Title: PSI 2017/2018 - Quantum Gravity - Lecture 12

Date: Apr 03, 2018 10:15 AM

URL: http://pirsa.org/18040009

Abstract:

3D

$S_{EH} = \int R\sqrt{g}\, d^3x$ $\longrightarrow$ First order formulation $S_{Palatini} = \int e \wedge F\, d^3x$ $\longrightarrow$ Canonical analysis $S = \int (E^a_j \partial_0 A^j_a + N^j \mathcal{F}_j + \Lambda^j \mathcal{G}_j) d^3x$

vacuum Einstein eq $R_{\mu\nu} = 0$

$\begin{cases} g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij} \\ \omega^j_{\mu k} \end{cases}$

$\begin{cases} F = 0 \\ T = 0 \end{cases}$

$\{ A^j_a(x), E^b_k(y) \} = \delta^j_k \delta^b_a \delta(x,y)$

$H = -\int d^3x \left(N^j \mathcal{F}_j + \Lambda^j \mathcal{G}_j \right)$

Where are we ?

- Created Hilbert space
- Found some states
- Imposed Gauss constraint

We now need to implement flatness constraint.
For a given path γ, we saw

$$h_\gamma = \mathbb{I} - \int_S F + \mathcal{O}(\epsilon^3).$$

Requiring flatness $F = 0$ is equivalent to requiring a trivial holonomy $h_\gamma = \mathbb{I}$

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Particular example

Triangulation of the sphere : Tetrahedron

- 6 vertices → 6 group holonomy elements
- 4 faces → 4 flatness constraints

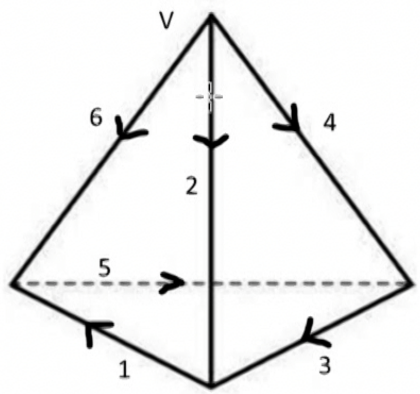
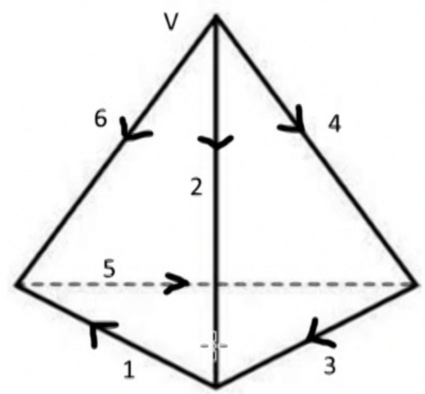


Fig. : https://www.goldendawnshop.com/product/tetrahedron-of-fire/

Particular example



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We have the 4 following constraints :

$$H_1 = g_4^{-1} g_5 g_6, \quad H_2 = g_2^{-1} g_1^{-1} g_6, \quad H_3 = g_4^{-1} g_3^{-1} g_2, \text{ and } \quad H_4 = g_5^{-1} g_3^{-1} g_1^{-1}.$$

But H_4 actually depends on the others : $H_4 = g_6 H_1^{-1} H_3 H_2 g_6^{-1}$.
We can simply impose $H_1 = H_2 = h_3 = \mathbb{I}$.

Inner product

We use the δish-function to create *physical* state (which satisfies the constraint). The inner product should be

$$\langle \prod \delta(H_i)\Psi_2 | \prod \delta(H_j)\Psi_1 \rangle_{phys.} .$$

The naughty physical scalar product makes this computation annoying. Fortunately, we can use

$$\delta(H)^\dagger = \delta(H), \quad \delta(H)^2 = \delta(H),$$

and a trick to use the kinematical scalar product.
We take a particular spin network state. Each Ψ_1 is associated with representations j_i on each edge (with some constraints).

Inner product

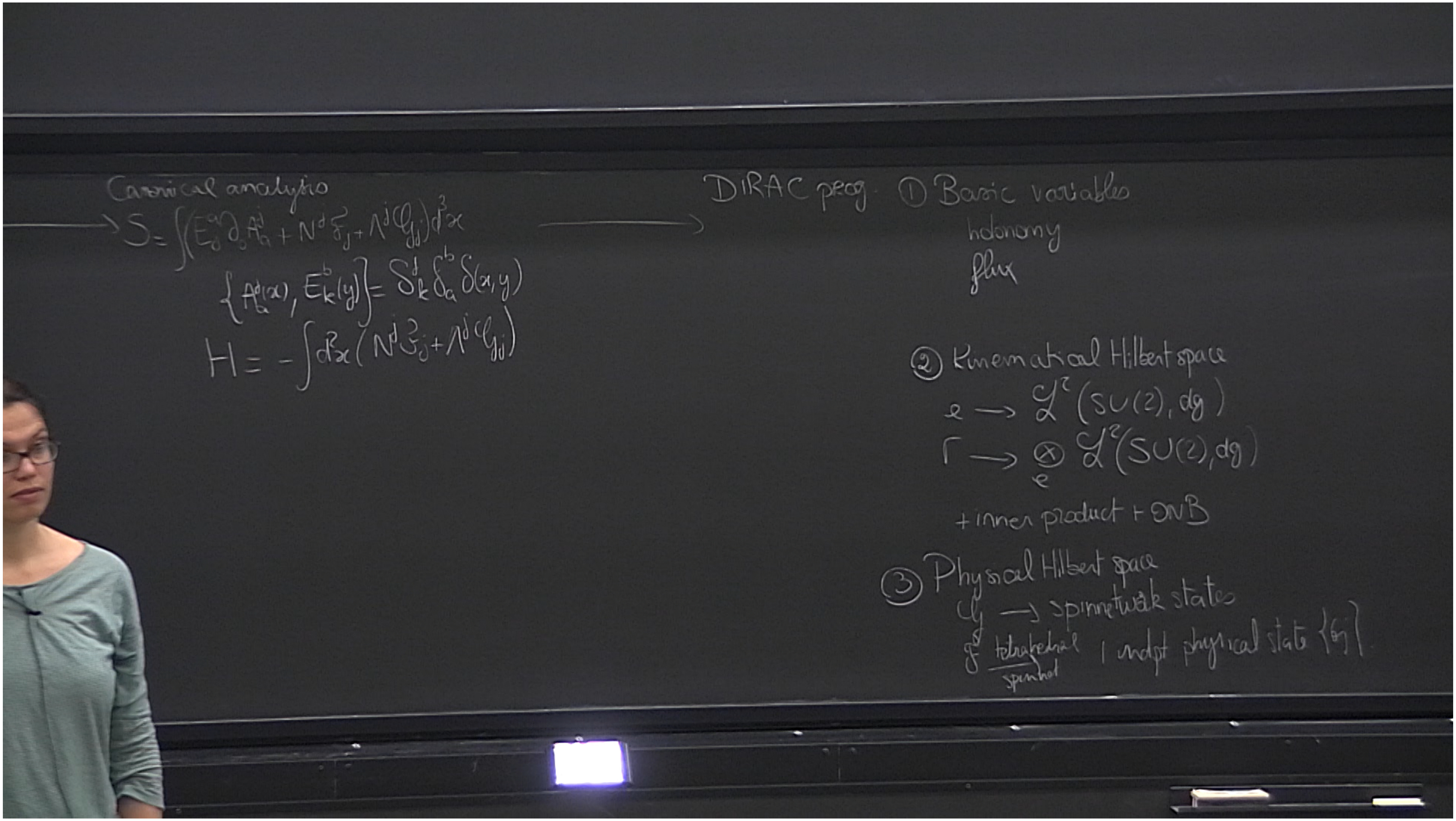
We get for the transition amplitude (after inserting integration over several $SU(2)$ and various recoupling's identities) :

$$\langle \Psi_2 | \prod \delta(H_j) \Psi_1 \rangle = \sqrt{\langle \Psi_1 | \Psi_1 \rangle} \sqrt{\langle \Psi_2 | \Psi_2 \rangle}$$

The Hilbert space is one dimensional ! (Recall high school definition of scalar product)
This means that the Hilbert space is one dimensional, but the phase space is zero dimensional.

Conclusion

For a given graph, the state is fully determined : No personal choice, and gravity has 0 degreee of freedom.


