Title: PSI 2017/2018 - Cosmology - Lecture 6

Date: Apr 16, 2018 10:15 AM

URL: http://pirsa.org/18040018

Abstract:

- BAYESIAN

1) CHOOSING A PRIOR LIKELIHOOD

$p(\theta)$

2) COMPUTE CONDITIONAL LIKELIHOOD

$p(d|\theta)$ — PDF ON d.

3) COMPUTE POSTERIOR LIKELIHOOD

$p(\theta|d) = p(\theta)\, p(d|\theta)$

- BAYESIAN

1) CHOOSING A PRIOR LIKELIHOOD

$p(\theta)$

2) COMPUTE CONDITIONAL LIKELIHOOD

$p(d|\theta)$ ~~— PDF ON d~~

3) COMPUTE POSTERIOR LIKELIHOOD

$$p(\theta|d) = \frac{p(\theta)\,p(d|\theta)}{\int d^N\theta\; p(\theta)\,p(d|\theta)}$$

4) INTERPRET POSTERIOR

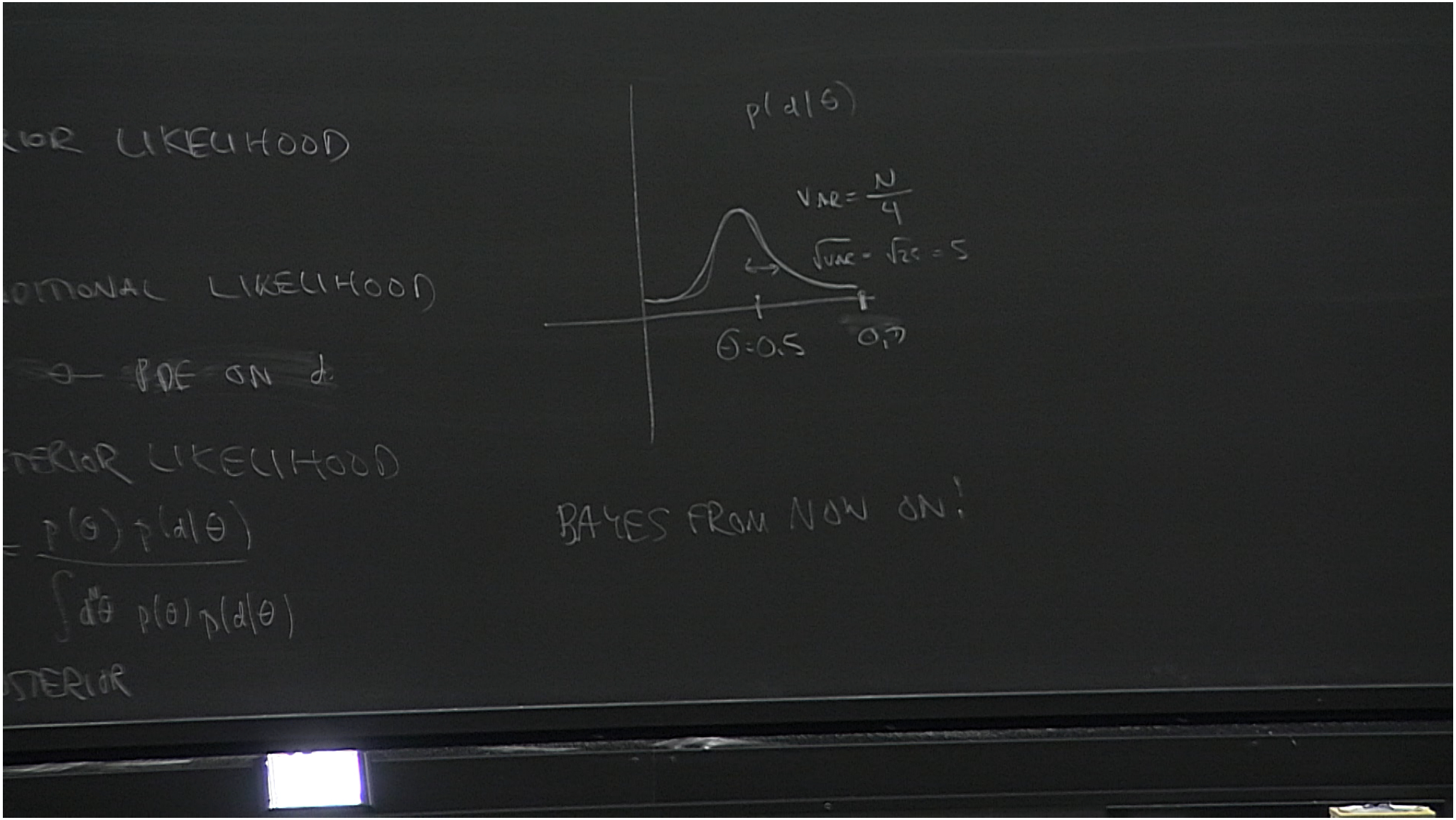
STATISTICS

MODEL PARAMETERS Θ_a $\xrightarrow{p(d|\Theta_a)}$ d

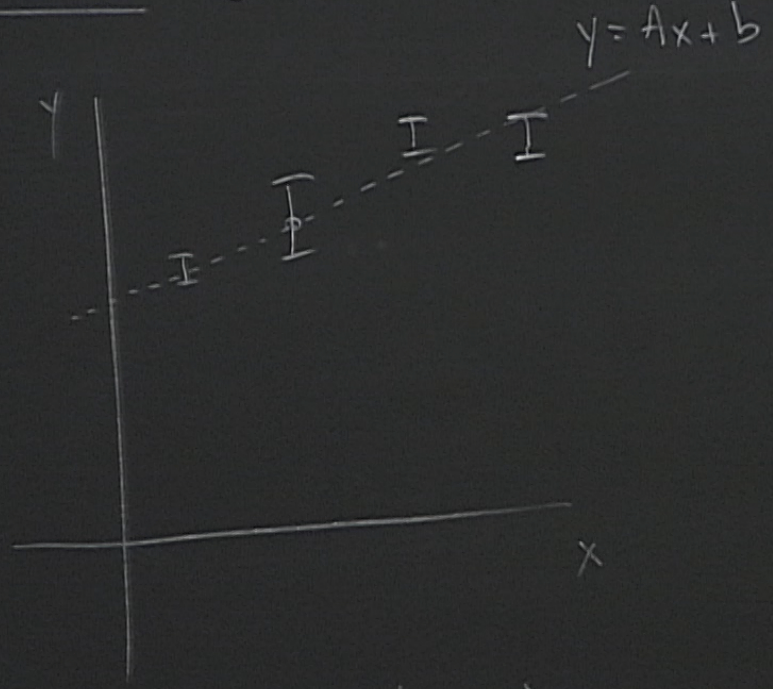
- FREQUENTIST

$p(\Theta)$ = "P-VALUE"

