Title: PSI 2017/2018 - Cosmology - Lecture 3

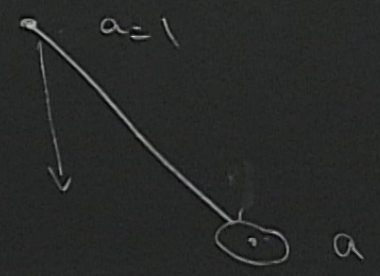
Date: Apr 11, 2018 10:15 AM

URL: http://pirsa.org/18040015

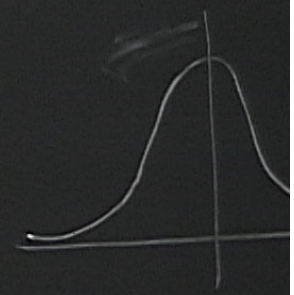
Abstract:

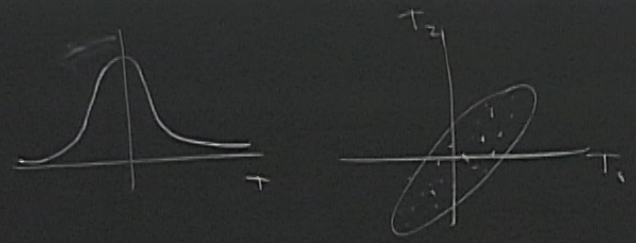
- TIME COORDINATES: t, τ, a, z

$$a = \frac{1}{1+z}$$



$$\nu \to a^{-1}\nu = (1+z)\nu$$





RANDOM VARIABLES

X_i DENOTES A MULTIVARIATE RANDOM VARIABLE $(i=1,\dots,N)$

MEAN $\overline{X}_i = \langle X_i \rangle$

COVARIANCE MATRIX $C_{ij} = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle$

