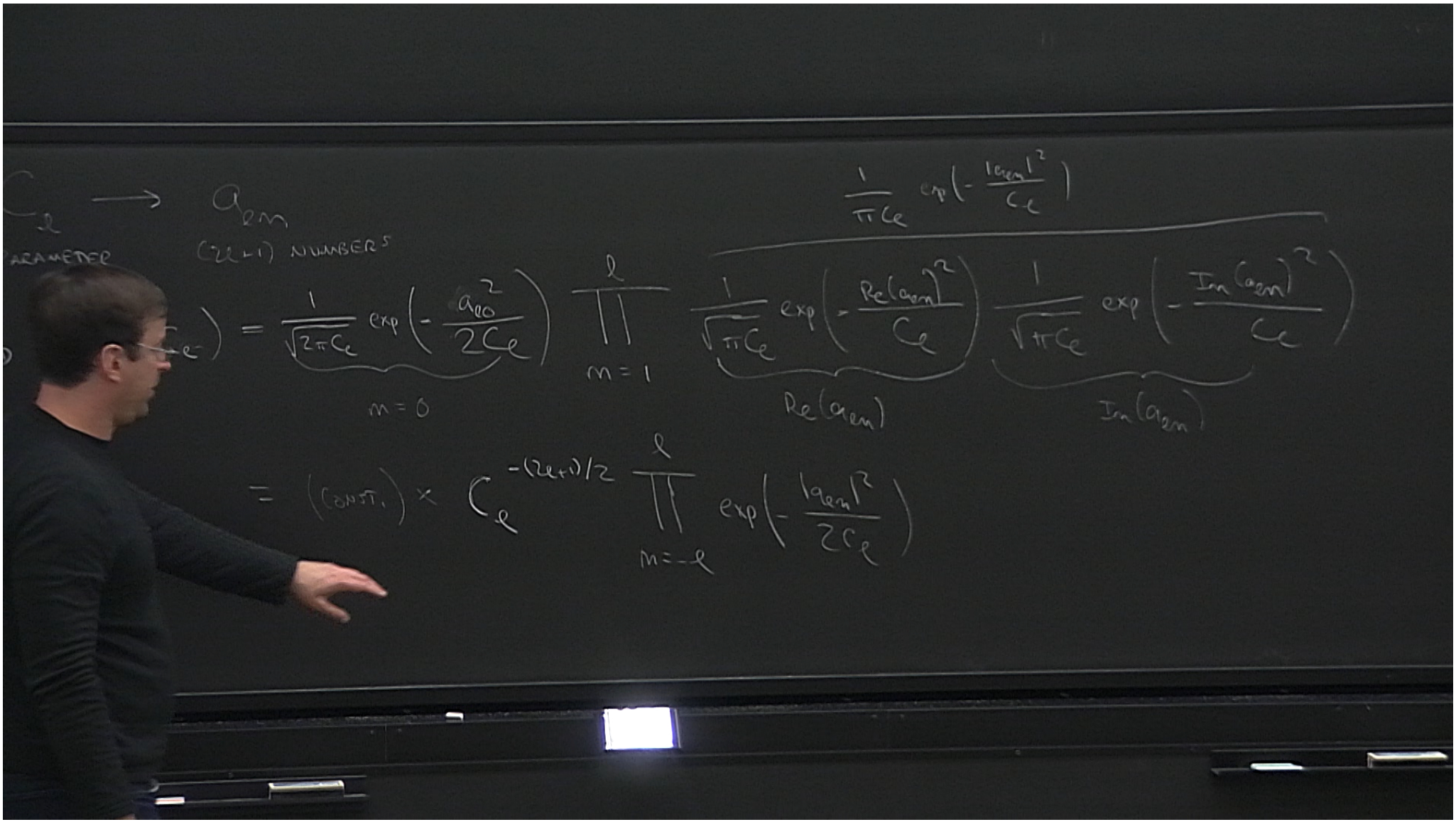
$C_\ell \longrightarrow a_{\ell m}$

PARAMETER $\quad$ $(2\ell+1)$ NUMBERS

$$\exp\left(-\frac{|a_{\ell m}|^2}{C_\ell}\right)$$

$$P(a_{\ell m} \mid C_\ell) = \underbrace{\frac{1}{\sqrt{2\pi C_\ell}} \exp\left(-\frac{a_{\ell 0}^2}{2C_\ell}\right)}_{m=0} \prod_{m=1}^{\ell} \underbrace{\frac{1}{\sqrt{\pi C_\ell}} \exp\left(-\frac{\mathrm{Re}(a_{\ell m})^2}{C_\ell}\right)}_{\mathrm{Re}(a_{\ell m})} \underbrace{\frac{1}{\sqrt{\pi C_\ell}} \exp\left(-\frac{\mathrm{Im}(a_{\ell m})^2}{C_\ell}\right)}_{\mathrm{Im}(a_{\ell m})}$$

$$= (\text{CONST.}) \times C_\ell^{-(2\ell+1)/2}$$


$C_\ell \longrightarrow a_{\ell m}$
PARAMETER
$(2\ell+1)$ NUMBERS
$\frac{1}{\pi C_\ell} \exp\left(-\frac{|a_{\ell m}|^2}{C_\ell}\right)$
$\left(\ldots C_\ell\right) = \underbrace{\frac{1}{\sqrt{2\pi C_\ell}} \exp\left(-\frac{a_{\ell 0}^2}{2C_\ell}\right)}_{m=0} \prod_{m=1}^{\ell} \underbrace{\frac{1}{\sqrt{\pi C_\ell}} \exp\left(-\frac{\mathrm{Re}(a_{\ell m})^2}{C_\ell}\right)}_{\mathrm{Re}(a_{\ell m})} \underbrace{\frac{1}{\sqrt{\pi C_\ell}} \exp\left(-\frac{\mathrm{Im}(a_{\ell m})^2}{C_\ell}\right)}_{\mathrm{Im}(a_{\ell m})}$
$= (\text{CONST.}) \times C_\ell^{-(2\ell+1)/2} \prod_{m=-\ell}^{\ell} \exp\left(-\frac{|a_{\ell m}|^2}{2C_\ell}\right)$

1-BY-1 FISHER "MATRIX" [SINGLE ℓ]

$$F = -\left\langle \frac{\partial^2 \log p(a_{\ell m} | C_\ell)}{\partial C_\ell^2} \right\rangle_{a_{\ell m}}$$

$$= -\left\langle \frac{\partial^2}{\partial C_\ell^2} \left(-\frac{2\ell+1}{2} \log C_\ell - \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{2 C_\ell} \right) \right\rangle_{a_{\ell m}}$$

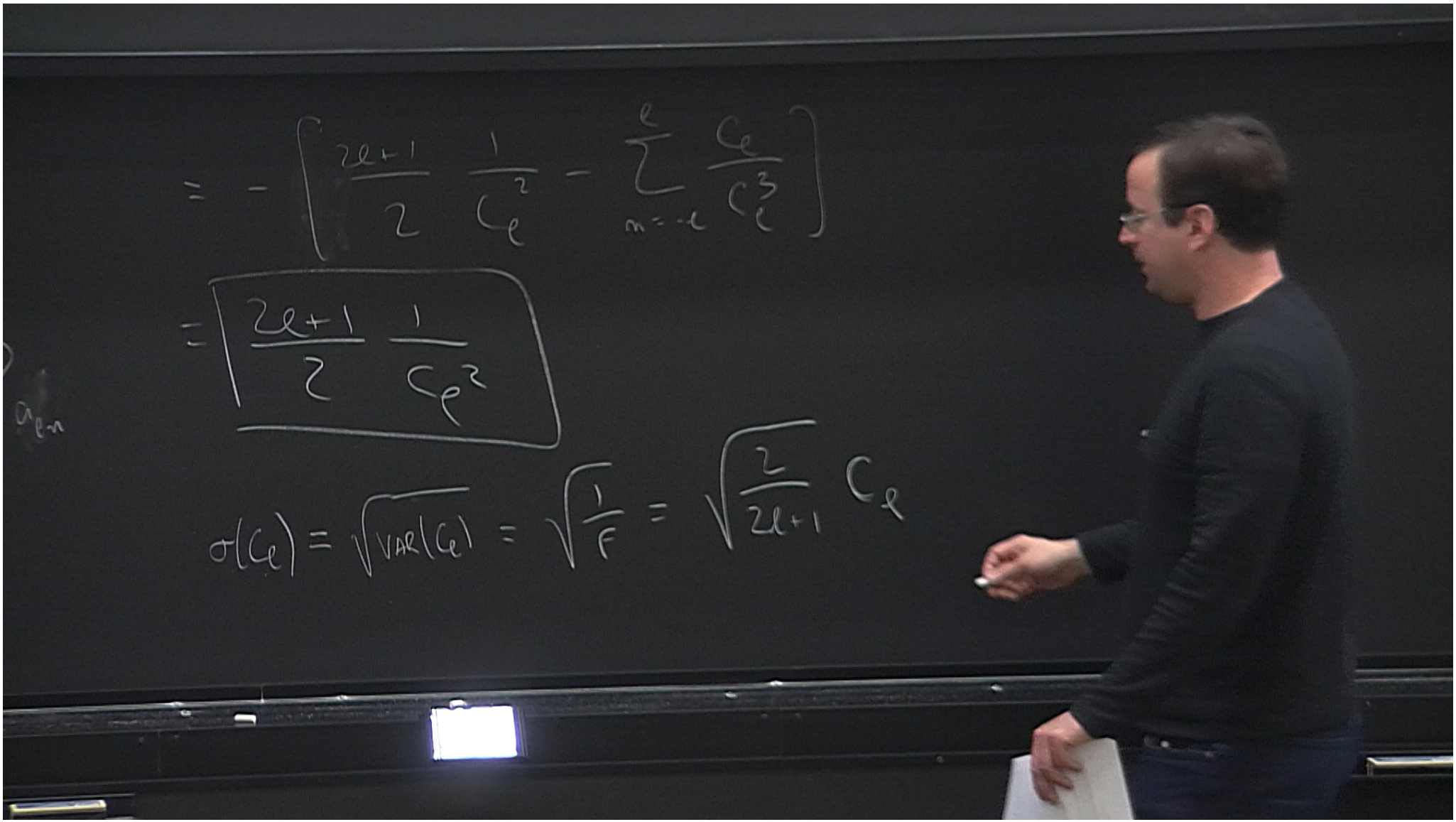
$$= -\left\langle \frac{2\ell+1}{2} \frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{C_\ell^3} \right\rangle_{a_{\ell m}}$$

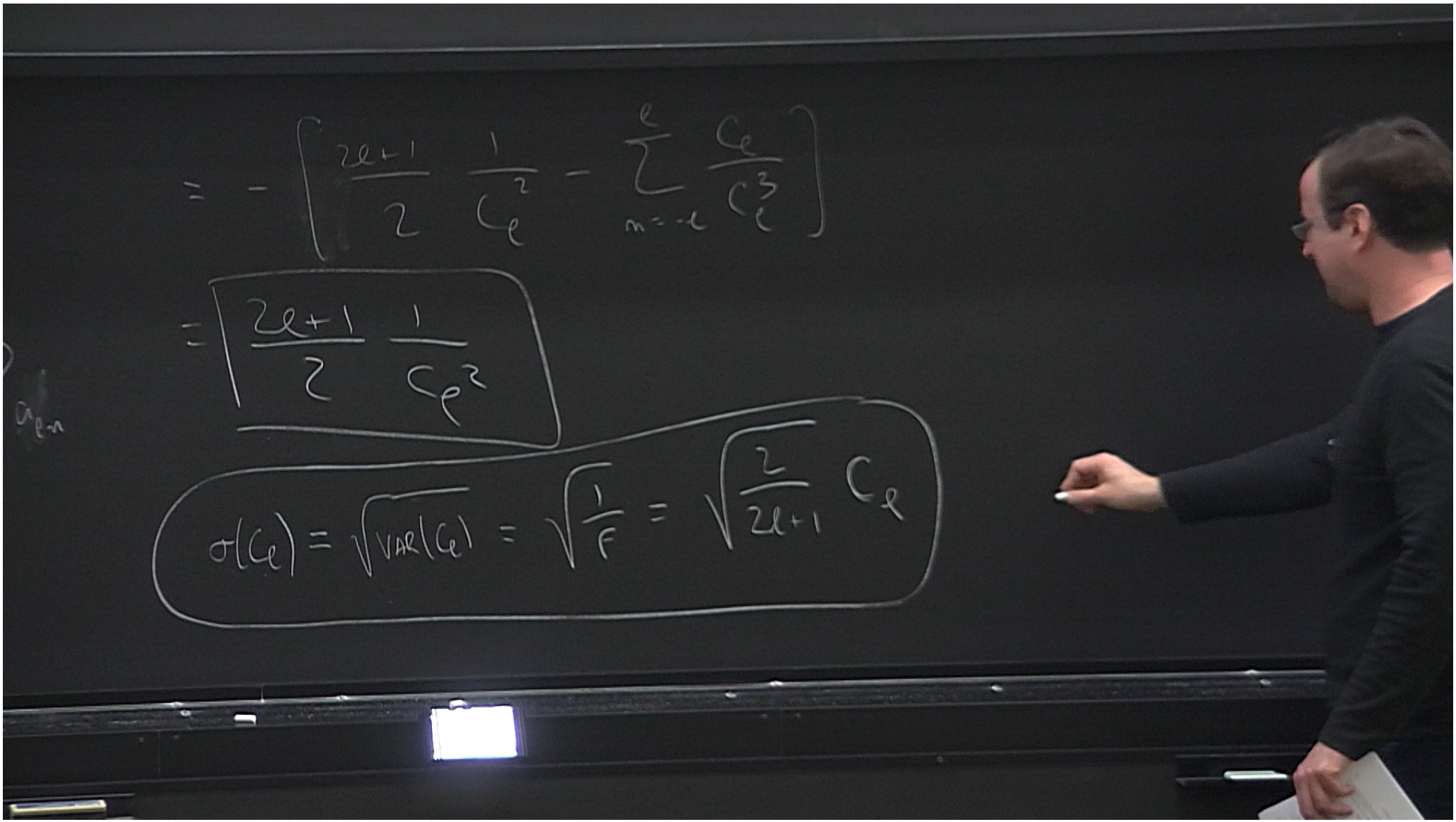
$$= -\left[\frac{2\ell+1}{2}\,\frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell}\frac{C_\ell}{C_\ell^3}\right]$$

$$=$$

$$= -\left[\frac{2\ell+1}{2}\frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell}\frac{C_\ell}{C_\ell^3}\right]$$

$$= \boxed{\frac{2\ell+1}{2}\frac{1}{C_\ell^2}}$$


$= -\left[\frac{2\ell+1}{2}\frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell}\frac{C_\ell}{C_\ell^3}\right]$
$= \boxed{\frac{2\ell+1}{2}\frac{1}{C_\ell^2}}$
$a_{\ell m}$
$\sigma(C_\ell) = \sqrt{\mathrm{VAR}(C_\ell)} = \sqrt{\frac{1}{F}} = \sqrt{\frac{2}{2\ell+1}}\, C_\ell$