= PROBABILITY OF A SIMULATION W/ MODEL PARAMETERS Θ PRODUCING A RESULT AS EXTREME AS THE DATA d


RIOR LIKELIHOOD
DITIONAL LIKELIHOOD
PDF ON d
TERIOR LIKELIHOOD
P(θ) P(d|θ)
∫dⁿθ P(θ) P(d|θ)
STERIOR
p(d|θ)
VAR = N/4
√VAR = √25 = 5
θ=0.5
0.7
BAYES FROM NOW ON!

EXAMPLE 2: CURVE FITTING



N POINTS $(X_1, Y_1) \cdots (X_N, Y_N)$

GAUSSIAN ERRORS $\sigma_1^2, \cdots, \sigma_N^2$

WHAT IS BEST-FIT LINE $y = Ax + b$?

MODEL PARAMETERS $\theta_a = (A, b)$

DATA: $(Y_1, \cdots, Y_N) = d$

STEP 1: UNIFORM PRIOR

$$p(\theta) \propto 1$$

- BAYESIAN

1) CHOOSING A PRIOR LIKELIHOOD

$p(\theta)$

2) COMPUTE CONDITIONAL LIKELIHOOD

$p(d|\theta)$ ← PDF ON d

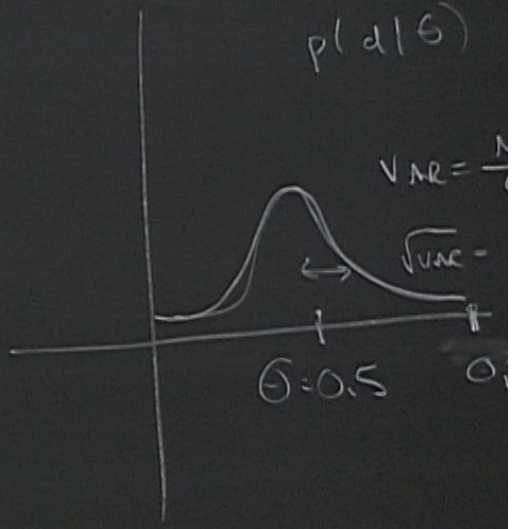
3) COMPUTE POSTERIOR LIKELIHOOD

$$p(\theta|d) = \frac{p(\theta)\,p(d|\theta)}{\int d^N\theta\; p(\theta)\,p(d|\theta)}$$

