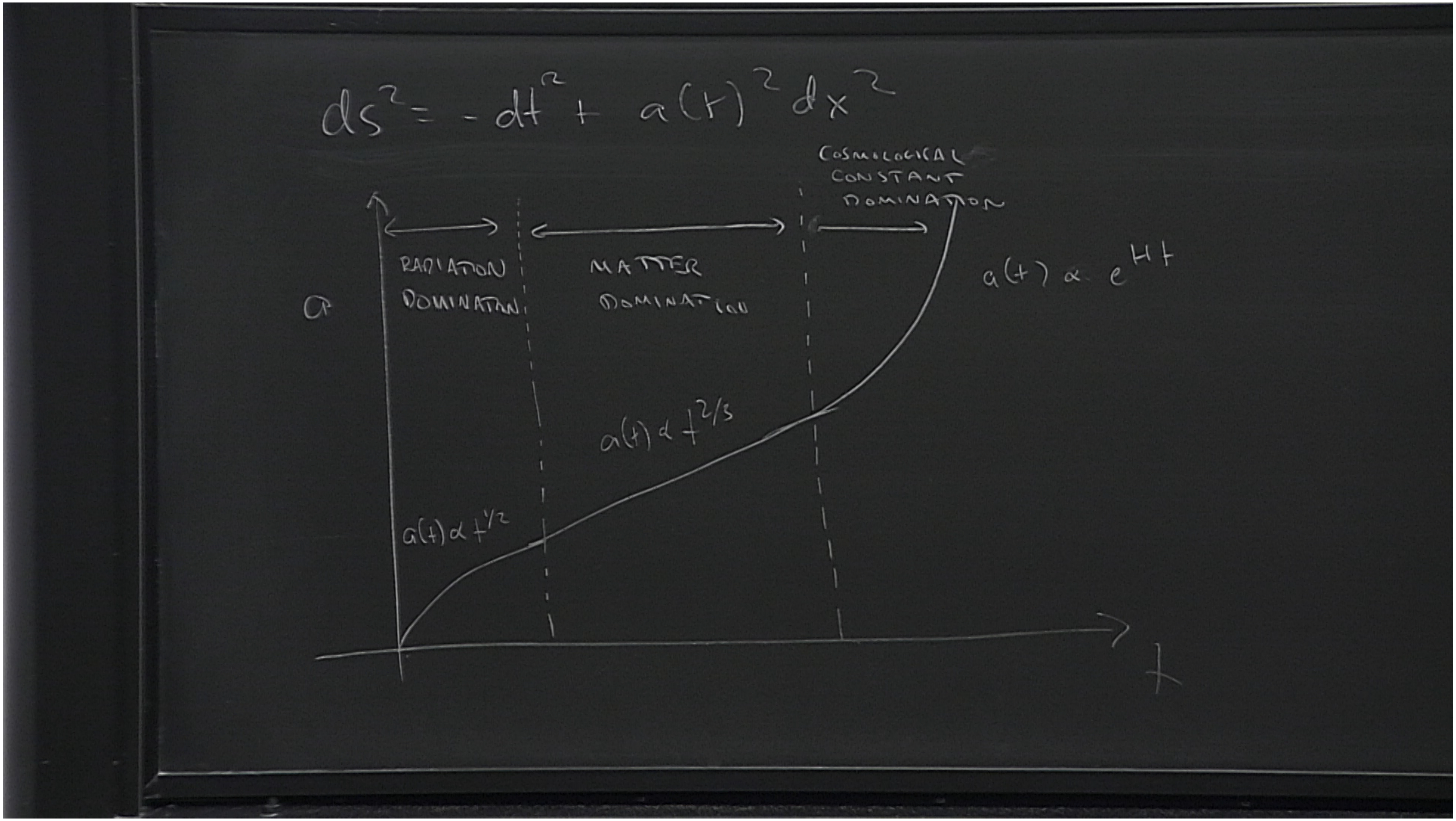
Title: PSI 2017/2018 - Cosmology - Lecture 2

Date: Apr 10, 2018 10:15 AM

URL: http://pirsa.org/18040014

Abstract:


$ds^2 = -dt^2 + a(t)^2 dx^2$
COSMOLOGICAL
CONSTANT
DOMINATION
RADIATION
DOMINATION
MATTER
DOMINATION
$a(t) \propto e^{Ht}$
a
$a(t) \propto t^{2/3}$
$a(t) \propto t^{1/2}$
t

$a(t) \propto e^{Ht}$

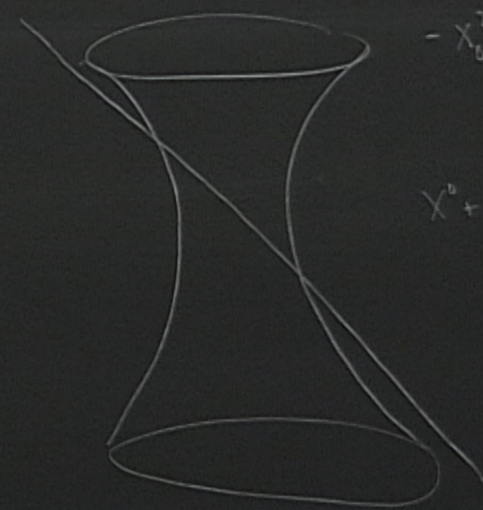
- SPATIAL ROTATIONS [3]
- SPATIAL TRANSLATIONS [3]

SPECIAL CASES WITH MORE SYMMETRY?

- MINKOWSKI $(a=1,\ H=0)$
  TIME TRANSLATIONS [1]
  BOOSTS [3]
- "FRW DE SITTER" $(a = e^{Ht},\ \dot{H} = 0)$
  4 EXTRA SYMMETRY

t

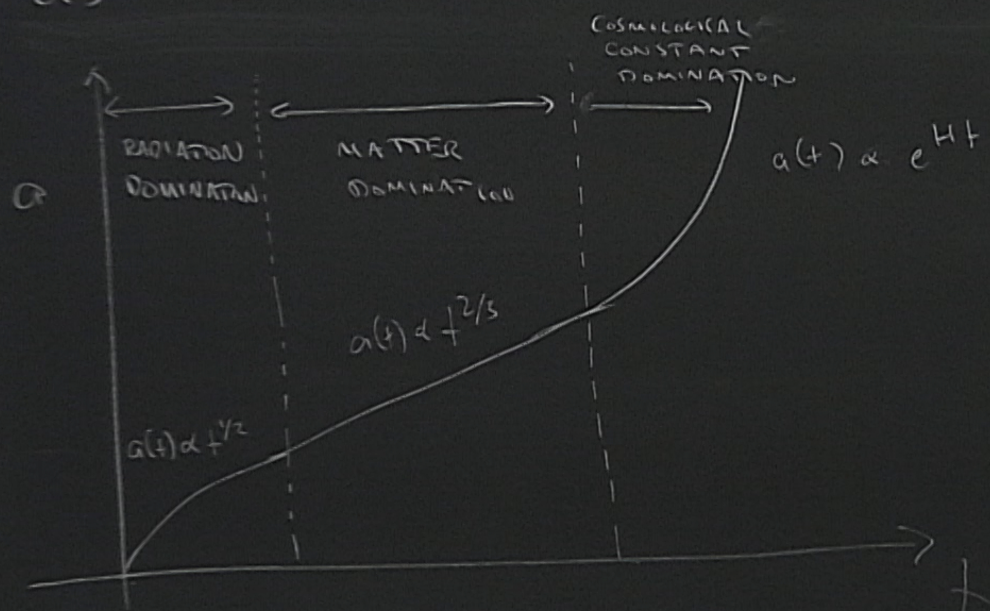
"global de Sitter"



$$-X_0^2 + \sum_{i=1}^{4} X_i^2 = H_0^{-2}$$

$$X^0 + X^1 \geq 0$$

$$ds^2 = -dt^2 + a(t)^2 dx^2$$



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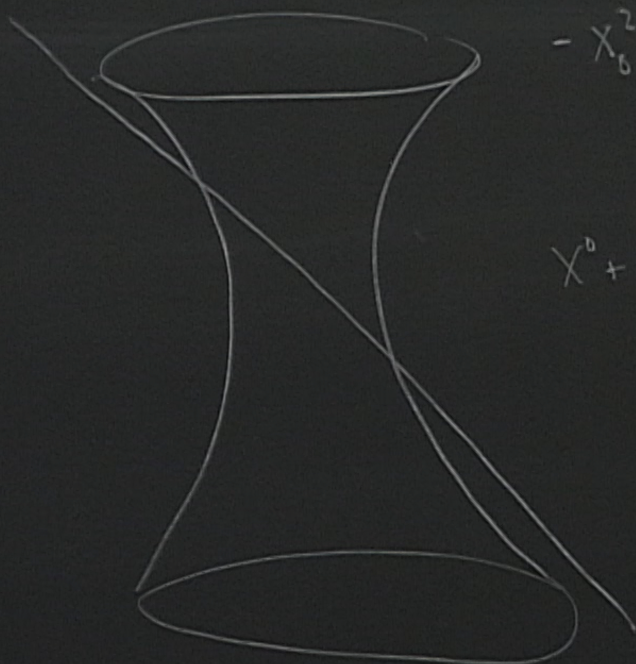

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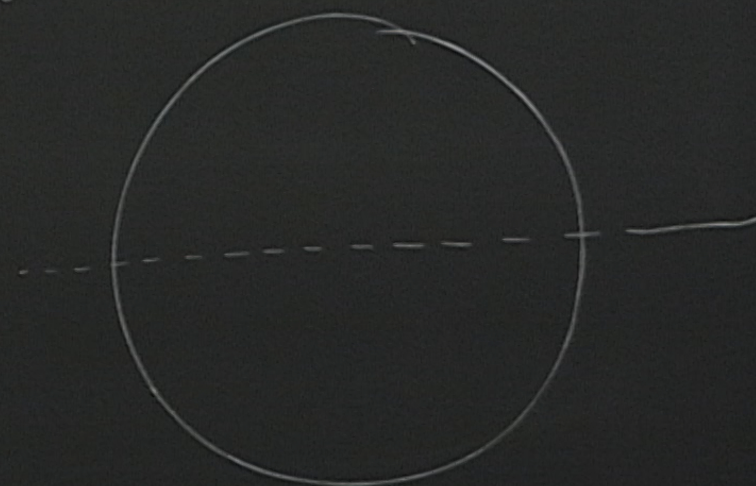


- UNIVERSE HAS A GLOBAL REST FRAME!