Canonical analysis
$S = \int (E^{a}_{j} \partial_0 A^{j}_{a} + N^{j} \mathcal{G}_{j} + \Lambda^{j} \mathcal{C}_{jj}) d^3x$
$\{A^{j}_{a}(x), E^{b}_{k}(y)\} = \delta^{j}_{k} \delta^{b}_{a} \delta(x,y)$
$H = -\int d^3x (N^{j} \mathcal{G}_{j} + \Lambda^{j} \mathcal{C}_{jj})$
DIRAC prog.
① Basic variables
holonomy
flux
② Kinematical Hilbert space
$e \longrightarrow \mathcal{L}^2(SU(2), dg)$
$\Gamma \longrightarrow \bigotimes_{e} \mathcal{L}^2(SU(2), dg)$
+ inner product + ONB
③ Physical Hilbert space
$\mathcal{C}_j \longrightarrow$ spinnetwork states
tetrahedral / spinnet | indpt physical state

$S_{EH} = \int R\sqrt{g}\, d^4x$ $\quad \begin{cases} g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij} \\ \omega^{IJ}_{\mu} \end{cases}$ $\quad S_{Palatini} = \int e \wedge F\, d^4x$

vacuum Einstein eq $R_{\mu\nu} = 0$ $\quad \begin{cases} F = 0 \\ T = 0 \end{cases}$

$\{A^i_a(x), E^b_k(y)\} = \delta^i_k \delta^b_a \delta(x,y)$

$H = -\int d^3x \left(N^a \mathcal{C}_a + N \mathcal{C} \right)$

$\boxed{4D}$

$S_{EH} \longrightarrow \left(S_{Palatini} = \frac{1}{2\kappa} \int_{\mathcal{M}} \epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL}(\omega) \right.$

Lorentz inv $\xrightarrow{}$ partial gauge fixing $\quad$ SU(2) gauge inv.

$\downarrow$ Canonical analysis

$\mathcal{M} = \Sigma \times \mathbb{R}$

$\boxed{\text{variables}}$

- q_{ab} spatial metric, metric on Σ
- conjugated momenta $\pi^{ab} = \frac{\partial L_{EH}}{\partial \dot{q}_{ab}}$

$\hookrightarrow$ related to the extrinsic curvature of Σ.

Constraints

variables · q_{ab} spatial metric
metric on Σ
· conjugated momenta $\pi^{ab} = \frac{\partial L_{EH}}{\partial \dot{q}_{ab}}$
↳ related to the extrinsic curvature of Σ

Constraints → the vector constraint V^a
(spatial diffeo)
1st class.
→ the scalar constraint S

$H^\nu = (S, V^a)$ generates diffeo.

ADM formalism – Dirac's program?
issue: H^ν: non polynomial functions of the space variabl…

A. Ashtekar: change of variables → triads.

$\boxed{q_{ab} = e^i_a e^j_b \delta_{ij}}$ → add… degr… freedom…

Canonical analysis (Lorentz inv $\longrightarrow$ partial gauge fixing; SO(2) gauge inv)

$\mathcal{M} = \Sigma \times \mathbb{R}$

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(spatial diffeo)

1st class

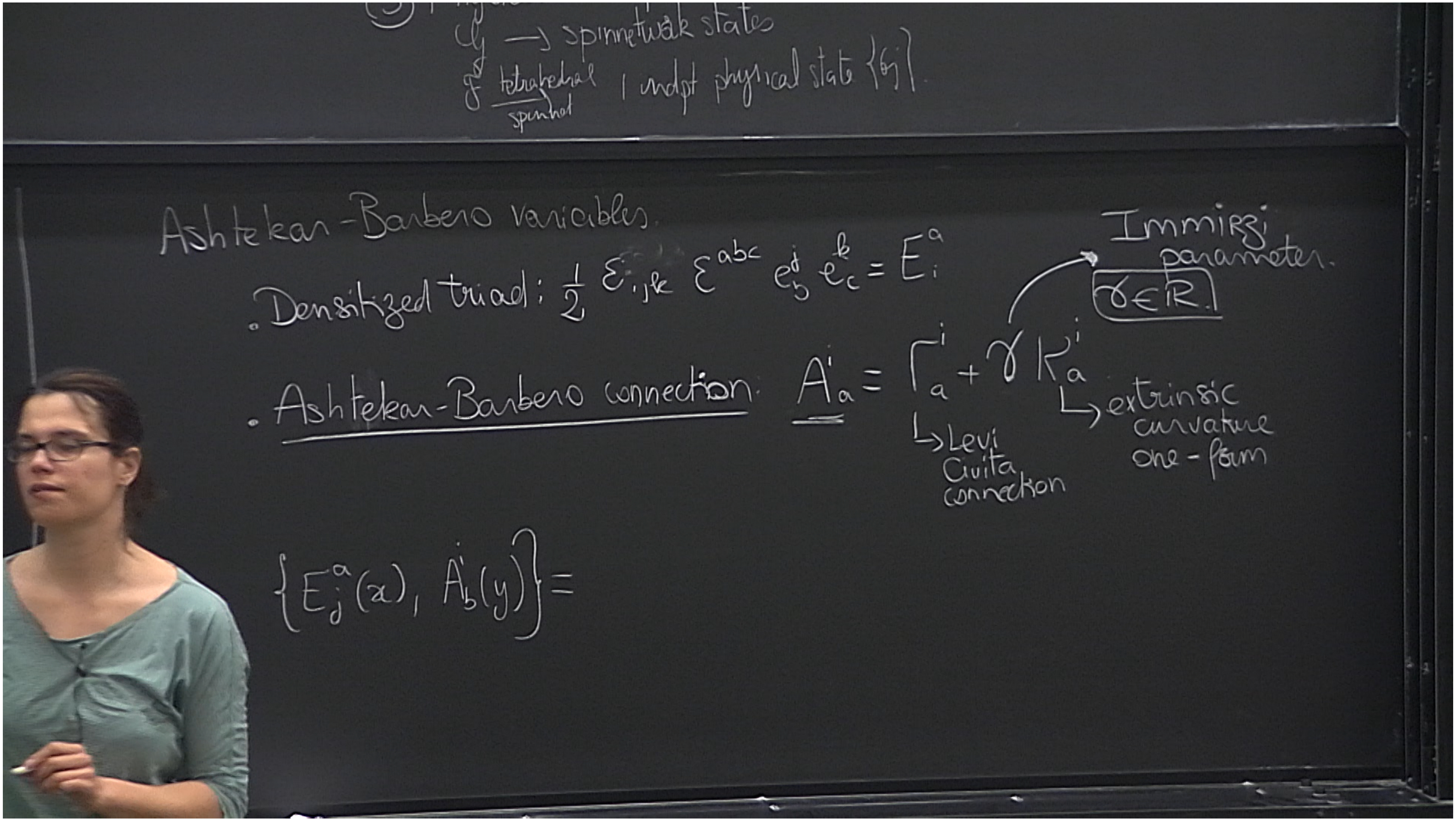
$\rightarrow$ the scalar constraint S] generates diffeo

ADM formalism – Dirac's program?

issue: H^{ν} : non polynomial functions of the phase space variables

A. Ashtekar: change of variables $\rightarrow$ triads

$\boxed{q_{ab} = e^i_a e^j_b \, \delta_{ij}}$ $\rightarrow$ additional degrees of freedom] SU(2) gauge symmetry



→ spinnetwork states
tetrahedral / indpt physical state
spinfoot
Ashtekar-Barbero variables.
Immirzi parameter.
• Densitized triad: ½ ε_ijk ε^abc e_b^j e_c^k = E_i^a
γ ∈ ℝ
• Ashtekar-Barbero connection: A_a^i = Γ_a^i + γ K_a^i
↳ Levi Civita connection
↳ extrinsic curvature one-form
{E_j^a(x), A_b^i(y)} =

spinfoam

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$$\{E_j^a(x), A_b^i(y)\} = k\gamma\,\delta_b^a\,\delta_j^i\,\delta(x,y).$$

+ constraints: $\begin{cases} G \\ V^a \\ S \end{cases}$ → smearing the constraints → constraint algebra.