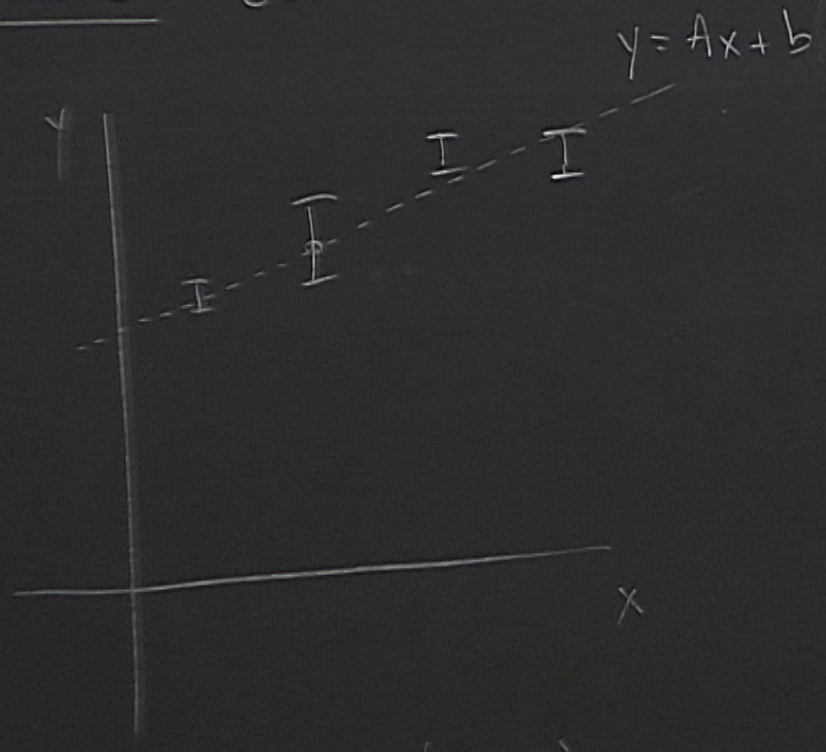
← PDF ON θ

BAYES FROM NOW

4) INTERPRET POSTERIOR



EXAMPLE 2: CURVE FITTING



N POINTS $(X_1, Y_1) \cdots (X_N, Y_N)$

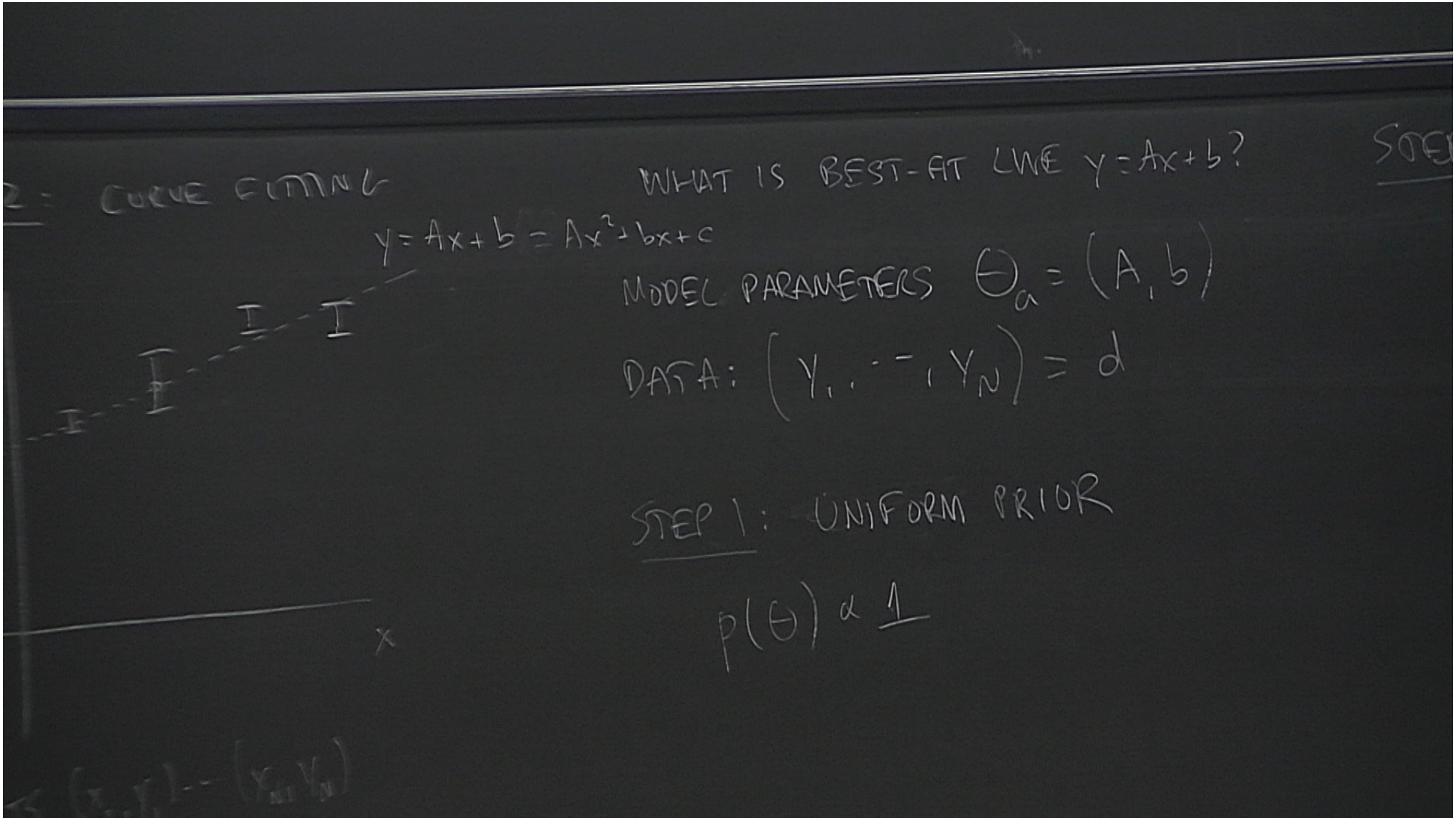
WHAT IS BEST-FIT LINE $y = Ax$

MODEL PARAMETERS $\theta_\alpha = (A, b$

DATA: $(Y_1, \cdots, Y_N) = d$

STEP 1: UNIFORM PRIOR

$$p(\theta) \propto 1$$


2: CURVE FITTING
WHAT IS BEST-FIT LINE y = Ax + b?
y = Ax + b = Ax² + bx + c
MODEL PARAMETERS θ = (A, b)
DATA: (Y₁, ···, Y_N) = d
STEP 1: UNIFORM PRIOR
p(θ) ∝ 1
x

STEP 2: COMPUTE $p(y_i|\theta_a)$

$$y_i = Ax_i + b + [\text{GAUSSIAN ERROR}]$$

$$= \sum_a \theta_a f_a(x_i) + [\text{GAUSSIAN ERROR}]$$

WHERE $f_a(x) = (x, 1)$

$$P(y_i|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{1}{2\sigma^2}\left(y_i - \sum_a \theta_a f_a(x)\right)^2\right]$$

