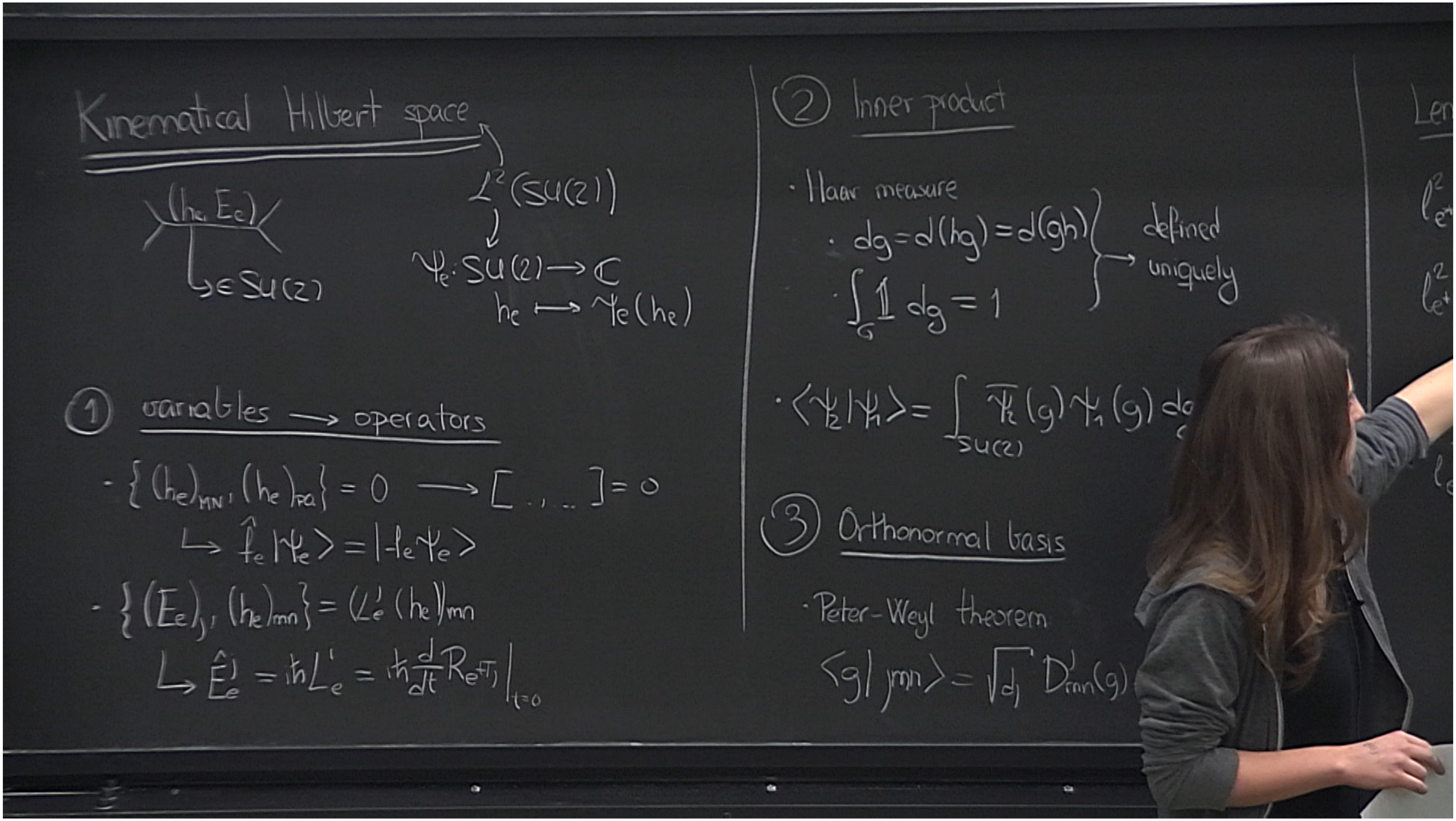
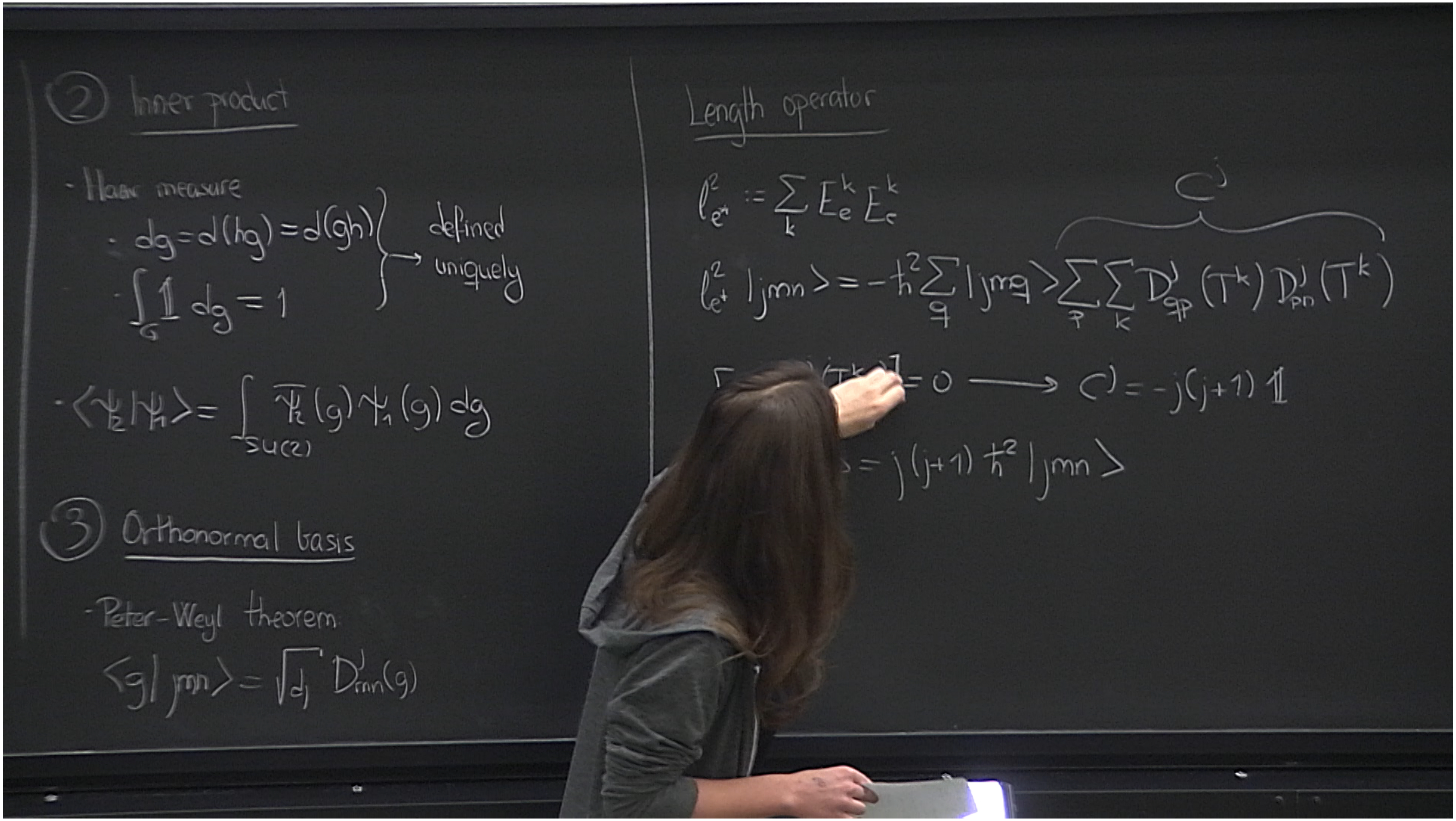
Title: PSI 2017/2018 - Quantum Gravity - Lecture 10