- FRW SCALE FACTOR CONVERTS BETWEEN COORDINATE DISTANCES ("COMOVING") AND PHYSICAL DISTANCES

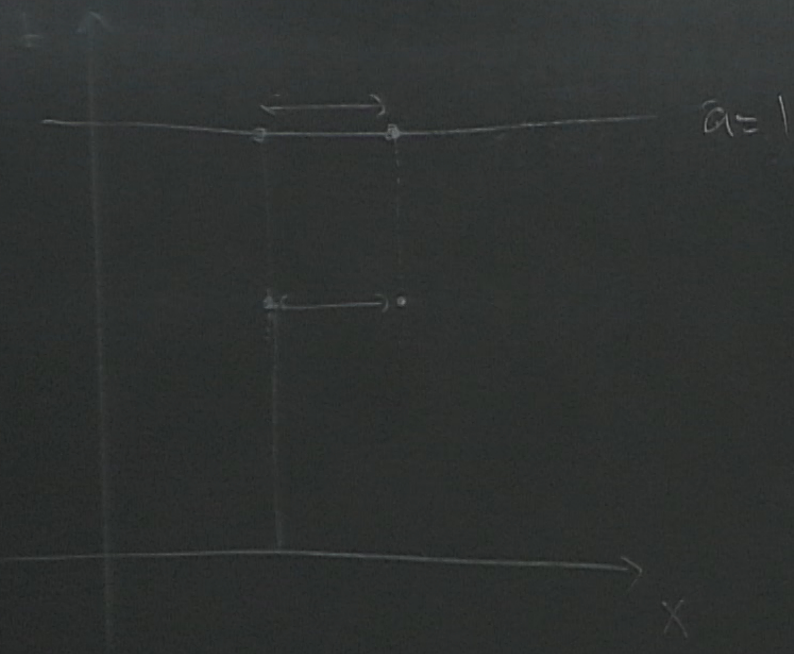
$$(\Delta x)_{phys} = a(t)\,(\Delta x)_{coord}$$

- OBSERVERS AT REST ARE "COMOVING"

$$T_{\mu\nu} = \begin{pmatrix} \rho(t) & 0 \\ 0 & a(t)^2 p(t)\,\delta_{ij} \end{pmatrix}$$

$\rho(t)$ = PHYSICAL ENERGY DENSITY

$p(t)$ = PRESSURE



- EQUATION OF STATE $w(t) = \dfrac{p(t)}{\rho(t)}$

- HUBBLE PARAMETER $H(t) = \dfrac{1}{a}\dfrac{da}{dt}$

EINSTEIN'S EQS ($G_{\mu\nu} = 8\pi G T_{\mu\nu}$)

"THE" FRIEDMANN EQUATIONS

$$\star \quad H(t)^2 = \frac{8\pi G}{3}\rho(t)$$

FRIEDMANN'S SECOND EQ.

$$\dot{H} = -4\pi G(\rho + p)$$

CONTINUITY EQS ($\nabla_\mu T^{\mu\nu} = 0$):

$$\star \quad \frac{d\log\rho}{d\log a} = -3(1+w)$$

TO CLOSE THESE EQS, WE NEED TO KNOW SOMETHING ABOUT HOW $\rho(t)$ EVOLVES.

SIMPLEST EXAMPLE: CONSTANT-w UNIVERSE

$$w = \frac{P}{\rho} \text{ IS CONSTANT.}$$

$$\xRightarrow{\text{CONTINUITY}} \rho \propto a^{-3(1+w)}$$

$$w = \begin{cases} \frac{1}{3} & \text{RADIATION DOMINATED} \quad [\rho \propto a^{-4}] \\ 0 & \text{MATTER DOMINATED} \quad [\rho \propto a^{-3}] \\ -1 & \text{COSMOLOGICAL CONSTANT DOMINATED} \quad [\rho \propto a^{0}] \end{cases}$$

TO CLOSE THESE EQS, WE NEED TO KNOW SOMETHING ABOUT HOW $\rho(t)$ EVOLVES.

$W =$

EQUATIONS

SECOND EQ.

SIMPLEST EXAMPLE: CONSTANT-w UNIVERSE

$$w = \frac{P}{\rho} \text{ IS CONSTANT.}$$

CONTINUITY

$$\Longrightarrow \rho \propto a^{-3(1+w)}$$

FRIEDMANN

$\Longrightarrow$ HOMEWORK

ΛCDM: $\rho(t) = \rho_{r0}\, a^{-4} + \rho_{m0}\, a^{-3} + \Lambda$

$$p(t) = \frac{1}{3}\rho_{r0}\, a^{-4} + 0\, \rho_{m0}\, a^{-3} - \Lambda$$

$(\cdot)_0 =$ "EVALUATED TODAY [AT $a=1$]"

ΛCDM: $\rho(t) = \rho_{r0}\, a^{-4} + \rho_{m0}\, a^{-3} + \Lambda$

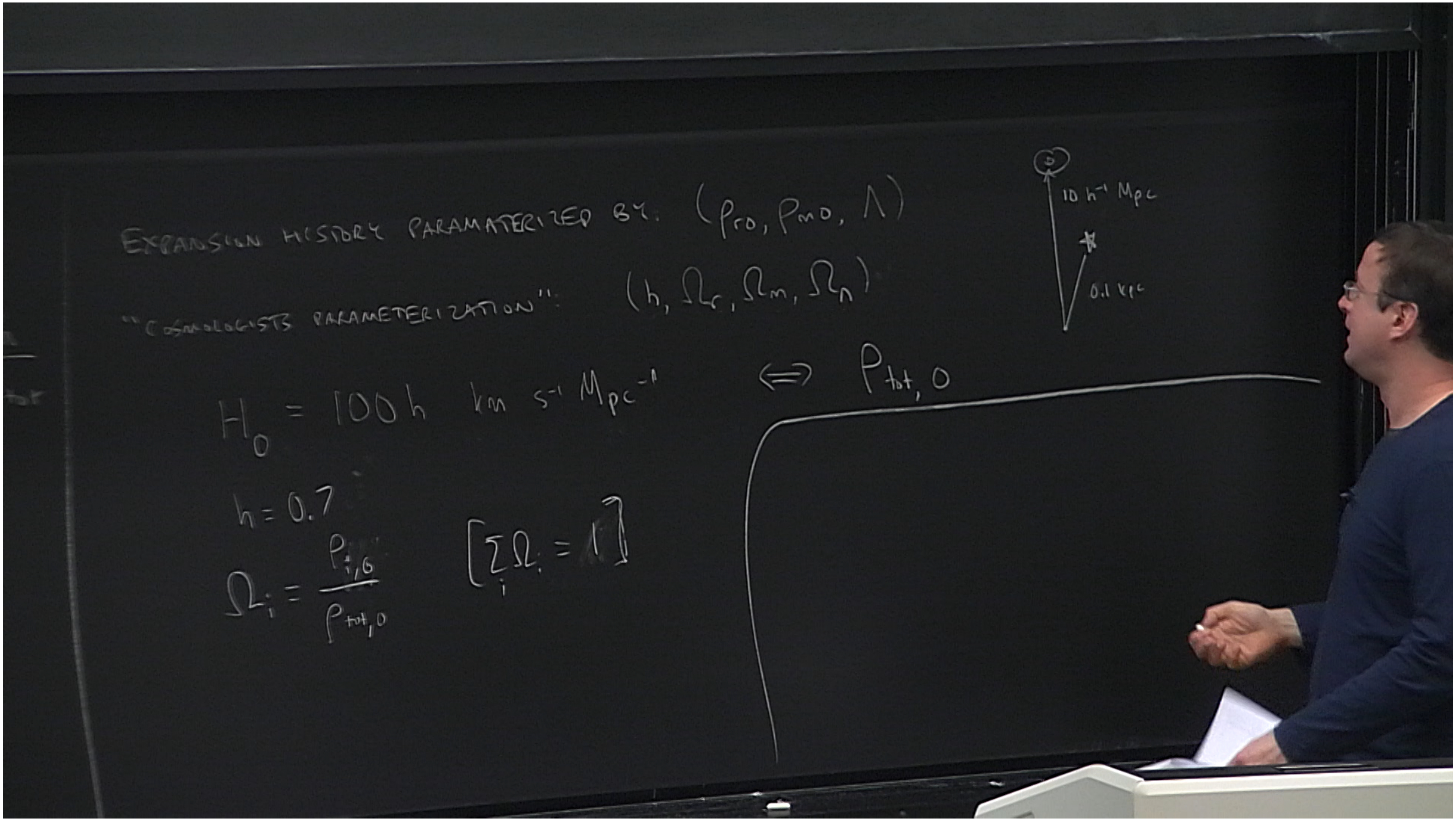
$$p(t) = \frac{1}{3}\rho_{r0}\, a^{-4} + 0\, \rho_{m0}\, a^{-3} - \Lambda$$

$\Lambda = \rho_\Lambda$

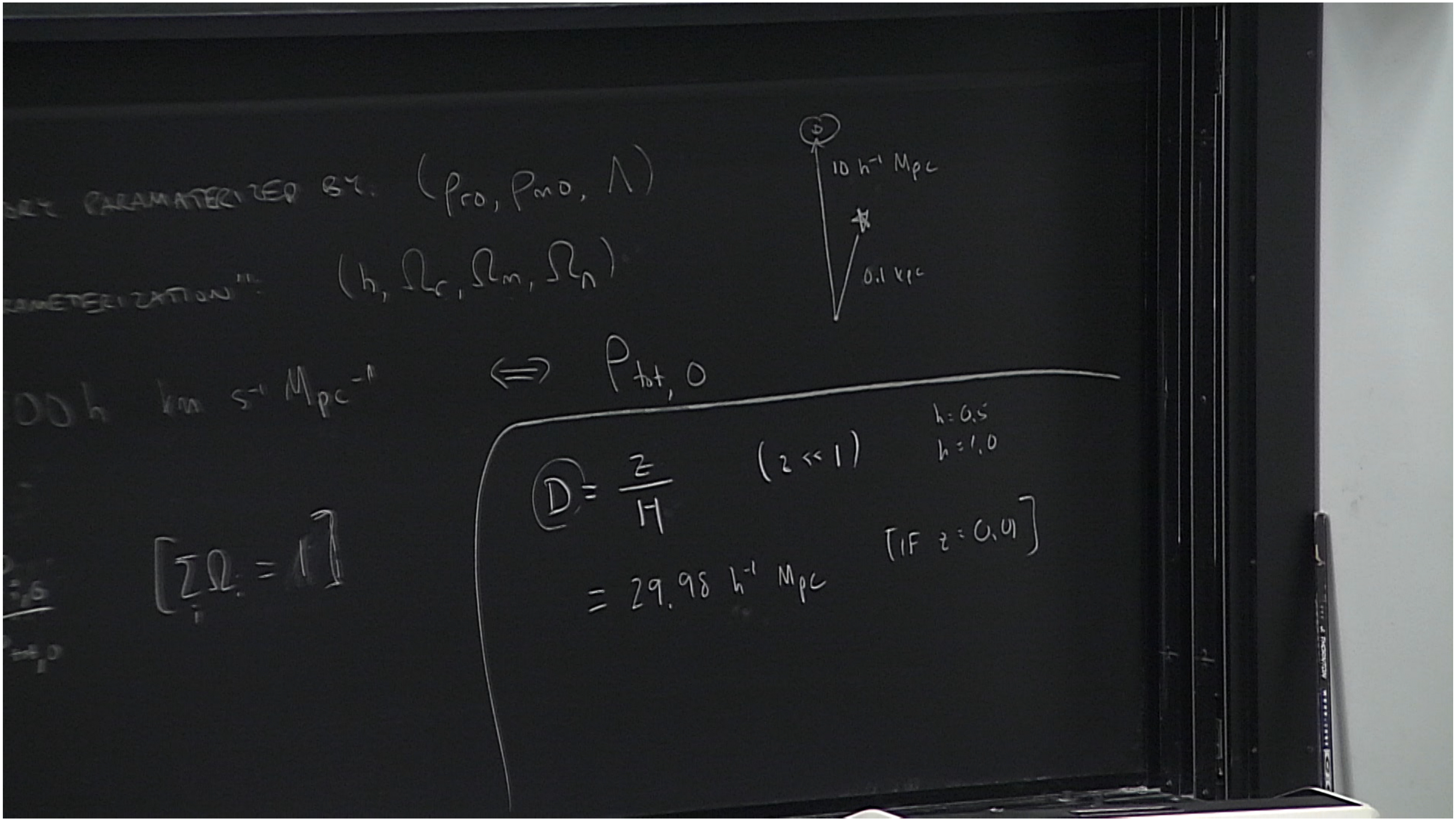
$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{tot}}$$