$= \langle (X_i - \overline{X}_i)(X_j - \overline{X}_j) \rangle$

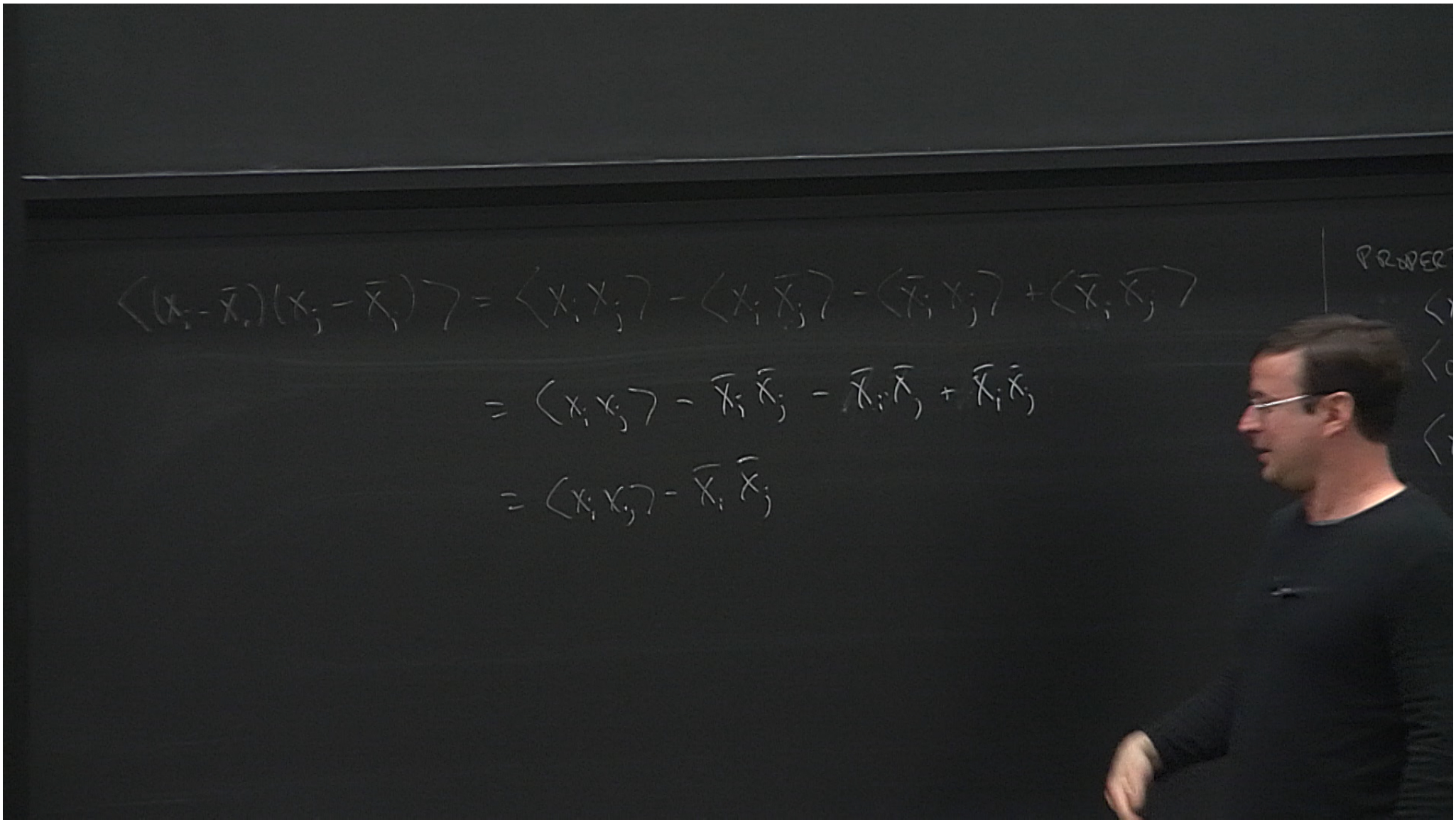
$= \langle \bar{X}_i X_j \rangle$

PROPERTIES OF $\langle \cdot \rangle$:

$\langle X \pm Y \rangle = \langle X \rangle \pm \langle Y \rangle$

$\langle cX \rangle = c \langle X \rangle$ IF c IS A CONSTANT (I.E. NONRANDOM)

$\langle XY \rangle = \langle X \rangle \langle Y \rangle$ IF X, Y ARE STATISTICALLY INDEPENDENT


$\langle (x_i - \bar{x}_i)(x_j - \bar{x}_j) \rangle = \langle x_i x_j \rangle - \langle x_i \bar{x}_j \rangle - \langle \bar{x}_i x_j \rangle + \langle \bar{x}_i \bar{x}_j \rangle$
$= \langle x_i x_j \rangle - \bar{x}_i \bar{x}_j - \bar{x}_i \bar{x}_j + \bar{x}_i \bar{x}_j$
$= \langle x_i x_j \rangle - \bar{x}_i \bar{x}_j$

VARIANCE $\mathrm{VAR}(X_i) = \mathrm{Cov}(X_i, X_i)$

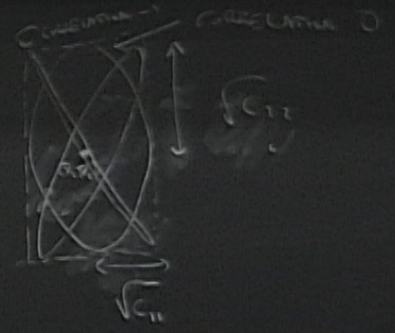
$$= \langle (X_i - \bar{X}_i)^2 \rangle$$

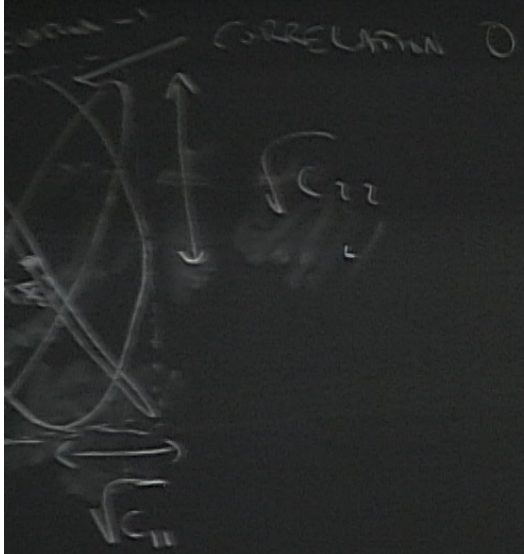
$\sqrt{\mathrm{VAR}(X_i)}$ = "TYPICAL" SIZE OF FLUCTUATIONS AROUND THE MEAN

CORRELATION COEFFICENT $\mathrm{CORR}(X_i, X_j) = \dfrac{\mathrm{Cov}(X_i, X_j)}{\sqrt{\mathrm{VAR}(X_i)\,\mathrm{VAR}(X_j)}}$ IS BETWEEN -1 AND 1

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}$$

$$-\sqrt{C_{11}C_{22}} \le C_{12} \le \sqrt{C_{11}C_{22}}$$





LET X_i BE A R.V.