STEP 3:

$$p(\theta|y) \propto \prod_{i=1}^{n} \exp\left(-\frac{(y_i - \theta_a f_a(x_i))^2}{2\sigma_i^2}\right)$$

$$\propto \exp\left(-\sum_i \frac{y_i^2}{2\sigma_i^2} + \sum_{ia} \frac{\theta_a f_a(x_i) y_i}{\sigma_i^2} - \sum_{iab} \frac{\theta_a \theta_b f_a(x_i) f_b(x_i)}{2\sigma_i^2}\right)$$

$$\propto \exp\left(-\frac{1}{2}\theta^T F \theta + \theta^T V\right)$$

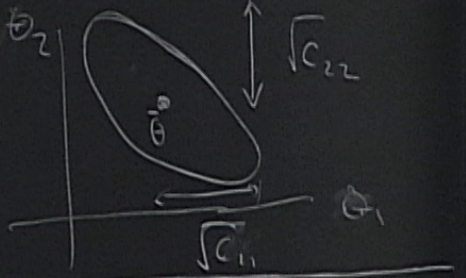
$$F_{ab} = \sum_i \frac{f_a(x_i) f_b(x_i)}{\sigma_i^2}$$

$$V_a = \sum_i \frac{f_a(x_i) y_i}{\sigma_i^2}$$

STEP 4: $p(\theta|y) \propto \exp\left[-\frac{1}{2}(\theta - \underbrace{F^{-1}V}_{\bar{\theta}})^T F (\theta - F^{-1}V)\right] \exp\left(\frac{1}{2}V^T F^{-1} V\right)$