Date: Apr 02, 2018 10:15 AM

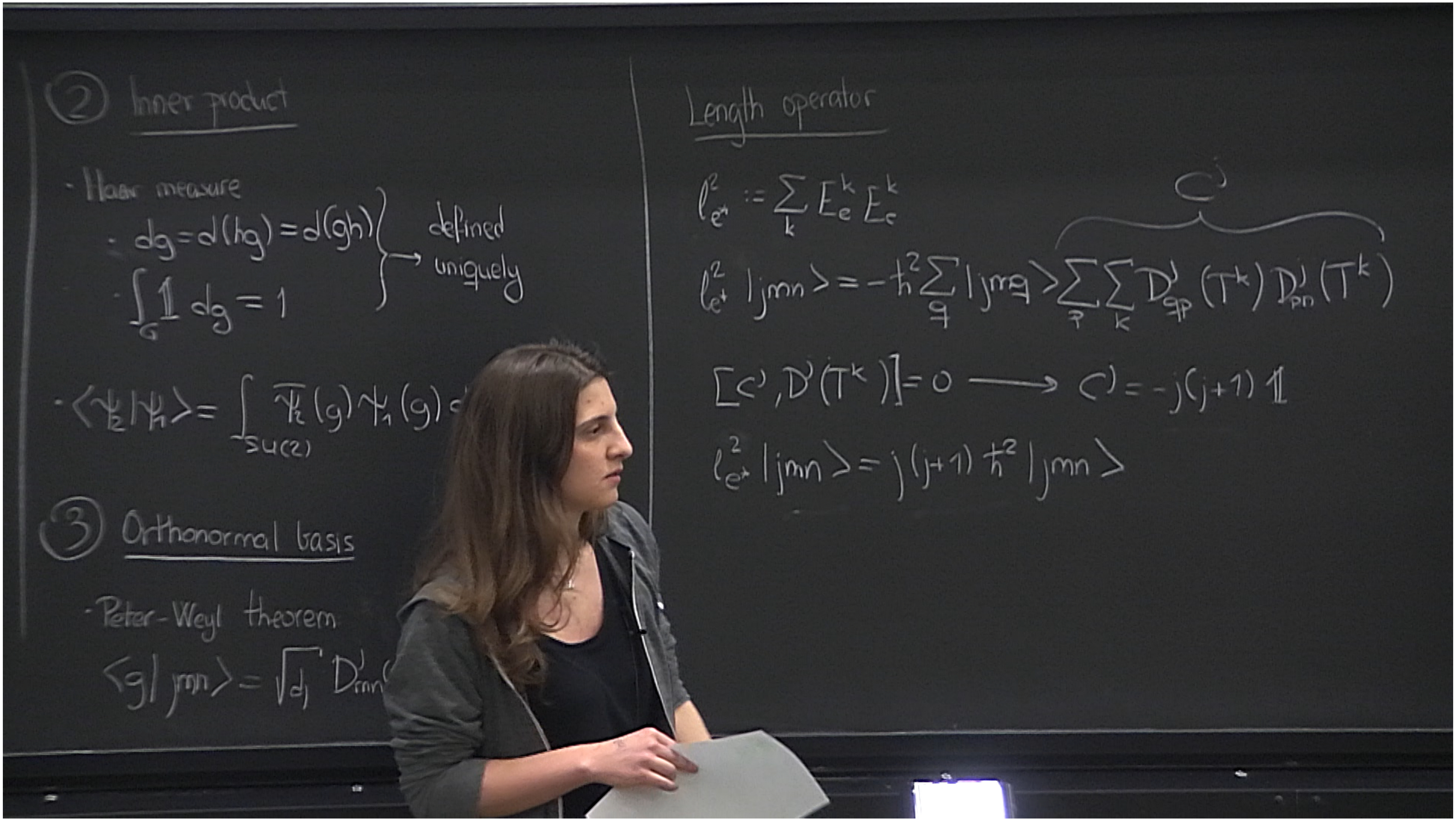
URL: http://pirsa.org/18040008

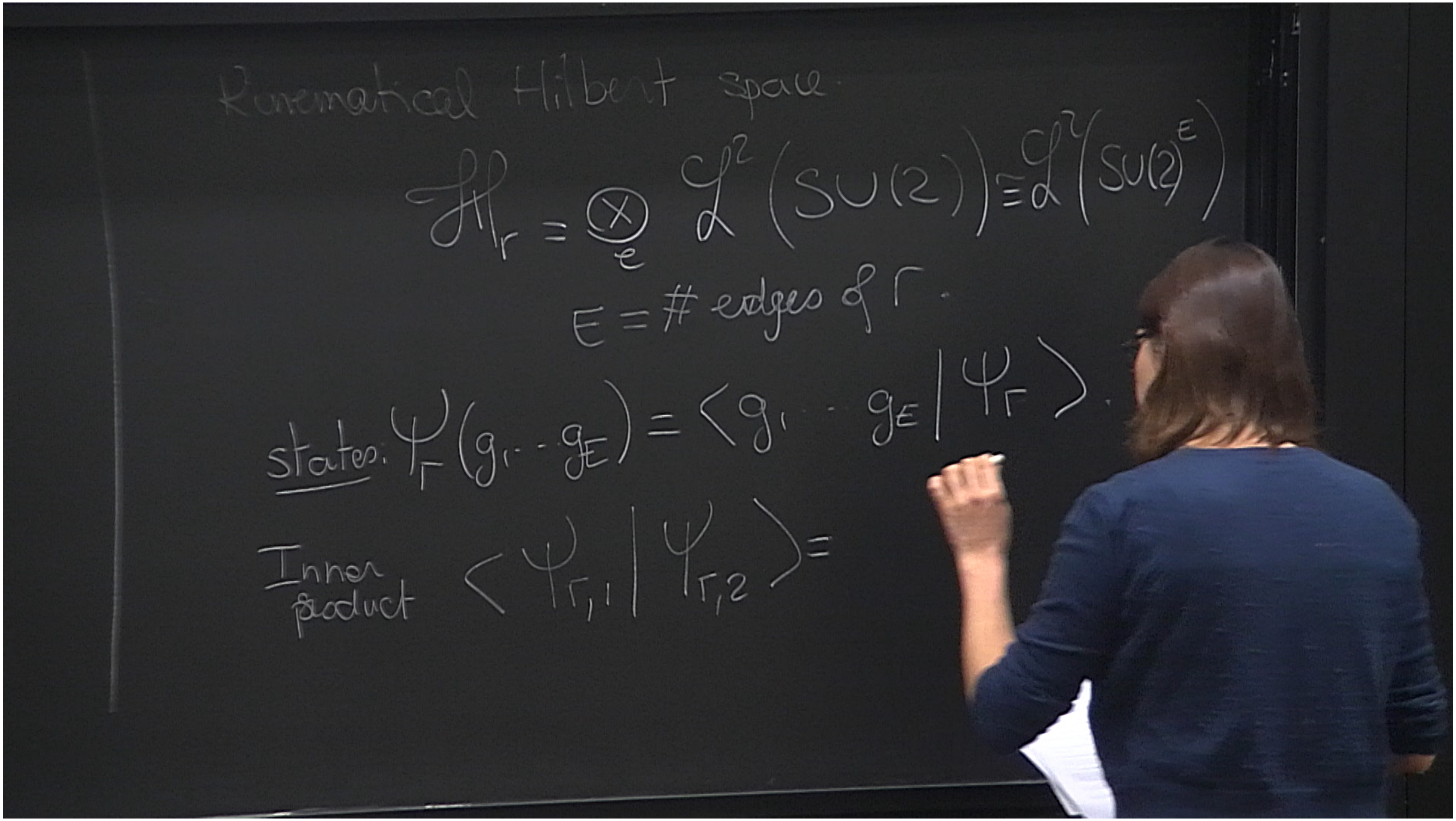
Abstract:


Kinematical Hilbert space
$\mathcal{L}^2(SU(2))$
(h_e, E_e)
$\in SU(2)$
$\psi_e : SU(2) \to \mathbb{C}$
$h_e \mapsto \psi_e(h_e)$
① variables → operators
$\cdot \{(h_e)_{MN}, (h_e)_{PQ}\} = 0 \longrightarrow [\,\cdot\,,\,\cdot\,] = 0$
$\hookrightarrow \hat{f}_e |\psi_e\rangle = |f_e \psi_e\rangle$
$\cdot \{(E_e)_j, (h_e)_{mn}\} = (L^j_e (h_e))_{mn}$
$\hookrightarrow \hat{E}^j_e = i\hbar L^j_e = i\hbar \frac{d}{dt} R_{e^{tT_j}} \Big|_{t=0}$
② Inner product
· Haar measure
$\cdot\ dg = d(hg) = d(gh)$
$\cdot \int_G \mathbb{1}\, dg = 1$
defined uniquely
$\cdot \langle \psi_2 | \psi_1 \rangle = \int_{SU(2)} \overline{\psi_2}(g)\, \psi_1(g)\, dg$
③ Orthonormal basis
· Peter-Weyl theorem
$\langle g | jmn \rangle = \sqrt{d_j}\, D^j_{mn}(g)$
Len
ℓ^2_e



2 Inner product
- Haar measure
· $dg = d(hg) = d(gh)$
· $\int_G \mathbb{1}\, dg = 1$
defined uniquely
· $\langle \psi_2 | \psi_1 \rangle = \int_{SU(2)} \overline{\psi_2}(g)\, \psi_1(g)\, dg$
3 Orthonormal basis
- Peter-Weyl theorem
$\langle g | jmn \rangle = \sqrt{d_j}\, D^j_{mn}(g)$
Length operator
$l^2_{e^*} := \sum_k E^k_e E^k_e$
C^j
$l^2_{e^*} |jmn\rangle = -\hbar^2 \sum_q |jmq\rangle \sum_p \sum_k D^j_{qp}(T^k)\, D^j_{pn}(T^k)$
$= 0 \longrightarrow C^j = -j(j+1)\,\mathbb{1}$
$= j(j+1)\,\hbar^2\, |jmn\rangle$


② Inner product
· Haar measure
· dg = d(hg) = d(gh)
· ∫_G 𝟙 dg = 1
defined uniquely
· ⟨ψ_2|ψ_1⟩ = ∫_SU(2) ψ̄_2(g) ψ_1(g) d
③ Orthonormal basis
· Peter-Weyl theorem
⟨g|jmn⟩ = √d_j D^j_mn(
Length operator
l²_e* := Σ_k E^k_e E^k_e
C^j
l²_e* |jmn⟩ = −ħ² Σ_q |jmq⟩ Σ_p Σ_k D^j_qp(T^k) D^j_pn(T^k)
[C^j, D^j(T^k)] = 0 ⟶ C^j = −j(j+1) 𝟙
l²_e* |jmn⟩ = j(j+1) ħ² |jmn⟩


Kinematical Hilbert space.
$\mathcal{H}_\Gamma = \bigotimes_e \mathcal{L}^2(SU(2)) = \mathcal{L}^2(SU(2)^E)$
$E = \#$ edges of Γ.
states: $\psi_\Gamma(g_1 \dots g_E) = \langle g_1 \dots g_E | \psi_\Gamma \rangle$.
Inner product $\langle \psi_{\Gamma,1} | \psi_{\Gamma,2} \rangle =$

Kinematical Hilbert space.

$$\mathcal{H}_\Gamma = \bigotimes_e \mathcal{L}^2(SU(2)) \equiv \mathcal{L}^2(SU(2)^E)$$

$E = \#$ edges of Γ.

<u>states</u>: $\Psi_\Gamma(g_1 \ldots g_E) = \langle g_1 \ldots g_E | \Psi_\Gamma \rangle$.

Inner product $\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle = \int_{SU(2)^E} \overline{\Psi_1(g_1 \ldots g_E)}\, \Psi_2(g_1 \ldots g_E)\, dg_1 \ldots dg_E$

- **states:** $\Psi_\Gamma(g_1 \cdots g_E) = \langle g_1 \cdots g_E | \Psi_\Gamma \rangle$.
- **Inner product** $\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle = \int_{SU(2)^E} \overline{\Psi_1(g_1 \cdots g_E)}\, \Psi_2(g_1 \cdots g_E)\, dg_1 \cdots dg_E$
- **ONB** $|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \cdots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$.

$\hookrightarrow \hat{f}_e|\Psi_e\rangle = |f_e\Psi_e\rangle$

· $\{(E_e)_j, (h_e)_{mn}\} = (L^j_e\,(h_e))_{mn}$

$\hookrightarrow \hat{E}^j_e = i\hbar L^j_e = i\hbar \frac{d}{dt} R_{e^{tT_j}}\Big|_{t=0}$