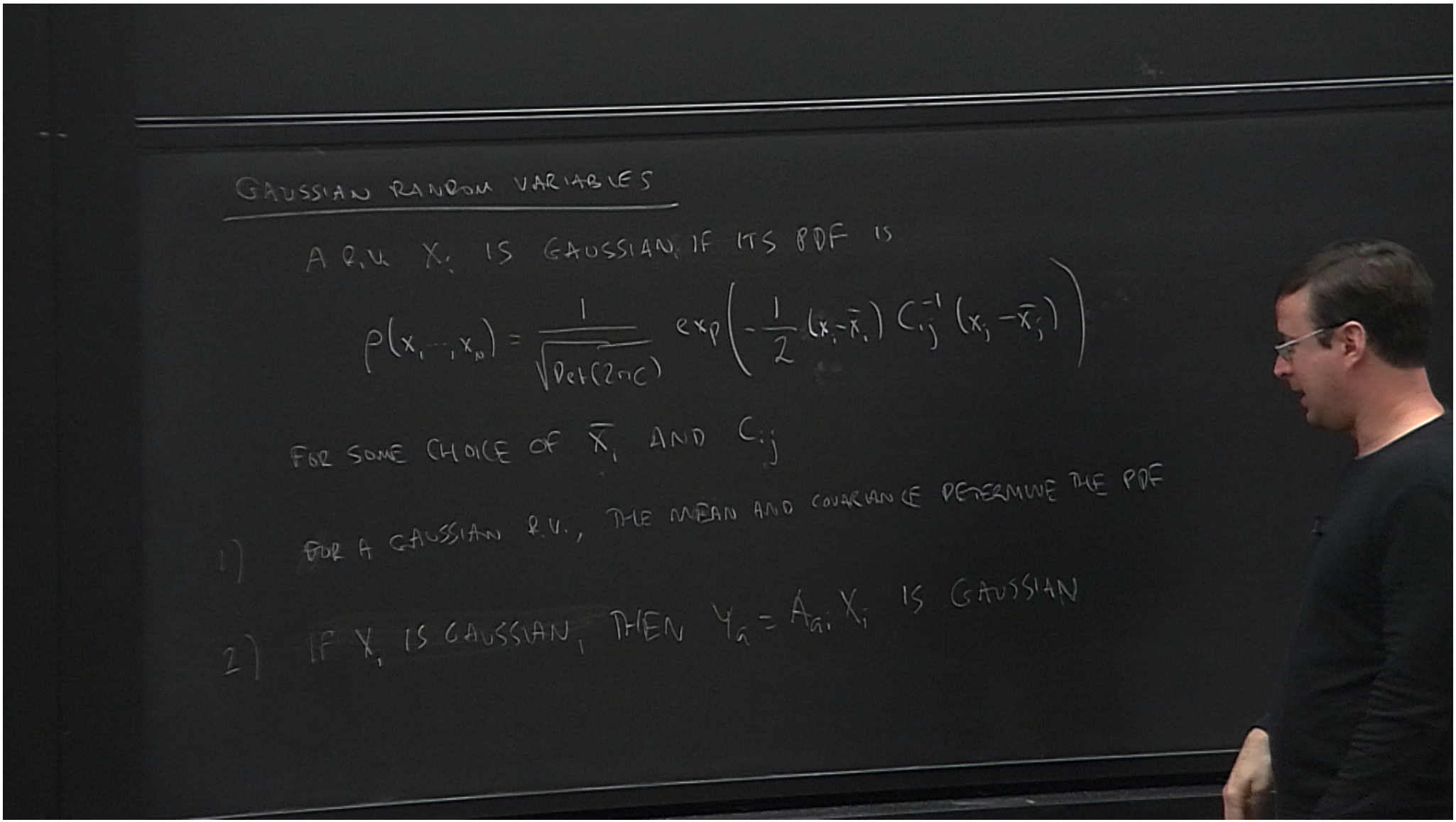
DEFINE $Y_a = A_{ai} X_i$

HOW ARE THE MEAN & COVARIANCE OF Y ARE RELATED TO THOSE OF X?

$$\bar{Y}_a = A_{ai} \bar{X}_i$$

$$C^Y_{ab} = A_{ai} A_{bj} C^X_{ij}$$

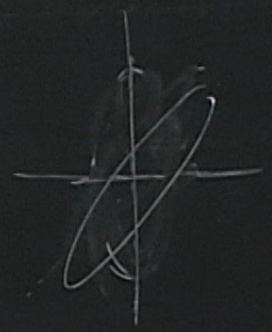
$$\boxed{\bar{Y} = A\bar{X} \qquad C_Y = A C_X A^T}$$


GAUSSIAN RANDOM VARIABLES
A R.V. X_i IS GAUSSIAN IF ITS PDF IS
$\rho(x_1, \dots, x_n) = \frac{1}{\sqrt{\mathrm{Det}(2\pi C)}} \exp\left(-\frac{1}{2}(x_i - \bar{x}_i) C^{-1}_{ij} (x_j - \bar{x}_j)\right)$
FOR SOME CHOICE OF $\bar{X}_i$ AND C_{ij}
1) FOR A GAUSSIAN R.V., THE MEAN AND COVARIANCE DETERMINE THE PDF
2) IF X_i IS GAUSSIAN, THEN $Y_a = A_{ai} X_i$ IS GAUSSIAN

- SIMULATING A MULTIVARIATE GAUSSIAN

GIVEN $C_{ij} \Rightarrow$ SAMPLE X_i

$N=2$ $\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}$



DIAGONALIZE

$C = R \Delta R^{-1}$ R = ROTATION $(R^{-1} = R^T)$

Δ = DIAGONAL

X_i HAS COVARIANCE $\Delta \Rightarrow Y = RX$ HAS COVARIANCE $R\Delta R^T = R\Delta R^{-1}$
$= C$