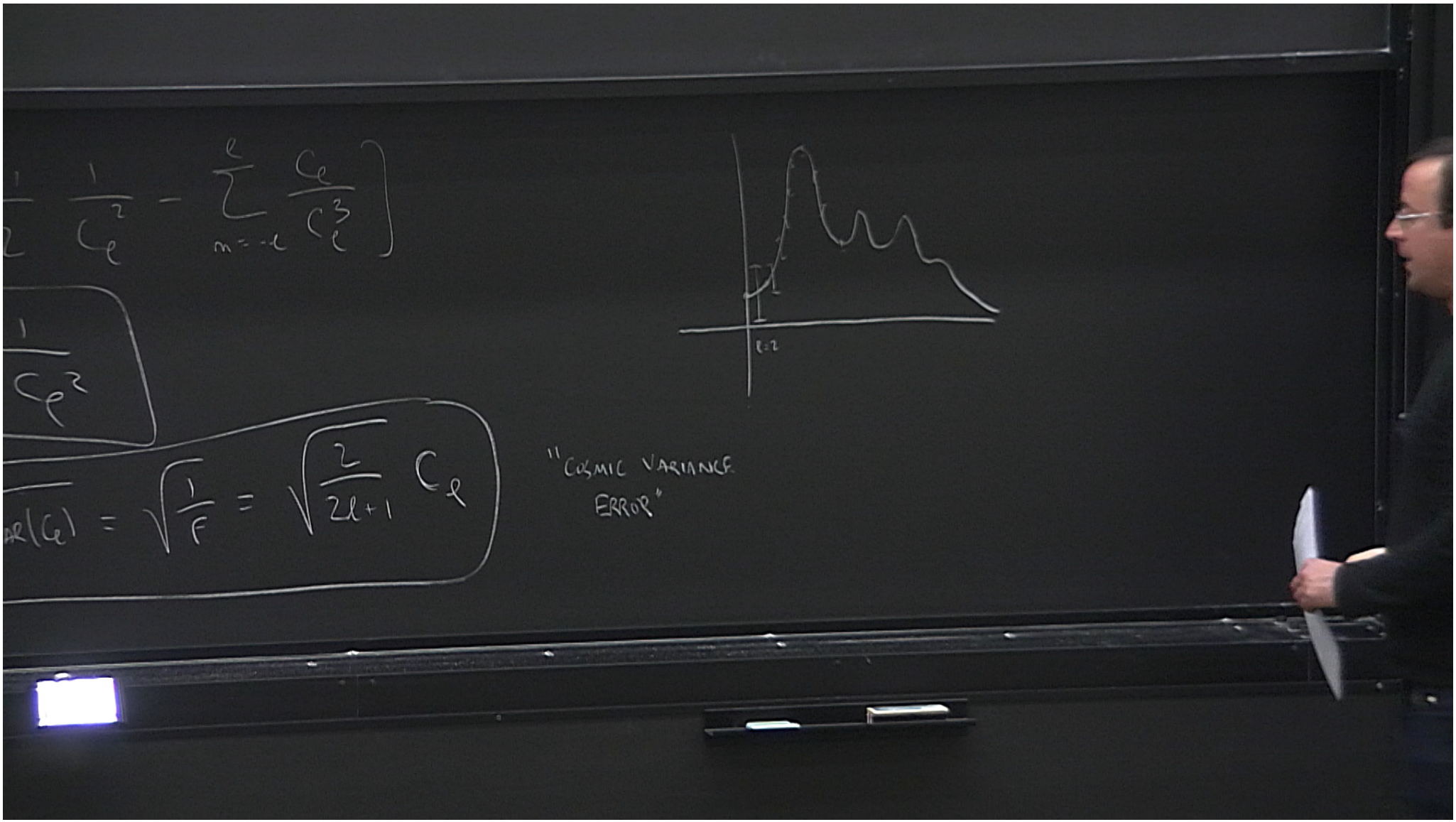

$= -\left[\frac{2\ell+1}{2}\frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell}\frac{C_\ell}{C_\ell^3}\right]$
$= \boxed{\frac{2\ell+1}{2}\frac{1}{C_\ell^2}}$
$a_{\ell m}$
$\sigma(C_\ell) = \sqrt{\mathrm{VAR}(C_\ell)} = \sqrt{\frac{1}{F}} = \sqrt{\frac{2}{2\ell+1}}\, C_\ell$

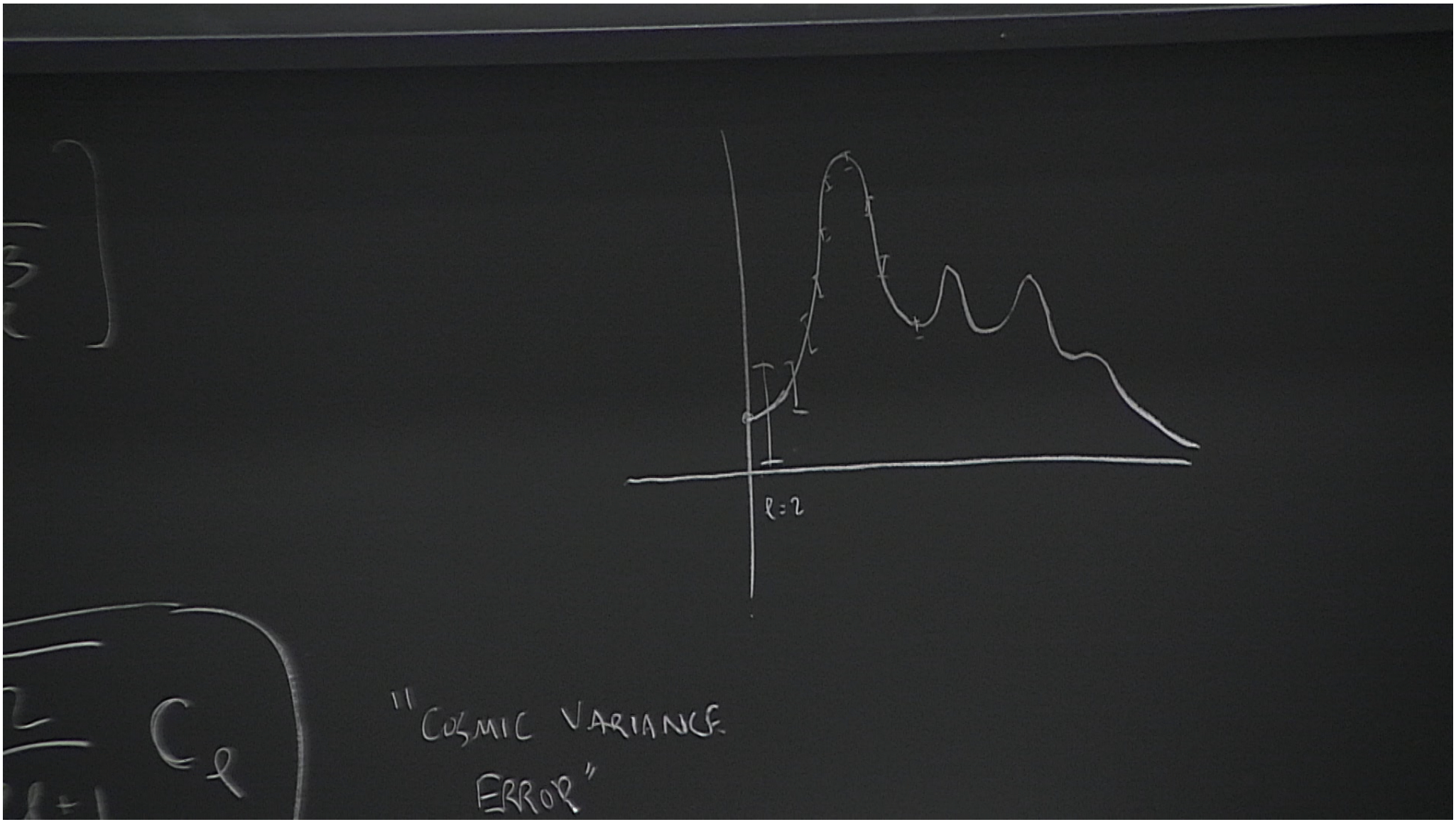
$$= -\left[\frac{2\ell+1}{2}\frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell}\frac{C_\ell}{C_\ell^3}\right]$$

$$= \boxed{\frac{2\ell+1}{2}\frac{1}{C_\ell^2}}$$

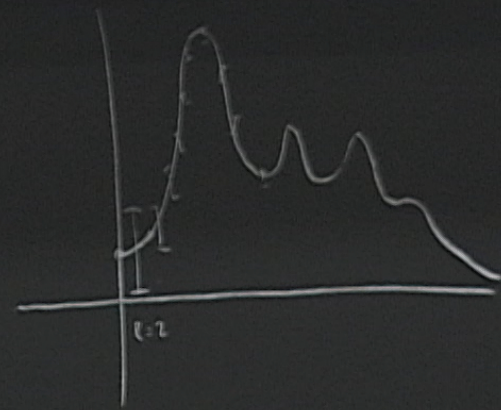
$$\sigma(C_\ell) = \sqrt{\mathrm{VAR}(C_\ell)} = \sqrt{\frac{1}{F}} = \sqrt{\frac{2}{2\ell+1}}\,C_\ell$$

"COSMIC VARIANCE ERROR"


$\frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell} \frac{C_\ell}{C_\ell^3}$
$\frac{1}{C_\ell^2}$
$\sqrt{\frac{1}{F}} = \sqrt{\frac{2}{2\ell+1}}\, C_\ell$
"COSMIC VARIANCE ERROR"
$\ell=2$

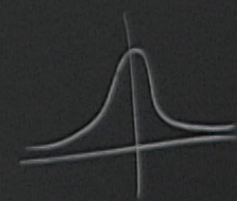

$\ell = 2$
C_ℓ
"COSMIC VARIANCE ERROR"