③ Orthonormal basis

· Peter-Weyl theorem

$\langle g | jmn\rangle = \sqrt{d_j}\, D^j_{mn}(g)$

④ The physical Hilbert space

The smeared Gauss constraint

$\mathcal{G} \sim$

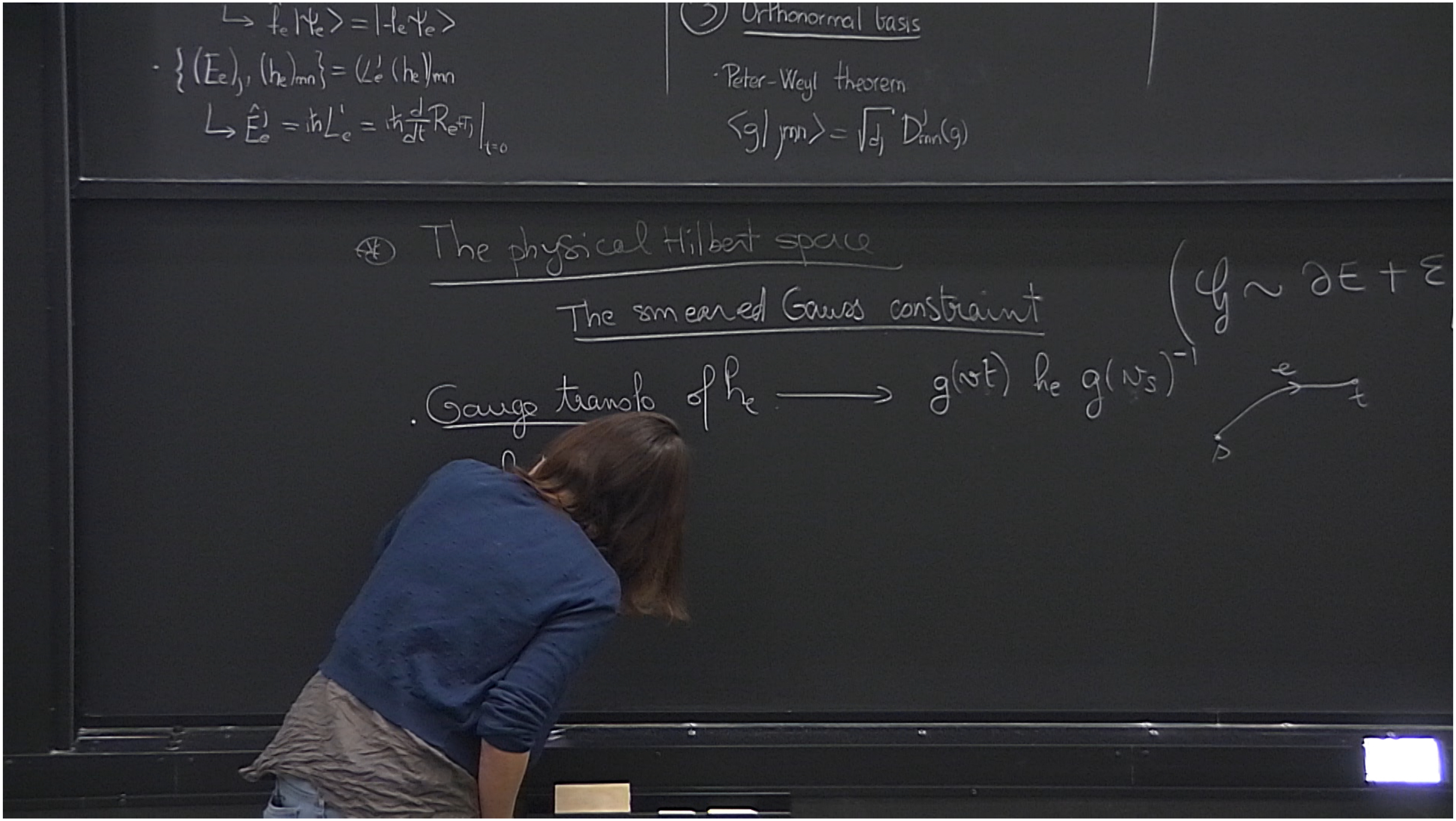
• Inner product $\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle$ $SU(2)^E$

$dg_1 \ldots$

• ONB $|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \ldots \otimes |j_{e_E} m \ldots$

$$\left(\mathcal{G} \sim \partial E + \varepsilon A E \right)$$

$\hookrightarrow$ not easy to quantize $\longrightarrow$ indirect way.



↳ $f_e|\psi_e\rangle = |f_e\psi_e\rangle$
· $\{(E_e)_j, (h_e)_{mn}\} = (L_e^j(h_e))_{mn}$
↳ $\hat{E}_e^j = i\hbar L_e^j = i\hbar \frac{d}{dt} R_{e^{t\tau_j}}\big|_{t=0}$
3 Orthonormal basis
· Peter-Weyl theorem
$\langle g | jmn\rangle = \sqrt{d_j}\, D^j_{mn}(g)$
The physical Hilbert space
The smeared Gauss constraint
· Gauge transfo of $h_e \longrightarrow g(v_t)\, h_e\, g(v_s)^{-1}$
$(\mathcal{G} \sim \partial E + \varepsilon$
e
t
s

$|\Psi_e\rangle = |f_e \Psi_e\rangle$

$\}_{mn}\} = (\mathcal{L}_e^i (h_e))_{mn}$

$= i\hbar \mathcal{L}_e^i = i\hbar \frac{d}{dt} R_{e^{tT_i}} \Big|_{t=0}$

(3) Orthonormal basis

- Peter-Weyl theorem

$\langle g | jmn \rangle = \sqrt{d_j}\, D^j_{mn}(g)$

(4) The physical Hilbert space

The smeared Gauss constraint

$\left(\mathcal{G} \sim \partial E + \varepsilon A E \right)$ ⟶ not easy t

- Gauge transfo of $h_e \longrightarrow g(v_t)\, h_e\, g(v_s)^{-1}$

$s \xrightarrow{e} t$

- $\langle h_e | \Psi_e \rangle = \Psi_e(h_e) \longrightarrow \Psi_e\left(g(v_t)\, h_e\, g(v_s)^{-1}\right)$

$= \langle h_e | L_{g^{-1}(v_t)} R_{g^{-1}(v_s)} | \Psi_e \rangle$

product $\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle = \int_{SU(2)^E} dg_1 \ldots dg_E$

- ONB $|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \ldots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$

$+ \varepsilon AE)$

$\hookrightarrow$ not easy to quantize $\longrightarrow$ indirect way.

$(N_S) | \Psi_e \rangle$

Left action: $(L_h f)(g) = f(h^{-1} g)$

Right action $(R_h f)(g) = f(g h)$.

• Inner product $\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle \ \int_{SU(2)^E} dg_1 \dots dg_E$

• ONB $|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \dots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$.

$E + \varepsilon AE)$

$\hookrightarrow$ not easy to quantize $\longrightarrow$ indirect way.

$(N_S) \, |\Psi_e\rangle$

Left action: $(L_h f)(g) = f(h^{-1} g)$.

Right action $(R_h f)(g) = f(g h)$.

$$\begin{cases} L_{h_1 h_2} = L_{h_1} \circ L_{h_2} \\ R_{h_1 h_2} = R_{h_1} \circ R_{h_2} \end{cases}$$

$\Psi_e\rangle = |\text{-}l_e \Psi_e\rangle$

$_{mn}\} = (L_e^j(h_e))_{mn}$

$i\hbar L_e^i = i\hbar \frac{d}{dt} R_{e^{tT_i}}\Big|_{t=0}$

③ Orthonormal basis

· Peter-Weyl theorem

$\langle g | jmn\rangle = \sqrt{d_j}\, D^j_{mn}(g)$

④ The physical Hilbert space

The smeared Gauss constraint

$\left(\mathcal{G} \sim \partial E + \varepsilon A E\right)$ ⟶ not easy to

· Gauge transfo of $h_e \longrightarrow g(v_t)\, h_e\, g(v_s)^{-1}$

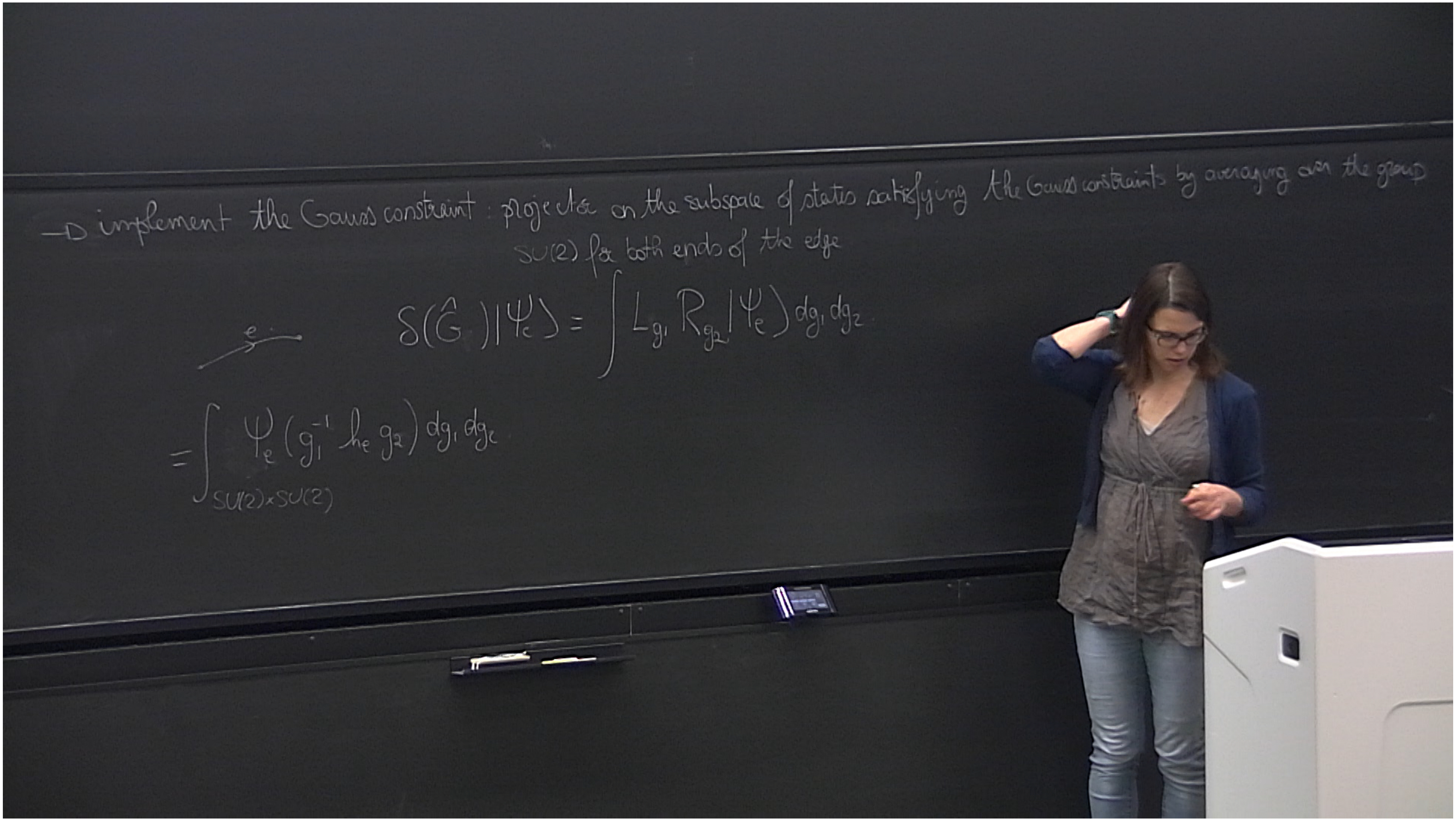
$\overset{s}{\bullet}\xrightarrow{\;e\;}\overset{t}{\bullet}$

· $\langle h_e | \Psi_e\rangle = \Psi_e(h_e) \longrightarrow \Psi_e\left(g(v_t)\, h_e\, g(v_s)^{-1}\right)$