Constraint algebra

$$\{G[\alpha], G[\beta]\} = G([\alpha,\beta])$$

$$\{G[\alpha], V(N^a)\} = -G[\mathcal{L}_N(\alpha)]$$

$$\{G[\alpha], S[N]\} = 0$$

$$\{V(N^a), V(M^a)\} = -S(\mathcal{L}_N M)$$

$$\{S[N], S[M]\} = V(S^a) + \text{terms proportional to the Gauss constraint}$$

This al

field, phase space function
given by the inverse of the metric → non

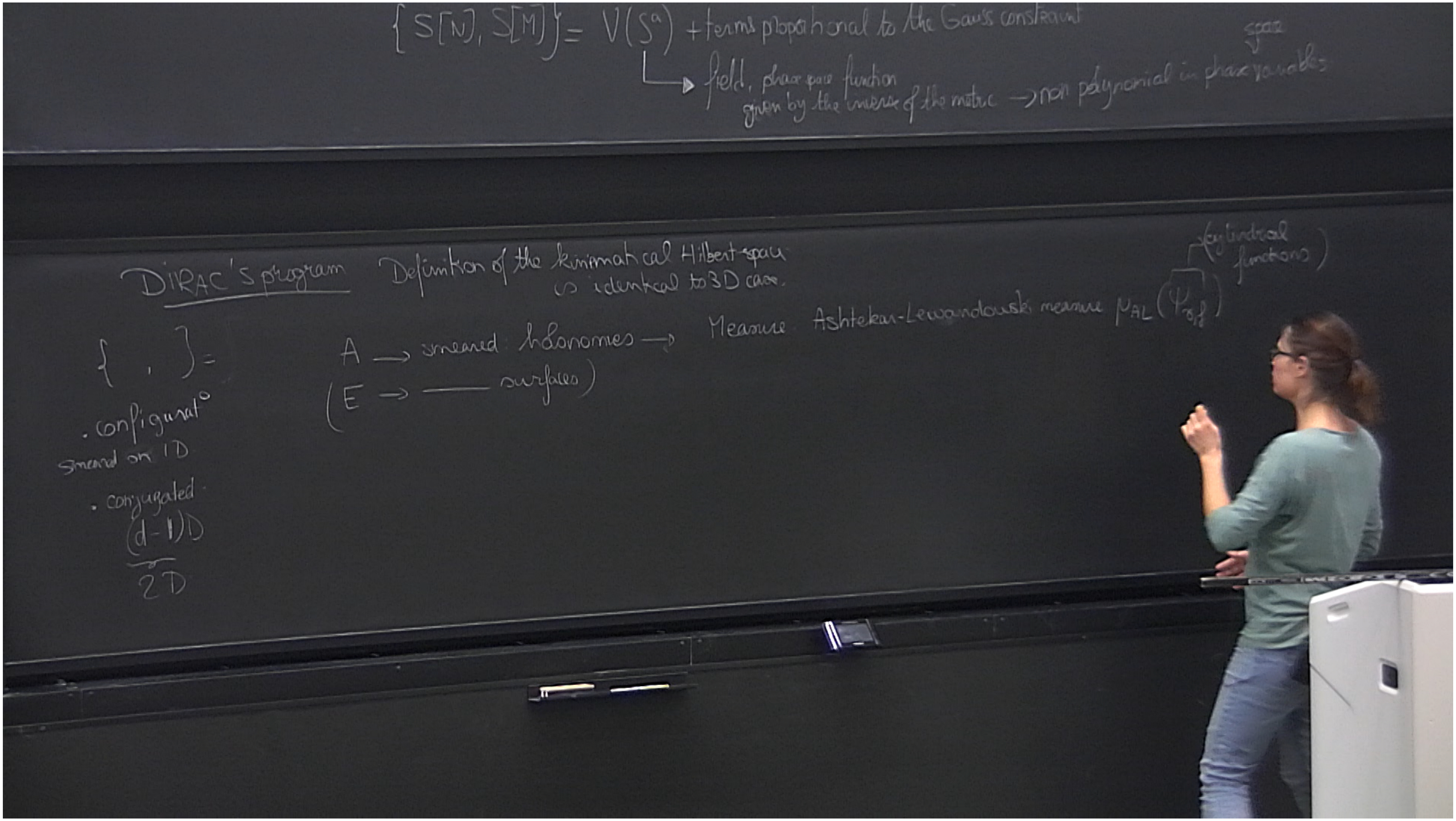
DIRAC's program   Definition of the kinematical Hilbert space
is identical to 3D case.

{ , } =

. configurat°
smeared on 1D

. conjugated
(d-1)D
~
2D

A → smeared: holonomies →
(E → ——— surfaces)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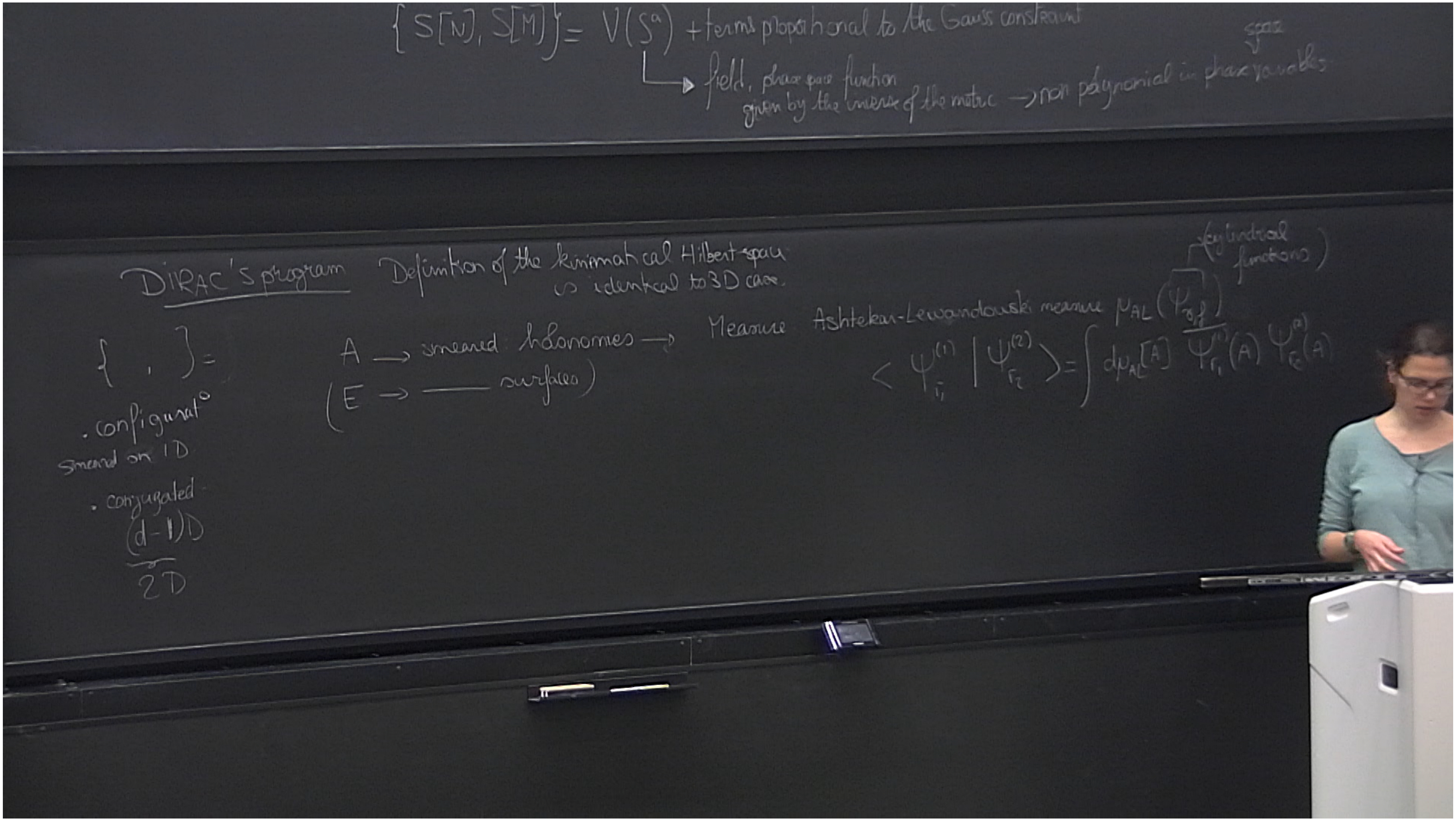
Measure Ashtekar-Lewandowski measure μ_AL (Ψ_γ,f)
(cylindrical functions)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DIRAC's program

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$A \longrightarrow$ smeared: holonomies $\longrightarrow$