MULTIVARIATE GAUSSIAN WITH MEAN $\boxed{\bar{\theta} = F^{-1}V}$

COVARIANCE MATRIX $C = F^{-1}$

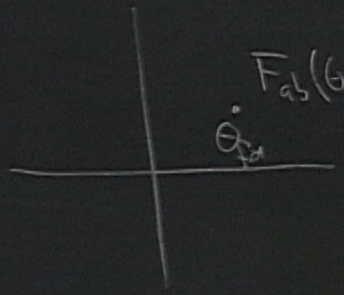


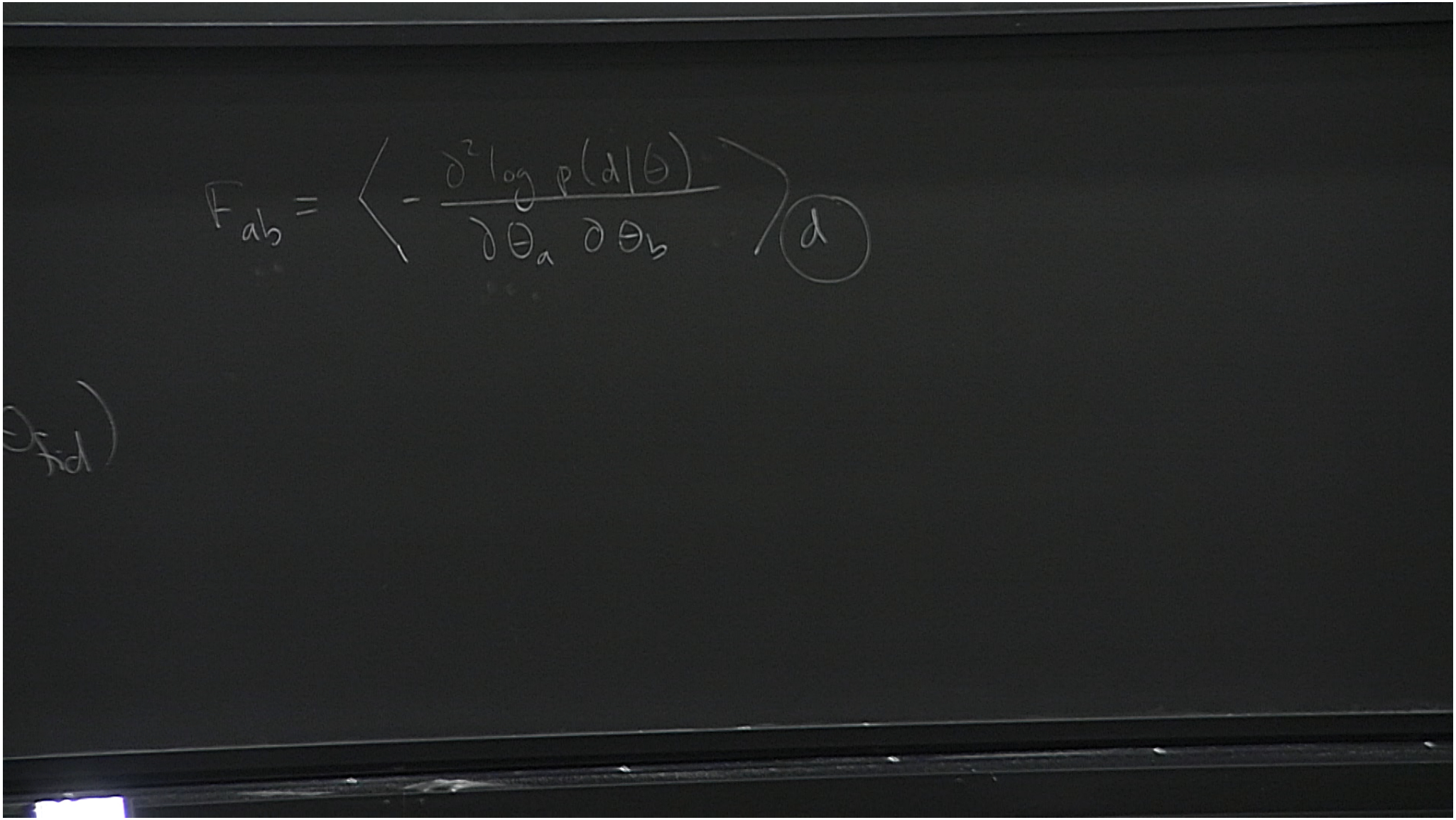
$f(\theta) = \theta^T C^{-1} \theta$ $\quad \theta = (\theta_1, \theta_2)$

$\exp\left[-\underbrace{\theta^T C^{-1} \theta}\right]$

4) INTERPRET POSTERIOR

- SIMULATION: GIVEN θ → RANDOM d ?
- ANALYSIS: GIVEN d → WHICH θ ARE ALLOWED?
- FORECASTING: WHAT ARE "TYPICAL" CONSTRAINTS ON θ
  (NO REALIZATION OF d PROVIDED,
  BUT DEPENDS ON CHOICE OF "FIDUCIAL" MODEL θ_{fid})




$F_{ab} = \left\langle -\frac{\partial^2 \log p(d|\theta)}{\partial\theta_a \partial\theta_b} \right\rangle_d$
θ_{fid}

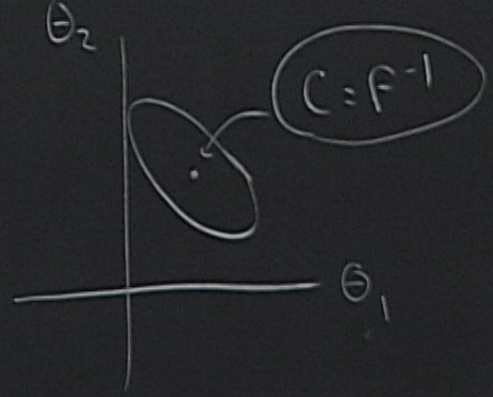
$$p(y|\theta) = \prod_{i=1}^{n} \frac{1}{(2\pi\sigma_i^2)^{1/2}} \exp\left[-\frac{(y_i - \theta_a f_a(x_i))^2}{2\sigma_i^2}\right]$$