$(\cdot)_0 =$ "EVALUATED TODAY [AT $a=1$]"

NUMERICALLY INTEGRATE TO GET $a(t)$



EXPANSION HISTORY PARAMATERIZED BY: $(\rho_{r0}, \rho_{m0}, \Lambda)$
"COSMOLOGISTS PARAMETERIZATION": $(h, \Omega_r, \Omega_m, \Omega_\Lambda)$
$H_0 = 100h$ km s^{-1} Mpc^{-1}
$\Longleftrightarrow \rho_{tot,0}$
$h = 0.7$
$\Omega_i = \frac{\rho_{i,0}}{\rho_{tot,0}}$
$[\sum_i \Omega_i = 1]$
10 h^{-1} Mpc
0.1 kpc


PARAMATERIZED BY: (ρr0, ρm0, Λ)
(h, Ωr, Ωm, ΩΛ)
⟺ ρtot, 0
km s⁻¹ Mpc⁻¹
[Σ Ωi = 1]
10 h⁻¹ Mpc
0.1 kpc
D = z / H
(z << 1)
h = 0.5
h = 1.0
[IF z = 0.01]
= 29.98 h⁻¹ Mpc

TIME COORDINATES

- t = PROPER TIME
- a = SCALE FACTOR

$$H(a)^2 = \frac{8\pi G}{3}\left(\rho_{r0} a^{-4} + \rho_{m0} a^{-3} + \Lambda\right)$$

$$= H_0^2\left(\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_\Lambda\right)$$

- CONFORMAL TIME τ

DEFINED BY DIFF. EQ. $\frac{d\tau}{dt} = \frac{1}{a(t)}$

IN CONFORMAL TIME, LIGHTCONES ARE AT 45°

TIME COORDINATES

- t = PROPER TIME
- a = SCALE FACTOR

$$H(a)^2 = \frac{8\pi G}{3}\left(\rho_{r0}\, a^{-4} + \rho_{m0}\, a^{-3} + \Lambda\right)$$

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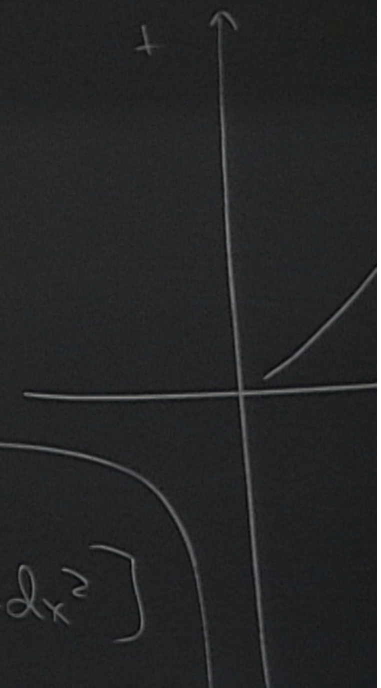
- CONFORMAL TIME τ

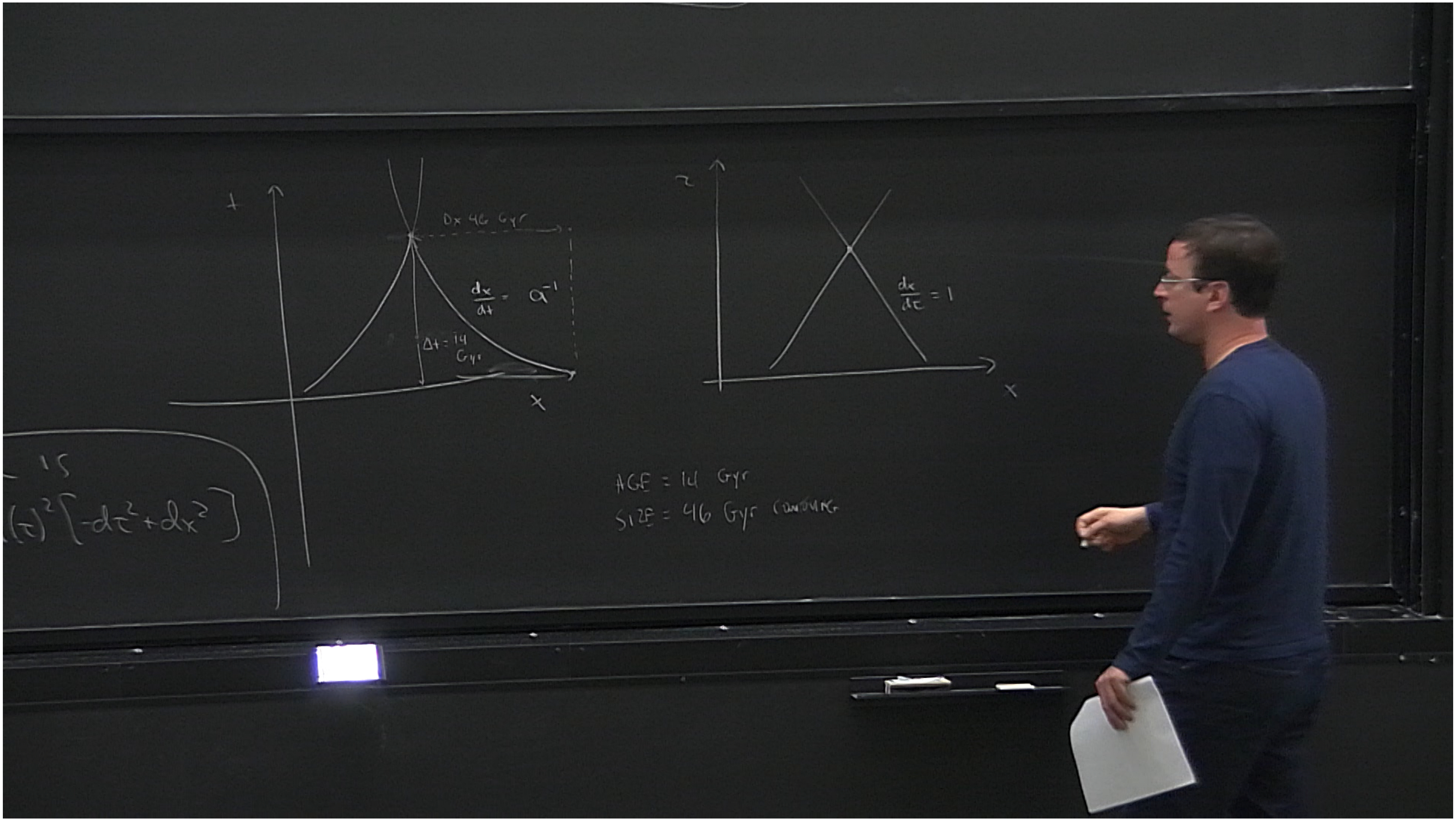
  DEFINED BY DIFF. EQ $\frac{d\tau}{dt} = \frac{1}{a(t)}$

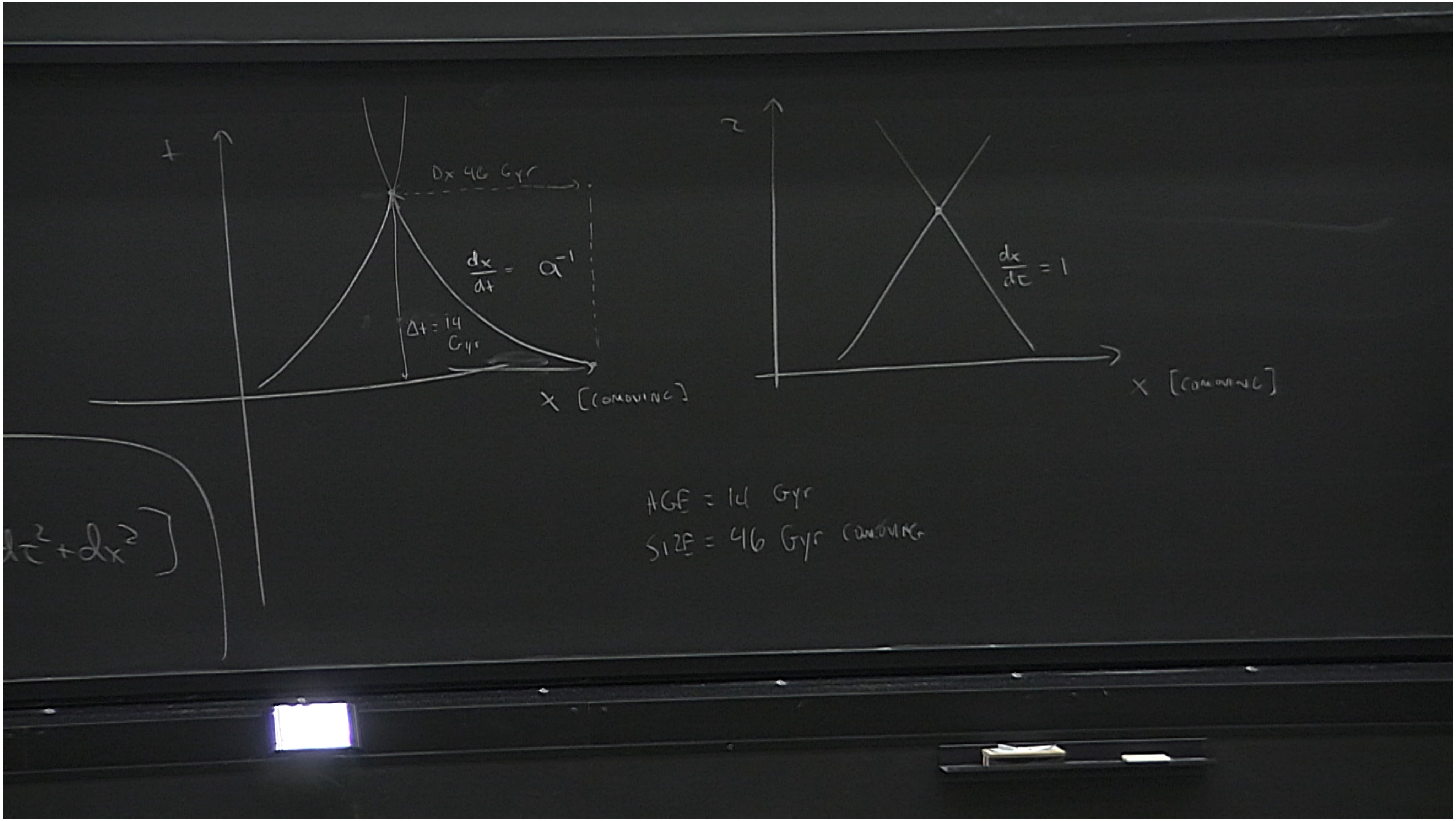
  IN CONFORMAL TIME, LIGHTCONES ARE AT 45°