- $C = LL^T$ CHOLESKY

X HAS COVARIANCE I

$\Rightarrow Y = LX$ HAS COVARIANCE

$LIL^T = C$

THEOREM: C_{ij} IS A POSITIVE SEMIDEFINITE MATRIX

(ALL EIGENVALUES ARE ≥ 0)

$N=1$ $\quad C = \left(\mathrm{var}(x_1) \right) \geq 0$

$N>1$ $\quad \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \Rightarrow \lambda_1 \geq 0$

VARIANCE $\mathrm{VAR}(x_i) = \mathrm{Cov}(x_i, x_i)$

$$= \langle (x_i - \bar{x}_i)^2 \rangle$$

$\sqrt{\mathrm{VAR}(x_i)}$ = "TYPICAL" SIZE OF FLUCTUATIONS AROUND THE MEAN

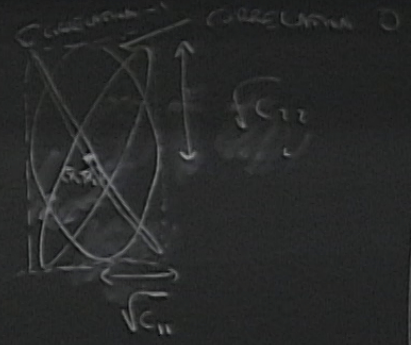
CORRELATION COEFFICENT $\mathrm{Corr}(x_i, x_j) = \dfrac{\mathrm{Cov}(x_i, x_j)}{\sqrt{\mathrm{VAR}(x_i)\,\mathrm{VAR}(x_j)}}$ IS BETWEEN -1 AND 1