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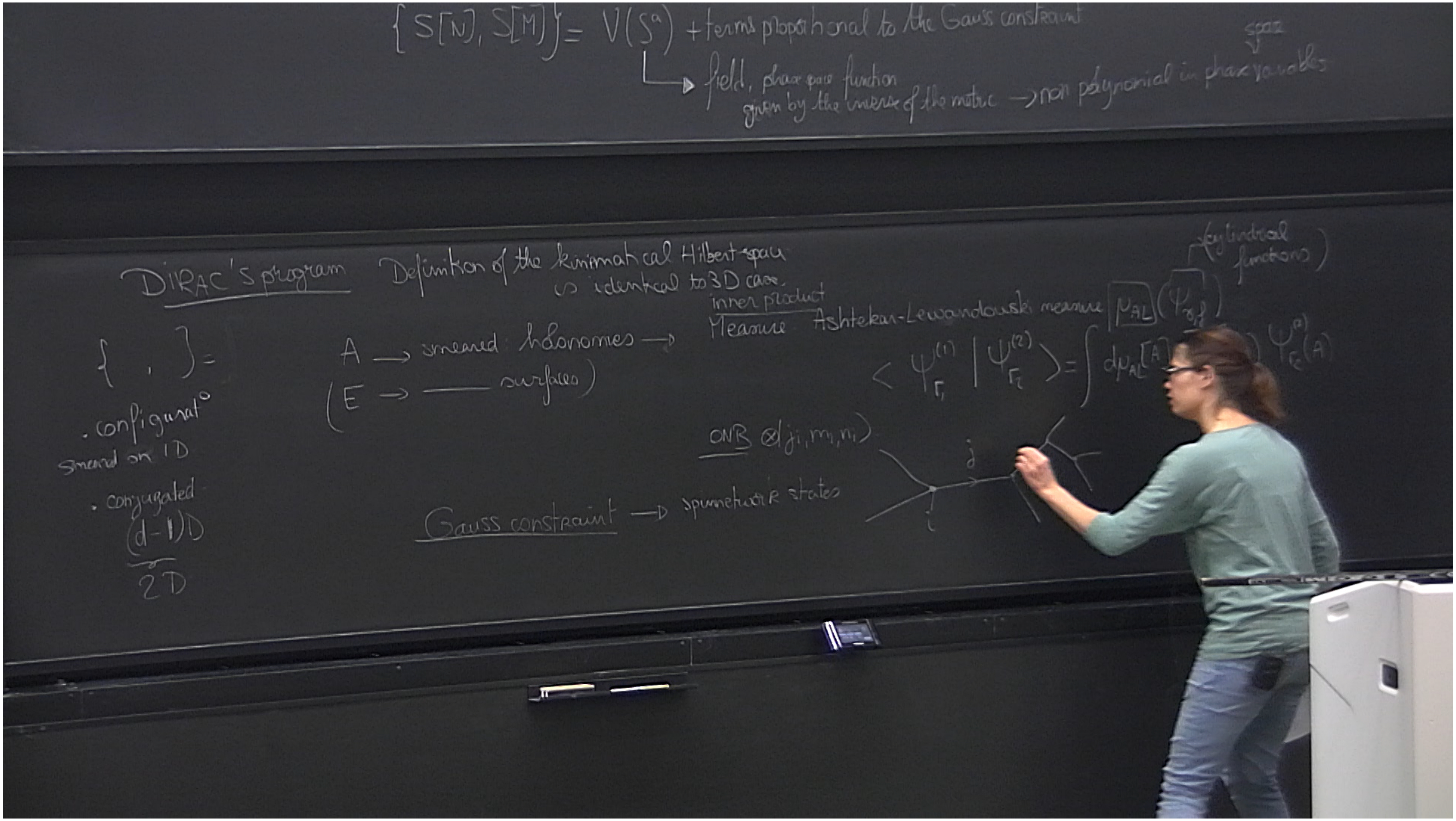
Measure: Ashtekar-Lewandowski measure $\boxed{\mu_{AL}}(\Psi_{\gamma,f})$ (cylindrical functions)

$$\langle \Psi^{(1)}_{\Gamma_1} | \Psi^{(2)}_{\Gamma_2} \rangle = \int d\mu_{AL}[A]\ \overline{\Psi^{(1)}_{\Gamma_1}}(A)\ \Psi^{(2)}_{\Gamma_2}(A)$$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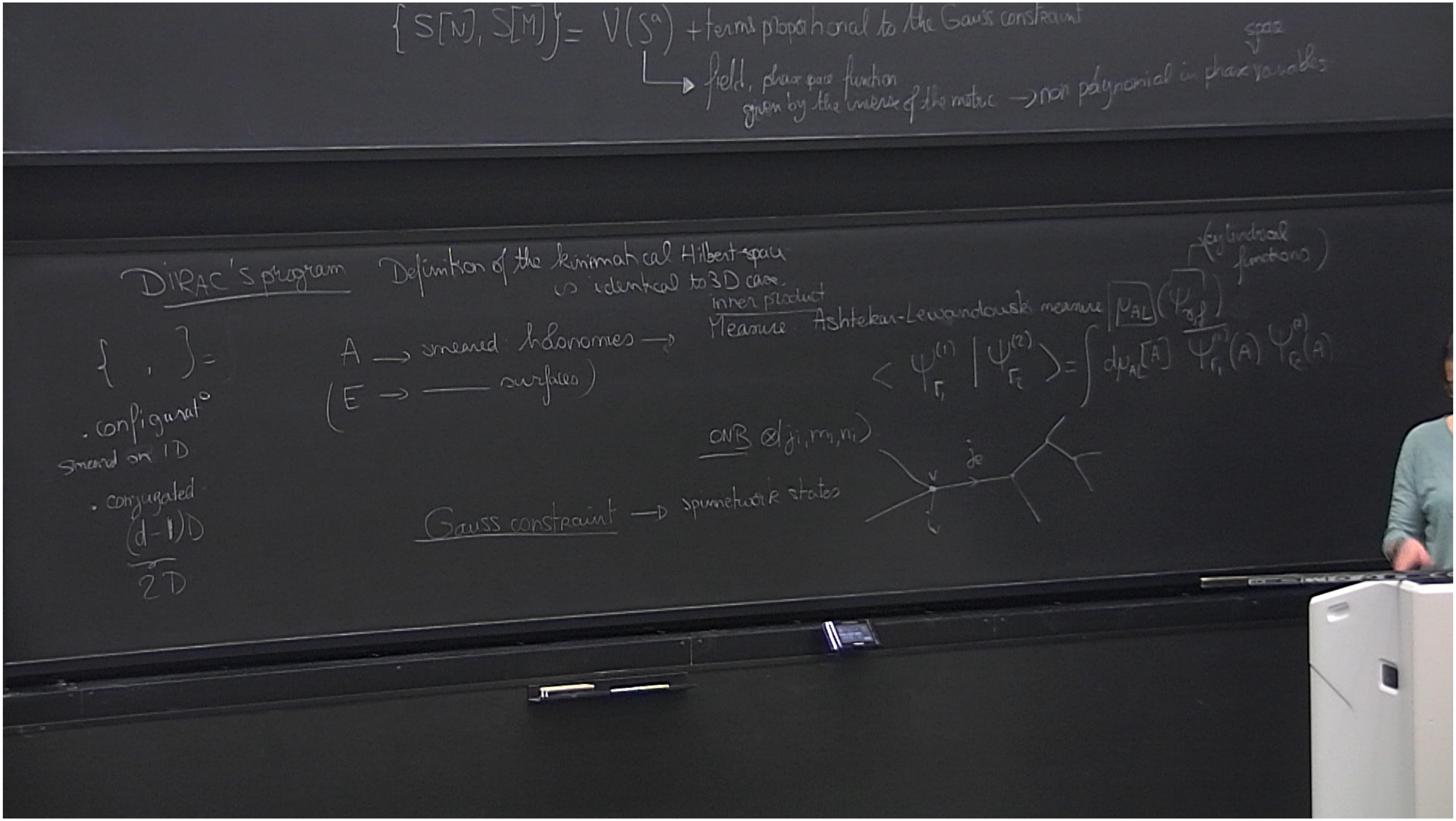
ONB ⊗(ji, mi, ni)
Gauss constraint → spinnetwork states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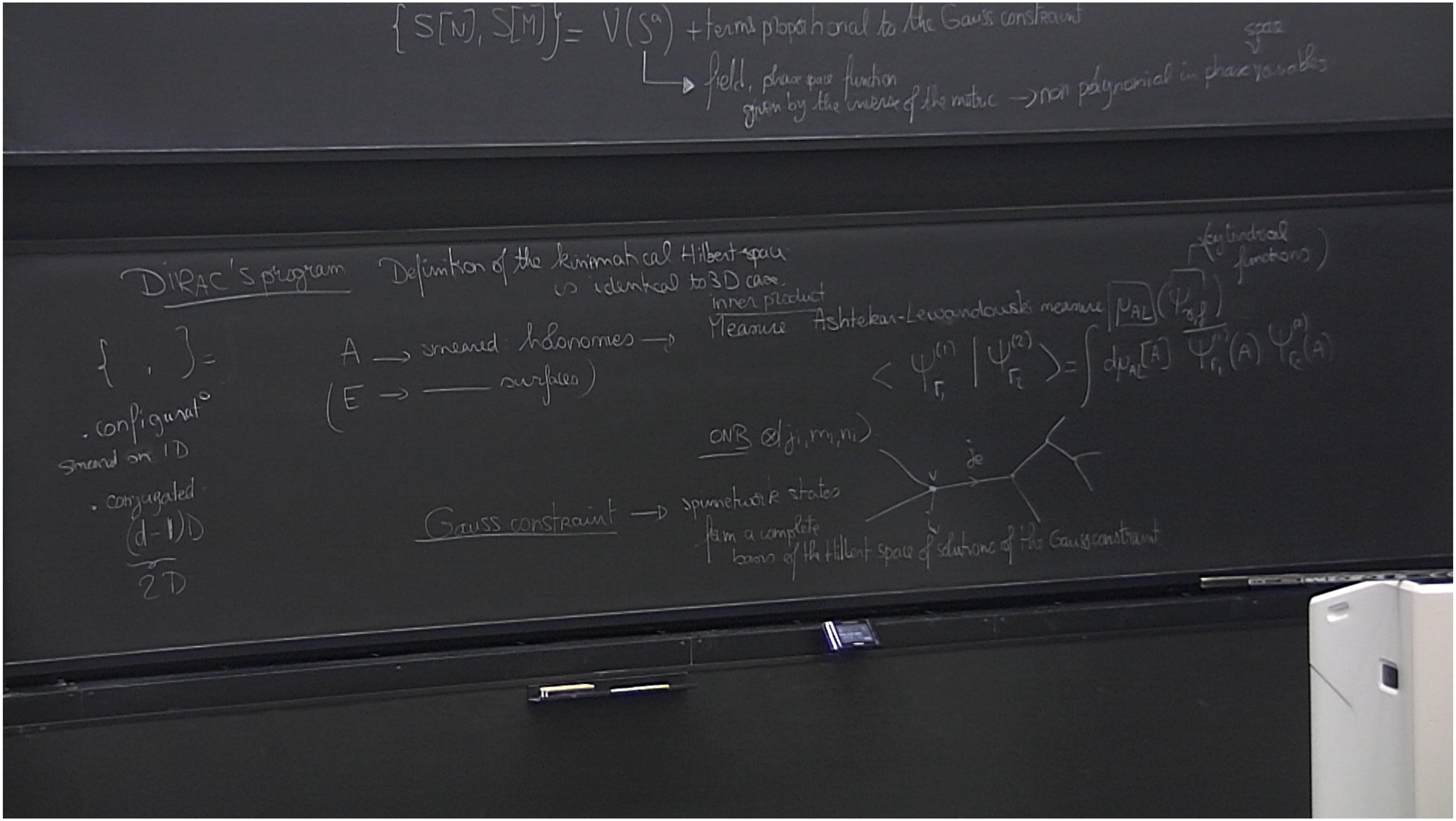
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ONB $\otimes(j_i, m_i, n_i)$
j_e
Gauss constraint → spinnetwork states
form a complete
basis of the Hilbert space of solutions of the Gauss constraint