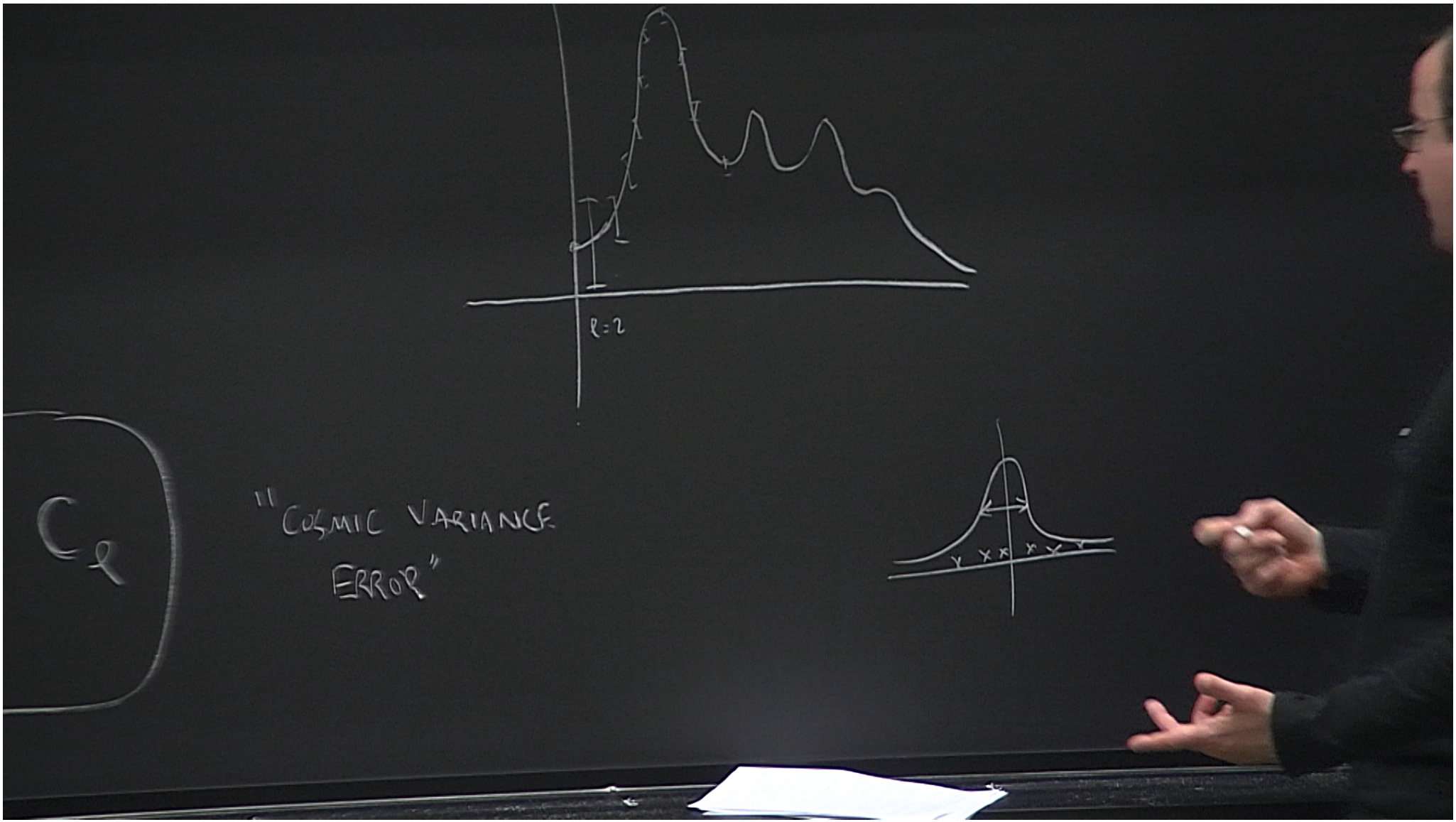
$$\frac{1}{C_\ell^2} - \sum_{m=-\ell}^{\ell} \frac{C_\ell}{C_\ell^3}\Big]$$

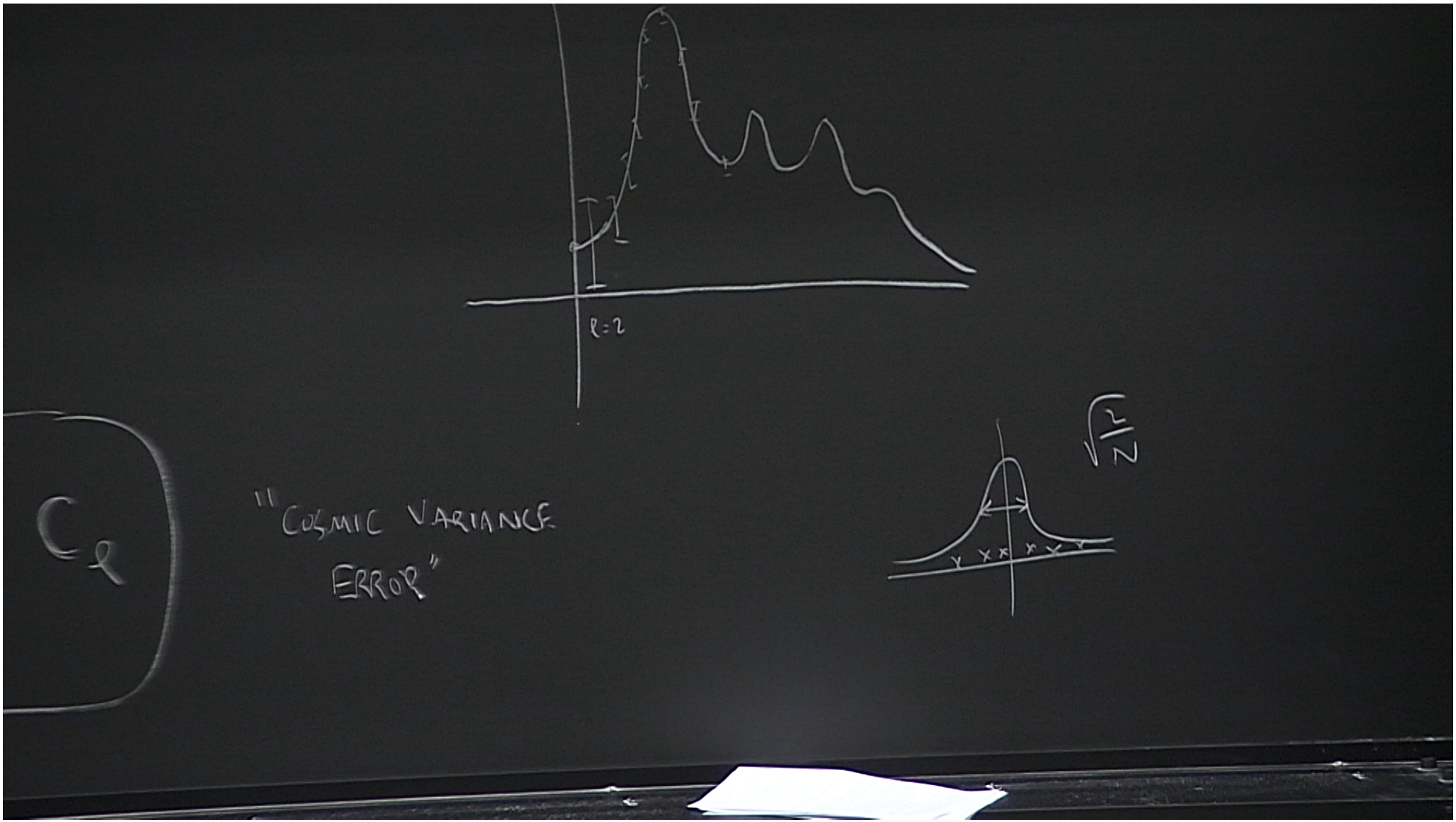


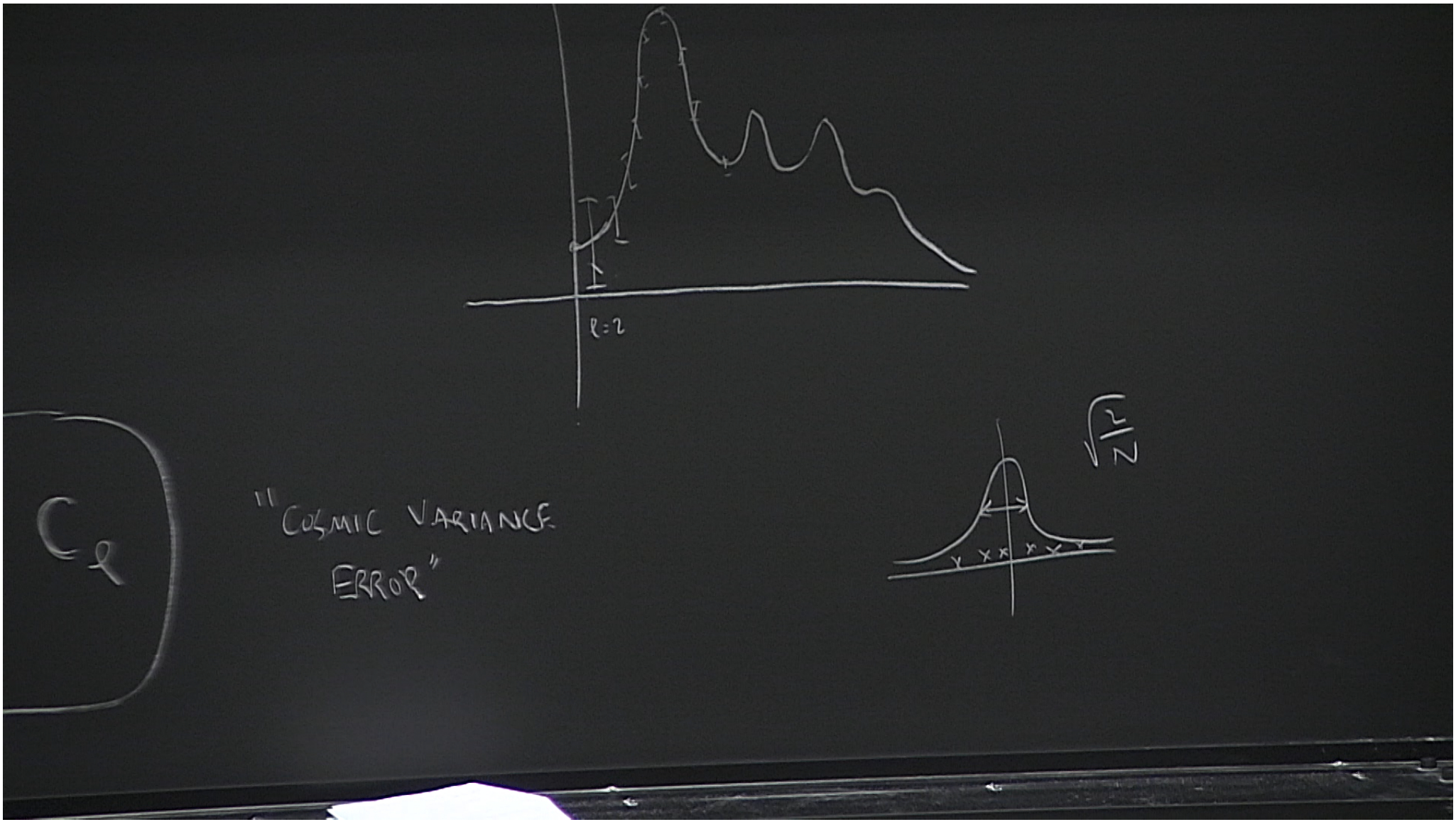
$$\boxed{\sqrt{\frac{1}{F}} = \sqrt{\frac{2}{2\ell+1}}\, C_\ell}$$

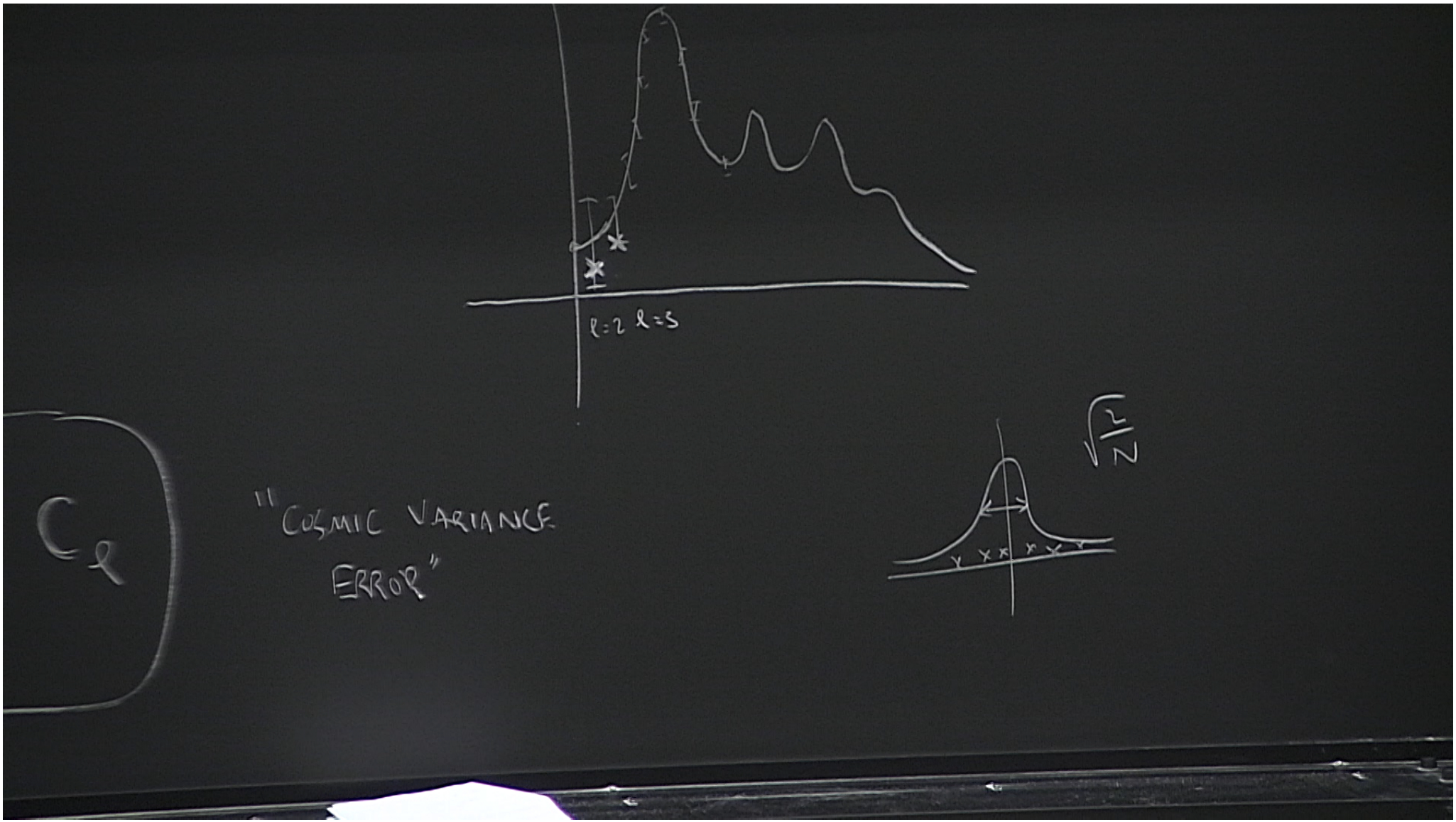
"COSMIC VARIANCE ERROR"