$\Delta \geq 0$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}$$

$$-\sqrt{C_{11}C_{22}} \leq C_{12} \leq \sqrt{C_{11}C_{22}}$$

$$C_{12}^2 \leq C_{11}C_{22}$$



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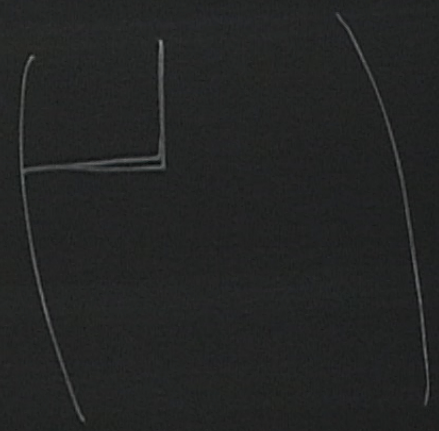
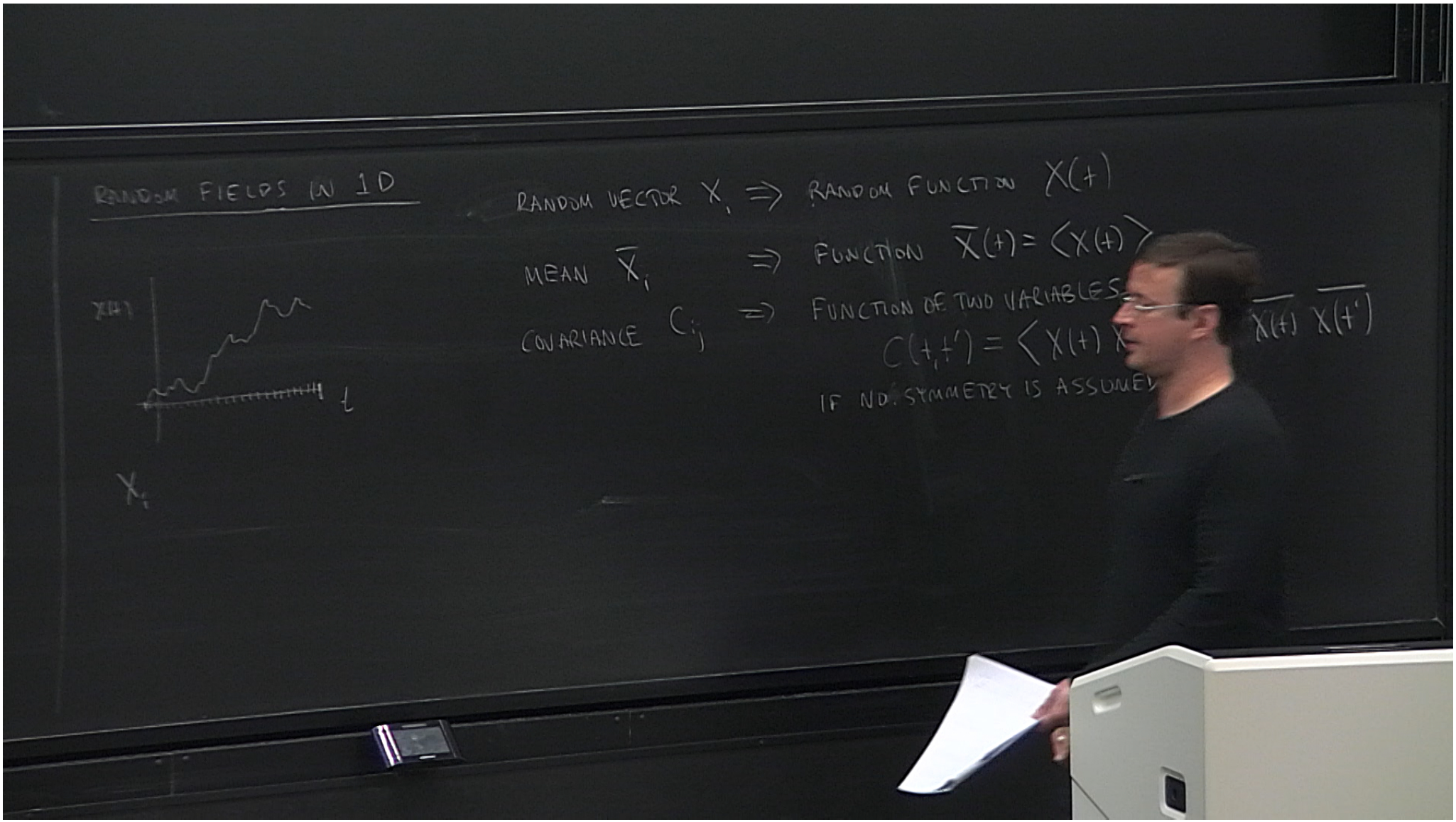
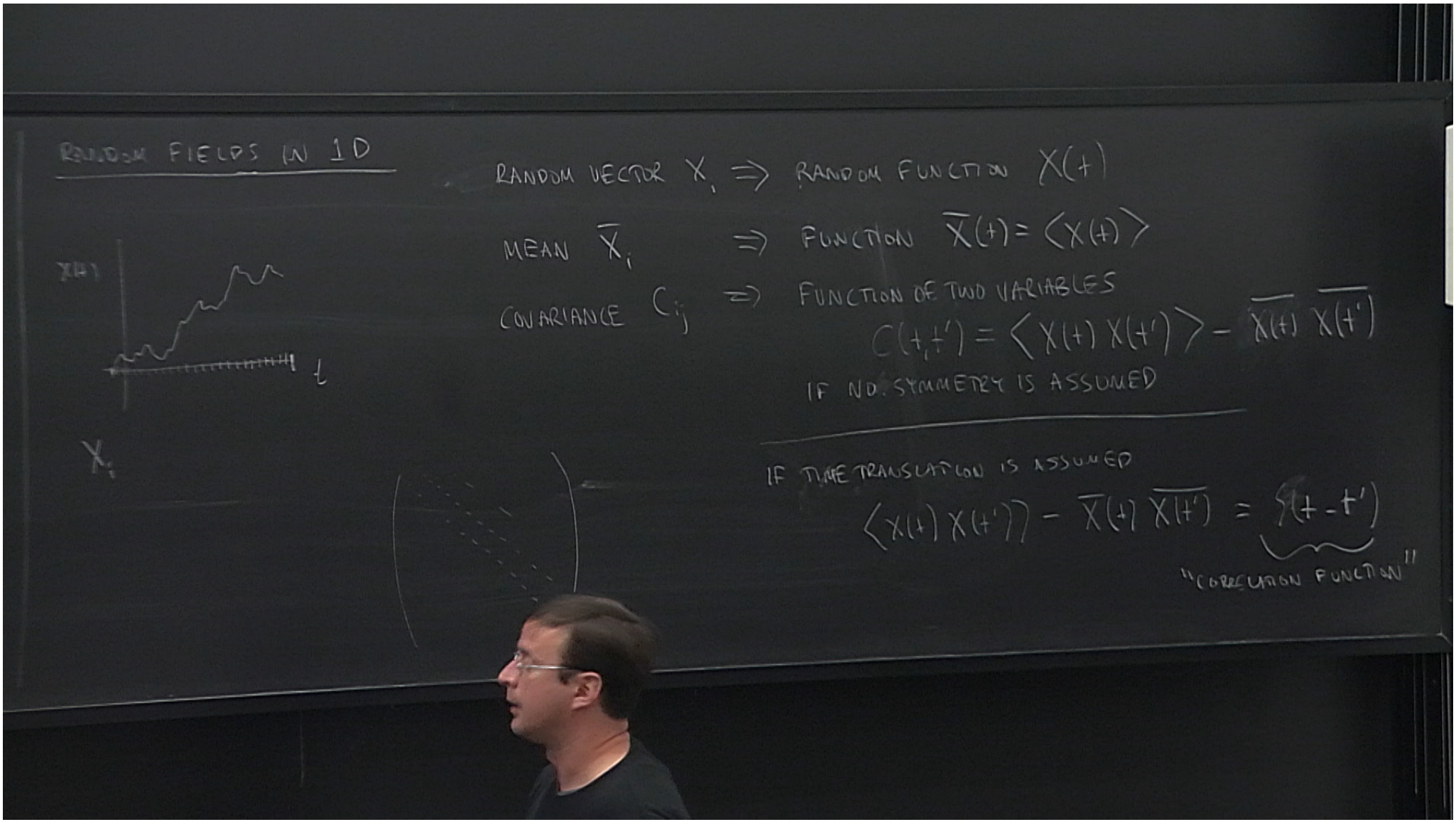