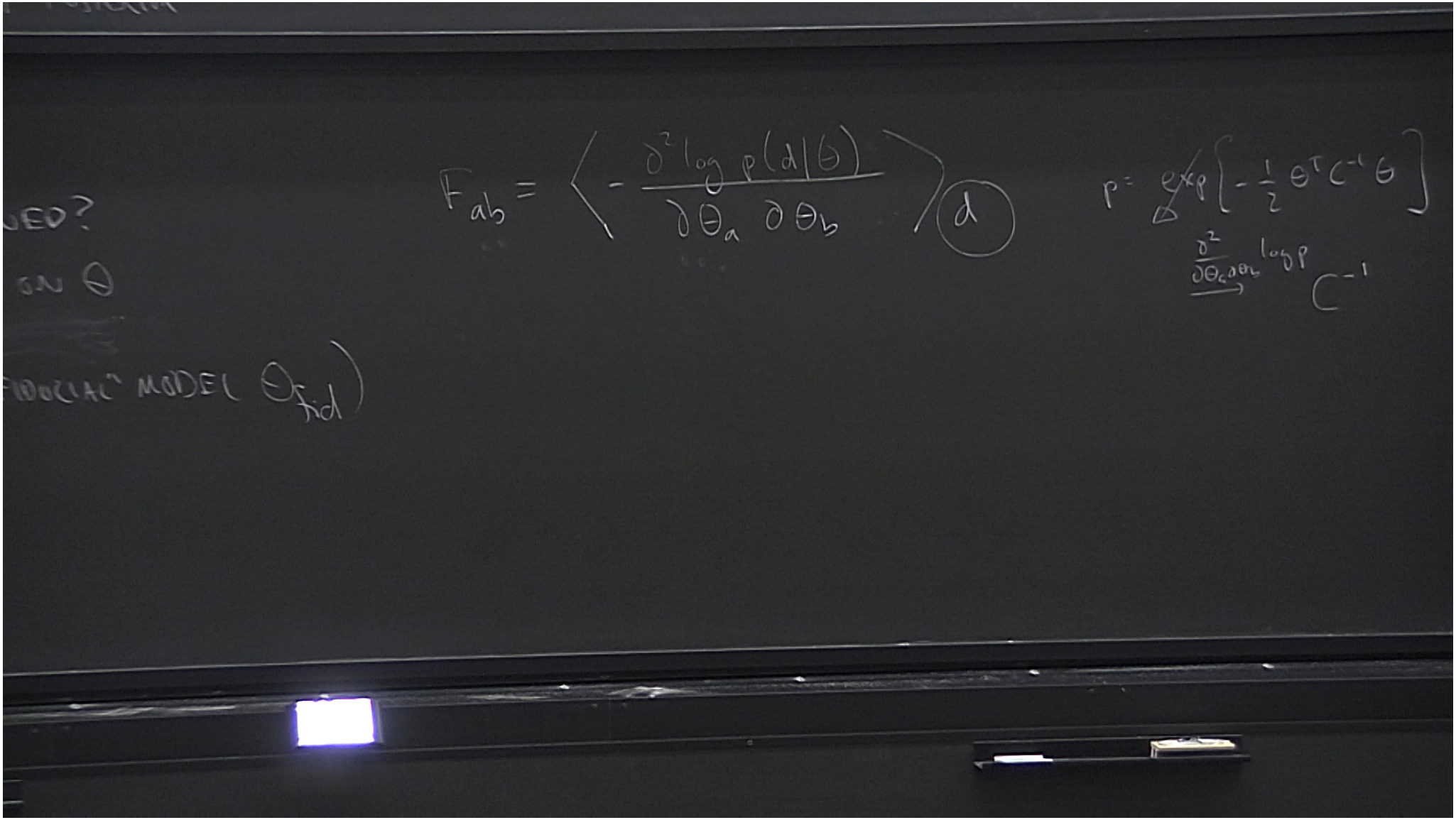
$$F_{ab} = -\left\langle \frac{\partial^2 \log p(y|\theta)}{\partial\theta_a \partial\theta_b} \right\rangle_d$$

$$= -\left\langle \frac{\partial^2}{\partial\theta_a \partial\theta_b} \sum_i \left(-\frac{1}{2}\log(2\pi\sigma_i^2) - \frac{1}{2\sigma_i^2}(y_i - \theta_c f_c(x_i))^2\right)\right\rangle_y$$

$$= -\left\langle \sum_i \left(-\frac{f_a(x_i)\, f_b(x_i)}{\sigma_i^2} \right) \right\rangle_y$$

$$= \sum_i \frac{f_a(x_i)\, f_b(x_i)}{\sigma_i^2}$$




$F_{ab} = \left\langle -\frac{\partial^2 \log p(d|\theta)}{\partial\theta_a \partial\theta_b} \right\rangle_d$
$p = \exp\left[-\frac{1}{2}\theta^T C^{-1}\theta\right]$
$\frac{\partial^2}{\partial\theta_a\partial\theta_b}\log p$
C^{-1}
θ_{fid}