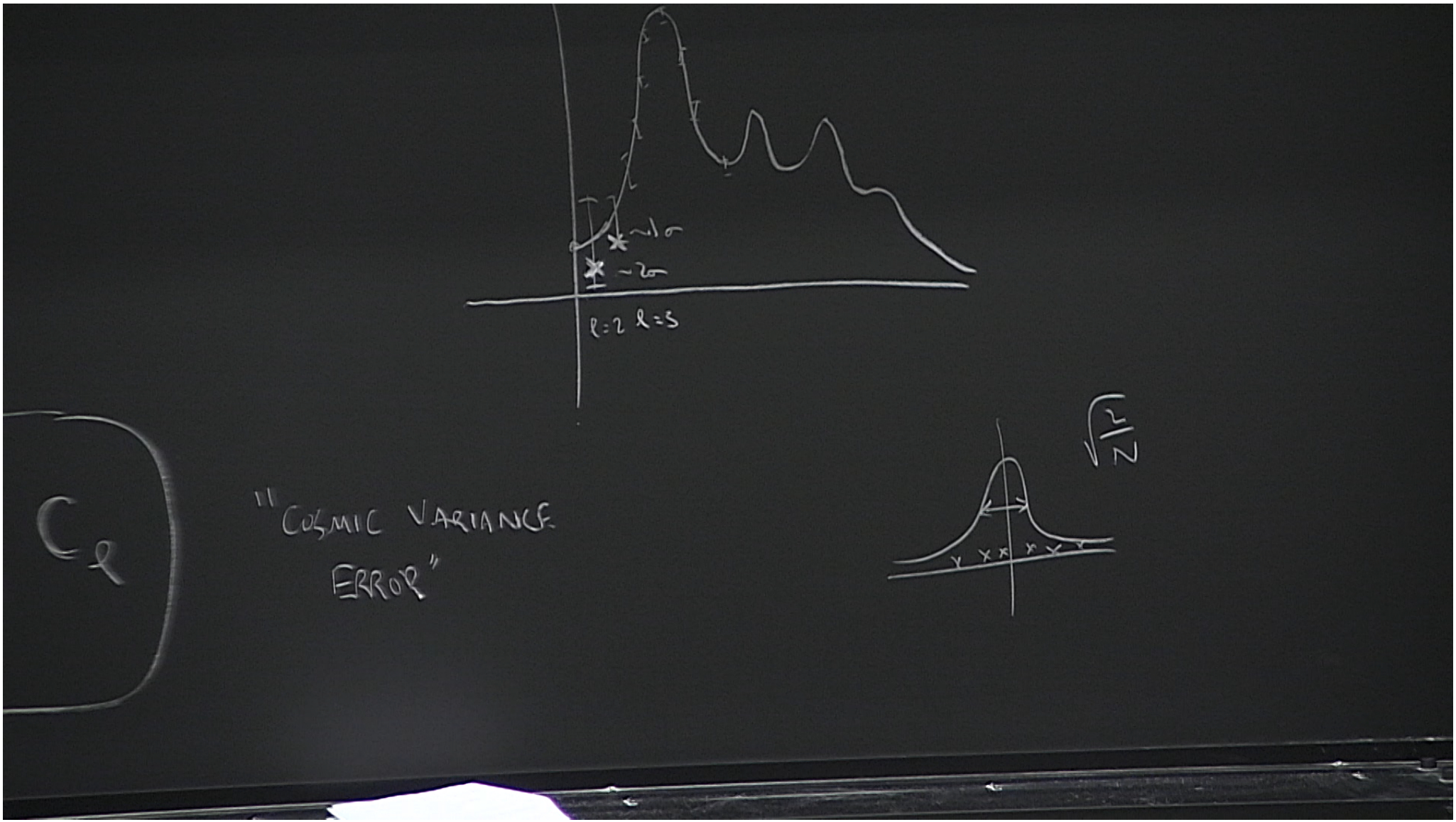


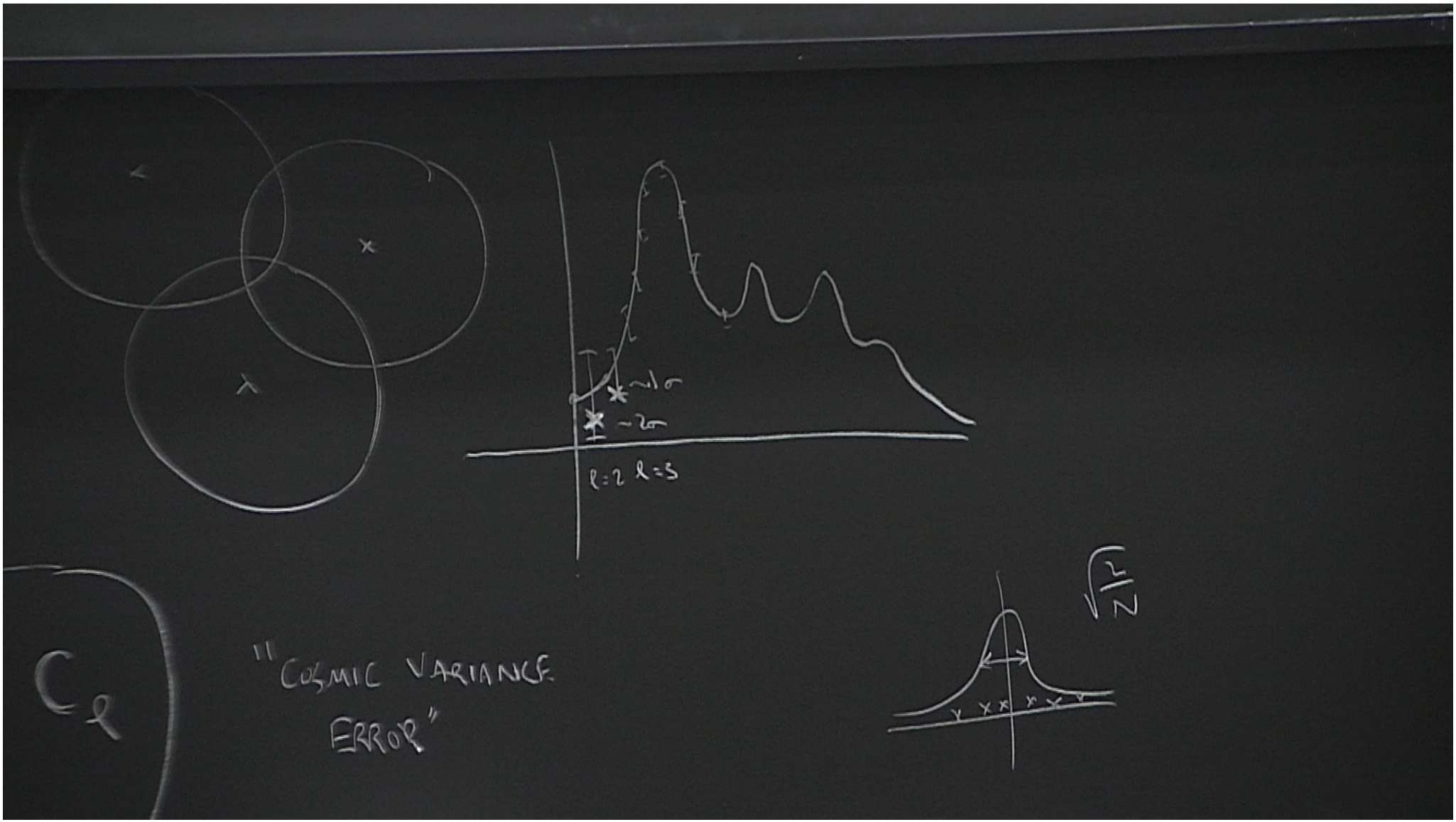

C_ℓ
"COSMIC VARIANCE ERROR"
$\ell = 2$


C_ℓ
"COSMIC VARIANCE ERROR"
$\ell = 2$
$\sqrt{\frac{2}{N}}$

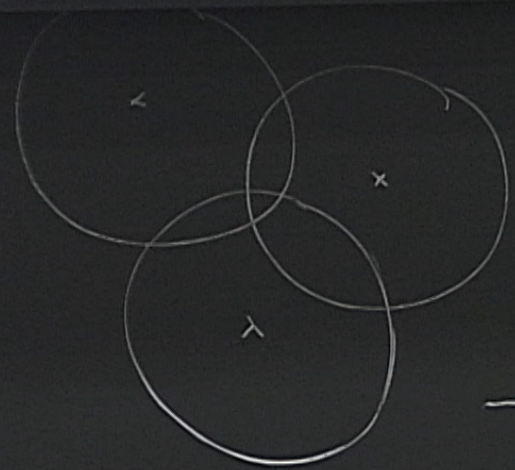

C_ℓ
"COSMIC VARIANCE ERROR"
$\ell = 2$
$\sqrt{\frac{2}{N}}$

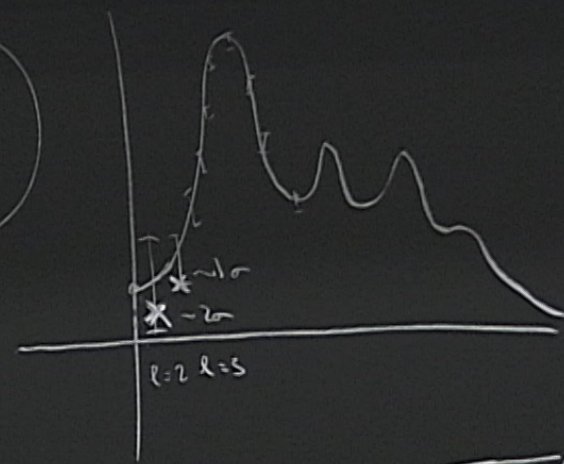

C_ℓ
"COSMIC VARIANCE ERROR"
$\ell=2$ $\ell=3$
$\sqrt{\frac{2}{N}}$


~1σ
~2σ
ℓ=2 ℓ=3
C_ℓ
"COSMIC VARIANCE ERROR"
$\sqrt{\frac{2}{N}}$


~1σ
~2σ
ℓ=2 ℓ=3
C_ℓ
"COSMIC VARIANCE ERROR"
$\sqrt{\frac{2}{N}}$

$$\sum_{m=-\ell}^{\ell} \frac{C_\ell}{C_\ell^3}$$





SKY FRACTION

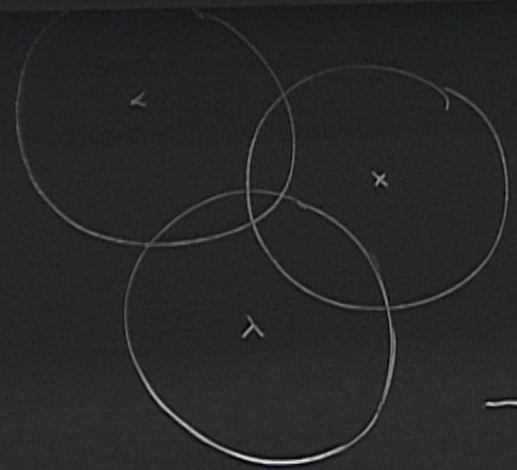
"NOISE POWER SPECTRUM"

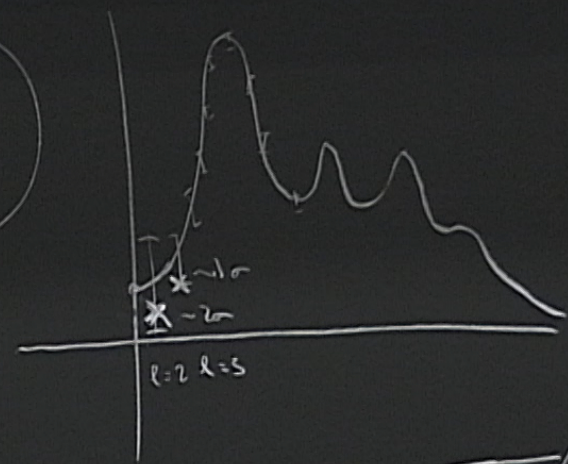
$$\frac{1}{F} = \sqrt{\frac{2}{2\ell+1}}\, C_\ell$$