First order formulation

$\left\{\begin{array}{l} g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij} \\ \omega^{ij}_\mu \end{array}\right.$

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③ Physical Hilbert space

$\mathcal{C}_j \rightarrow$ spinnetwork states

$\mathcal{C}$ tetrahedral / spinfoam ... physical state $\{\mathcal{C}_j\}$

issue H non polynomial functions of the phase space variables

Ashtekar change of variables $\rightarrow$ triads

$\boxed{q_{ab} = e^i_a e^j_b \delta_{ij}} \rightarrow$ additional degrees of freedom $\rightarrow$ SU(2) gauge symmetry $\rightarrow \mathcal{C}_j$

Levi-Civita connection

one-form

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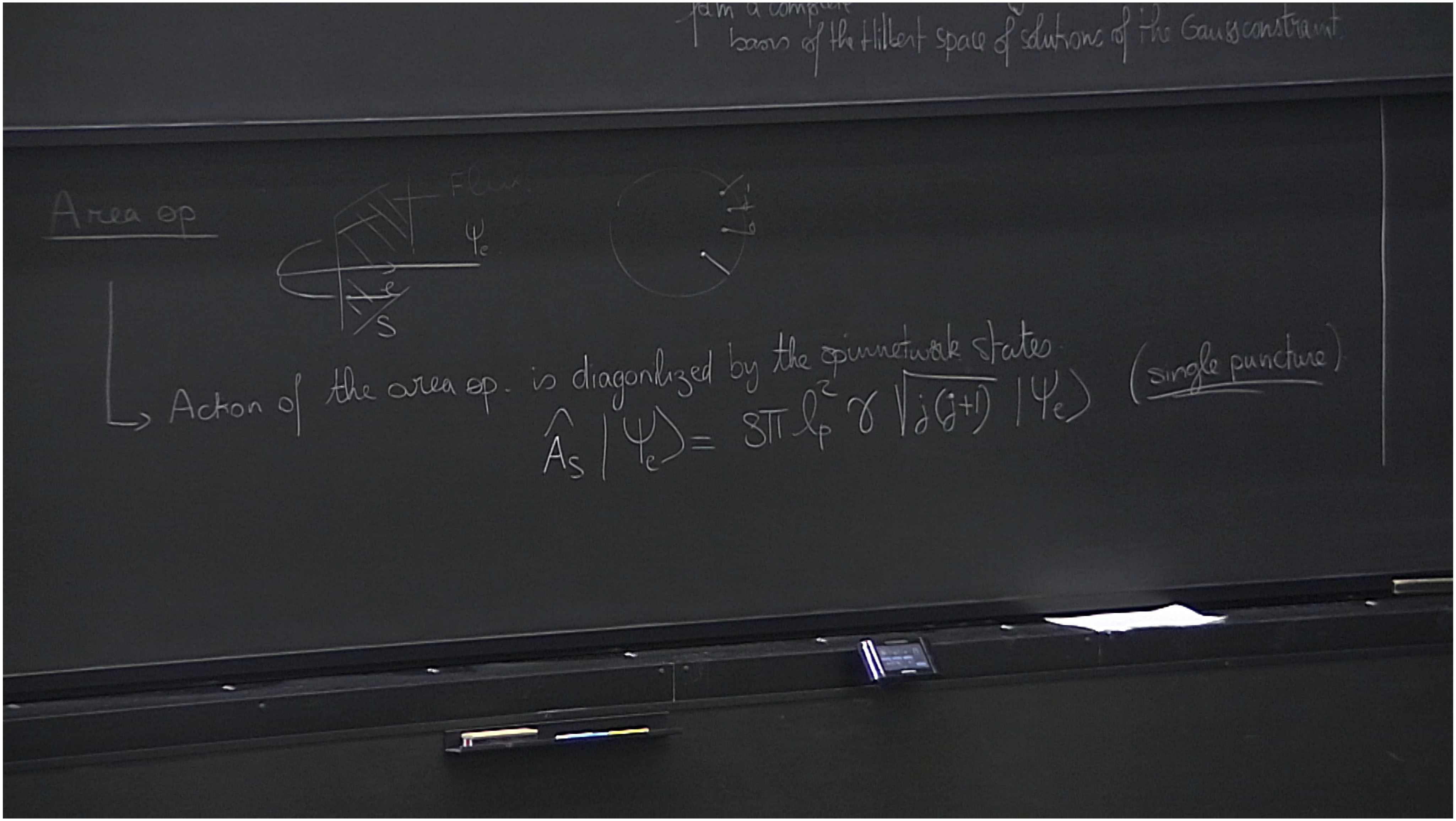
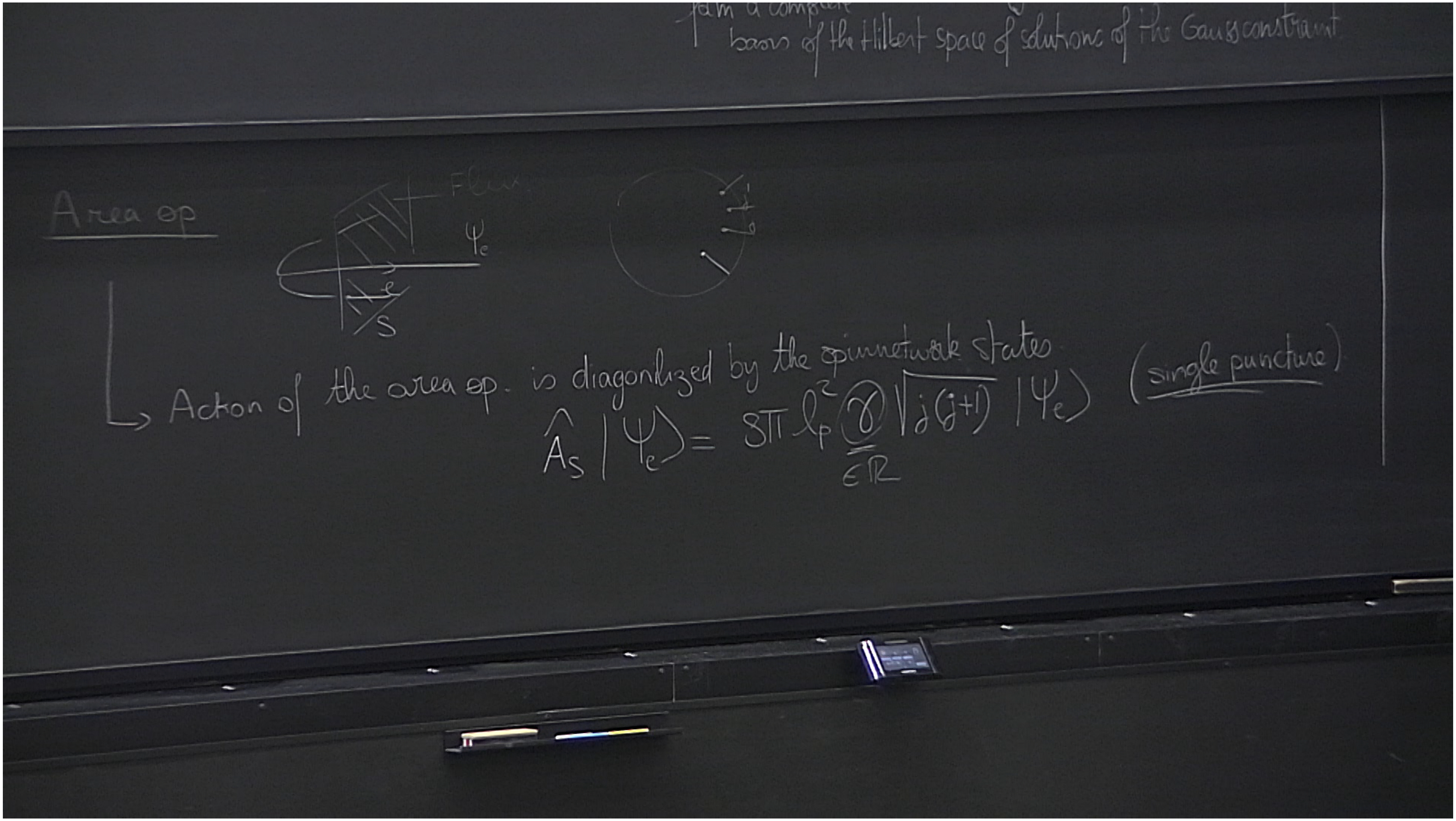