FRW METRIC IS

$$ds^2 = a(\tau)^2\left[-d\tau^2 + dx^2\right]$$




t
Dx 46 Gyr
dx/dt = a^-1
Δt = 14 Gyr
x
τ
dx/dτ = 1
x
is
(τ)^2 [-dτ^2 + dx^2]
AGE = 14 Gyr
SIZE = 46 Gyr


t
Dx 46 Gyr
dx/dt = a^-1
Δt = 14 Gyr
x [COMOVING]
τ
dx/dτ = 1
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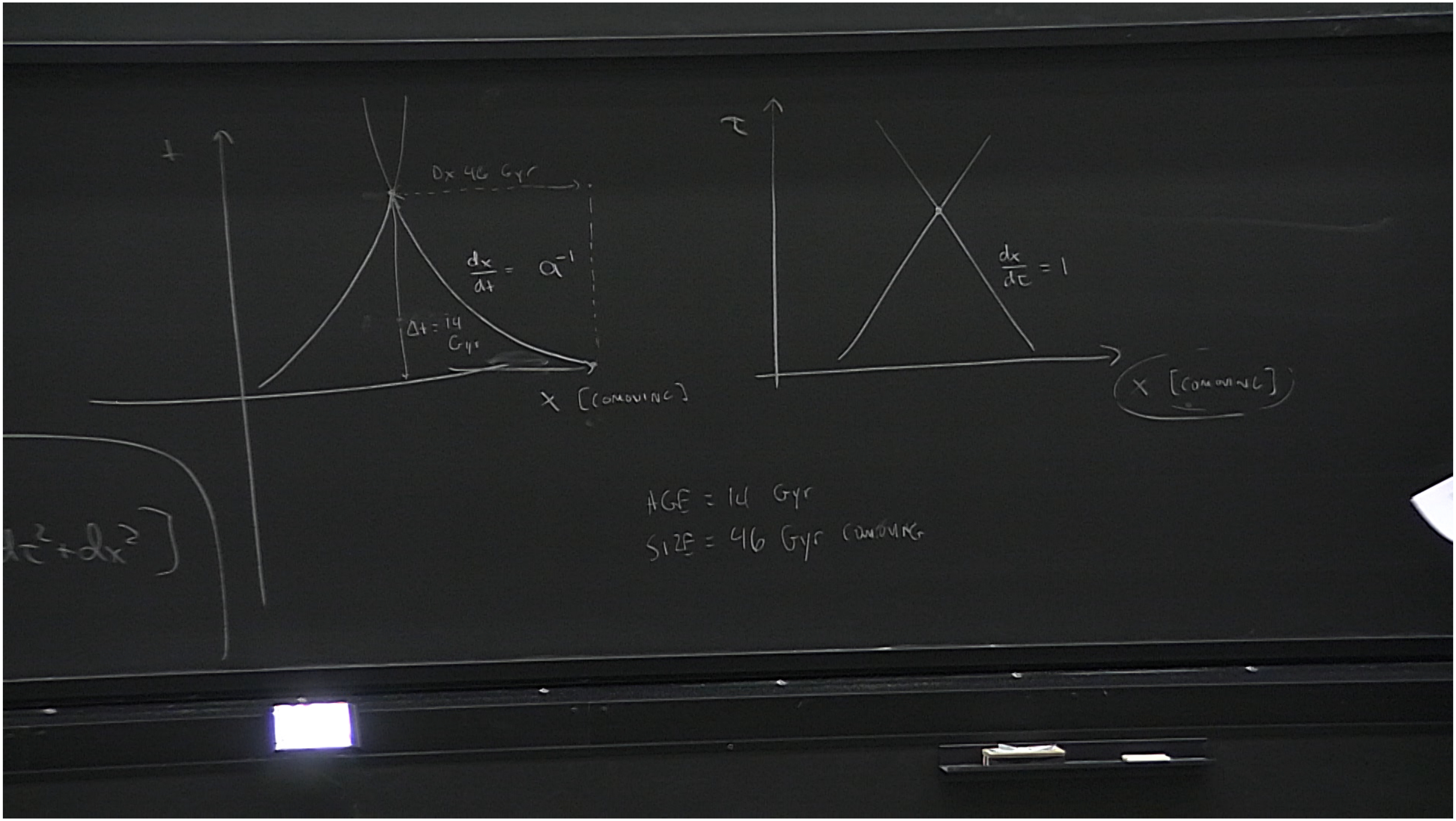
SIZE = 46 Gyr COMOVING
dτ² + dx²]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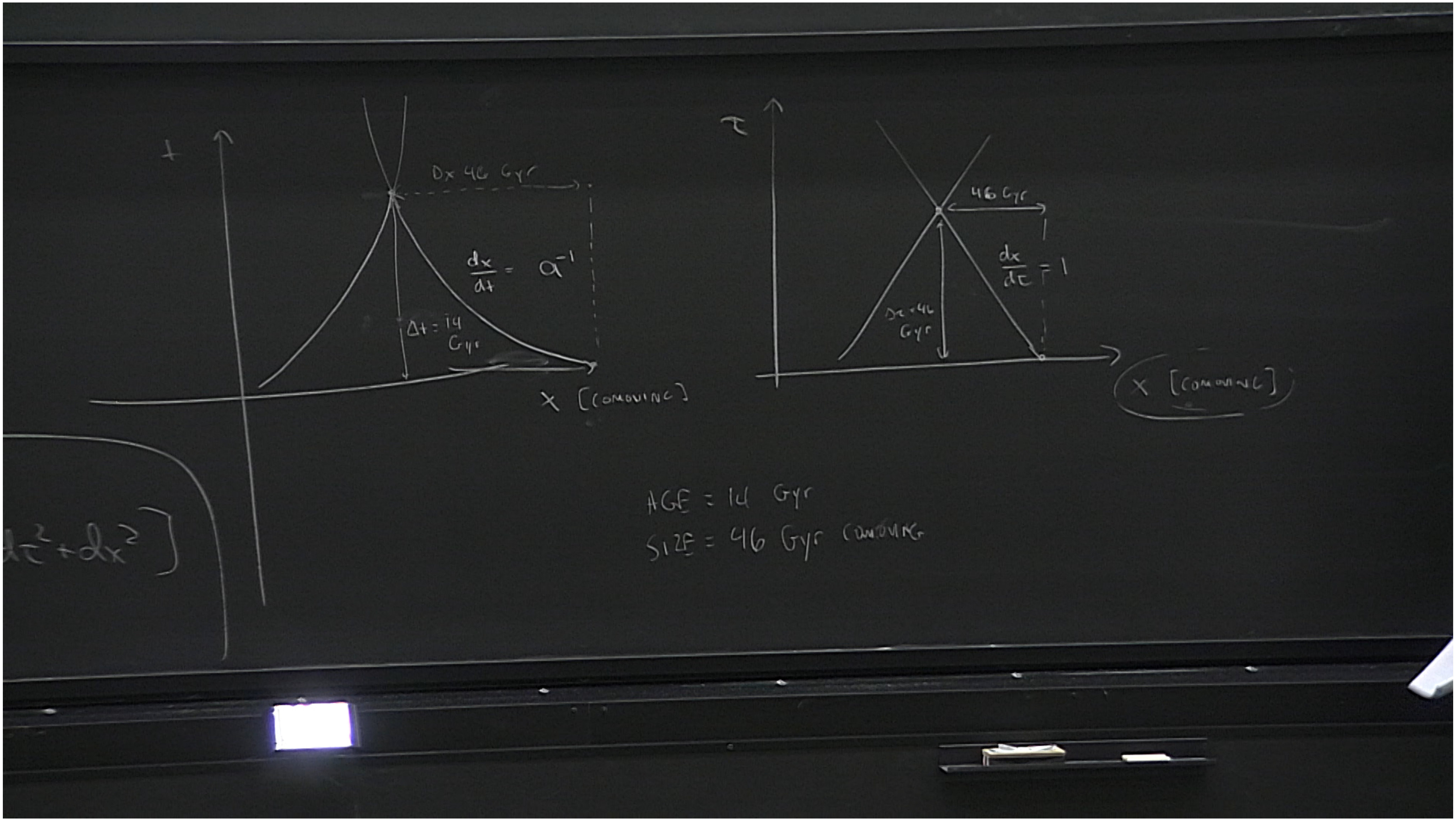
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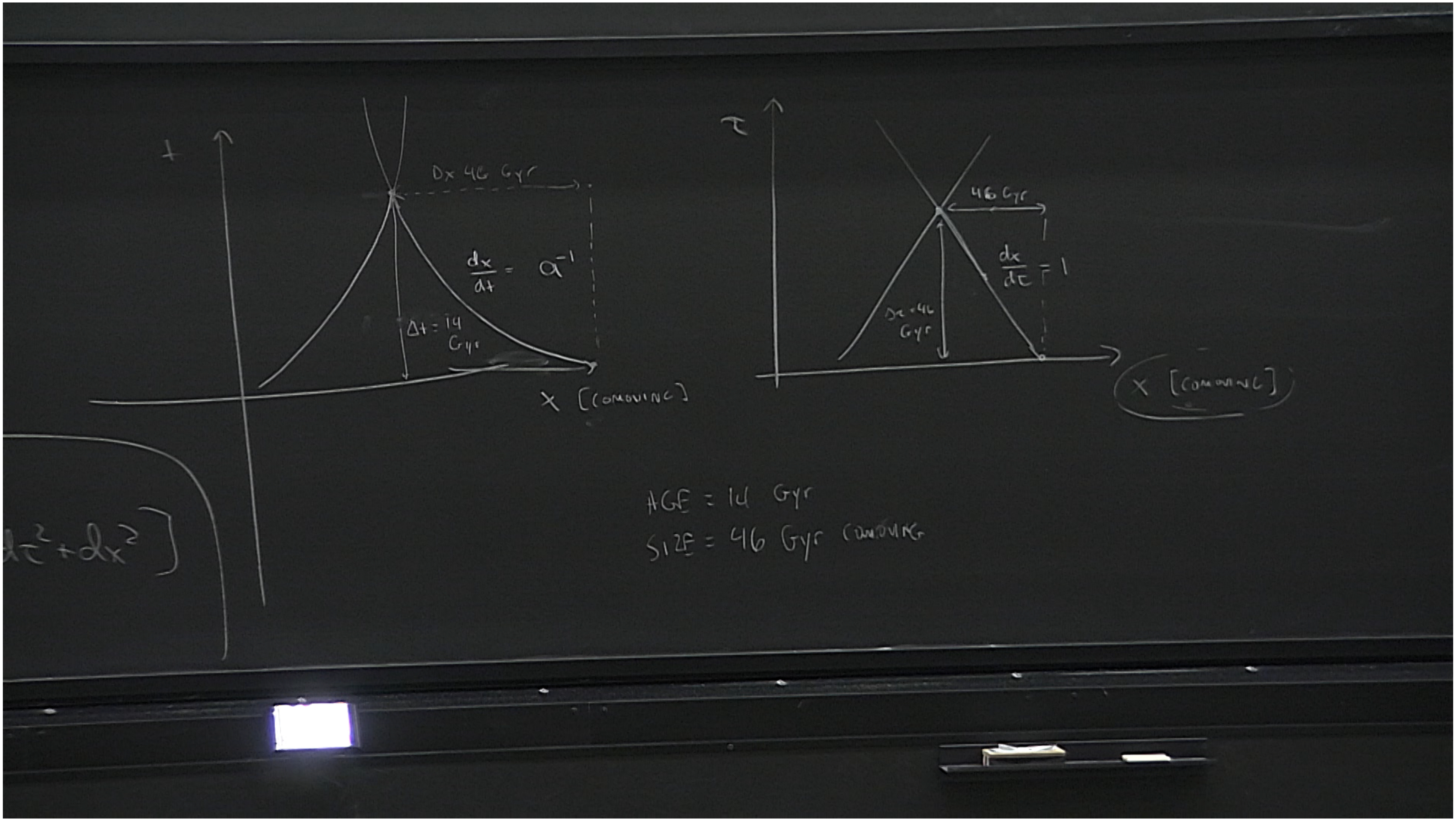
dτ² + dx²]



t
Dx 46 Gyr
$\frac{dx}{dt} = a^{-1}$
Δt = 14 Gyr
x [COMOVING]
τ
46 Gyr
$\frac{dx}{d\tau} = 1$
Δx = 46 Gyr
x [COMOVING]
$d\tau^2 + dx^2$]
AGE = 14 Gyr
SIZE = 46 Gyr COMOVING