EUCLIDEAN 2D FIELD $\phi(\vec{x})$

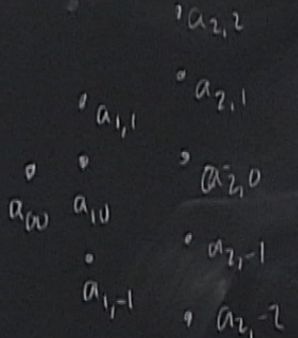
SPHERICAL 2D FIELD $\phi(\hat{n})$

$$\phi(\vec{x}) = \int \frac{d^2\ell}{(2\pi)^2} \tilde{\phi}(\ell)\, e^{i\ell\cdot x} \iff \phi(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}\, Y_{\ell m}(\theta,\phi)$$



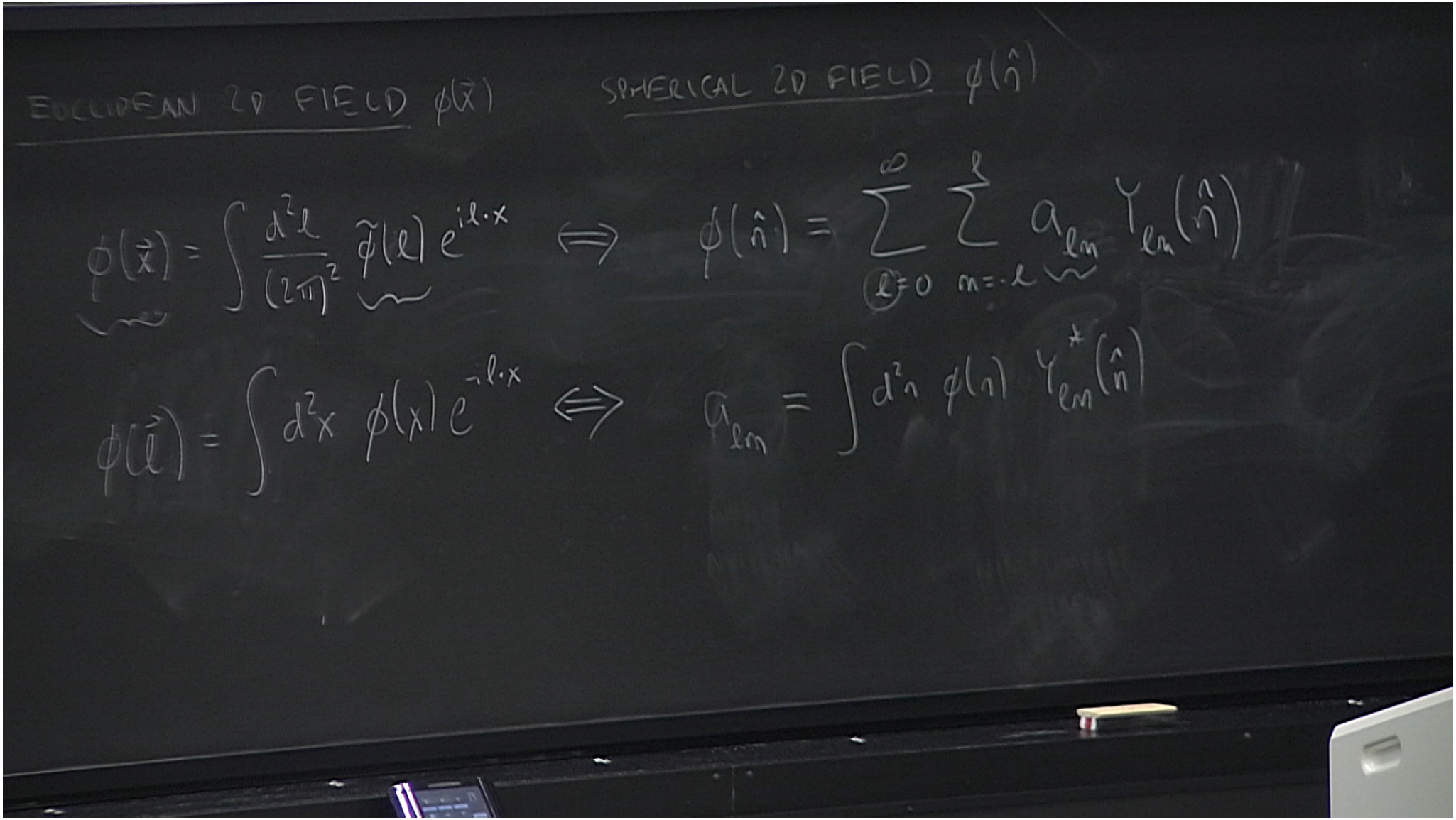
COMPACT

NON-DISCRETE



NON-COMPACT

DISCRETE


EUCLIDEAN 2D FIELD $\phi(\vec{x})$
SPHERICAL 2D FIELD $\phi(\hat{n})$
$\phi(\vec{x}) = \int \frac{d^2\ell}{(2\pi)^2} \tilde{\phi}(\ell) e^{i\ell\cdot x} \iff \phi(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{n})$
$\tilde{\phi}(\vec{\ell}) = \int d^2x\, \phi(x) e^{-i\ell\cdot x} \iff a_{\ell m} = \int d^2\hat{n}\, \phi(\hat{n}) Y^*_{\ell m}(\hat{n})$