basis of the Hilbert space of solutions of the Gauss constraint
Area op
Flux
ψ_e
e
S
Action of the area op. is diagonlized by the spinnetwork states
$\hat{A}_S |\psi_e\rangle = 8\pi l_p^2 \gamma \sqrt{j(j+1)} |\psi_e\rangle$ (single puncture)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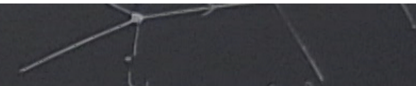
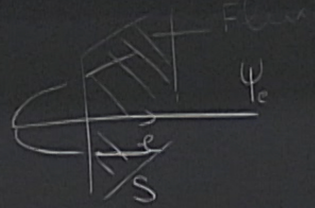
(d-1)D

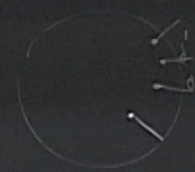
2D

Gauss constraint → spinnetwork states form a complete basis of the Hilbert space of solutions of the Gauss constraint



Area op

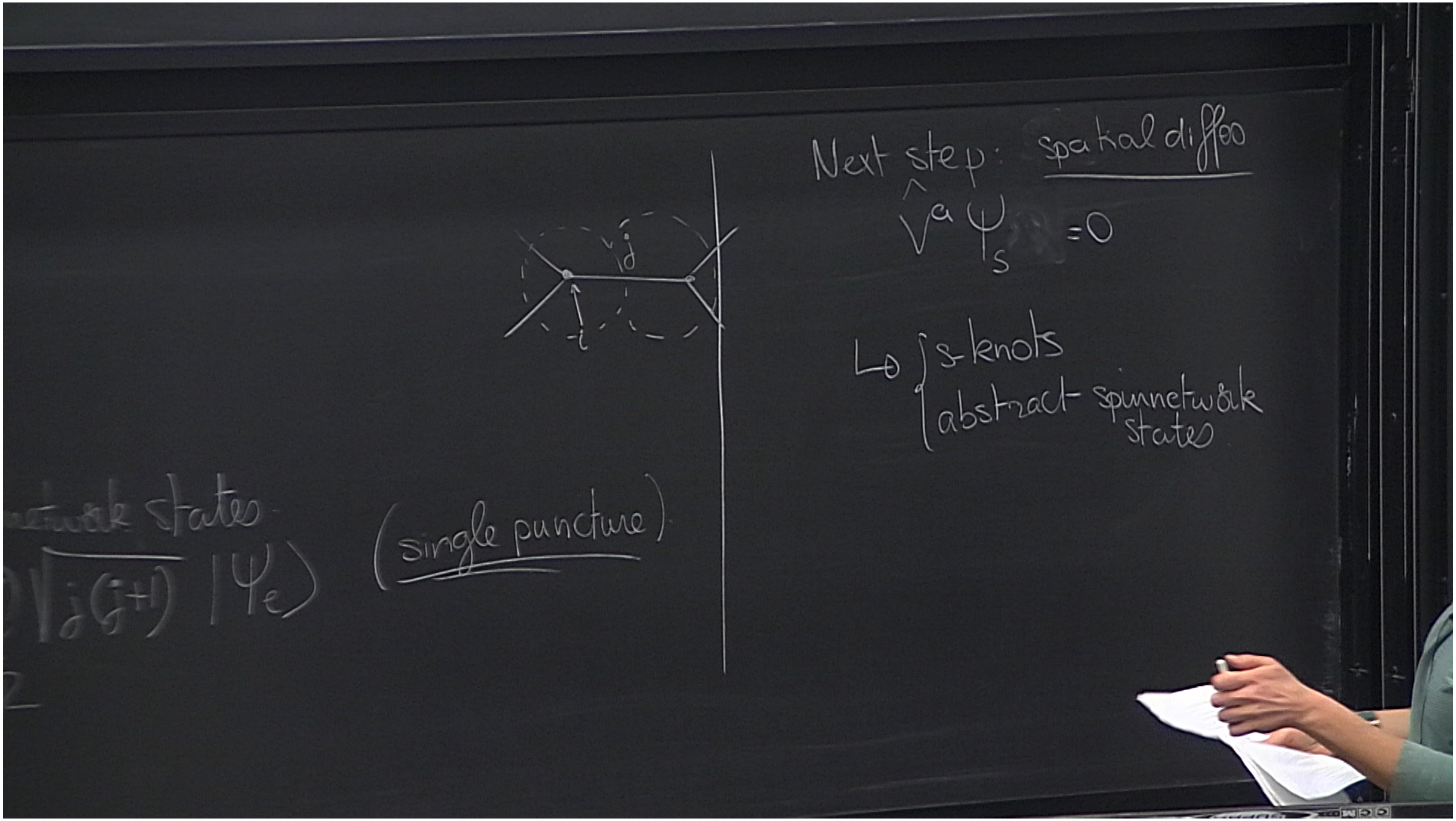


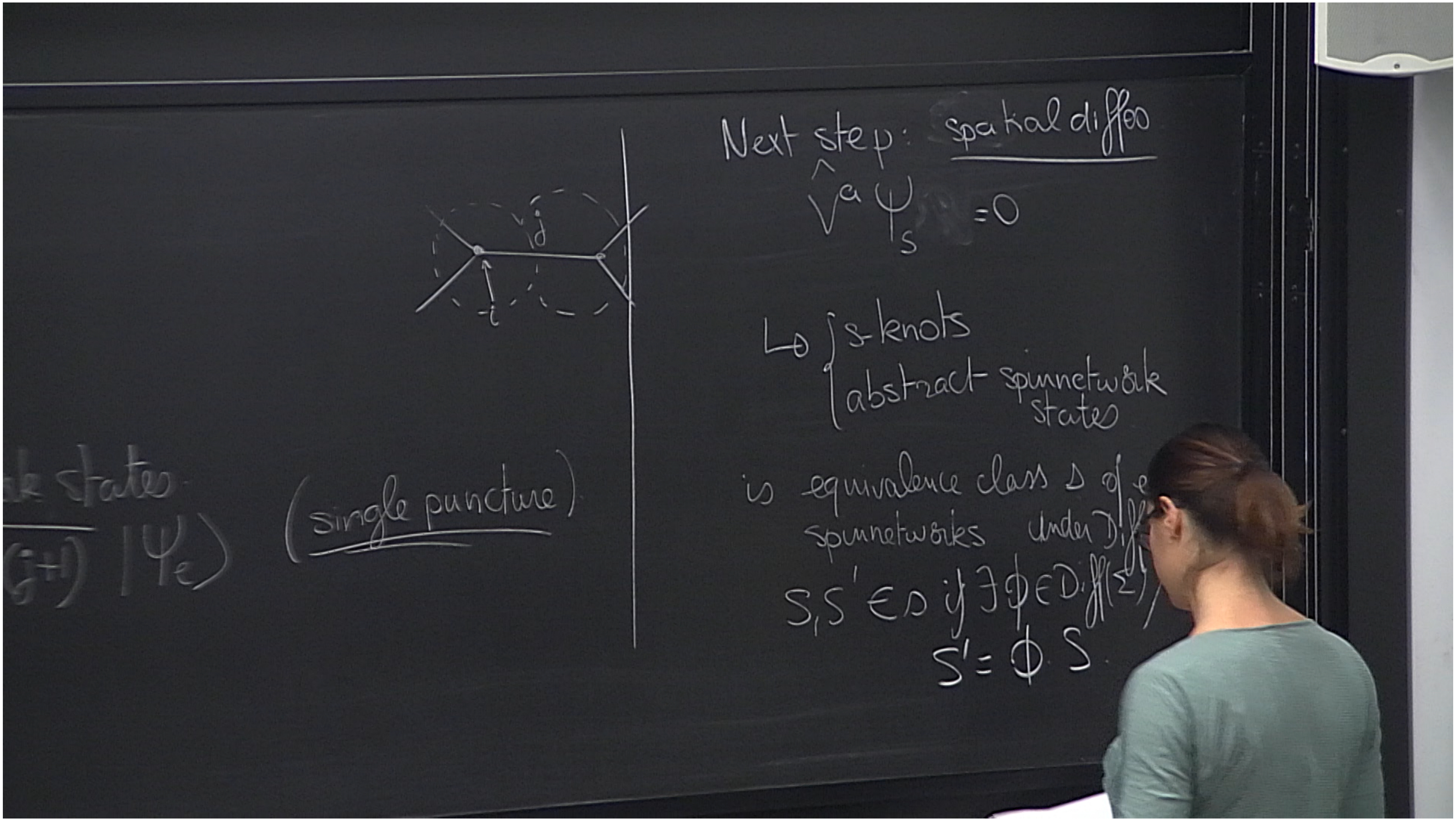


↳ Action of the area op. is diagonalized by the spinnetwork states (single puncture)