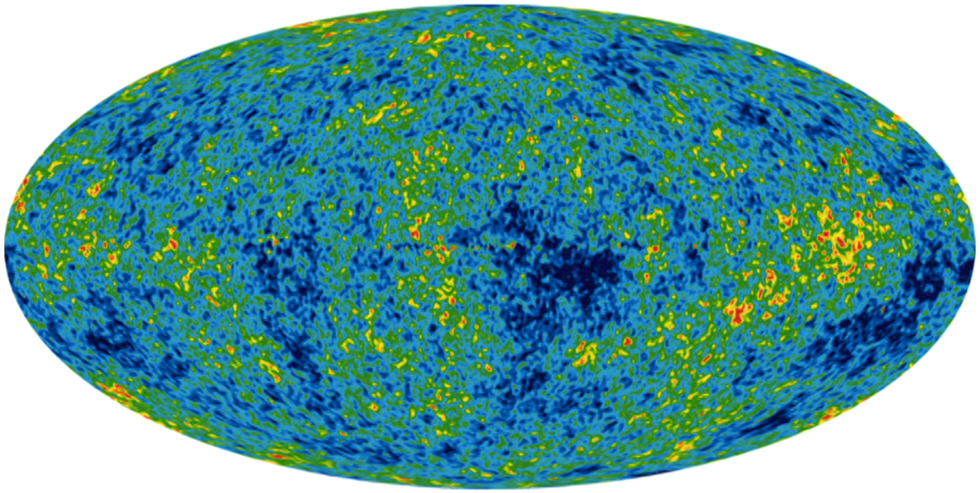
Random variables

Kendrick Smith
PSI 2018, Explorations in Cosmology

The standard model of cosmology is a probabilistic model.

For example, it can predict the probability of a given CMB realization occurring, but not the specific realization.

To specify the standard model precisely, we'll need to build up some machinery for working with random variables.



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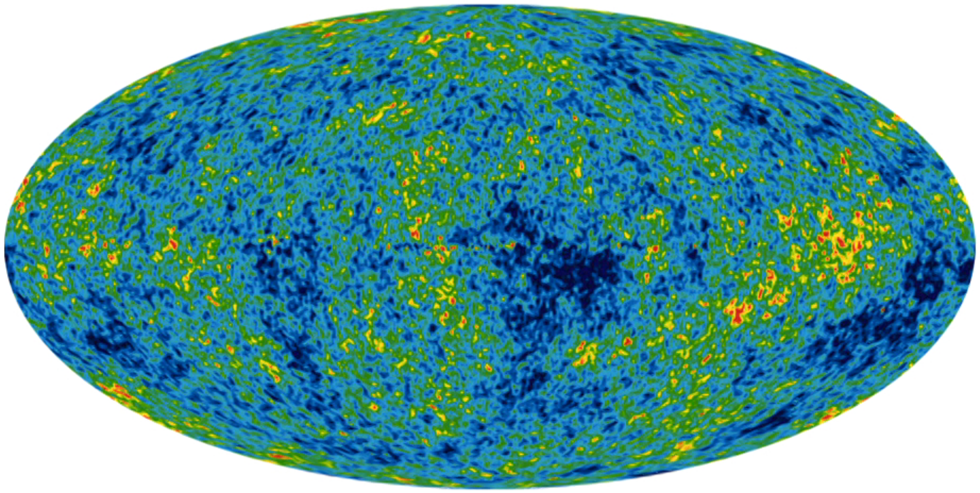
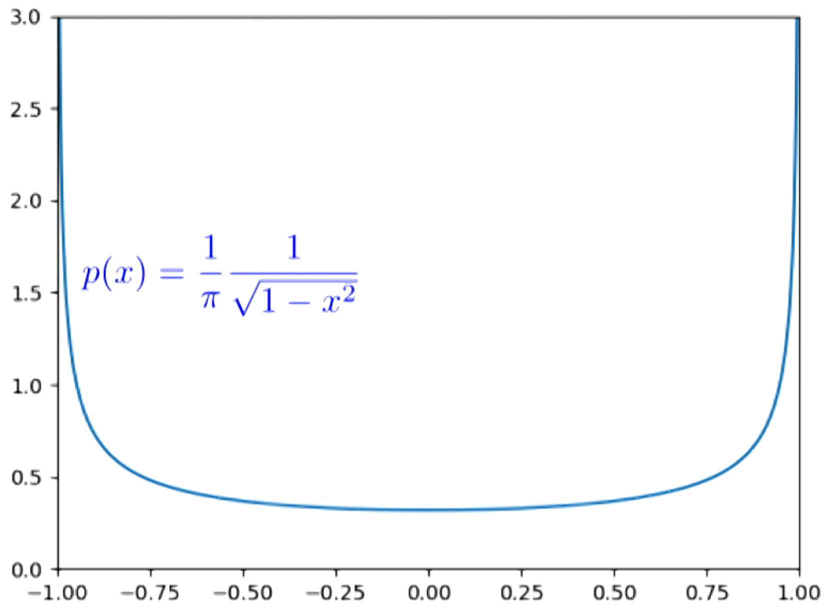


Physicist's definition of a one-dimensional random variable X: anything with a probability distribution function (PDF) p(x).

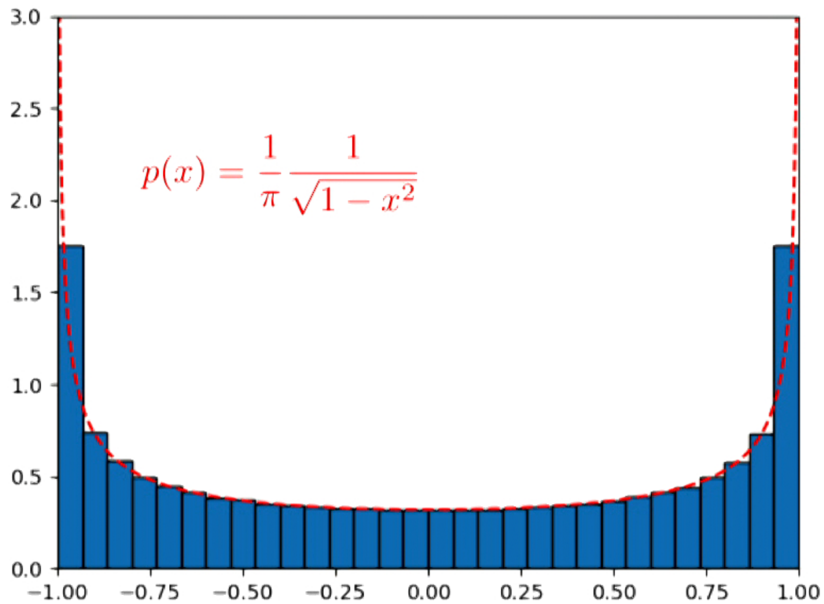
Here is an arbitrarily chosen example.



Histogram of 10^6 random samples in 30 bins, compared to the continuous PDF. The probability for the random variable X to be in bin [a,b] is:

$$\mathrm{Prob}\Big(a < X < b\Big) = \int_a^b dx\, p(x)$$

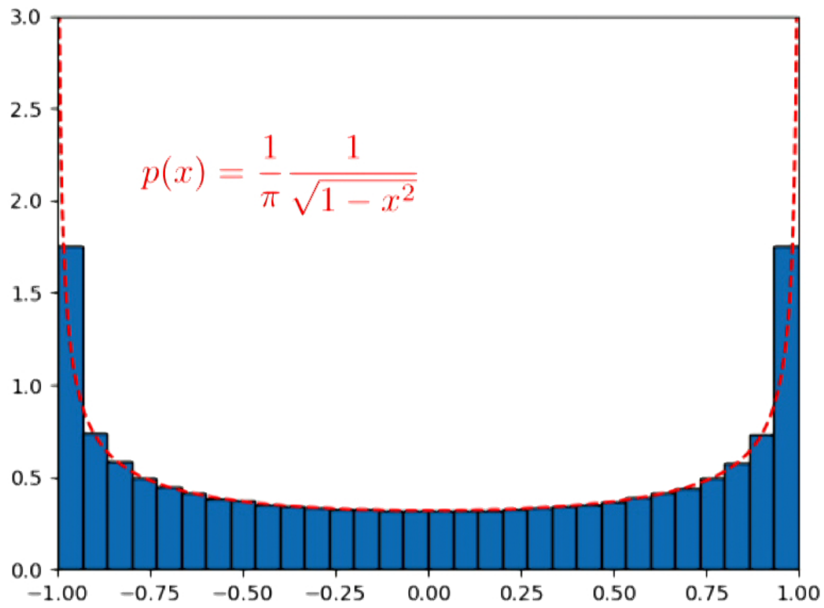
Note that the PDF must satisfy $\int_{-\infty}^{\infty} dx\, p(x) = 1$



The notation < . > denotes an expectation value over realizations of the random variable X. For example:

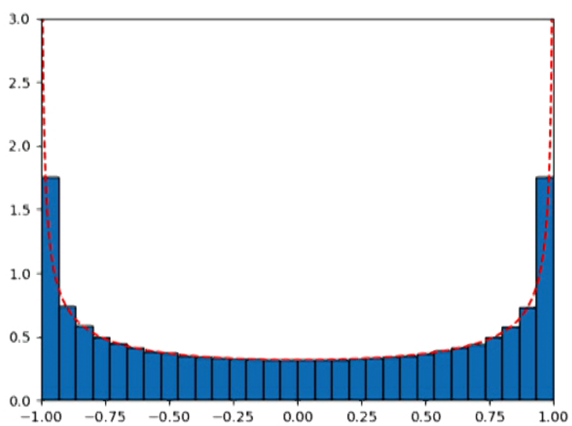
$$\langle X \rangle = \int_{-1}^{1} dx\, x\, p(x) = 0$$

$$\langle X^2 \rangle = \int_{-1}^{1} dx\, x^2\, p(x) = \frac{1}{2}$$



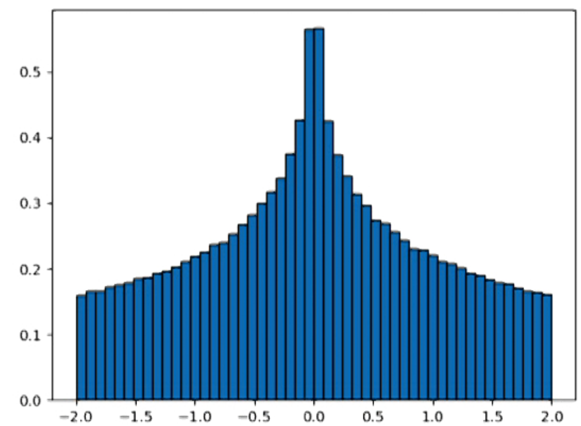
New random variables can be constructed from old ones.