RICAL 2D FIELD $\phi(\hat{n})$

$$\phi(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\hat{n})$$

$$a_{\ell m} = \int d^2\hat{n}\, \phi(\hat{n})\, Y^*_{\ell m}(\hat{n})$$

$$\int d^2\hat{n}\, Y_{\ell m}(\hat{n})\, Y^*_{\ell' m'}(\hat{n}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\phi(\hat{n}) \longleftrightarrow a_{\ell m}$$

EUCLIDEAN 2D FIELD $\phi(\vec{x})$ SPHERICAL 2D FIELD $\phi(\hat{n})$

$$\phi(\vec{x}) = \int \frac{d^2\ell}{(2\pi)^2} \tilde{\phi}(\ell)\, e^{i\ell\cdot x} \iff \phi(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}\, Y_{\ell m}(\hat{n}) \qquad \int d^2\hat{n}\, Y_{\ell m}(\hat{n}) = \delta_{\ell\ell'}$$

$$\tilde{\phi}(\vec{\ell}) = \int d^2x\, \phi(x)\, e^{-i\ell\cdot x} \iff a_{\ell m} = \int d^2\hat{n}\, \phi(\hat{n})\, Y^*_{\ell m}(\hat{n}) \qquad \phi(\hat{n}) \leftrightarrow a_{\ell}$$

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10} = \sqrt{\frac{3}{\delta_\pi}}\cos(\theta)$$

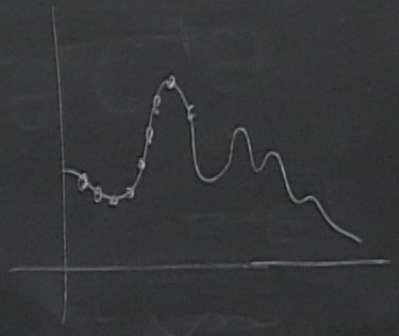
STEP 4: $p(\theta|y) \propto \exp\left[-\frac{1}{2}(\theta - F^{-1}V)^T F (\theta - F^{-1}V)\right]$

$$\langle \phi(\vec{x})\,\phi(\vec{y}) \rangle = \xi(|\vec{x}-\vec{y}|) \iff \langle \phi(\hat{n})\,\phi(\hat{n}') \rangle = \xi(\theta_{\hat{n}\hat{n}'})$$

$$\langle \phi(\vec{\ell})\,\phi(\vec{\ell}')^* \rangle = P(\ell)(2\pi)^2 \delta^2(\ell-\ell') \iff \langle a_{\ell m}\, a^*_{\ell' m'} \rangle = C_\ell\, \delta_{\ell\ell'}$$

POWER SPECTRUM

POWER SPECTRUM



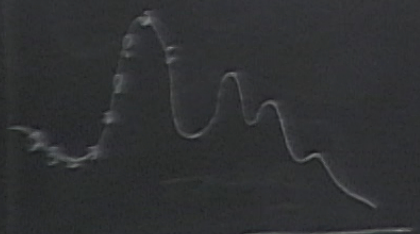
$$\Leftrightarrow \langle \phi(\hat{n})\,\phi(\hat{n}') \rangle = \zeta(\theta_{\hat{n}\hat{n}'}) \qquad \theta_{nn'} = \cos^{-1}(n \cdot n')$$

$$(l-l') \Leftrightarrow \langle a_{lm}\, a^{*}_{l'm'} \rangle = C_l\, \delta_{ll'}\, \delta_{mm'}$$

[NOT STRAIGHTFORWARD]

WIGNER-ECKHART

POWER SPECTRUM



$\tilde{\phi}(\ell)^* = \tilde{\phi}(-\ell)$ $\Longleftrightarrow$ $a^*_{\ell m} = (-1)^m a_{\ell,-m}$

IF $\phi(x)$ REAL-VALUED IN REAL SPACE

IF $\phi(n)$ REAL-VALUED IN REAL SPACE

$(2\ell+1)$ REAL NUMBERS FOR EA

$\{ \mathrm{Re}(a_{\ell 0}),\ \mathrm{Re}(a_{\ell 1}),\ \mathrm{Im}(a_{\ell 1}$