$= \langle h_e | L_{g^{-1}(v_t)} R_{g^{-1}(v_s)} | \Psi_e\rangle$

L

R

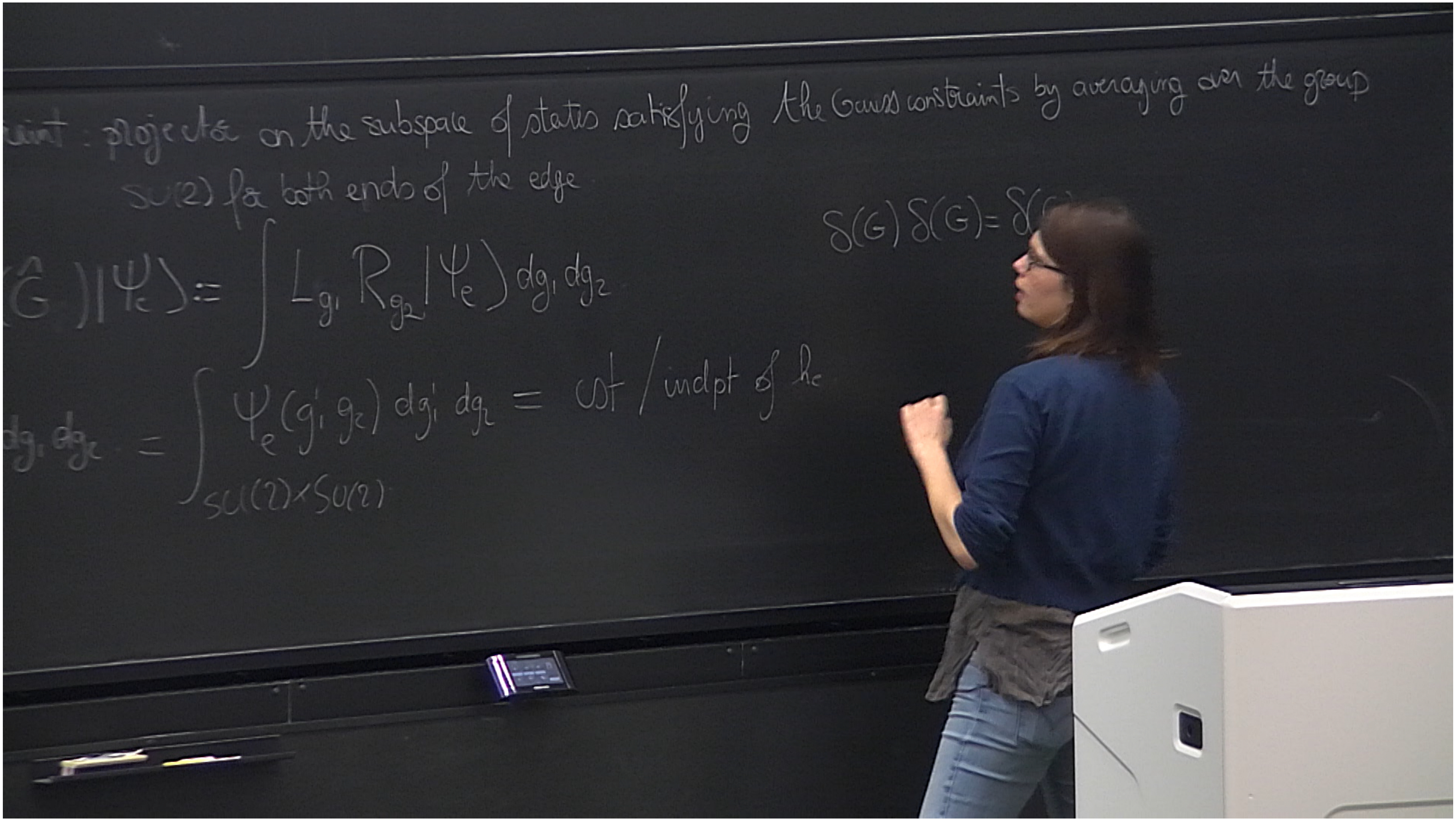

→ implement the Gauss constraint: projector on the subspace of states satisfying the Gauss constraints by averaging over the group SU(2) for both ends of the edge
e
$\delta(\hat{G})|\Psi_e\rangle = \int L_{g_1} R_{g_2} |\Psi_e\rangle \, dg_1 \, dg_2$
$= \int_{SU(2)\times SU(2)} \Psi_e(g_1^{-1} h_e g_2) \, dg_1 \, dg_2$

→ implement the Gauss constraint: project on the subspace of states satisf[ying]
SU(2) for both ends of the edge

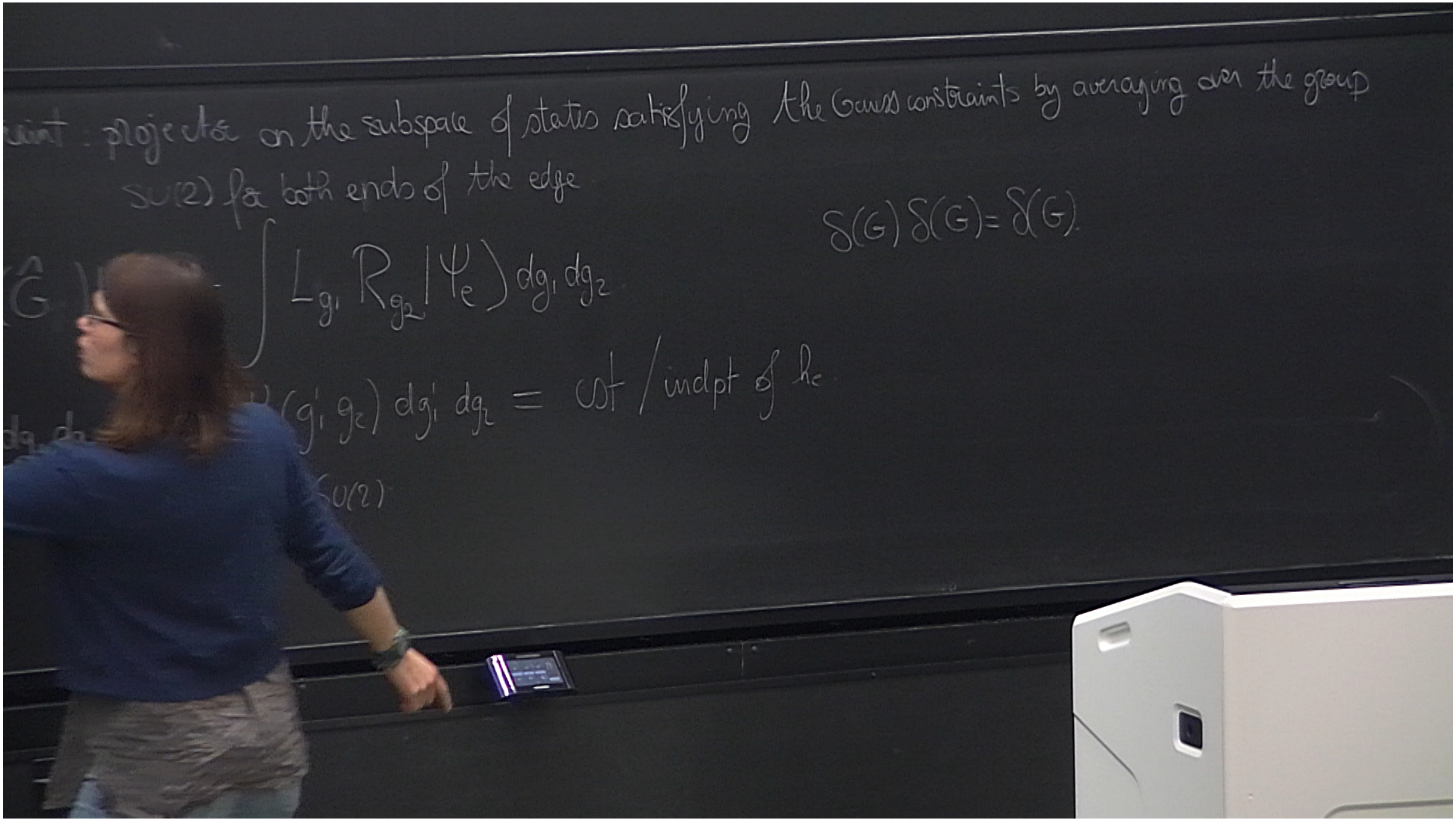
e

$$\delta(\hat{G})|\Psi_e\rangle = \int L_{g_1} R_{g_2} |\Psi_e\rangle \, dg_1 \, dg_2$$

$$= \int_{SU(2)\times SU(2)} \Psi_e(\overbrace{g_1^{-1} h_e}^{g_1'} g_2) \, dg_1 \, dg_2 = \int_{SU(2)\times SU(2)} \Psi_e(g_1' g_2) \, dg_1' \, dg_2 =$$



int : projector on the subspace of states satisfying the Gauss constraints by averaging over the group
SU(2) for both ends of the edge
δ(G) δ(G) =
(Ĝ)|Ψe⟩ := ∫ Lg1 Rg2 |Ψe⟩ dg1 dg2
dg1 dg2 = ∫ SU(2)×SU(2) Ψe(g1 g2) dg1 dg2 = cst / indpt of he

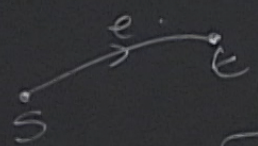

projector on the subspace of states satisfying the Gauss constraints by averaging over the group
SU(2) for both ends of the edge
δ(G) δ(G) = δ(G).
∫ L_{g1} R_{g2} |Ψ_e) dg1 dg2
(g1' g2) dg1' dg2 = cst / indpt of h_e
SU(2)

…mplement the Gauss constraint: projector on the subspace of states satisfying the Gauss constraint