RANDOM FIELDS IN 1D
X(t)
t
X_i
RANDOM VECTOR X_i ⇒ RANDOM FUNCTION $X(t)$
MEAN $\bar{X}_i$ ⇒ FUNCTION $\bar{X}(t) = \langle X(t) \rangle$
COVARIANCE C_{ij} ⇒ FUNCTION OF TWO VARIABLES
$C(t,t') = \langle X(t) X$
$\overline{X(t)}\ \overline{X(t')}$
IF NO SYMMETRY IS ASSUME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X(t)
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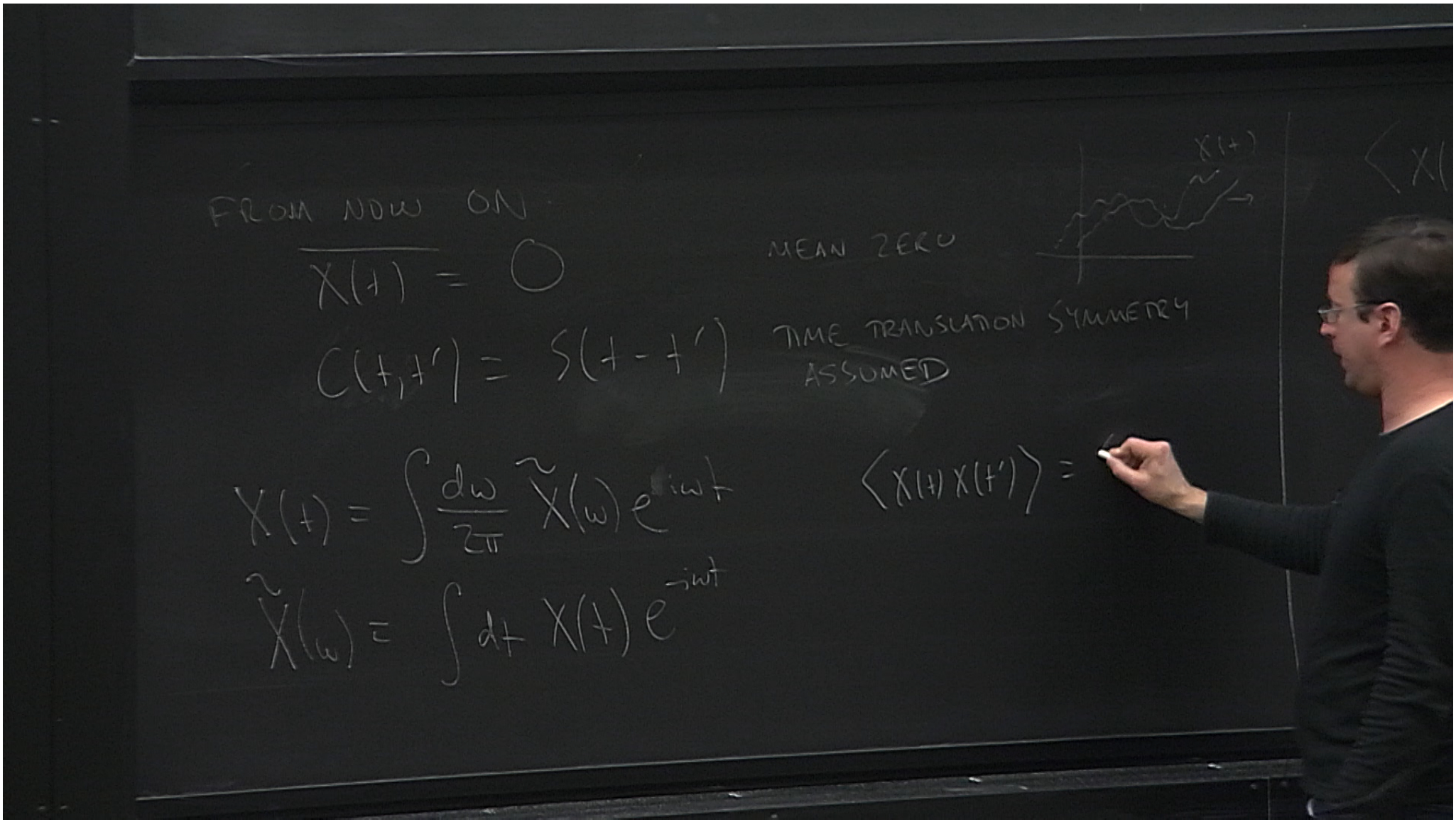
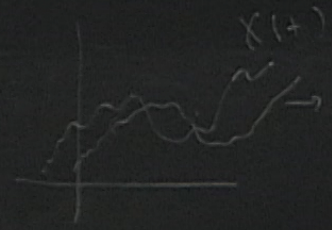
RANDOM VECTOR X_i ⇒ RANDOM FUNCTION $X(t)$
MEAN $\bar{X}_i$ ⇒ FUNCTION $\bar{X}(t) = \langle X(t) \rangle$
COVARIANCE C_{ij} ⇒ FUNCTION OF TWO VARIABLES
$C(t,t') = \langle X(t) X(t') \rangle - \overline{X(t)}\,\overline{X(t')}$
IF NO SYMMETRY IS ASSUMED
IF TIME TRANSLATION IS ASSUMED
$\langle X(t) X(t') \rangle - \bar{X}(t)\,\overline{X(t')} = \underbrace{f(t-t')}$
"CORRELATION FUNCTION"