For example, define Y = (X_1 + X_2), where X_1,X_2 are independent random variables with the same PDF as before, $p(x) = \frac{1}{\pi\sqrt{1-x^2}}$



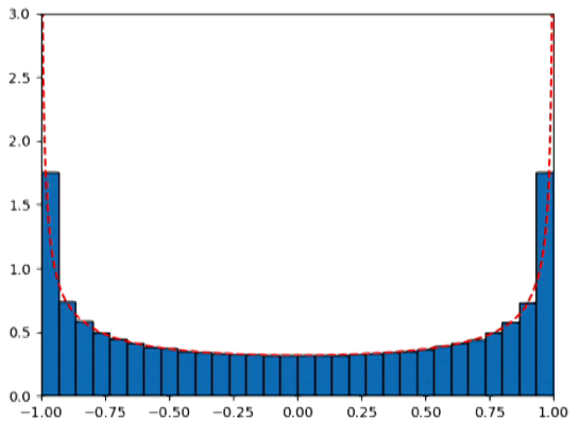
X

$\Rightarrow$



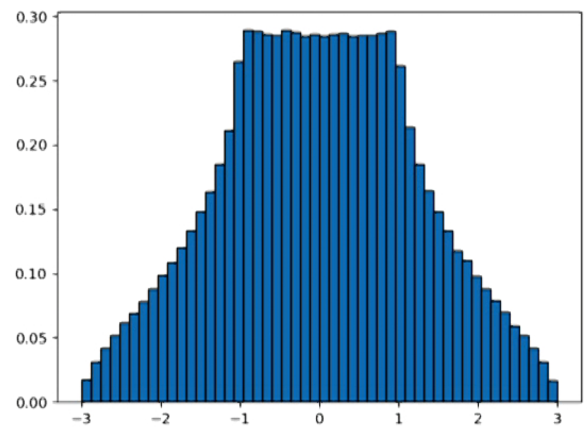
$Y = X_1 + X_2$

Three X's added together: $Y = X_1 + X_2 + X_3$



X

$\Rightarrow$

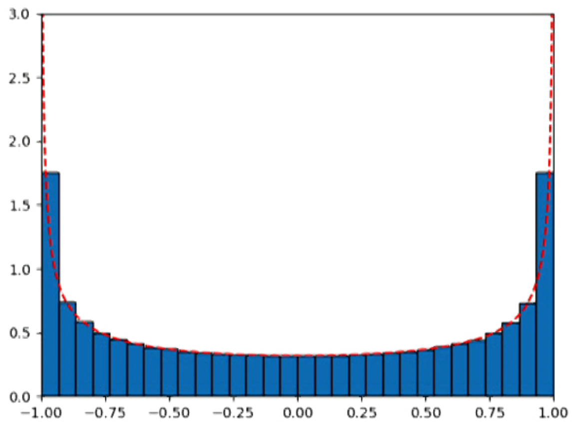


$Y = X_1 + X_2 + X_3$

Twenty X's added together: $Y = \sum_{i=1}^{20} X_i$

In the next few slides, we'll explain where the PDF
$p(x) = \frac{1}{\sqrt{20\pi}} e^{-x^2/20}$ comes from (including factors of 20, pi).

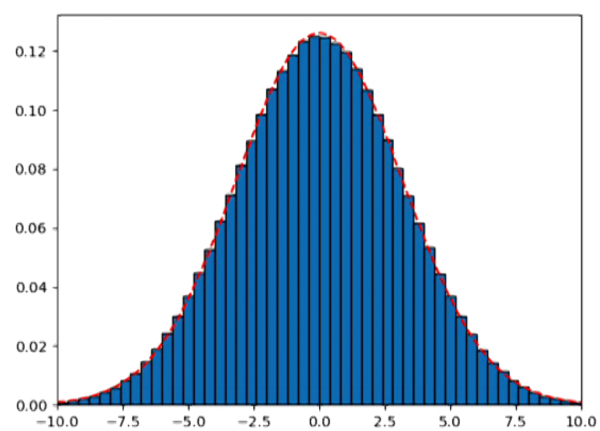
$$p(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}$$



X

$$\Rightarrow$$

$$p(x) = \frac{1}{\sqrt{20\pi}} e^{-x^2/20}$$



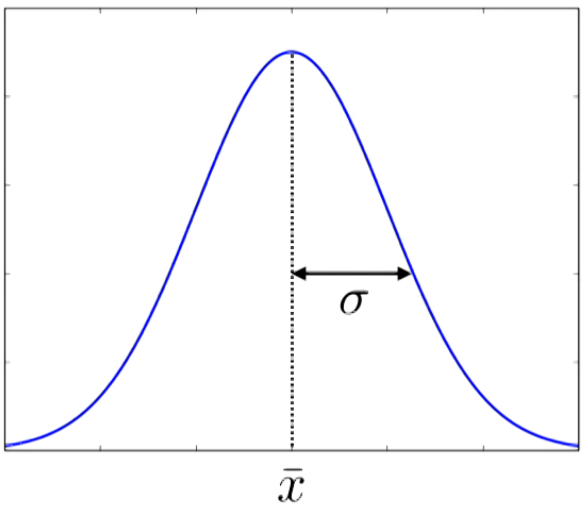
$$Y = \sum_{i=1}^{20} X_i$$

Central limit theorem: the sum of a large number of independent, identically distributed random variables has a PDF which is approximately Gaussian.

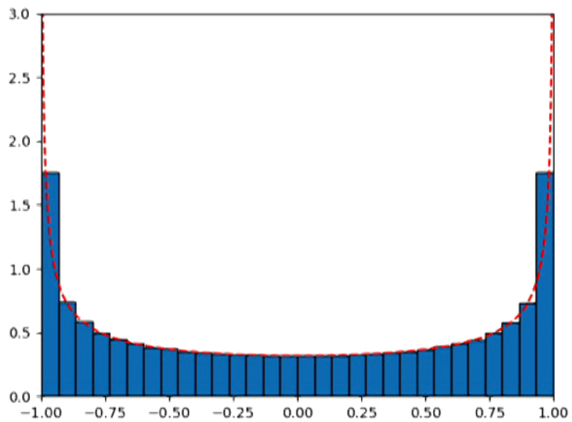
The Gaussian PDF is defined by:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\bar{x})^2}{2\sigma^2}\right)$$

and has two parameters: a mean $\bar{x}$ and a width σ.

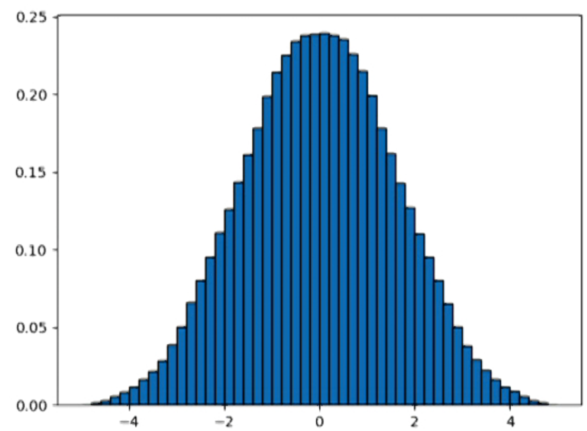


Five X's added together: $Y = X_1 + X_2 + X_3 + X_4 + X_5$



X

$\Rightarrow$



$Y = X_1 + X_2 + X_3 + X_4 + X_5$

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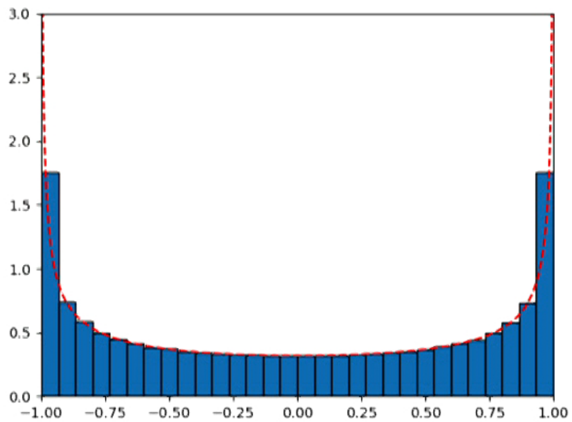
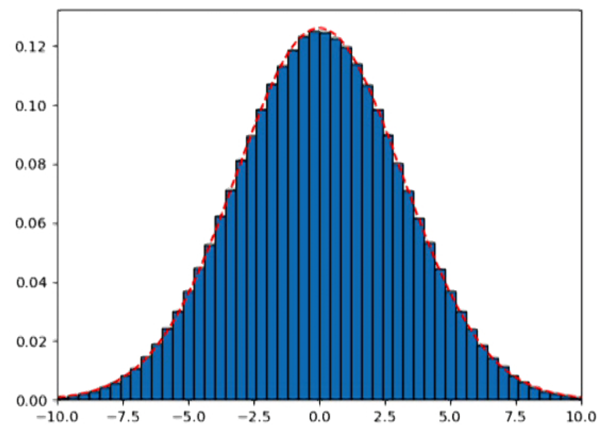
$$p(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$$



X

$\Rightarrow$

$$p(x) = \frac{1}{\sqrt{20\pi}} e^{-x^2/20}$$



$$Y = \sum_{i=1}^{20} X_i$$

Some definitions: the mean and variance of a random variable X are defined by:

$$\bar{X} = \langle X \rangle \qquad \text{[mean]}$$

$$\begin{aligned} \mathrm{Var}(X) &= \langle X^2 \rangle - \langle X \rangle^2 \\ &= \langle (X - \bar{X})^2 \rangle \end{aligned} \qquad \text{[variance]}$$

$\sqrt{\mathrm{Var}(X)}$ can be interpreted as the "typical" size of fluctuations around the mean.