SU(2) for both ends of the edge

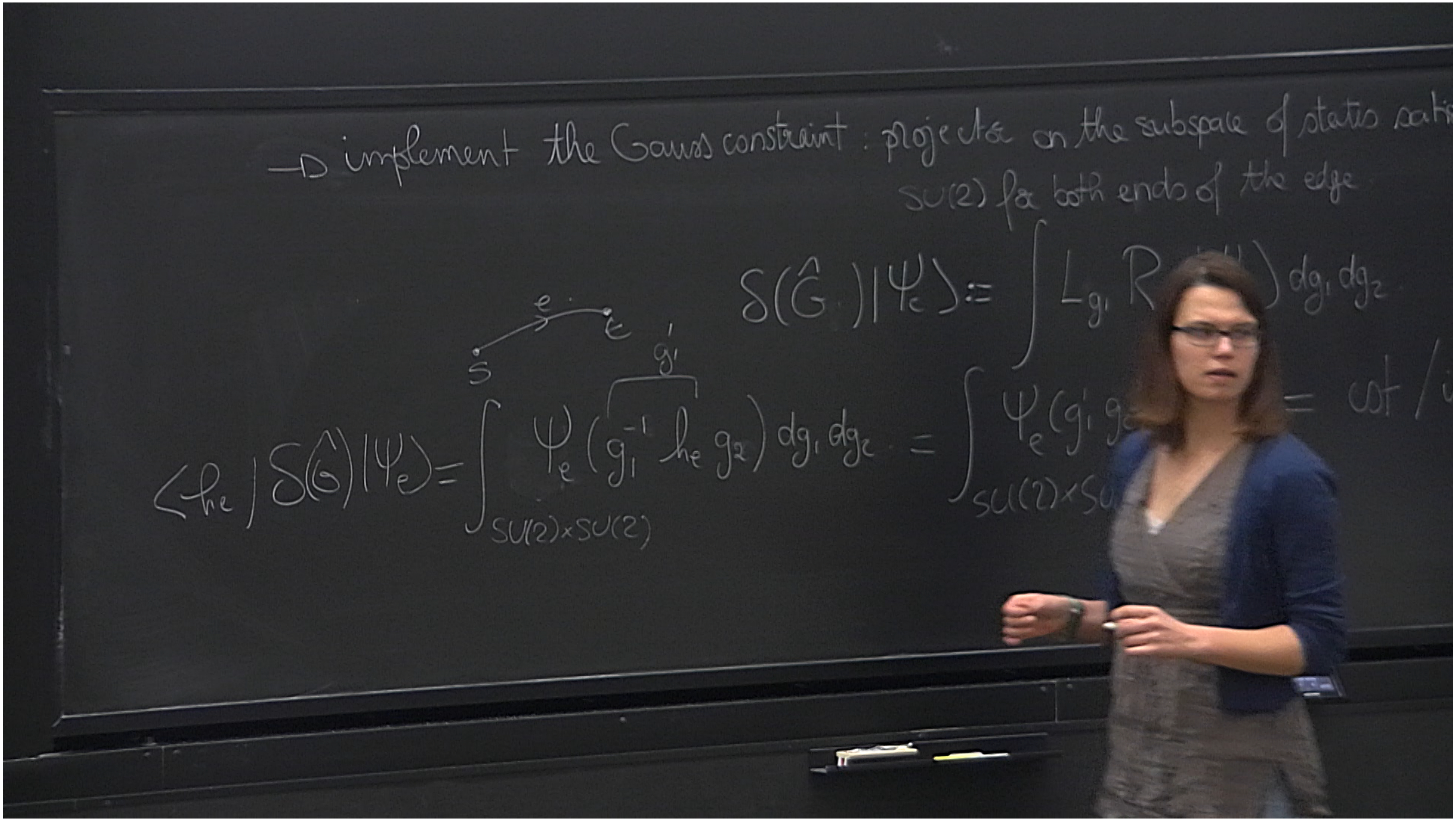
$\delta(G)\,\delta(G) =$

$$\delta(\hat{G})|\Psi_e\rangle := \int L_{g_1} R_{g_2} |\Psi_e\rangle \, dg_1 \, dg_2$$



$$= \int_{SU(2)\times SU(2)} \Psi_e\big(\overbrace{g_1^{-1} h_e}^{g_1'}\, g_2\big) \, dg_1 \, dg_2 = \int_{SU(2)\times SU(2)} \Psi_e(g_1' g_2) \, dg_1' \, dg_2 = \text{cst / indpt of } h_e$$

Orthonormal basis
· Peter-Weyl theorem
The physical Hilbert space
The smeared Gauss constraint
· Gauge transfo of h_e
· $\langle h_e | \Psi_e \rangle = \Psi_e(h_e)$
$|\Psi_e\rangle$


→ implement the Gauss constraint: projector on the subspace of states sati
SU(2) for both ends of the edge
δ(Ĝ)|Ψe⟩ := ∫ Lg1 R
dg1 dg2
e
s
t
g1'
⟨he | δ(Ĝ) | Ψe⟩ = ∫SU(2)×SU(2) Ψe(g1^-1 he g2) dg1 dg2 = ∫SU(2)×SU Ψe(g1' g
= cst /

$E + \varepsilon AE)$

↳ not easy to quantize $\longrightarrow$ indirect way.

t

Left action: $(L_h f)(g) = f(h^{-1} g)$

Right action $(R_h f)(g) = f(g h)$.

$$\begin{cases} L_{h_1 h_2} = L_{h_1} \circ L_{h_2} \\ R_{h_1 h_2} = R_{h_1} \circ R_{h_2} \end{cases}$$

$(L_h f)(g) = f(gh)$

$\tilde{f} = f(\cdot\, h)$.

$(NS)\, |\Psi_e\rangle$

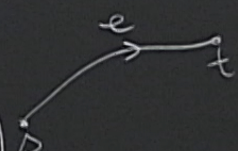
...ert space

...ed Gauss constraint

$$\left(\mathcal{G} \sim \partial E + \varepsilon AE\right)$$

$\hookrightarrow$ not easy to quantize $\longrightarrow$ indire...

$$h_e \longrightarrow g(v_t)\, h_e\, g(v_s)^{-1}$$



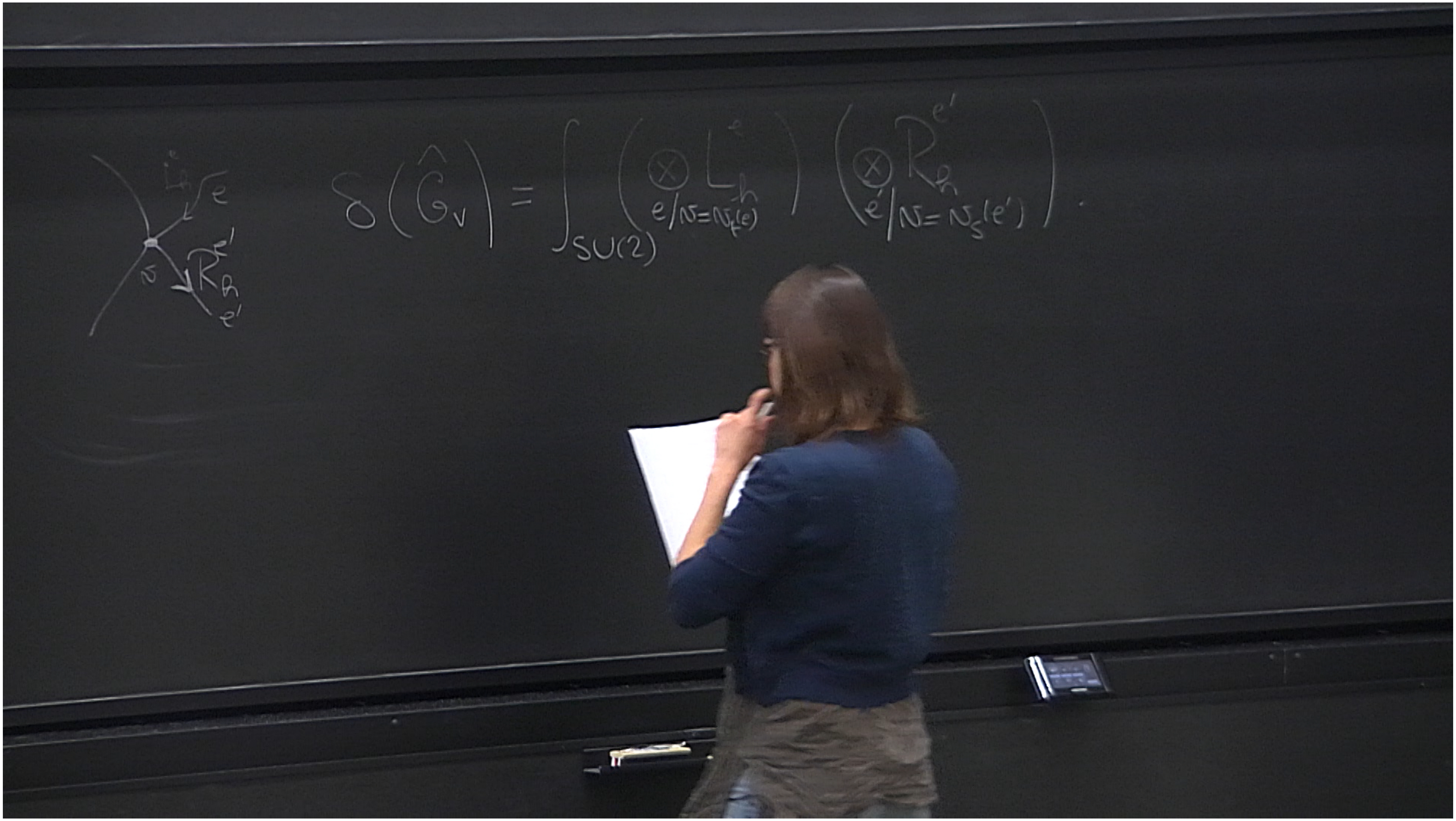
$$\Psi(h_e) \longrightarrow \Psi_e\left(g(v_t)\, h_e\, g(v_s)^{-1}\right)$$