"COSMIC VARIANCE ERROR"

$$\sigma(C_\ell) = \sqrt{\frac{2 f_{sky}}{2\ell+1}}\,(C_\ell + N_\ell)$$

$$\sum_{m=-\ell}^{\ell} \frac{C_\ell}{C_\ell^3}$$





SKY FRACTION

"NOISE POWER SPECTRUM"

$$\frac{1}{F} = \sqrt{\frac{2}{2\ell+1}\, C_\ell}$$

"COSMIC VARIANCE ERROR"

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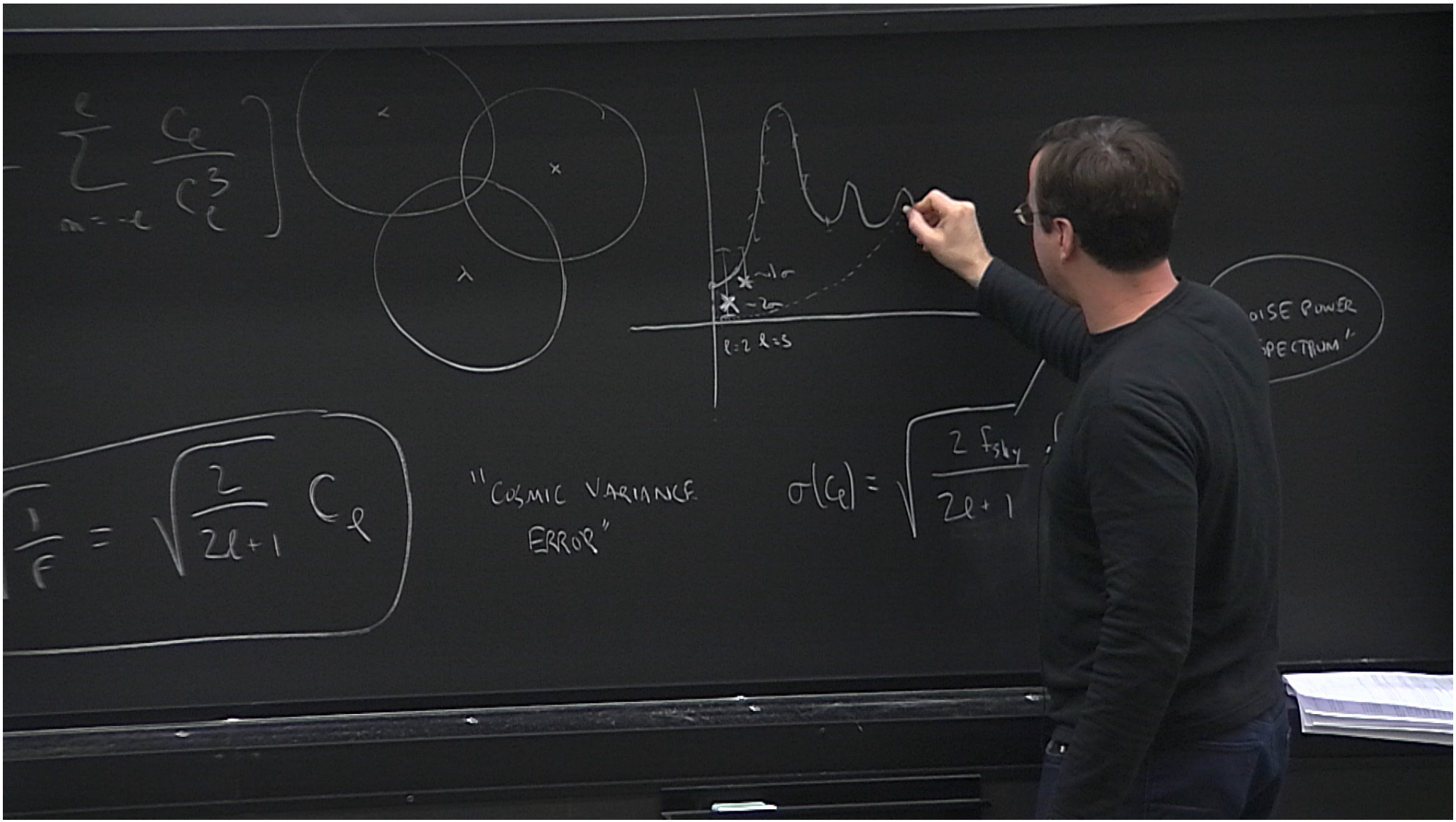
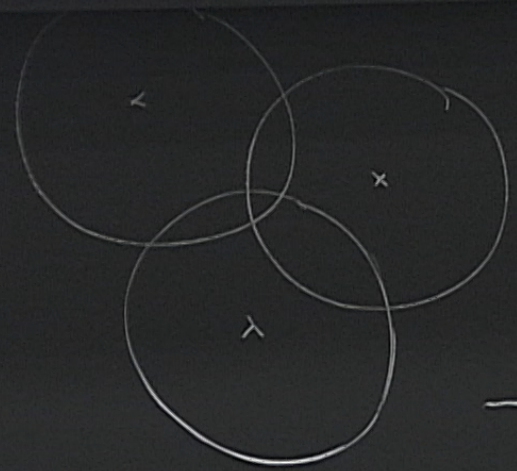
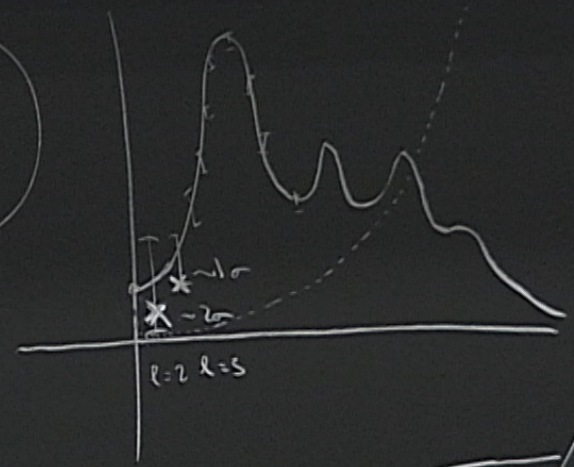

"COSMIC VARIANCE ERROR"
$\sigma(C_\ell) = \sqrt{\frac{2 f_{sky}}{2\ell+1}}$
$\frac{1}{F} = \sqrt{\frac{2}{2\ell+1}} C_\ell$
$\ell=2$ $\ell=5$
NOISE POWER SPECTRUM

$$\sum_{m=-\ell}^{\ell} \frac{C_\ell}{C_\ell^3}$$





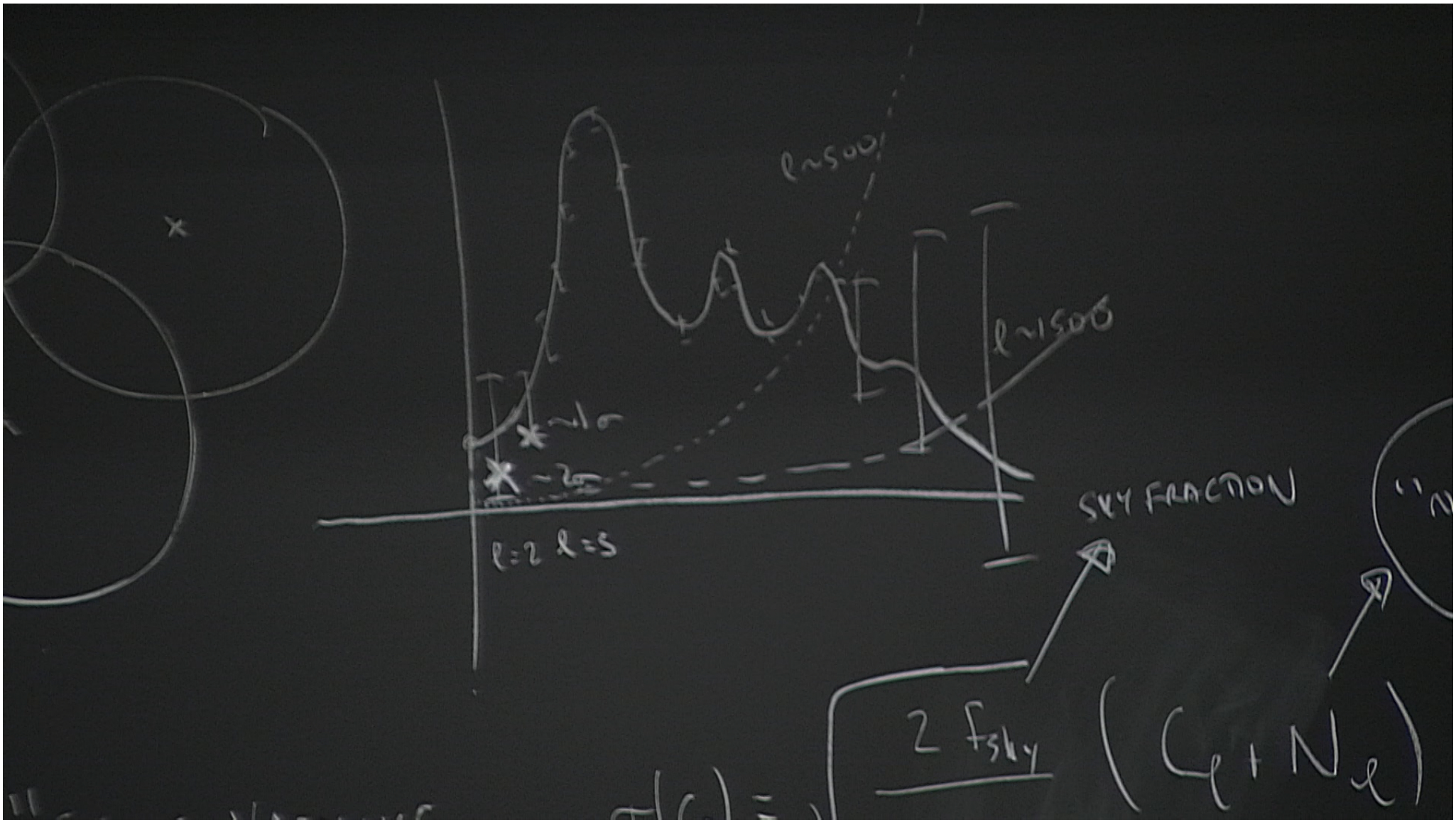
$$\frac{1}{F} = \sqrt{\frac{2}{2\ell+1} C_\ell}$$

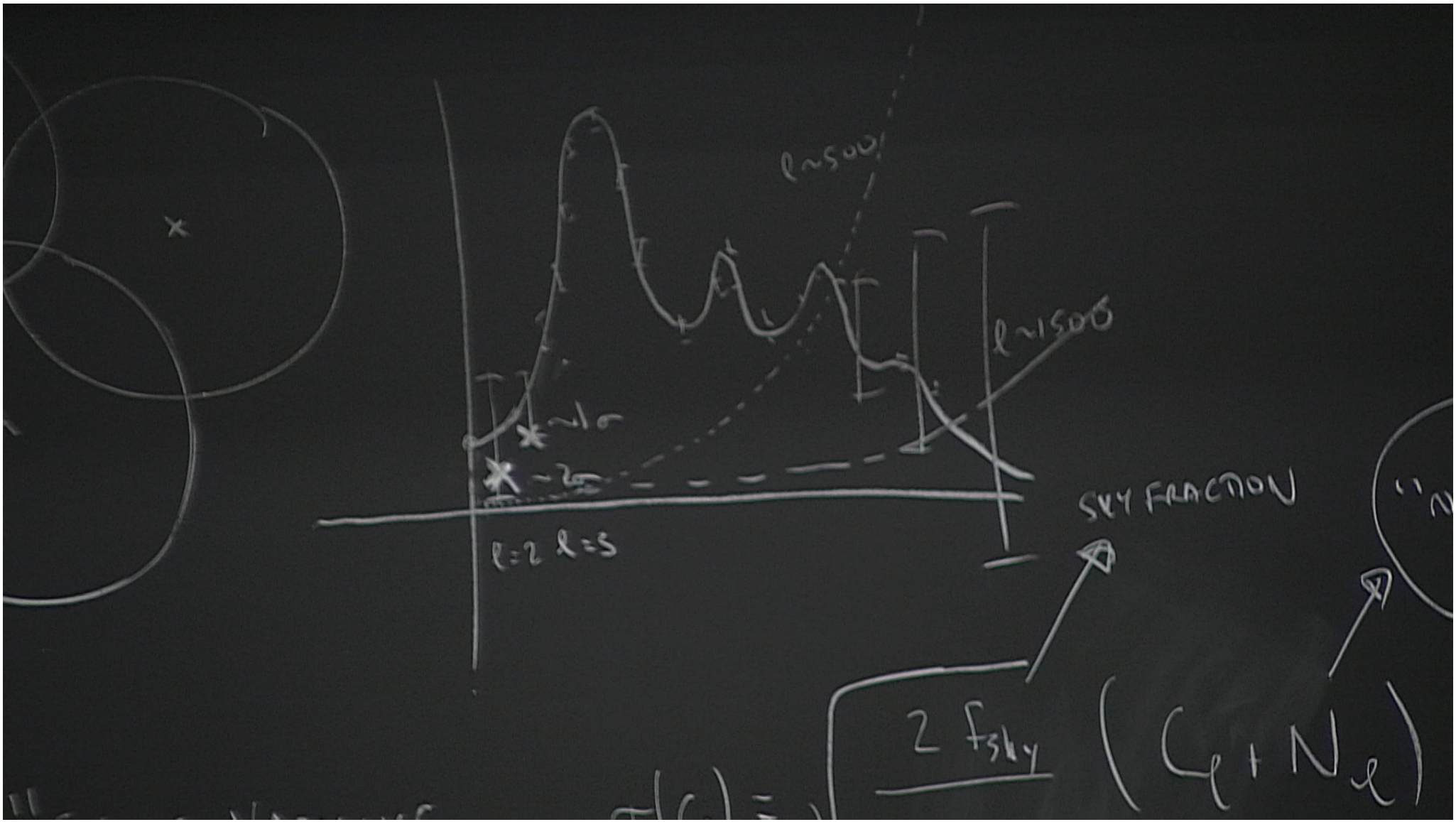
"COSMIC VARIANCE ERROR"