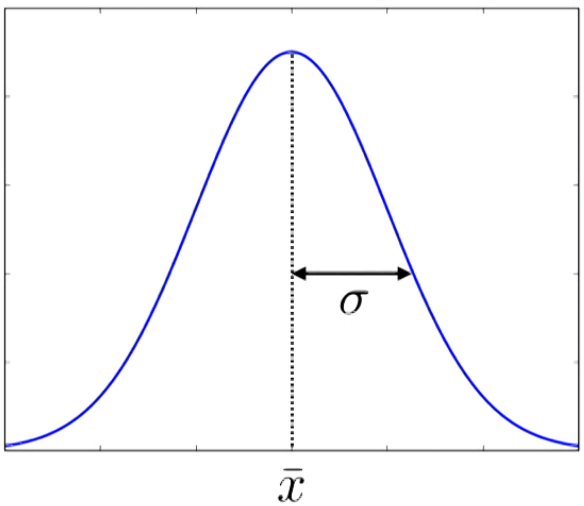
Example: For the Gaussian

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\bar{x})^2}{2\sigma^2}\right)$$

a short calculation shows:

$$\text{Mean} = \int_{-\infty}^{\infty} dx\, p(x)\, x = \bar{x}$$

$$\text{Variance} = \int_{-\infty}^{\infty} dx\, p(x)\, (x^2 - \bar{x}^2) = \sigma^2$$

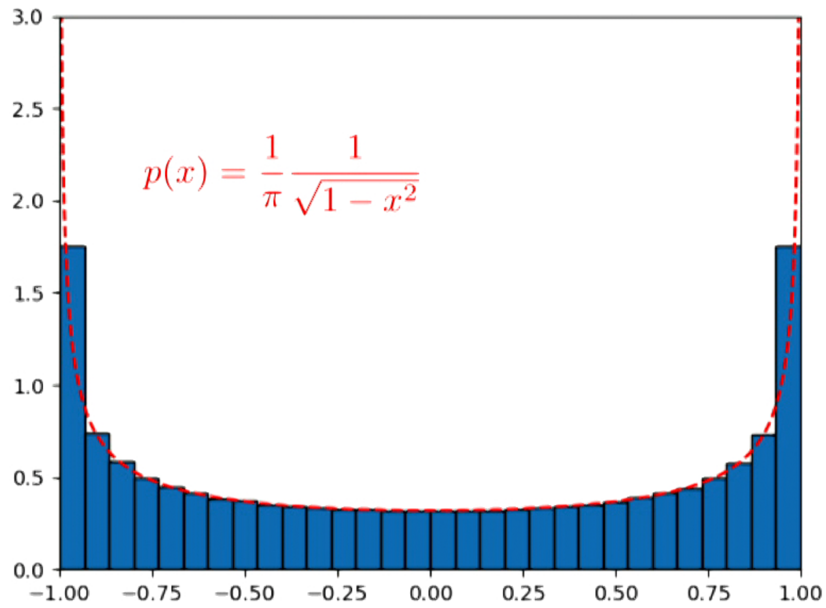


Example 2: for the PDF $p(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}$ considered previously,

$$\bar{X} = 0$$

$$\mathrm{Var}(X) = \langle (X - \bar{X})^2 \rangle = \frac{1}{2}$$

Next let's calculate mean and variance of $Y = \sum_{i=1}^{N} X_i$, where the X's are assumed to be independent samples.



Properties of expectation values:

$$\langle X \pm X' \rangle = \langle X \rangle \pm \langle X' \rangle$$

$$\langle cX \rangle = c\langle X \rangle$$ if c is a constant (not a random variable)

$$\langle X X' \rangle = \langle X \rangle \langle X' \rangle$$ if X, X' are independent random variables (not true in general!)

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Now we can calculate mean and variance of $Y = \sum_{i=1}^{N} X_i$

$$\bar{Y} = \sum_{i=1}^{N} \bar{X}_i = 0$$

$$\begin{aligned}
\mathrm{Var}(Y) &= \langle (Y - \bar{Y})^2 \rangle \\
&= \langle (\textstyle\sum_i X_i)^2 \rangle \\
&= \langle \textstyle\sum_i X_i^2 + \sum_{i \neq j} X_i X_j \rangle \\
&= \textstyle\sum_i \langle X_i^2 \rangle + \sum_{i \neq j} \langle X_i \rangle \langle X_j \rangle \\
&= N \left(\tfrac{1}{2} \right)
\end{aligned}$$

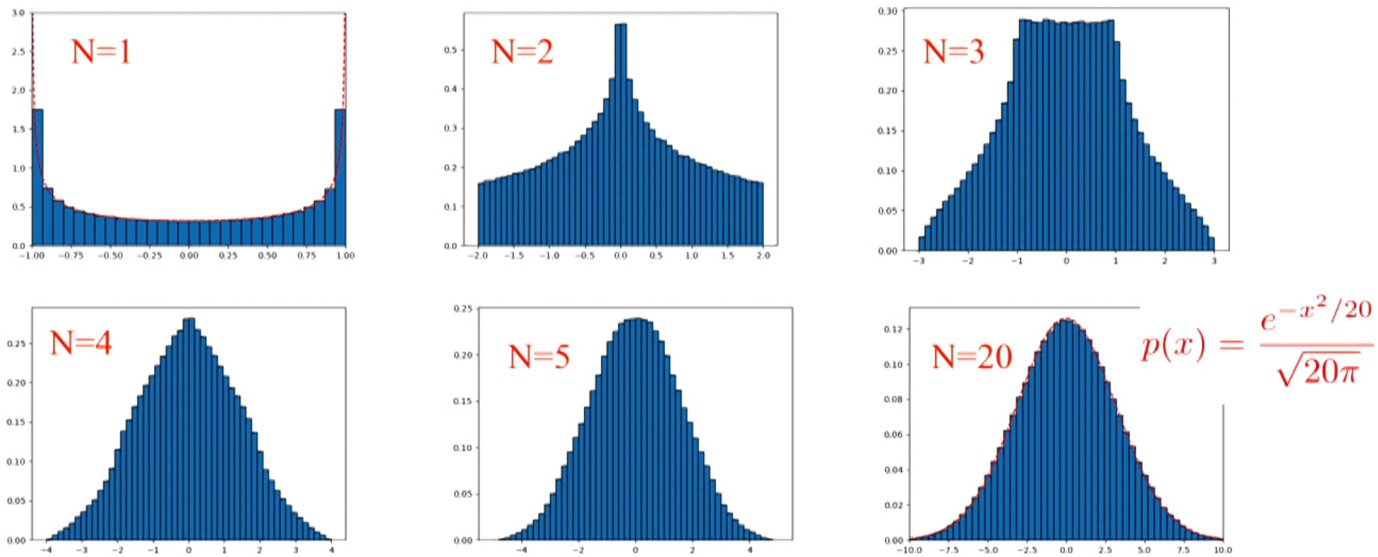
$$p(x, x') = p(x)\, p(x')$$

This calculation gives the mean and variance of $Y = \sum_{i=1}^{N} X_i$:

$$\bar{Y} = 0 \qquad \mathrm{Var}(Y) = N/2 \qquad\qquad \text{(for all N)}$$

In general, the mean and variance do not determine the PDF p(x). However, for a Gaussian PDF they do!

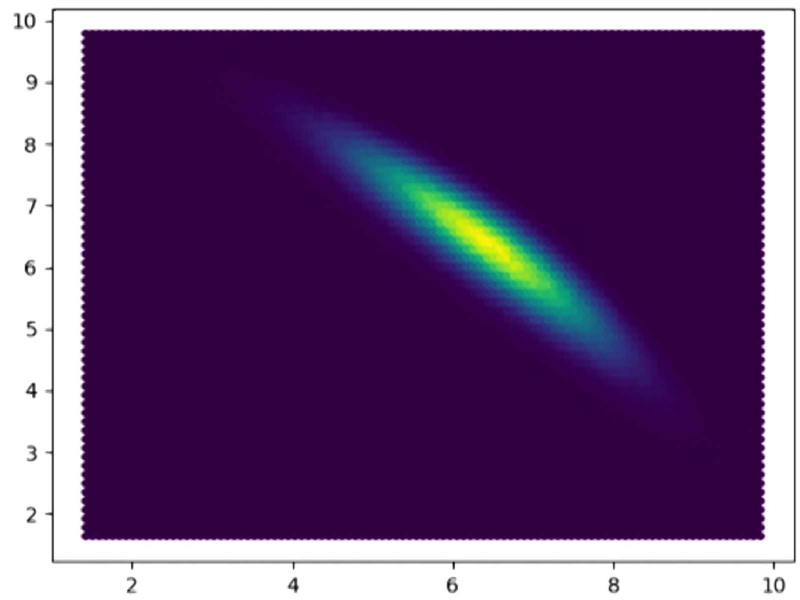
$$p(x) \approx \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\bar{x})^2/2\sigma^2} = \frac{1}{\sqrt{\pi N}} e^{-x^2/N} \qquad \text{(for N >> 1)}$$



Multivariate random variables: let's generalize to the case of N random variables (X_1, …, X_N) which are not assumed independent.

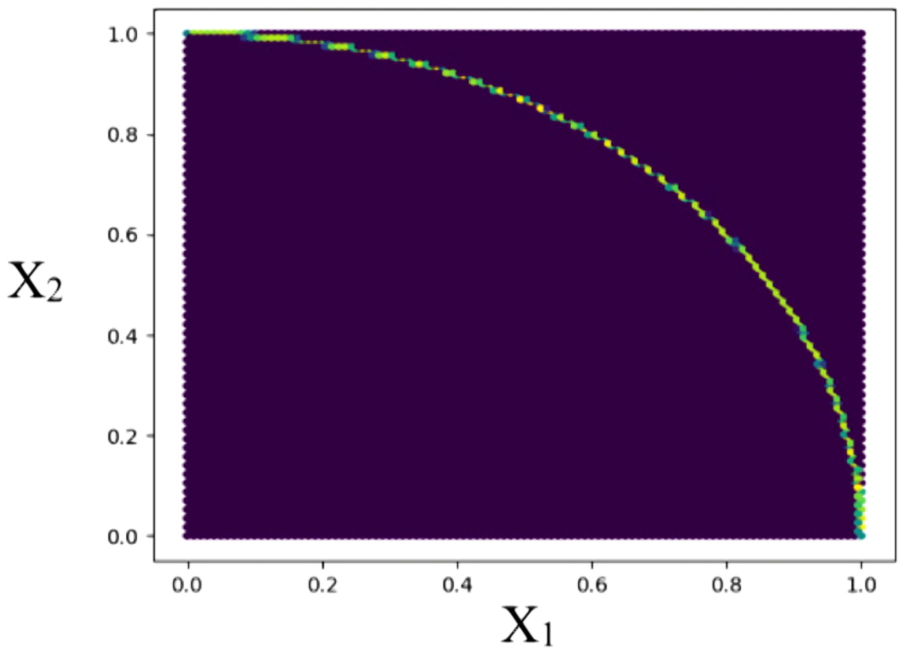
The PDF becomes a function of N variables $p(x_1,\ldots,x_N)$.

Example: a multivariate Gaussian (X_1, X_2) with a correlation between X_1 and X_2. (To be defined precisely in a few slides!)



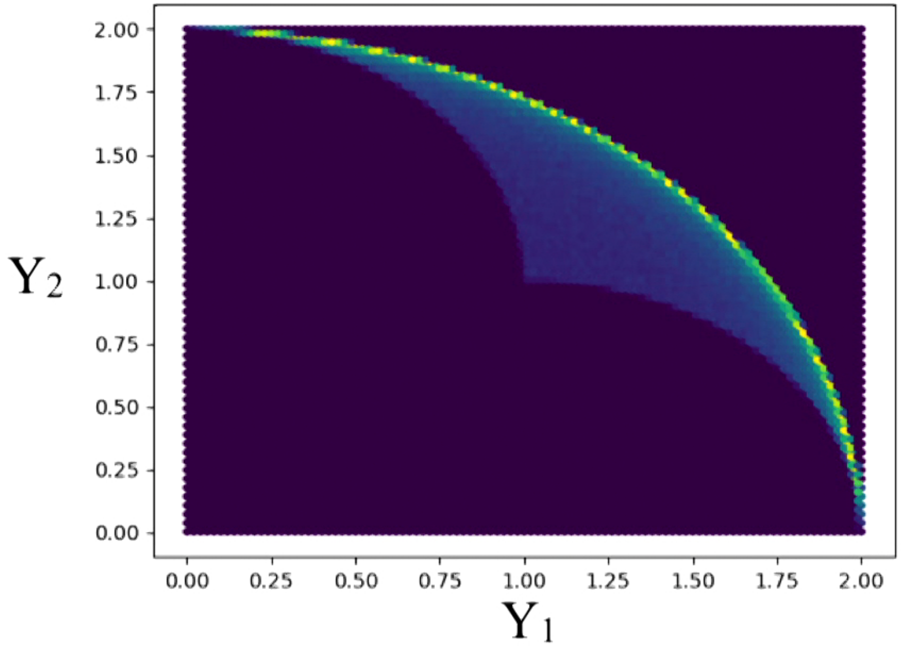
Example: $p(x_1, x_2) = \begin{cases} \frac{2}{\pi}\delta(\sqrt{x_1^2 + x_2^2} - 1) & \text{if } x_1, x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Just to show an extreme case where the variables x_1, x_2 are very non-independent!