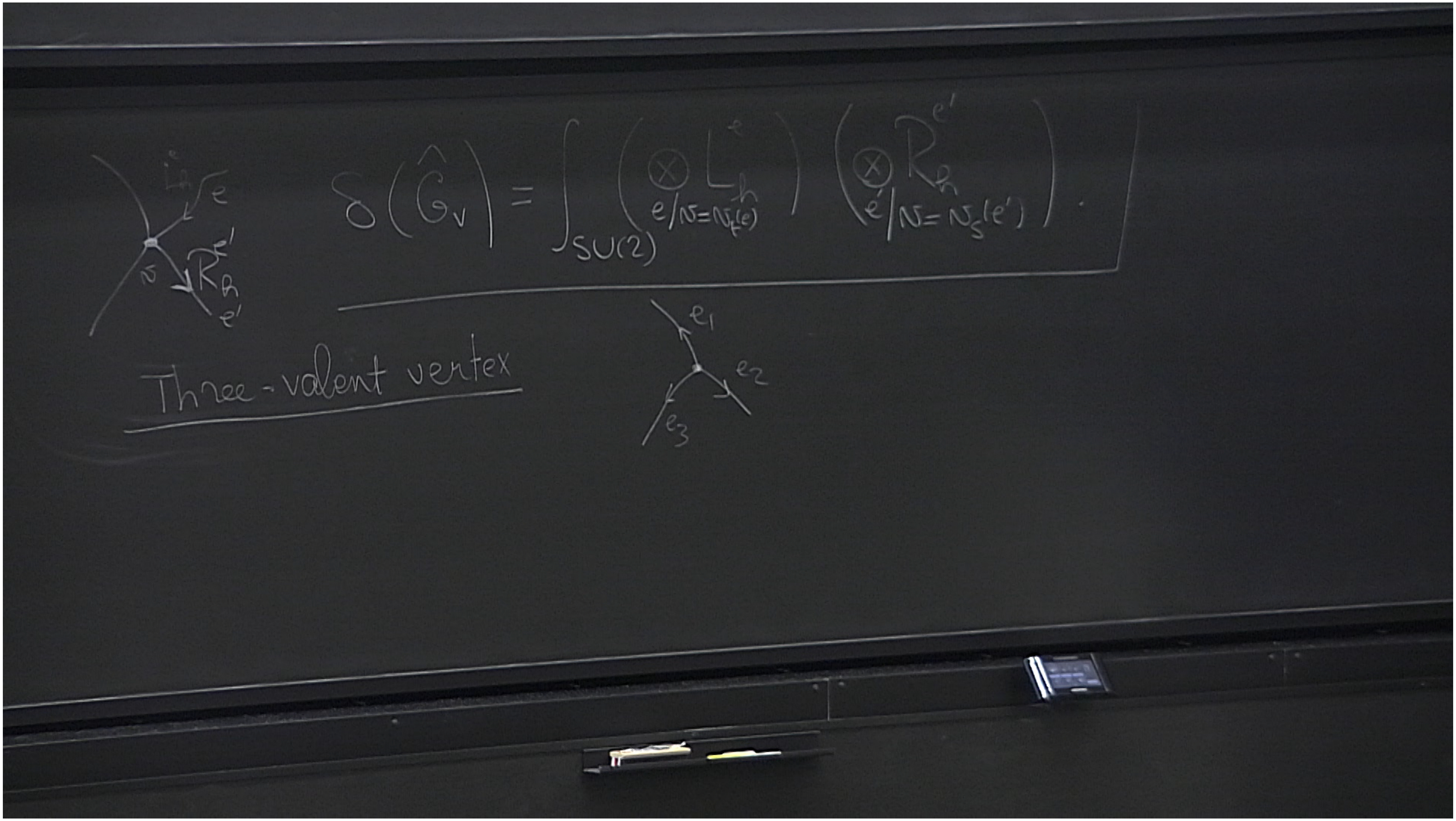
$$= \langle h_e | \mathbb{L}_{g^{-1}(v_t)} \underbrace{\mathbb{R}_{g^{-1}(v_s)} | \Psi_e \rangle}_{|\Psi_e\rangle}$$

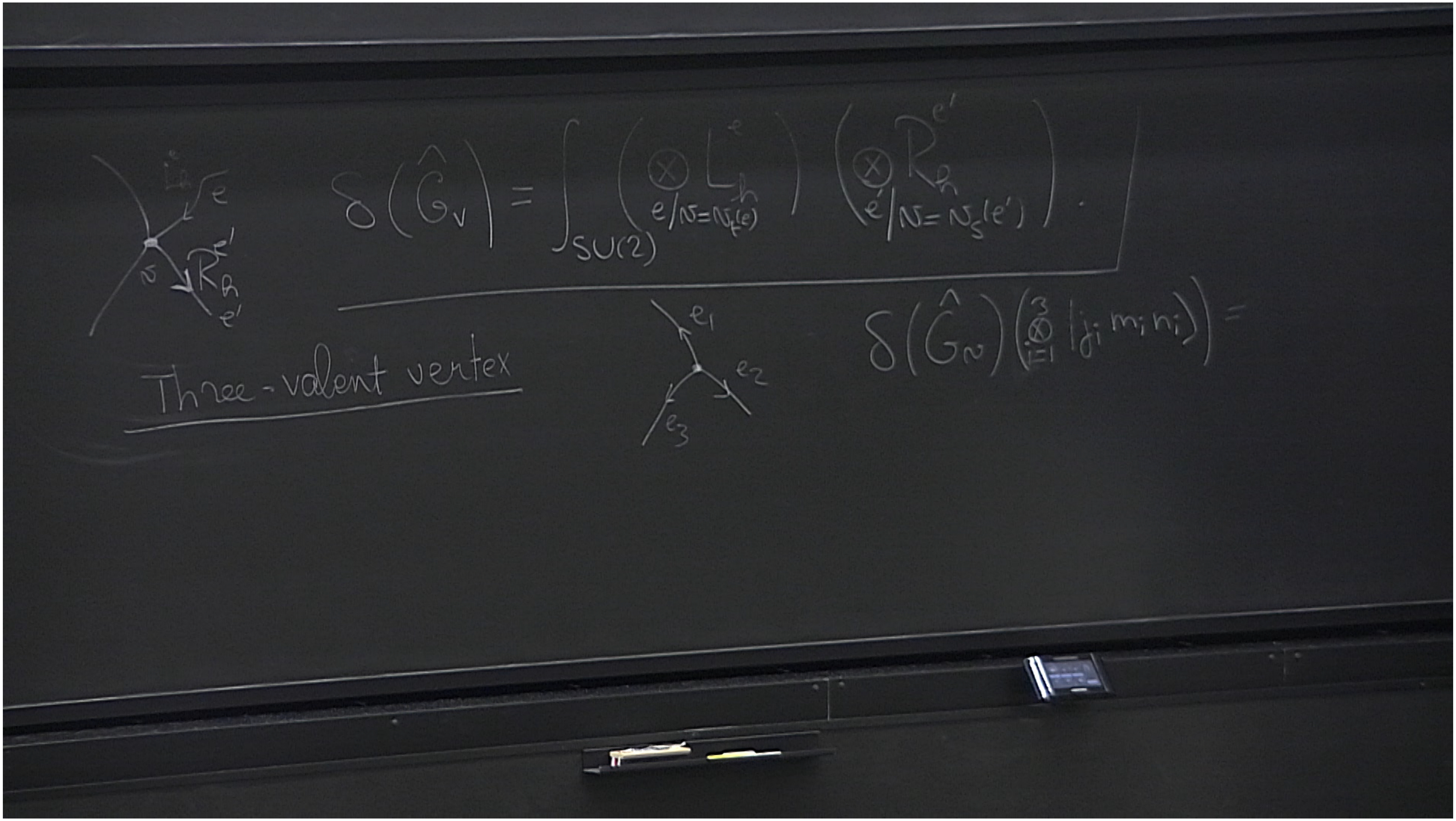
Left action: $(L_h f)(g) = \ldots$

Right action $(R_h f)(g) = \ldots$

$$\begin{cases} L_{h_1 h_2} = L_{h_1} \circ L_{h_2} \\ R_{h_1 h_2} = R_{h_1} \circ R_{h_2} \end{cases}$$


$\delta(\hat{G}_v) = \int_{SU(2)} \left(\bigotimes_{e / v = v_t(e)} L^e_h \right) \left(\bigotimes_{e' / v = v_s(e')} R^{e'}_h \right) .$
$R^{e'}_h$


$\delta(\hat{G}_v) = \int_{SU(2)} \left(\bigotimes_{e/v=v_t(e)} L^e_h \right) \left(\bigotimes_{e'/v=v_s(e')} R^{e'}_h \right).$
Three-valent vertex
e_1
e_2
e_3


$R_h^{e'}$
e
e'
v
$\delta(\hat{G}_v) = \int_{SU(2)} \left(\bigotimes_{e/v=v_t(e)} L_h^{e} \right) \left(\bigotimes_{e'/v=v_s(e')} R_h^{e'} \right) .$
Three-valent vertex
e_1
e_2
e_3
$\delta(\hat{G}_v)\left(\bigotimes_{i=1}^{3} |j_i\, m_i\, n_i\rangle \right) =$

$$\left(\bigotimes_{e'} R_h^{e'}\right)_{/ N = N_s(e')} .$$

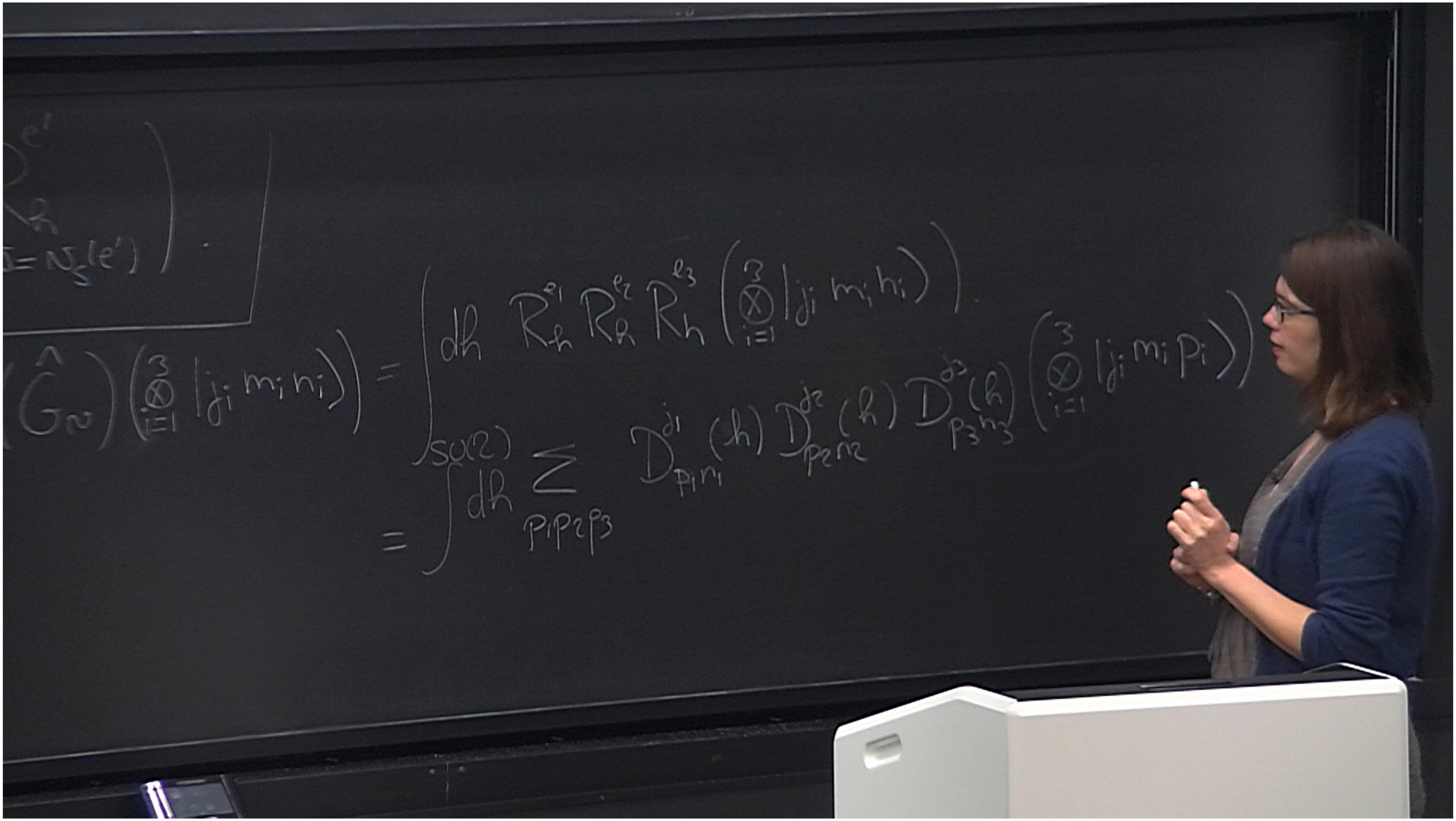
$$\delta(\hat{G}_N)\left(\bigotimes_{i=1}^{3} |j_i\, m_i\, n_i\rangle\right) = \int_{SU(2)} dh\; R_h^{e_1} R_h^{e_2} R_h^{e_3} \left(\bigotimes_{i=1}^{3} |j_i\, m_i\, h_i\rangle\right).$$