$$\hat{A}_S |\Psi_e\rangle = 8\pi l_p^2 \underset{\in \mathbb{R}}{\gamma} \sqrt{j(j+1)}\, |\Psi_e\rangle$$

Volume op


Next step: spatial diffeo
$\hat{V}^a \Psi_s = 0$
s-knots
abstract spinnetwork states
states
(single puncture)
j
i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V̂^a Ψ_s = 0
s-knots
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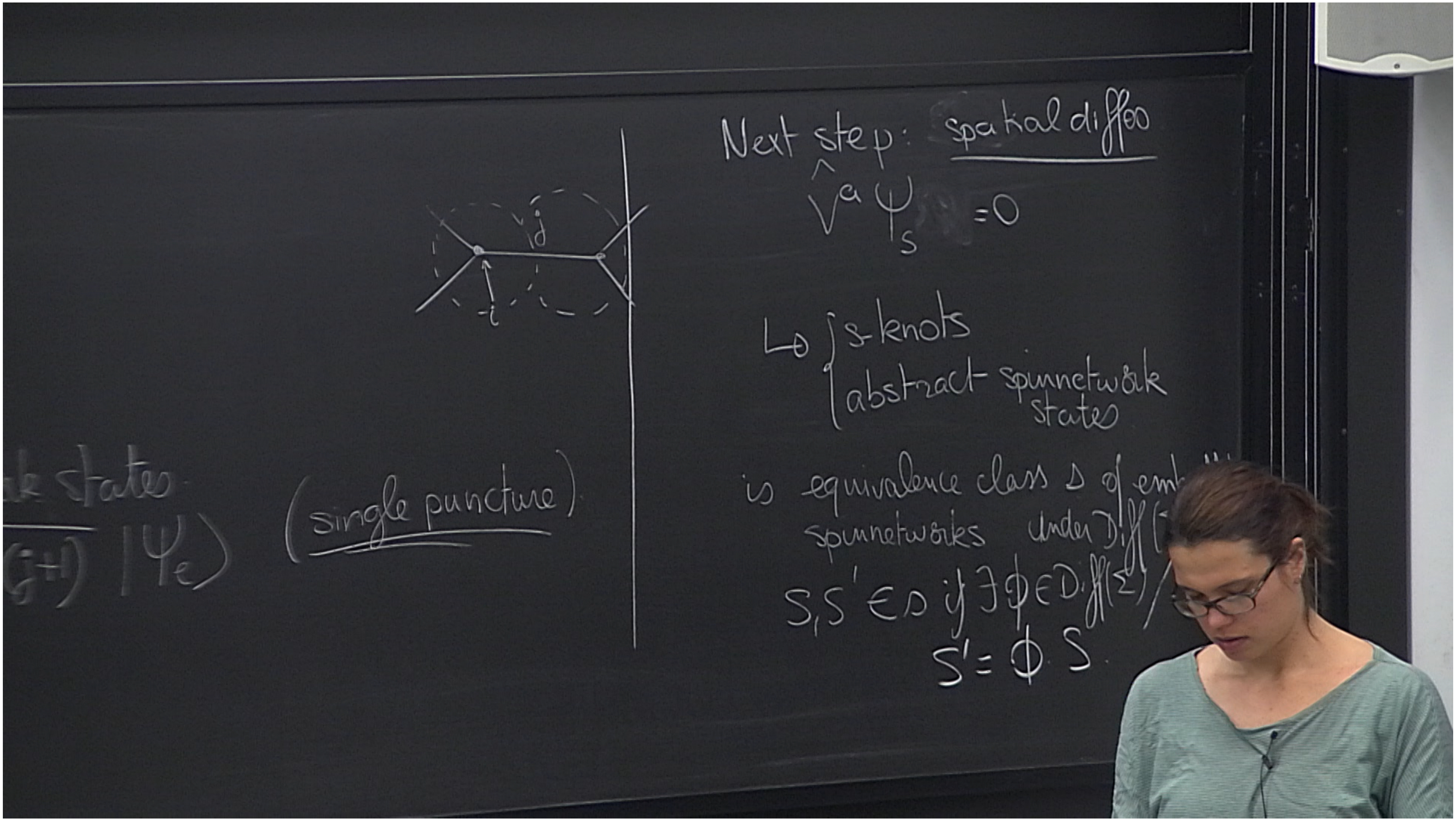
is equivalence class s of
spinnetworks under Diff
S, S' ∈ s if ∃ φ ∈ Diff(Σ)
S' = φ · S
(single puncture)
states



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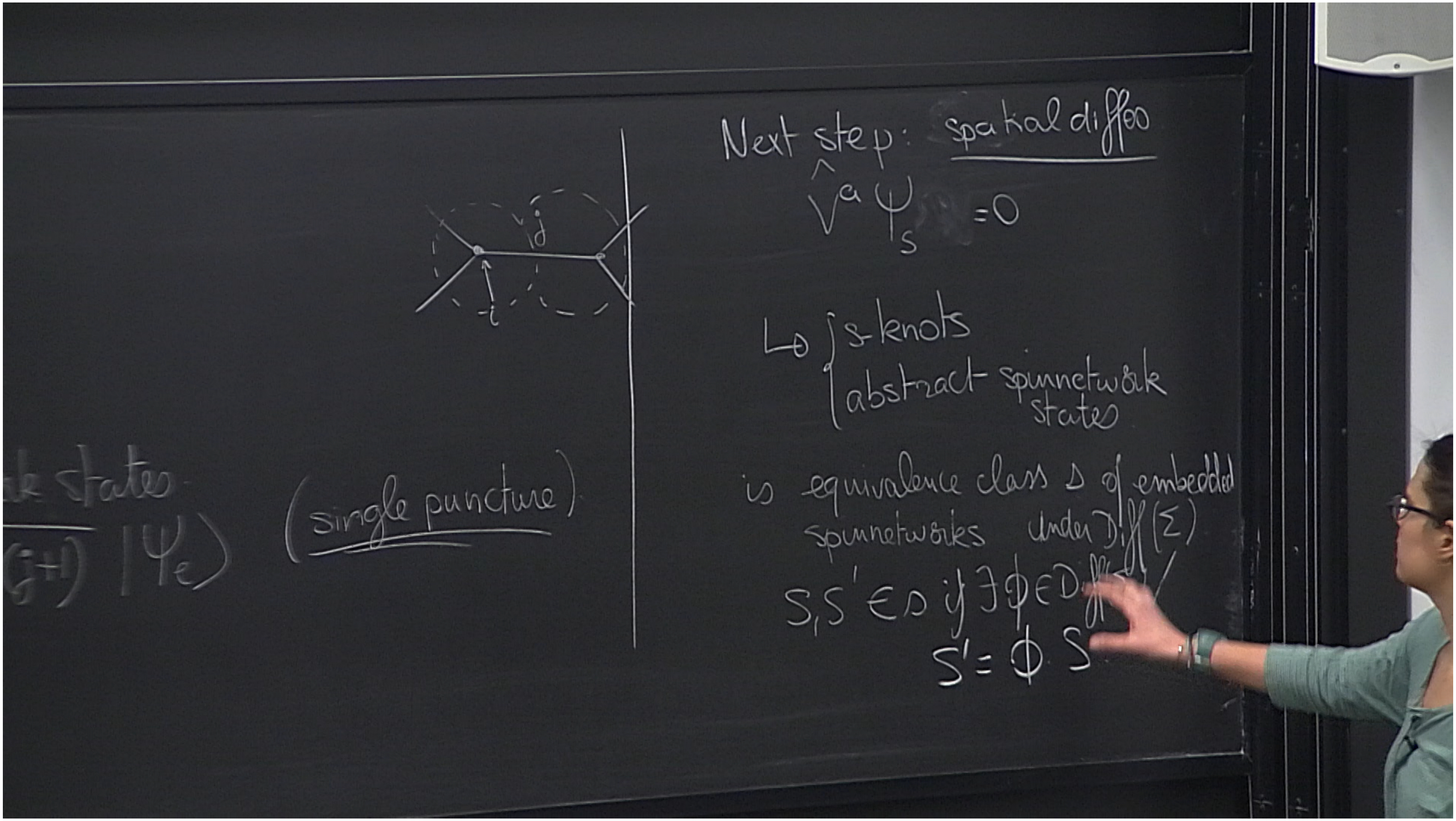