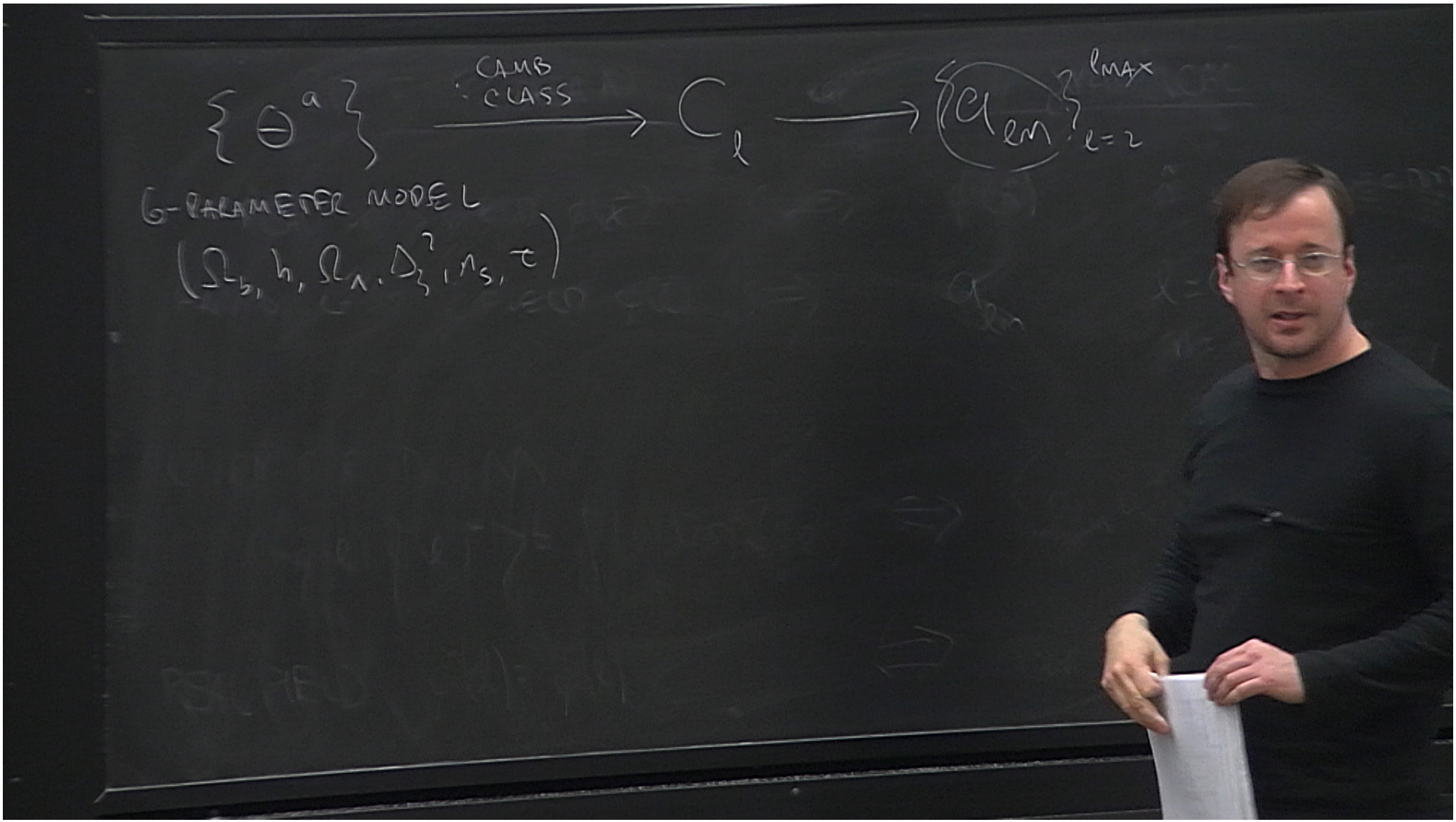
$$\sigma(C_\ell) = \sqrt{\frac{2 f_{sky}}{2\ell+1} \left(C_\ell + N_\ell \right)}$$

SKY FRACTION

"NOISE POWER SPECTRUM"


ℓ~500
ℓ~1500
~1σ
~2σ
ℓ=2 ℓ=5
SKY FRACTION
$\sqrt{\frac{2 f_{sky}}{}} (C_\ell + N_\ell)$


ℓ~500
ℓ~1500
~1σ
~2σ
ℓ=2 ℓ=5
SKY FRACTION
$\sqrt{\frac{2 f_{sky}}{}} (C_\ell + N_\ell)$


$\{\Theta^a\} \xrightarrow[\text{CLASS}]{\text{CAMB}} C_\ell \longrightarrow \{a_{\ell m}\}_{\ell=2}^{\ell_{MAX}}$
6-PARAMETER MODEL
$(\Omega_b, h, \Omega_\Lambda, \Delta_\zeta^2, n_s, \tau)$

$$\{\theta^a\} \xrightarrow[\text{CLASS}]{\text{CAMB}} C_\ell \longrightarrow \{a_{\ell m}\}_{\ell=2}^{\ell_{MAX}}$$

6-PARAMETER MODEL

$$(\Omega_b, h, \Omega_\Lambda, \Delta_\zeta^2, n_s, \tau)$$

$$\frac{\partial C_\ell}{\partial \theta^a} = \frac{C_\ell(\theta^a + \Delta\theta^a) - C_\ell(\theta^a)}{\Delta\theta^a}$$

$$\{\Theta^a\} \xrightarrow[\text{CLASS}]{\text{CAMB}} C_\ell \longrightarrow \{a_{\ell m}\}_{\ell=2}^{\ell_{MAX}}$$

6-PARAMETER MODEL

$$(\Omega_b, h, \Omega_\Lambda, \Delta_\zeta^2, n_s, \tau)$$

$$\frac{\partial C_\ell}{\partial \Theta^a} \approx \frac{C_\ell(\Theta^a + \Delta\Theta^a) - C_\ell(\Theta^a - \Delta\Theta^a)}{2(\Delta\Theta^a)}$$

$$\{\theta^a\} \xrightarrow{\text{CAMB / CLASS}} C_\ell \longrightarrow \{a_{\ell m}\}_{\ell=2}^{\ell_{MAX}}$$

6-PARAMETER MODEL

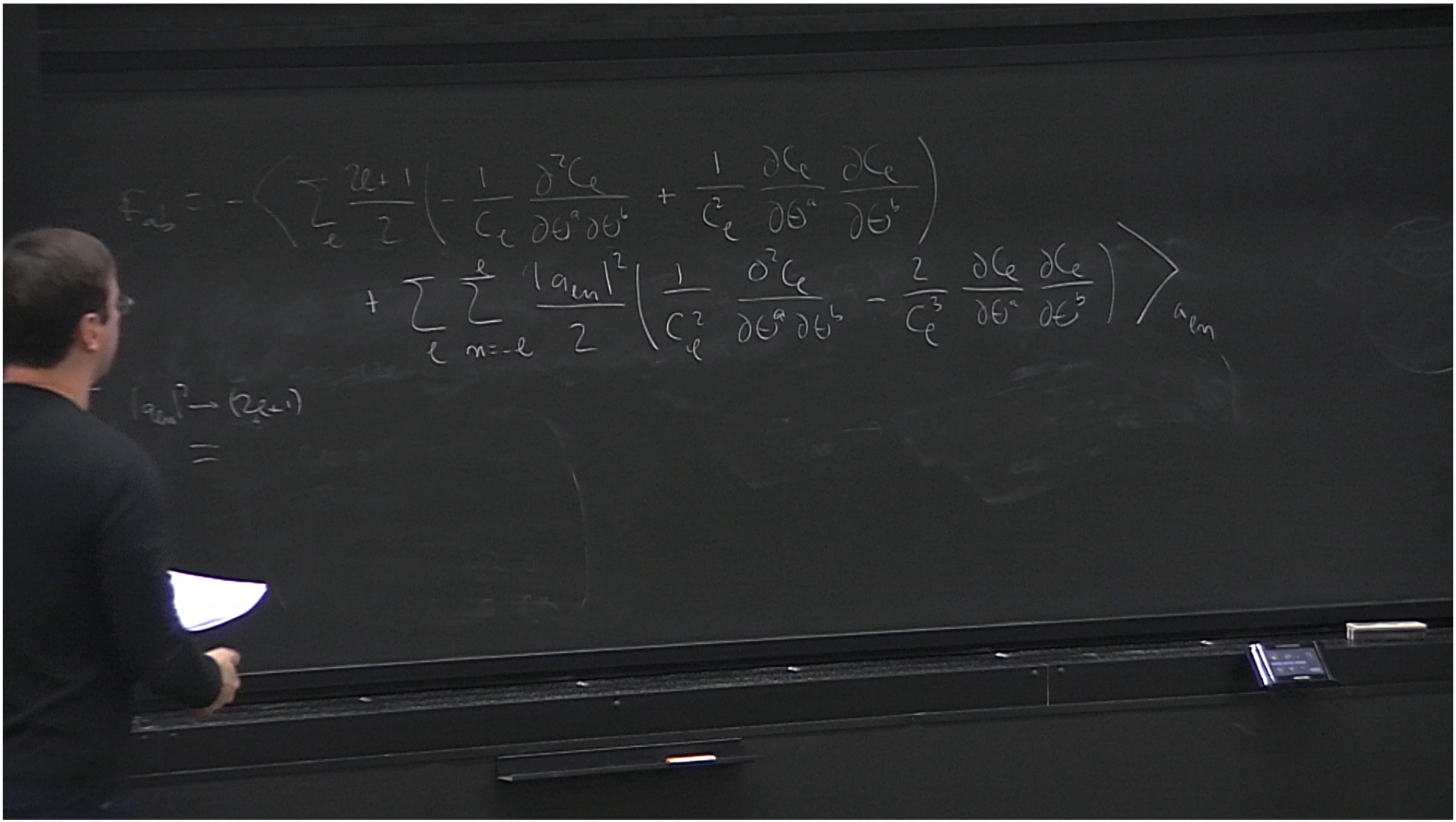
$$(\Omega_b, h, \Omega_\Lambda, \Delta_\zeta^2, n_s, \tau)$$

$$\frac{\partial C_\ell}{\partial \theta^a} \approx \frac{C_\ell(\theta^a + \Delta\theta^a) - C_\ell(\theta^a - \Delta\theta^a)}{2(\Delta\theta^a)}$$

$$P(a_{\ell m} | \theta^a) = [\text{CONST.}] \prod_{\ell} (C_\ell)^{-(2\ell+1)/2} \prod_{m=-\ell}^{\ell} \exp\left(-\frac{|a_{\ell m}|^2}{2C_\ell}\right)$$

$$F_{ab} = -\left\langle \frac{\partial^2 \log P(a_{\ell m} | \theta)}{\partial\theta^a \partial\theta^b} \right\rangle_{a_{\ell m}}$$

$$= -\left\langle \frac{\partial^2}{\partial\theta^a \partial\theta^b} \left[-\sum_{\ell} \frac{2\ell+1}{2} \log(C_\ell) - \sum_{\ell} \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{2C_\ell} \right] \right\rangle$$


$f_{ab} = -\left\langle \sum_\ell \frac{2\ell+1}{2}\left(-\frac{1}{C_\ell}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} + \frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right.$
$\left. + \sum_\ell \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{2}\left(\frac{1}{C_\ell^2}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} - \frac{2}{C_\ell^3}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right\rangle_{a_{\ell m}}$
$|a_{\ell m}|^2 \to (2\ell+1)$
$=$

$$f_{ab} = -\left\langle \sum_\ell \frac{2\ell+1}{2}\left(-\frac{1}{C_\ell}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} + \frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right) + \sum_\ell \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{2}\left(\frac{1}{C_\ell^2}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} - \frac{2}{C_\ell^3}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right\rangle$$

$$\sum_m |a_{\ell m}|^2 \to (2\ell+1)$$

$$= \boxed{\sum_\ell \frac{2\ell+1}{2}\frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}}$$

$$F_{ab} = -\left\langle \sum_\ell \frac{2\ell+1}{2}\left(-\frac{1}{C_\ell}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} + \frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right) + \sum_\ell \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{2}\left(\frac{1}{C_\ell^2}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} - \frac{2}{C_\ell^3}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right\rangle_{a_{\ell m}}$$

$$\sum_m |a_{\ell m}|^2 \to (2\ell+1)$$

$$= \boxed{\sum_\ell \frac{2\ell+1}{2}\frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}}$$