$$=$$

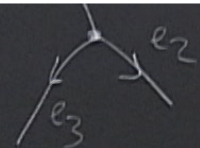
$$\left(\bigotimes_{e'/N=N_s(e')} R_h^{e'}\right).$$

$$\delta(\hat{G}_N)\left(\bigotimes_{i=1}^{3} |j_i\, m_i\, n_i\rangle\right) = \int_{SU(2)} dh\, R_h^{e_1} R_h^{e_2} R_h^{e_3}\left(\bigotimes_{i=1}^{3} |j_i\, m_i\, n_i\rangle\right).$$

$$= \int dh \sum_{p_1 p_2 p_3} D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h)\left(\bigotimes_{i=1}^{3} |j_i \right.$$


$= N_s(e')$
$(\hat{G}_N)\left(\bigotimes_{i=1}^{3} |j_i\, m_i\, n_i\rangle\right) = \int_{SU(2)} dh\, R_h^{e_1} R_h^{e_2} R_h^{e_3} \left(\bigotimes_{i=1}^{3} |j_i\, m_i\, h_i\rangle\right).$
$= \int dh \sum_{p_1 p_2 p_3} D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h) \left(\bigotimes_{i=1}^{3} |j_i\, m_i\, p_i\rangle\right)$

valent vertex



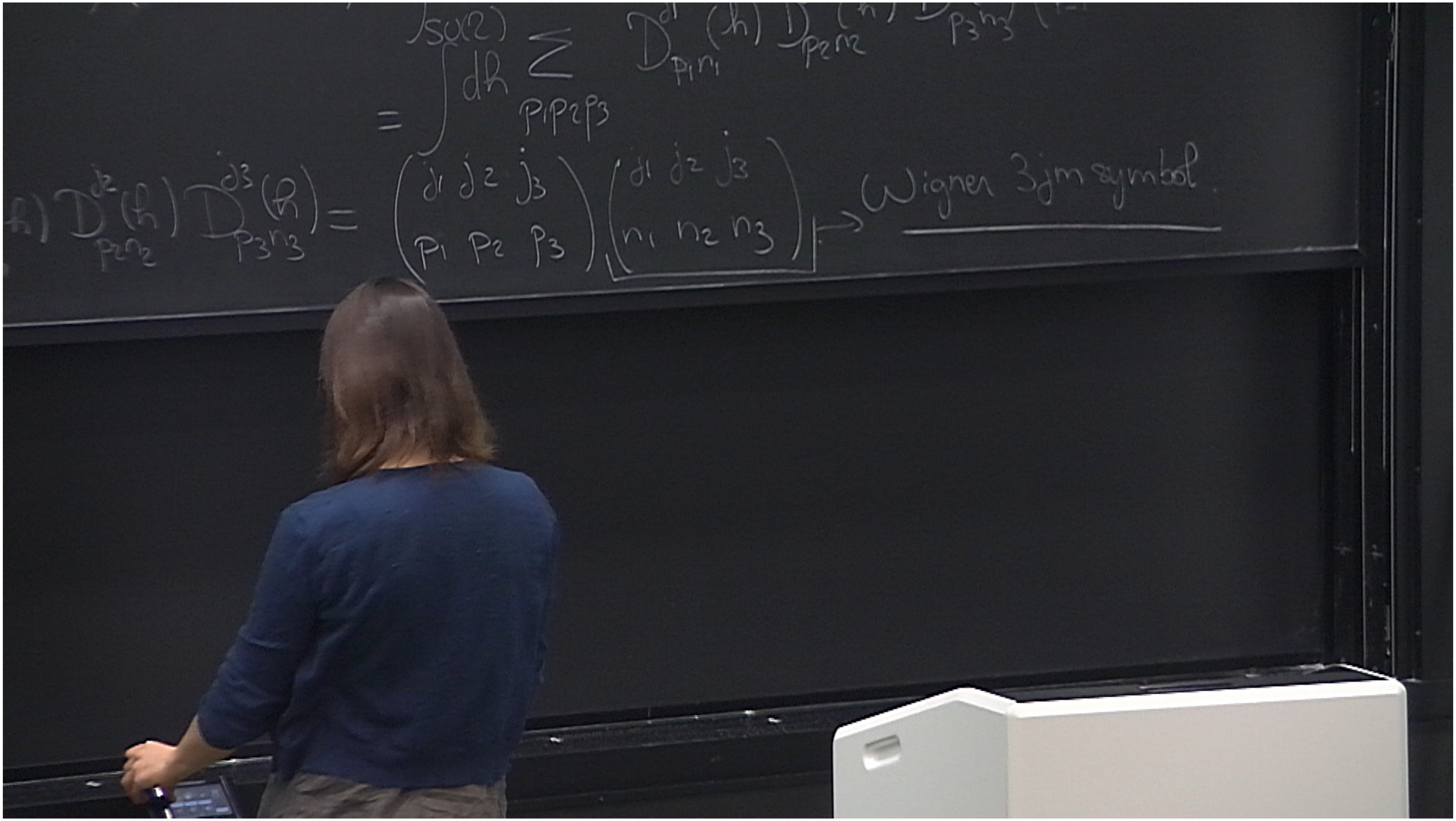
$\delta(G_N)(\ldots)$

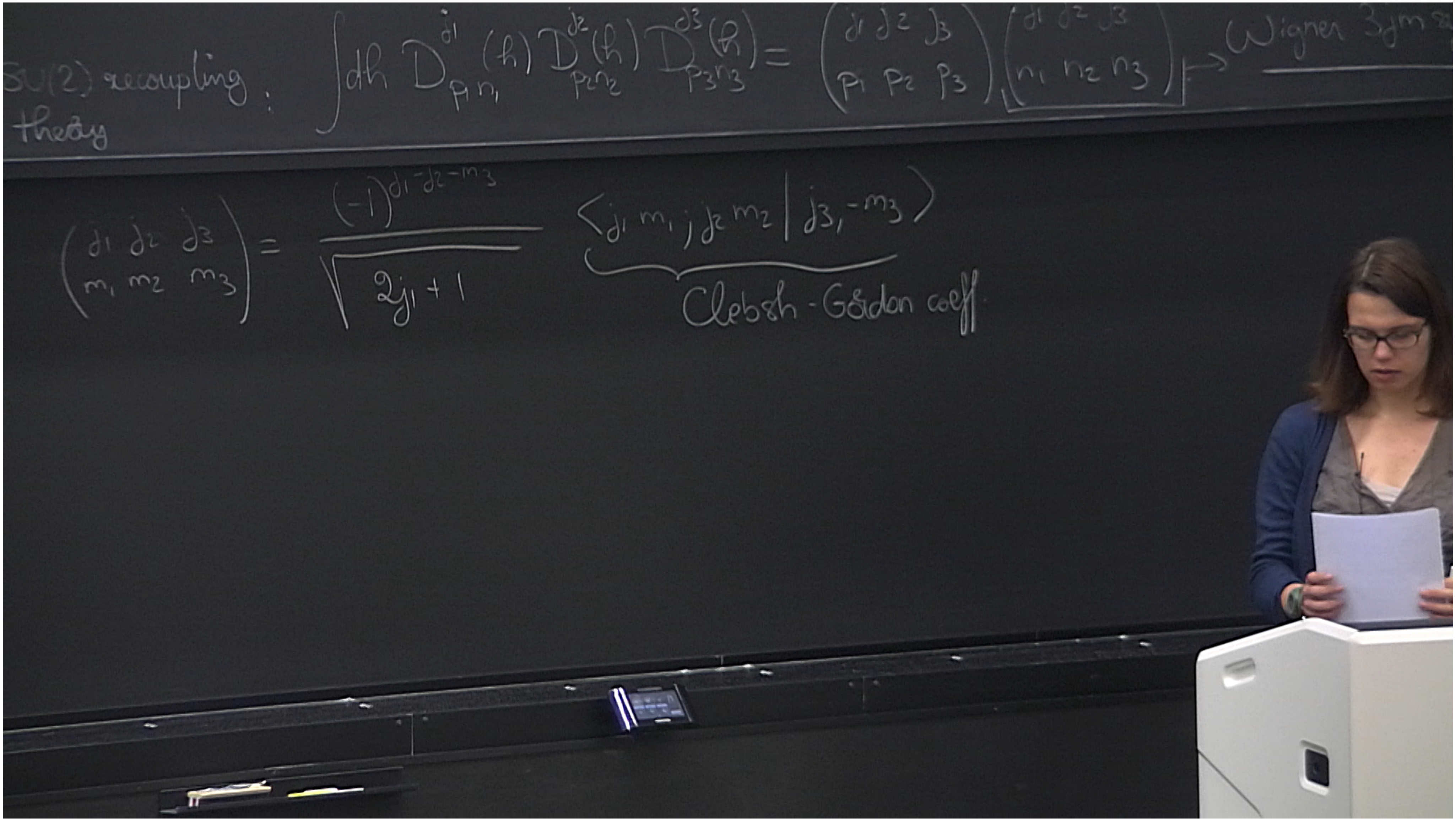
$\int_{SU(2)} \ldots D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h) \ldots$