Next step: spatial diffeo
$\hat{V}^a \Psi_s = 0$
s-knots
abstract spinnetwork states
is equivalence class s of embedded spinnetworks under Diff(Σ)
$S, S' \in s$ if $\exists \phi \in$ Diff(Σ)
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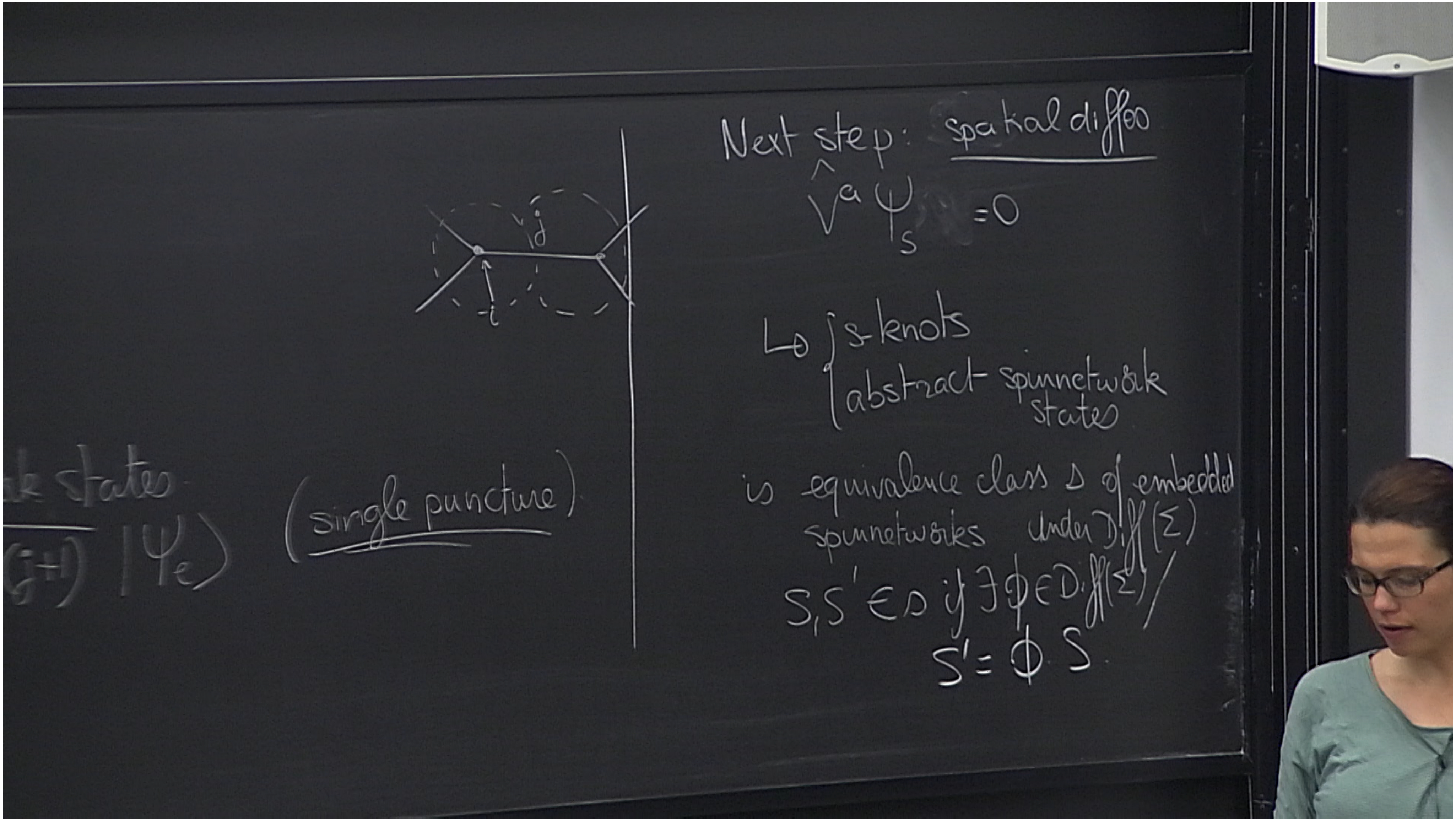
states
(single puncture)
$|\Psi_e\rangle$
j
i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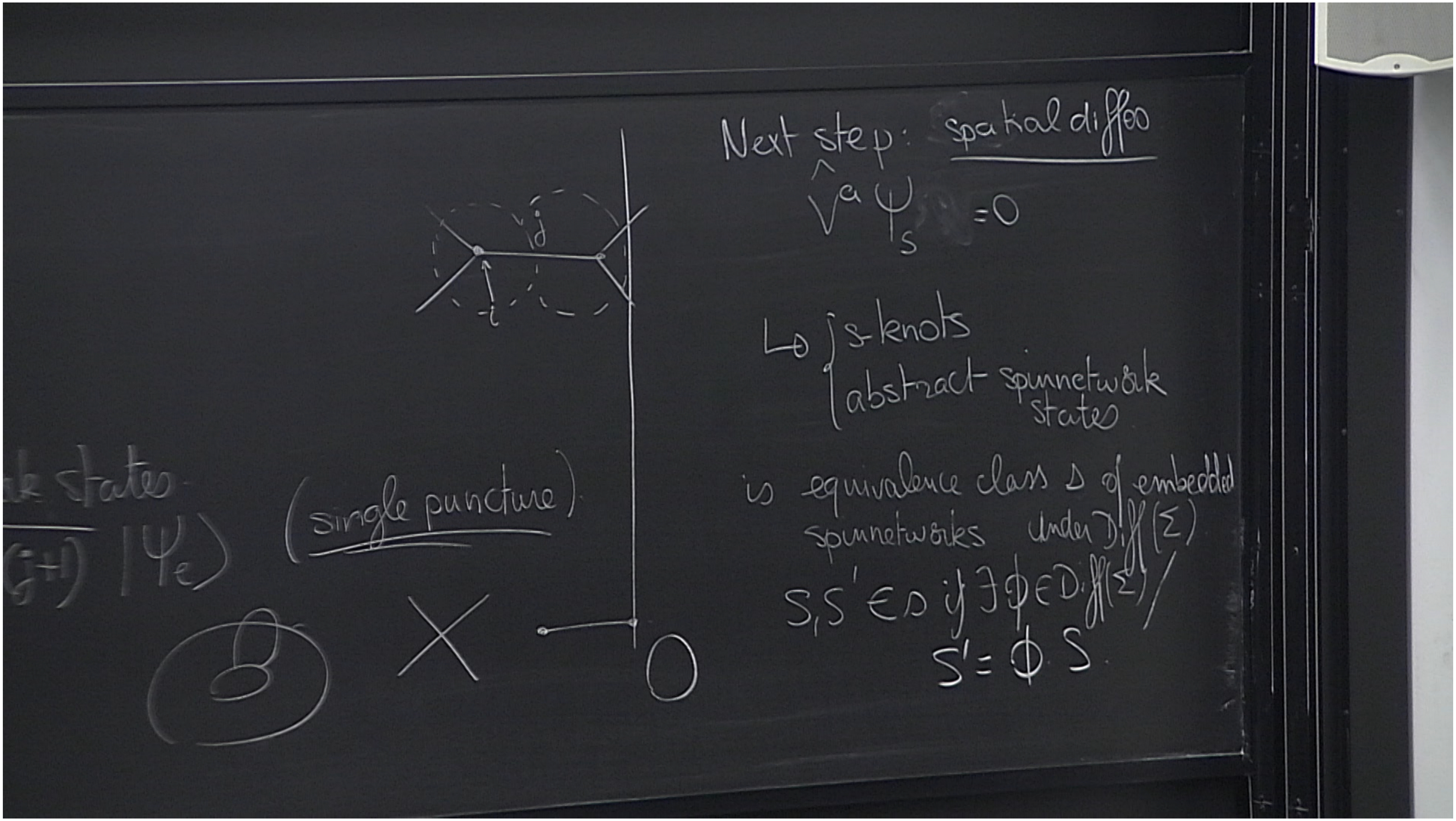
$S' = \phi \cdot S$
j
i
k states
(single puncture)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|Ψ_e)
j
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