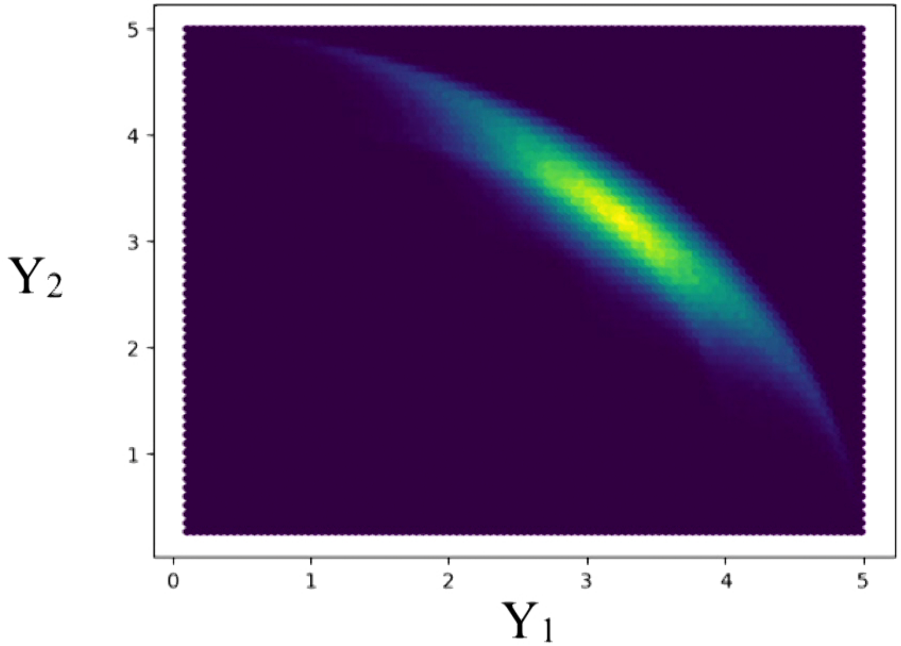


Does the central limit theorem still hold when the random variable is a vector X_i? (In this case, a two-component vector)

Two X's: $Y_i = X_i^{(1)} + X_i^{(2)}$



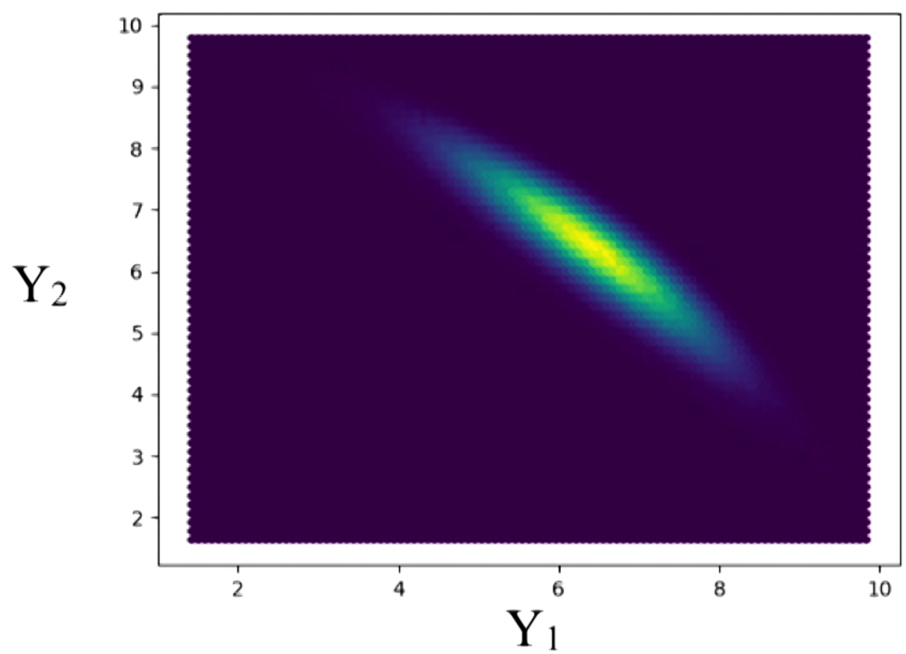
Five X's: $Y_i = \sum_{j=1}^{5} X_i^{(j)}$



Ten X's: $Y_i = \sum_{j=1}^{10} X_i^{(j)}$

The distribution has become a multivariate Gaussian.

In two variables, the multivariate Gaussian has five parameters: two "means", and three parameters describing the size and orientation.



In N variables, the mean becomes an N-component vector

$$\bar{X}_i = \langle X_i \rangle$$

The variance generalizes to an N-by-N covariance matrix:

$$\begin{aligned}\mathrm{Cov}(X_i, X_j) &= \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle \\ &= \langle (X_i - \bar{X}_i)(X_j - \bar{X}_j) \rangle\end{aligned}$$

The diagonal elements are variances $\mathrm{Cov}(X_i,X_i) = \mathrm{Var}(X_i)$ which have the same interpretation as before (that is, $\mathrm{Var}(X_i)^{1/2}$ is the characteristic size of fluctuations in X_i around its mean $\bar{X}_i$)

The off-diagonal elements $\mathrm{Cov}(X_i,X_j)$ quantify the correlation between random variables X_i, X_j. The correlation coefficient

$$\mathrm{Corr}(X_i, X_j) = \frac{\mathrm{Cov}(X_i, X_j)}{\sqrt{\mathrm{Var}(X_i)\,\mathrm{Var}(X_j)}}$$

is always between -1 and 1.

Now we can give the definition of a multivariate Gaussian PDF:

$$p(x_1, \cdots, x_N) = \frac{1}{\mathrm{Det}(2\pi C)^{1/2}} \exp\left(-\frac{1}{2}(x_i - \bar{x}_i) C^{-1}_{ij} (x_j - \bar{x}_j)\right)$$

The PDF of a multivariate Gaussian random variable is determined by its mean $\bar{X}_i$ and covariance matrix $C_{ij} = \mathrm{Cov}(X_i, X_j)$

In cosmology, we are usually interested in Gaussian random variables. Therefore, it suffices to keep track of the mean (a vector) and the covariance (a matrix).

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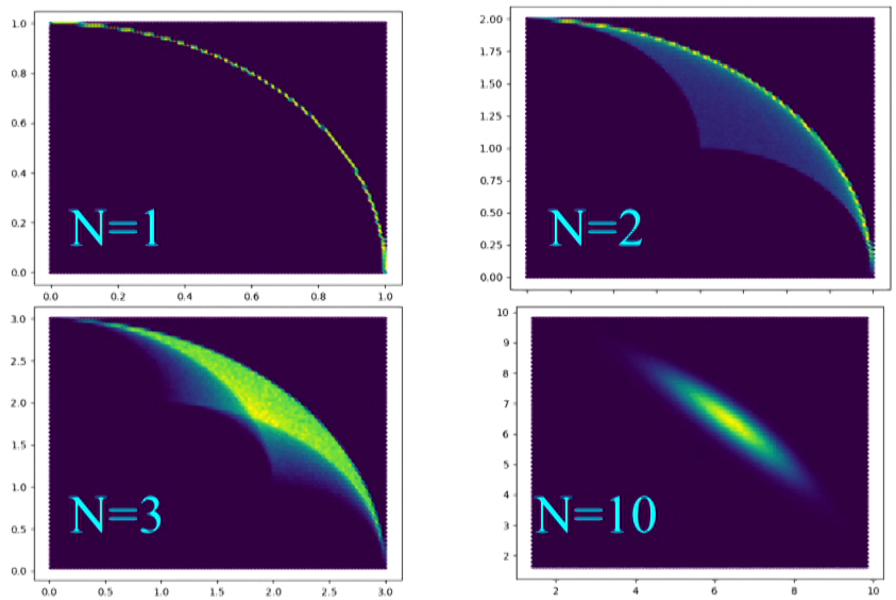
In this example, a short calculation gives the mean and covariance:

$$\begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix} = N \begin{pmatrix} 0.64 \\ 0.64 \end{pmatrix} \qquad \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} = N \begin{pmatrix} 0.095 & -0.087 \\ -0.087 & 0.095 \end{pmatrix}$$

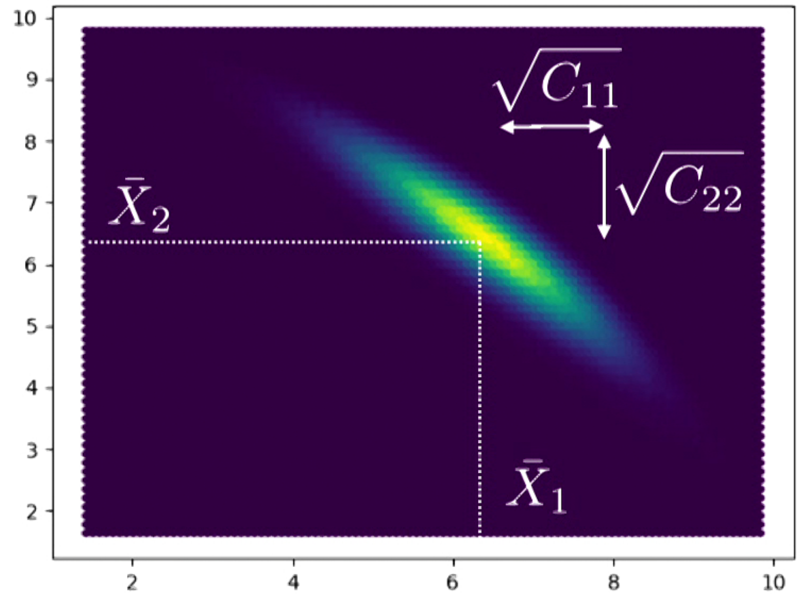
In the large-N limit, these determine the PDF (central limit theorem):

$$p(x_1, x_2) \approx \frac{1}{\mathrm{Det}(2\pi C)^{1/2}} \exp\left(-\frac{1}{2}(x_i - \bar{x}_i) C^{-1}_{ij} (x_j - \bar{x}_j) \right) \qquad (N >> 1)$$



In this example, the strong "tilt" of the ellipse means that X_1 and X_2 are strongly negatively correlated:

$$C_{ij} = \begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix} \qquad \frac{C_{12}}{\sqrt{C_{11}C_{22}}} \approx -0.9$$



The CMB is a multivariate Gaussian random variable!

If the map below is represented with $N=10^7$ pixels, then the statistics are described perfectly (as far as we know) by a multivariate Gaussian, whose N-by-N covariance matrix is calculable. (Numerically, not analytically!)