COSMIC VARIANCE LIMITED
CMB FISHER MATRIX

$$F_{ab} = -\left\langle \sum_\ell \frac{2\ell+1}{2}\left(-\frac{1}{C_\ell}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} + \frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right) + \sum_\ell \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{2}\left(\frac{1}{C_\ell^2}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} - \frac{2}{C_\ell^3}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right\rangle_{a_{\ell m}}$$

$$\sum_m |a_{\ell m}|^2 \to (2\ell+1)$$

$$= \boxed{\sum_{\ell=2}^{\ell_{max}} \frac{2\ell+1}{2}\frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}}$$

COSMIC VARIANCE LIMITED
CMB FISHER MATRIX

T_{CMB}

$$F_{ab} = -\left\langle \sum_\ell \frac{2\ell+1}{2}\left(-\frac{1}{C_\ell}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} + \frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right) + \sum_\ell \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{2}\left(\frac{1}{C_\ell^2}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} - \frac{2}{C_\ell^3}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right\rangle_{a_{\ell m}}$$

$$\sum_m |a_{\ell m}|^2 \to (2\ell+1)$$

$$= \boxed{\sum_{\ell=2}^{\ell_{max}} \frac{2\ell+1}{2}\frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}}$$

COSMIC VARIANCE LIMITED
CMB FISHER MATRIX

$T_{CMB} + \delta T_{CMB}$

$2.7K$ "

$$F_{ab} = -\left\langle \sum_{\ell} \frac{2\ell+1}{2}\left(-\frac{1}{C_\ell}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} + \frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right) + \sum_{\ell}\sum_{m=-\ell}^{\ell}\frac{|a_{\ell m}|^2}{2}\left(\frac{1}{C_\ell^2}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} - \frac{2}{C_\ell^3}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right\rangle_{a_{\ell m}}$$

$$\sum_m |a_{\ell m}|^2 \to (2\ell+1)$$

$$= \boxed{\sum_{\ell=2}^{\ell_{max}} \frac{2\ell+1}{2}\frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}}$$

COSMIC VARIANCE LIMITED
CMB FISHER MATRIX

$$T_{CMB} + \delta T_{CMB}$$

$$\overset{\shortparallel}{2.7\,K} \qquad \overset{\shortparallel}{100\,\mu K}$$

$$F_{ab} = -\left\langle \sum_{\ell} \frac{2\ell+1}{2}\left(-\frac{1}{C_\ell}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} + \frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right) + \sum_{\ell}\sum_{m=-\ell}^{\ell}\frac{|a_{\ell m}|^2}{2}\left(\frac{1}{C_\ell^2}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} - \frac{2}{C_\ell^3}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right\rangle_{a_{\ell m}}$$

$$\sum_m |a_{\ell m}|^2 \to (2\ell+1)$$

$$= \boxed{\sum_{\ell=2}^{\ell_{max}} \frac{2\ell+1}{2}\frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}}$$

COSMIC VARIANCE LIMITED
CMB FISHER MATRIX

$$T_{CMB} + \delta T_{CMB}$$

$$\boxed{2.7\,K} \qquad 100\,\mu K$$

$$F_{ab} = -\left\langle \sum_\ell \frac{2\ell+1}{2}\left(-\frac{1}{C_\ell}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} + \frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right) + \sum_\ell \sum_{m=-\ell}^{\ell} \frac{|a_{\ell m}|^2}{2}\left(\frac{1}{C_\ell^2}\frac{\partial^2 C_\ell}{\partial\theta^a\partial\theta^b} - \frac{2}{C_\ell^3}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}\right)\right\rangle_{a_{\ell m}}$$

$$\sum_m |a_{\ell m}|^2 \to (2\ell+1)$$

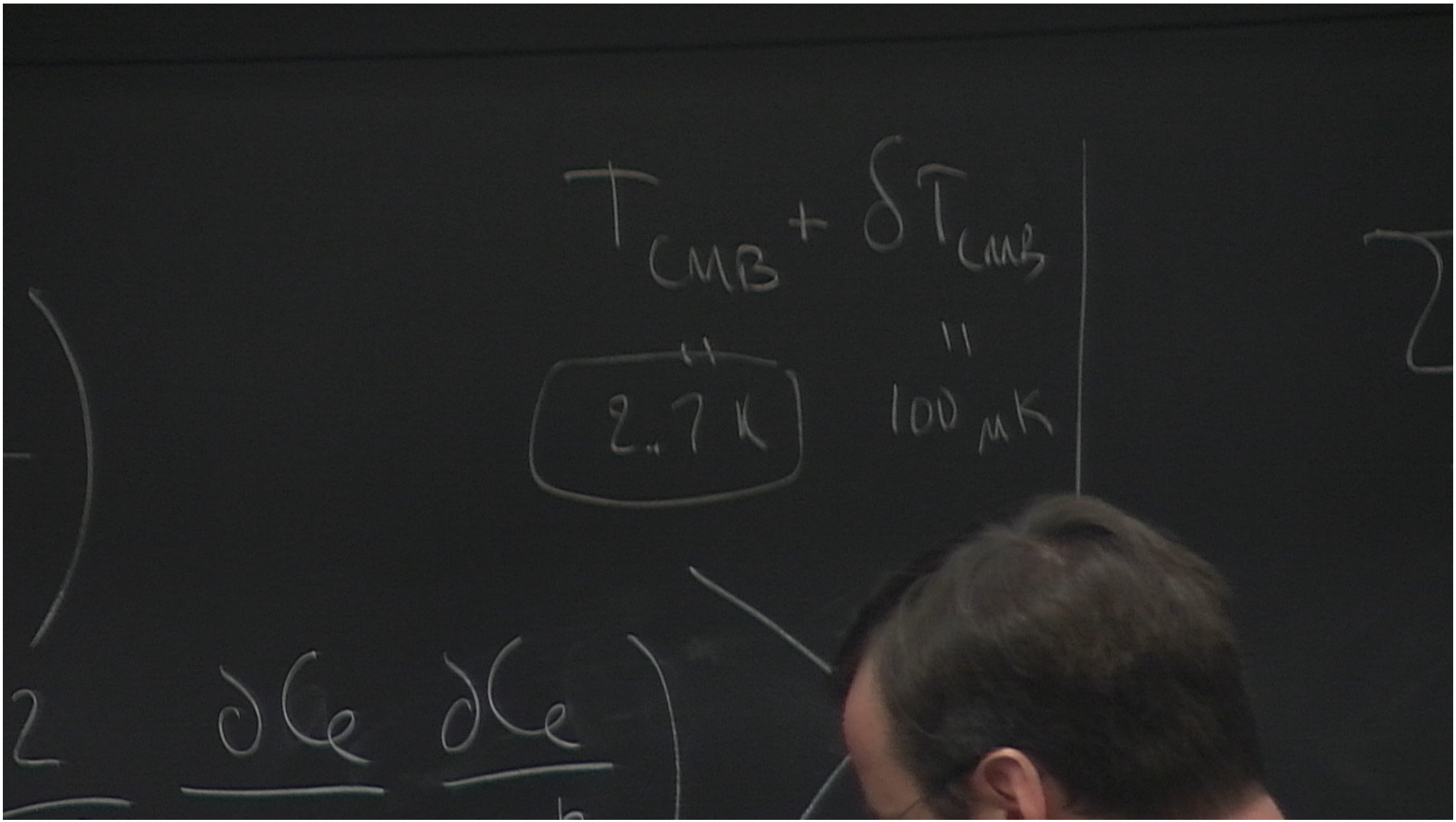
$$= \boxed{\sum_{\ell=2}^{\ell_{max}} \frac{2\ell+1}{2}\frac{1}{C_\ell^2}\frac{\partial C_\ell}{\partial\theta^a}\frac{\partial C_\ell}{\partial\theta^b}}$$

COSMIC VARIANCE LIMITED
CMB FISHER MATRIX

$$+ \sum_{l} \sum_{m=-l}^{l} \frac{|a_{lm}|^2}{2}$$

$$\sum_{m} |a_{lm}|^2 \to (2l+1)$$

$$= \boxed{\sum_{l=2}^{l_{MAX}} \frac{2l+1}{2} \frac{1}{C_l^2} \frac{\partial C}{\partial \theta}}$$