FROM NOW ON
$\overline{X(t)} = 0$
MEAN ZERO
$X(t)$
$C(t,t') = \delta(t-t')$
TIME TRANSLATION SYMMETRY ASSUMED
$X(t) = \int \frac{d\omega}{2\pi} \tilde{X}(\omega) e^{i\omega t}$
$\langle X(t) X(t') \rangle =$
$\tilde{X}(\omega) = \int dt\, X(t) e^{-i\omega t}$

FROM NOW ON:

$$\overline{X(t)} = 0$$

MEAN ZERO



$$C(t,t') = \delta(t-t')$$

TIME TRANSLATION SYMMETRY ASSUMED

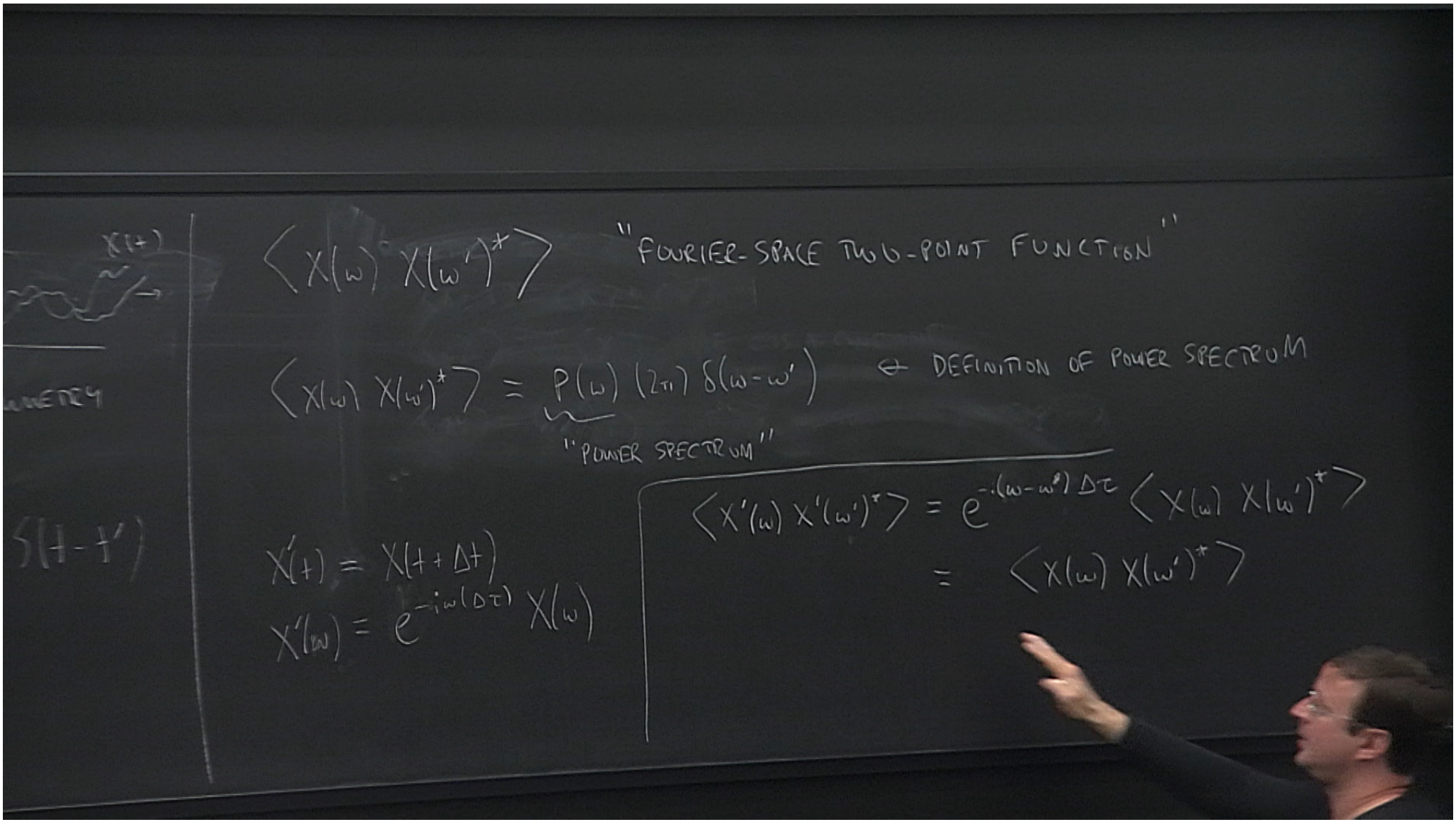
$$X(t) = \int \frac{d\omega}{2\pi} \tilde{X}(\omega) e^{i\omega t}$$

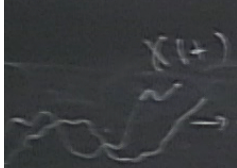
$$\langle X(t) X(t') \rangle = \delta(t-t')$$

$$\tilde{X}(\omega) = \int dt\, X(t) e^{-i\omega t}$$

$\langle X($

$\langle X(\omega$


X(t)
$\langle X(\omega)\, X(\omega')^* \rangle$
"FOURIER-SPACE TWO-POINT FUNCTION"
$\langle X(\omega)\, X(\omega')^* \rangle = P(\omega)\,(2\pi)\,\delta(\omega-\omega')$
← DEFINITION OF POWER SPECTRUM
"POWER SPECTRUM"
$X'(t) = X(t+\Delta t)$
$X'(\omega) = e^{-i\omega(\Delta\tau)}\, X(\omega)$
$\langle X'(\omega)\, X'(\omega')^* \rangle = e^{-i(\omega-\omega')\Delta\tau} \langle X(\omega)\, X(\omega')^* \rangle$
$= \langle X(\omega)\, X(\omega')^* \rangle$



$\delta(t-t')$

$$\langle X(\omega)\, X(\omega')^* \rangle$$ "FOURIER-SPACE TWO-POINT FUNCTION"

$$\langle X(\omega)\, X(\omega')^* \rangle = \underbrace{P(\omega)}_{\text{"POWER SPECTRUM"}} (2\pi)\, \delta(\omega-\omega')$$ ← DEFINITION OF POWER SPECTRUM

$$X'(t) = X(t+\Delta t)$$

$$X'(\omega) = e^{-i\omega(\Delta\tau)} X(\omega)$$

$$\langle X'(\omega)\, X'(\omega')^* \rangle = e^{-i(\omega-\omega')\Delta\tau} \langle X(\omega)\, X(\omega')^* \rangle$$

$$= \langle X(\omega)\, X(\omega')^* \rangle$$

$$\langle \tilde{X}(\omega)\, \tilde{X}(\omega')^* \rangle = \left\langle \left(\int dt\, X(t)\, e^{-i\omega t} \right) \left(\int dt'\, X(t')\, e^{+i\omega' t'} \right) \right\rangle$$

$$= \left\langle \int dt\, dt'\, e^{-i\omega t + i\omega' t'}\, X(t)\, X(t') \right\rangle$$

$$= \int dt\, dt'\, e^{-i\omega t + i\omega' t'} \langle X(t)\, X(t') \rangle$$

$$= \int dt\, dt'\, e^{-i\omega t + i\omega' t'}\, S(t - t') \qquad t' \to t'' = t - t'$$

$$= \int dt\, dt''\, e^{-i\omega t + i\omega'(t - t'')}\, S(t'')$$

$$= \int dt\, e^{-i(\omega - \omega')t}\, \tilde{S}(\omega')$$

$$= \underbrace{\tilde{S}(\omega)}\,(2\pi)\,\delta(\omega-\omega')$$

$$\boxed{P(\omega) = \tilde{S}(\omega)}$$

"WIENER - KHINCHIN THEOREM"

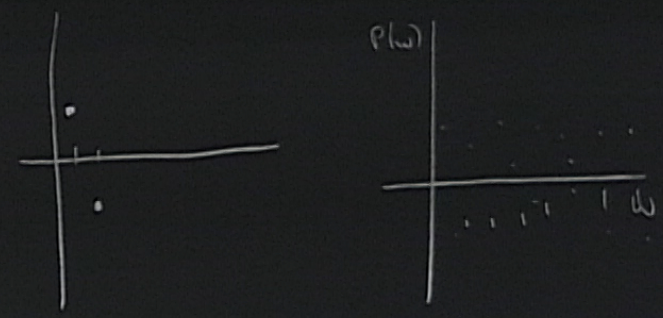
$X(t)$ RANDOM FIELD w/ POWER SPECTRUM $P_X(\omega)$

$$Y(t) = X'(t) \Rightarrow \boxed{P_Y(\omega) = \omega^2 P_X(\omega)}$$

$$Y(\omega) = i\omega X(\omega) \qquad \langle Y(\omega) Y(\omega')^* \rangle = P_Y(\omega)(2\pi)\delta(\omega-\omega')$$

$$= \omega^2 P_X(\omega)(2\pi)\delta(\omega-\omega')$$

"WHITE NOISE": $\xi(t-t') = \eta\,\delta(t-t')$



POWER SPECTRUM

$$P(\omega) = \int dt\, \xi(t)\, e^{-i\omega t}$$

$$= \int dt\, \left(\eta\,\delta(t)\right) e^{-i\omega t}$$

$$= \eta \qquad \text{[CONSTANT IN } \omega\text{]}$$