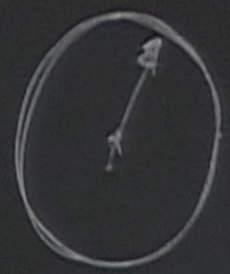

$T_{CMB} + \delta T_{CMB}$
$\boxed{2.7\,K}$
$100\,\mu K$

COSMIC VARIANCE LIMITED

CMB FISHER MATRIX

ρ_r

$\Omega_r h^2$



COSMIC VARIANCE LIMITED

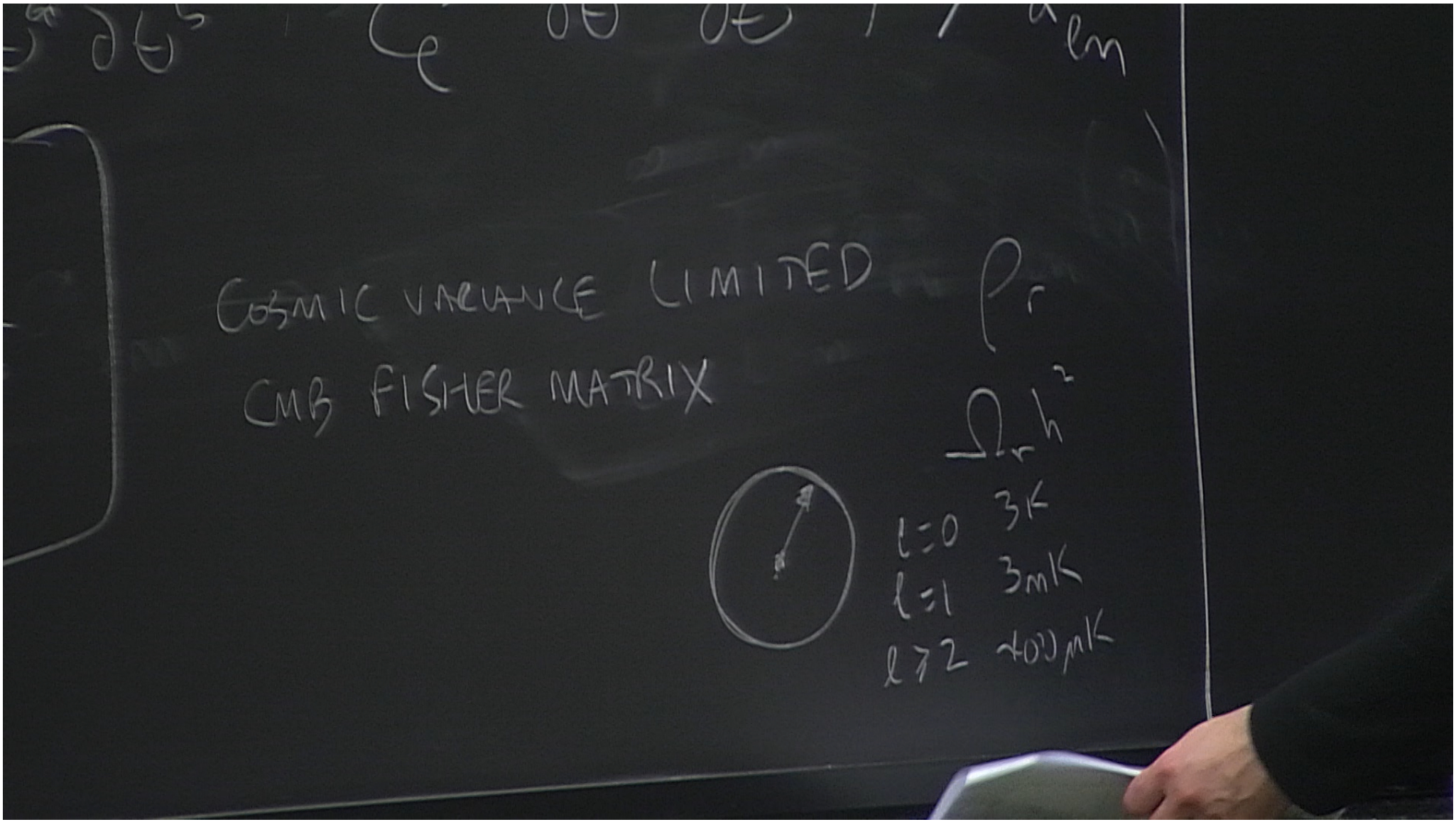
CMB FISHER MATRIX

ρ_r

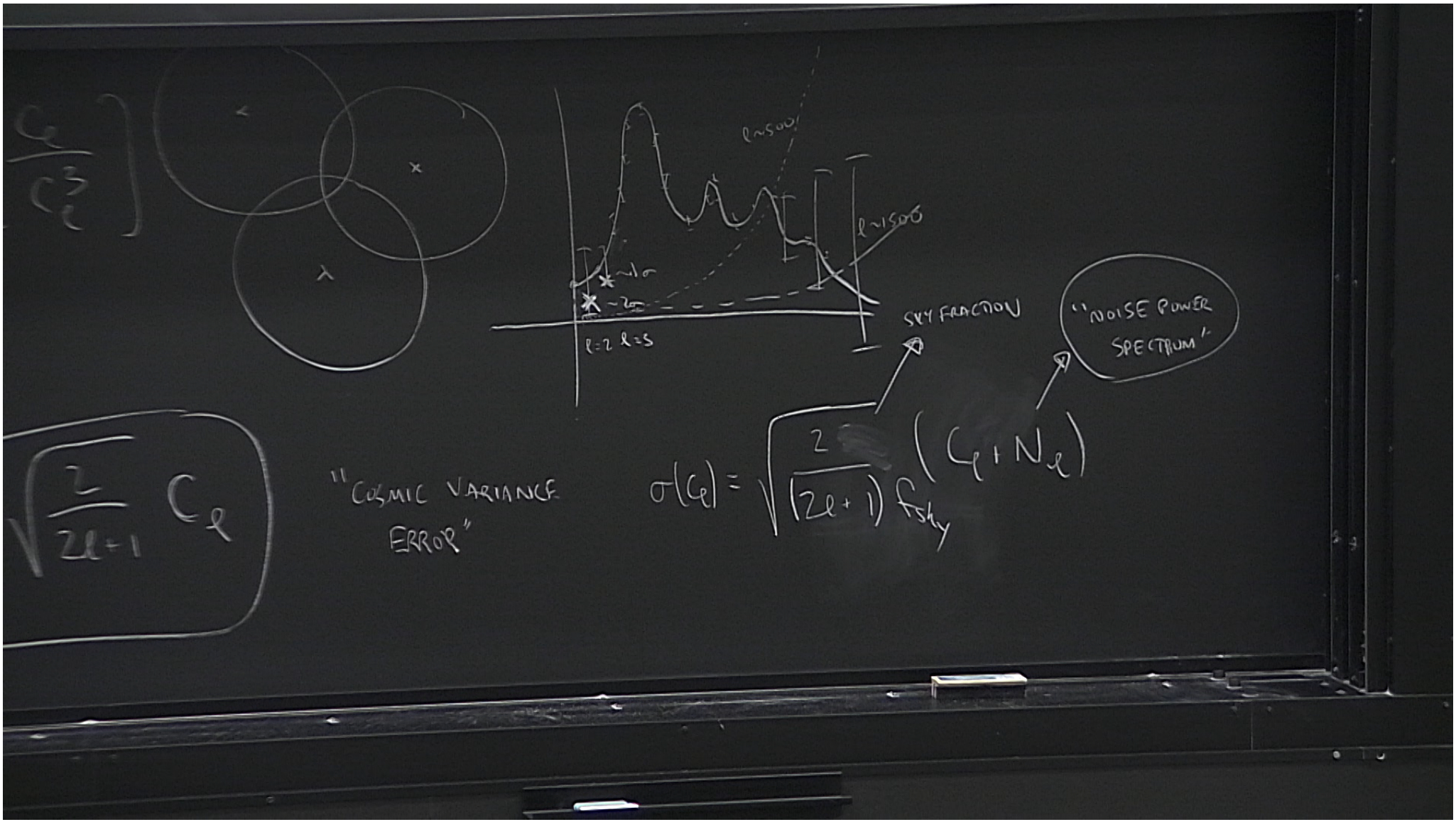
$\Omega_r h^2$

$l=0$ 3K

$l=1$ 3mK


COSMIC VARIANCE LIMITED
CMB FISHER MATRIX
ρ_r
$\Omega_r h^2$
$l=0$ 3K
$l=1$ 3mK
$l \geq 2$ 400 μK

$$f_{sky} \sum_{\ell=2}^{\infty} \frac{2\ell+1}{2} \frac{1}{(C_\ell + N_\ell)^2} \frac{\partial C_\ell}{\partial \theta^a} \frac{\partial C_\ell}{\partial \theta^b}$$


$\frac{C_\ell}{C_\ell^3}$
$\ell \sim 500$
$\ell \sim 1500$
$\sim 1\sigma$
$\sim 2\sigma$
$\ell = 2$ $\ell = 5$
SKY FRACTION
"NOISE POWER SPECTRUM"
$\sqrt{\frac{2}{2\ell+1} C_\ell}$
"COSMIC VARIANCE ERROR"
$\sigma(C_\ell) = \sqrt{\frac{2}{(2\ell+1) f_{sky}} (C_\ell + N_\ell)}$

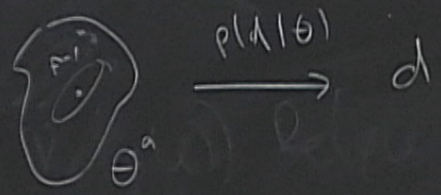
$$= -\left[\frac{2\ell+1}{2}\frac{1}{C_\ell^2} - \sum_{m=-\ell} \frac{C_\ell}{C_\ell^3}\right]$$

$$= \boxed{f_{sky}\frac{2\ell+1}{2}\frac{1}{C_\ell^2}}$$

$C_{\ell m}$

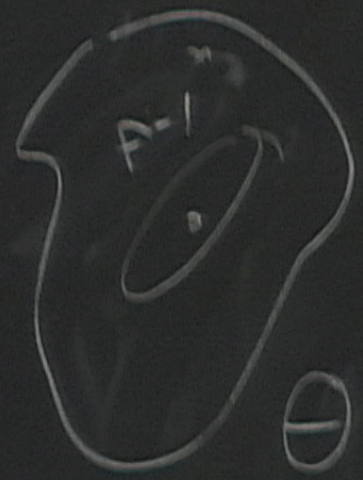
$$\sigma(C_\ell) = \sqrt{VAR(C_\ell)} = \sqrt{\frac{1}{F}} = \sqrt{\frac{2}{2\ell+1}}\, C_\ell$$

PROPERTIES OF THE FISHER MATRIX



1) IF DATASET d CONSISTS OF TWO INDEPENDENT PARTS $d = d_1 + d_2$

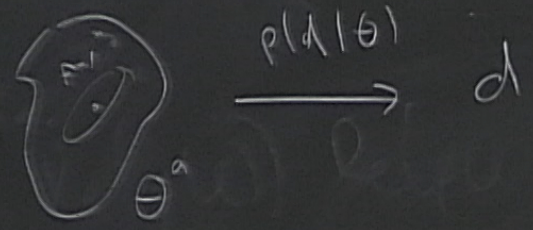
THEN $F = F_1 + F_2$



$$\xrightarrow{p(d|\theta)} d$$

1) IF DATASET d CONSISTS OF TWO

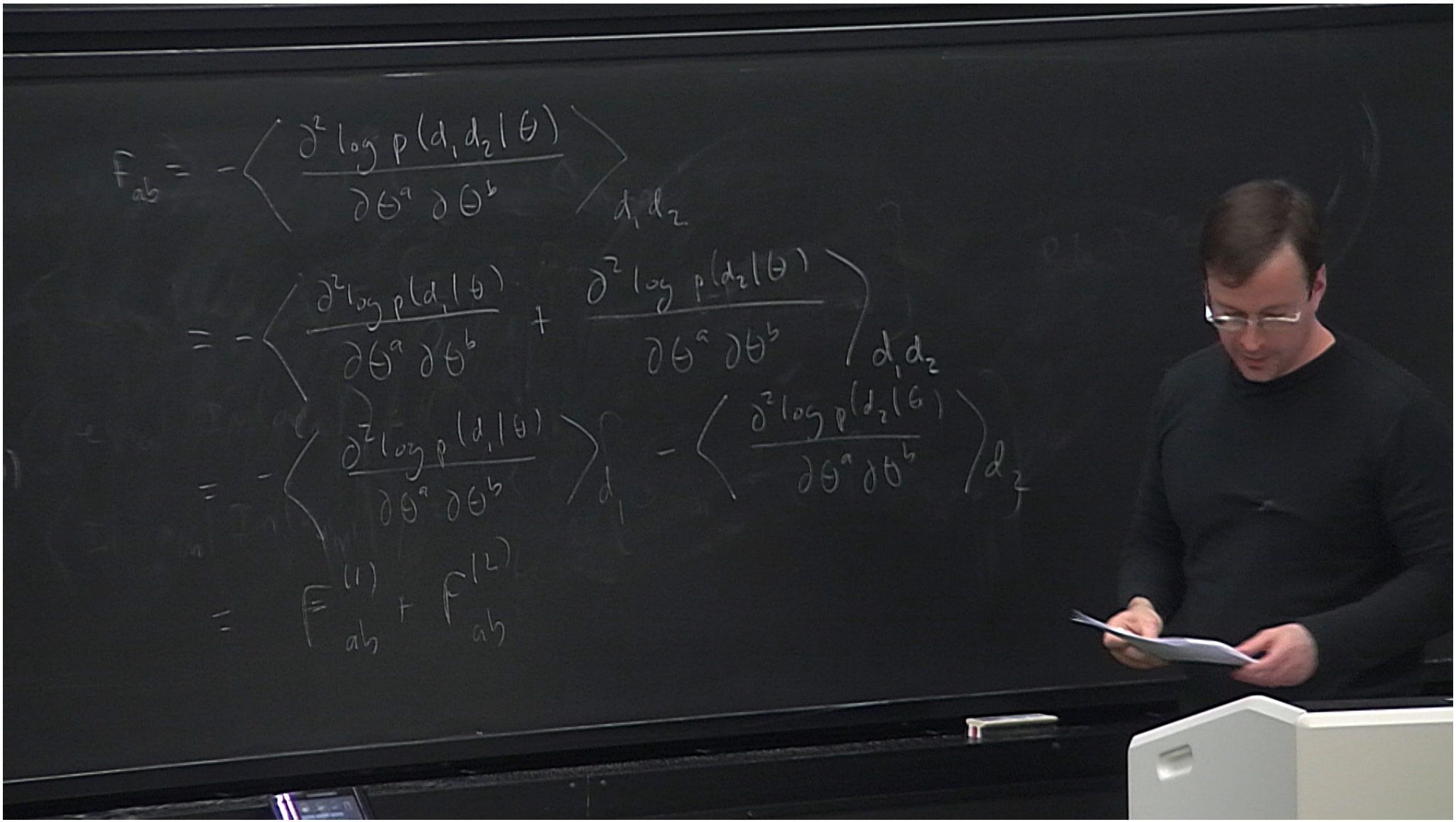
PROPERTIES OF THE FISHER MATRIX



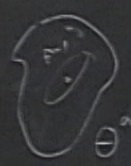
1) IF DATASET d CONSISTS OF TWO INDEPENDENT PARTS $d = d_1 + d_2$

$\hookrightarrow p(d_1, d_2 | \theta) = p(d_1|\theta)\, p(d_2|\theta)$

THEN $F = (F_1 + F_2)$


$F_{ab} = -\left\langle \frac{\partial^2 \log p(d_1 d_2 | \theta)}{\partial\theta^a \partial\theta^b} \right\rangle_{d_1 d_2}$
$= -\left\langle \frac{\partial^2 \log p(d_1 | \theta)}{\partial\theta^a \partial\theta^b} + \frac{\partial^2 \log p(d_2 | \theta)}{\partial\theta^a \partial\theta^b} \right\rangle_{d_1 d_2}$
$= -\left\langle \frac{\partial^2 \log p(d_1 | \theta)}{\partial\theta^a \partial\theta^b} \right\rangle_{d_1} - \left\langle \frac{\partial^2 \log p(d_2 | \theta)}{\partial\theta^a \partial\theta^b} \right\rangle_{d_2}$
$= F^{(1)}_{ab} + F^{(2)}_{ab}$

PROPERTIES OF THE FISHER MATRIX



$$\theta^a \xrightarrow{p(d|\theta)} d$$

$$F = F_1 + \cdots + F_N + F'_{N-1}$$

$$F_{ab} = -\left\langle \frac{\partial^2 \log p(d_1, d_2|}{\partial\theta^a \partial\theta^b} \right.$$

$$= -\left\langle \frac{\partial^2 \log p(d_1|\theta)}{\partial\theta^a \partial\theta^b} \right.$$

$$= -\left\langle \frac{\partial^2 \log p(d_1|}{\partial\theta^a \partial\theta^b} \right.$$

$$= F^{(1)}_{ab} + F$$

1) IF DATASET d CONSISTS OF TWO INDEPENDENT PARTS $d = d_1 + d_2$

$$\hookrightarrow p(d_1, d_2|\theta) = p(d_1|\theta)\, p(d_2|\theta)$$

THEN $\boxed{F = (F_1 + F_2)}$

PROPERTIES OF THE FISHER MATRIX

$$\theta^a \xrightarrow{p(d|\theta)} d$$

$$F = F_1 + \cdots + F_N + F'_{N+1}$$

$$f_{ab} = -\left\langle \frac{\partial^2 l}{\ldots} \right\rangle$$

$$= -\left\langle \frac{\partial^2 l}{\ldots} \right\rangle$$

$$= -\left\langle \ldots \right\rangle$$

$$= F_{ab}$$

1) IF DATASET d CONSISTS OF TWO INDEPENDENT PARTS $d = d_1 + d_2$

$$\hookrightarrow p(d_1, d_2 | \theta) = p(d_1|\theta)\, p(d_2|\theta)$$

THEN $\boxed{F = (F_1 + F_2)}$