$= \int dh \sum_{p_1 p_2 p_3}$

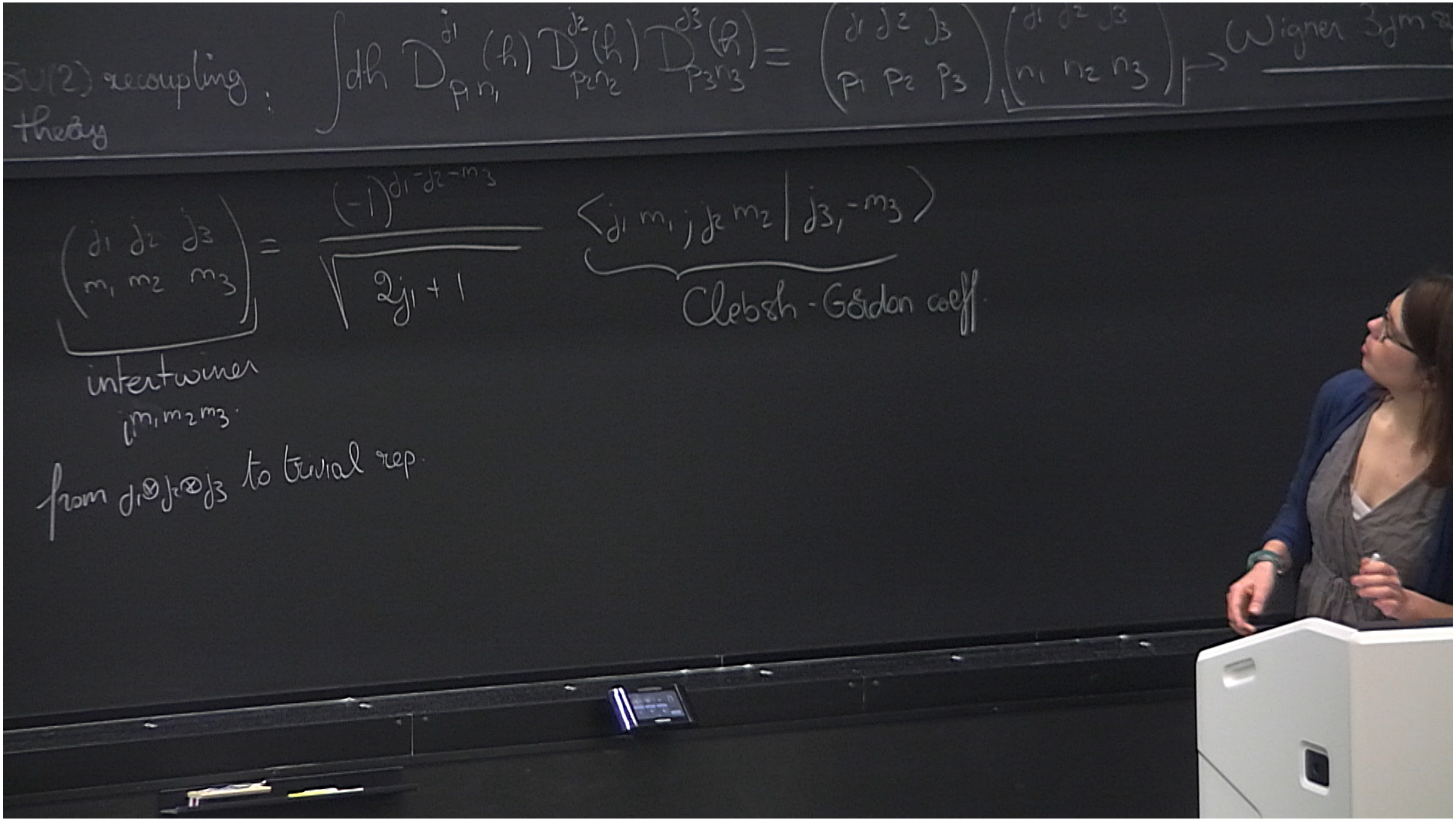
From SU(2) recoupling theory:

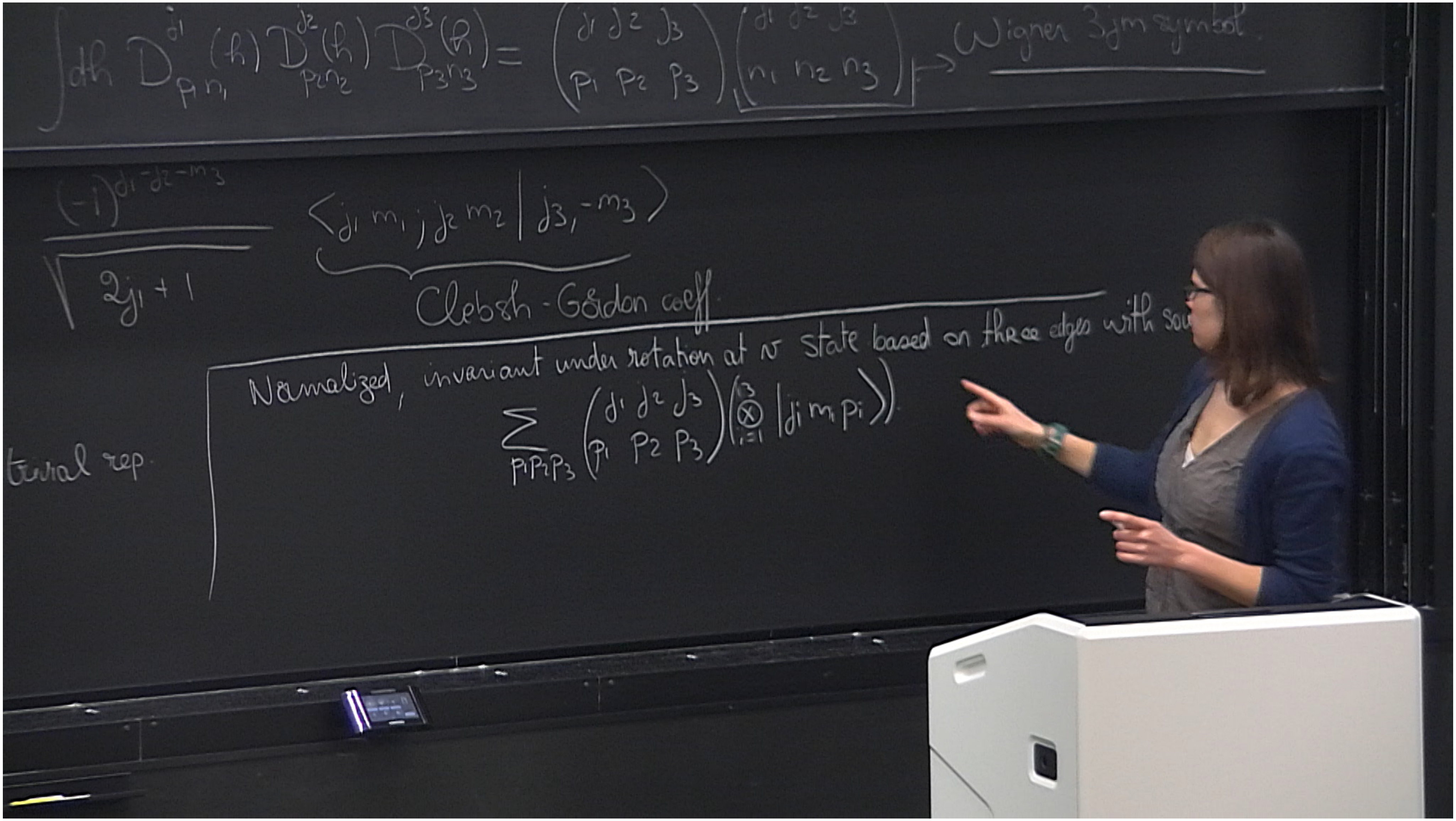
$$\int dh\, D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}.$$

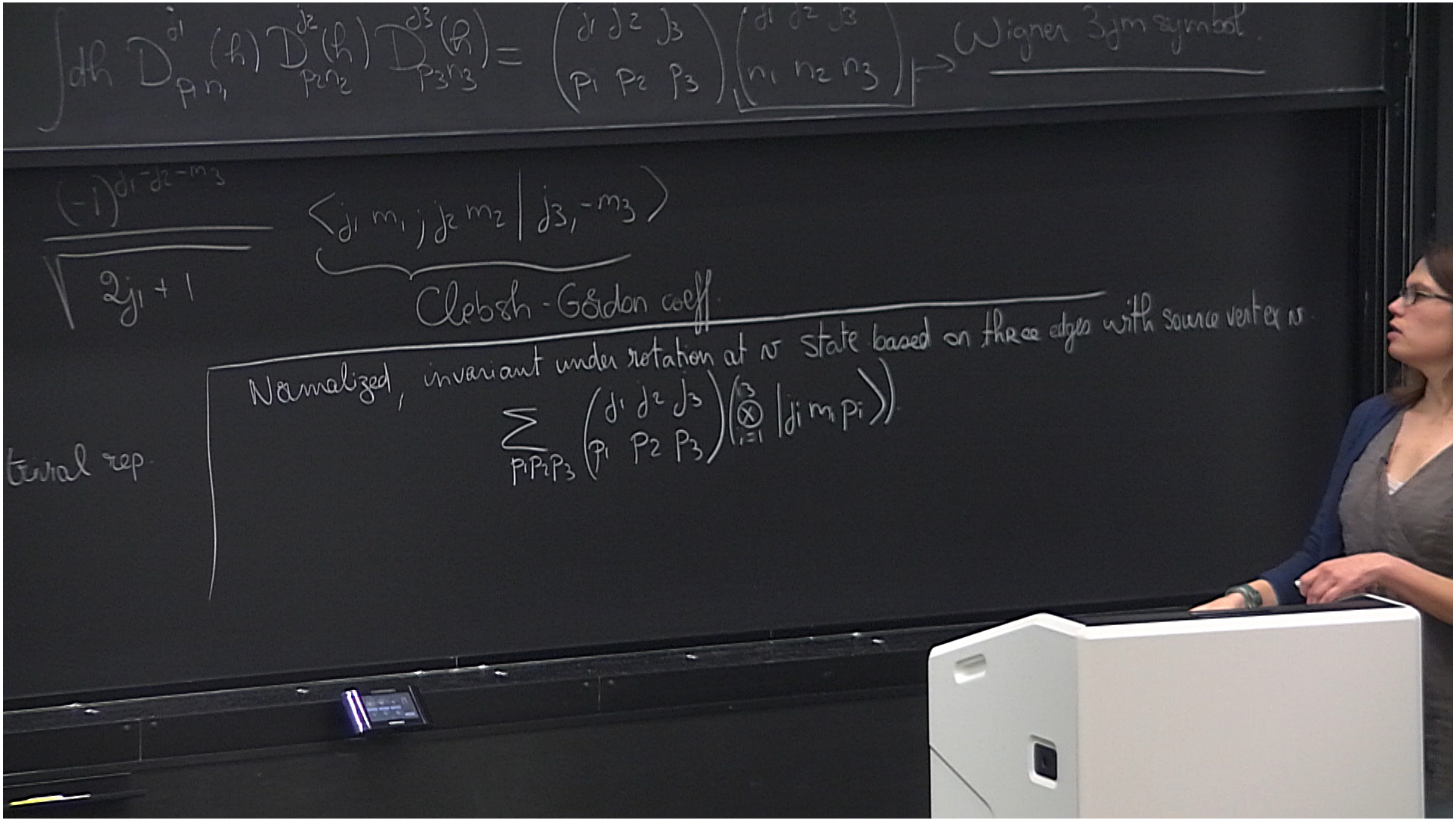

Wigner 3jm symbol

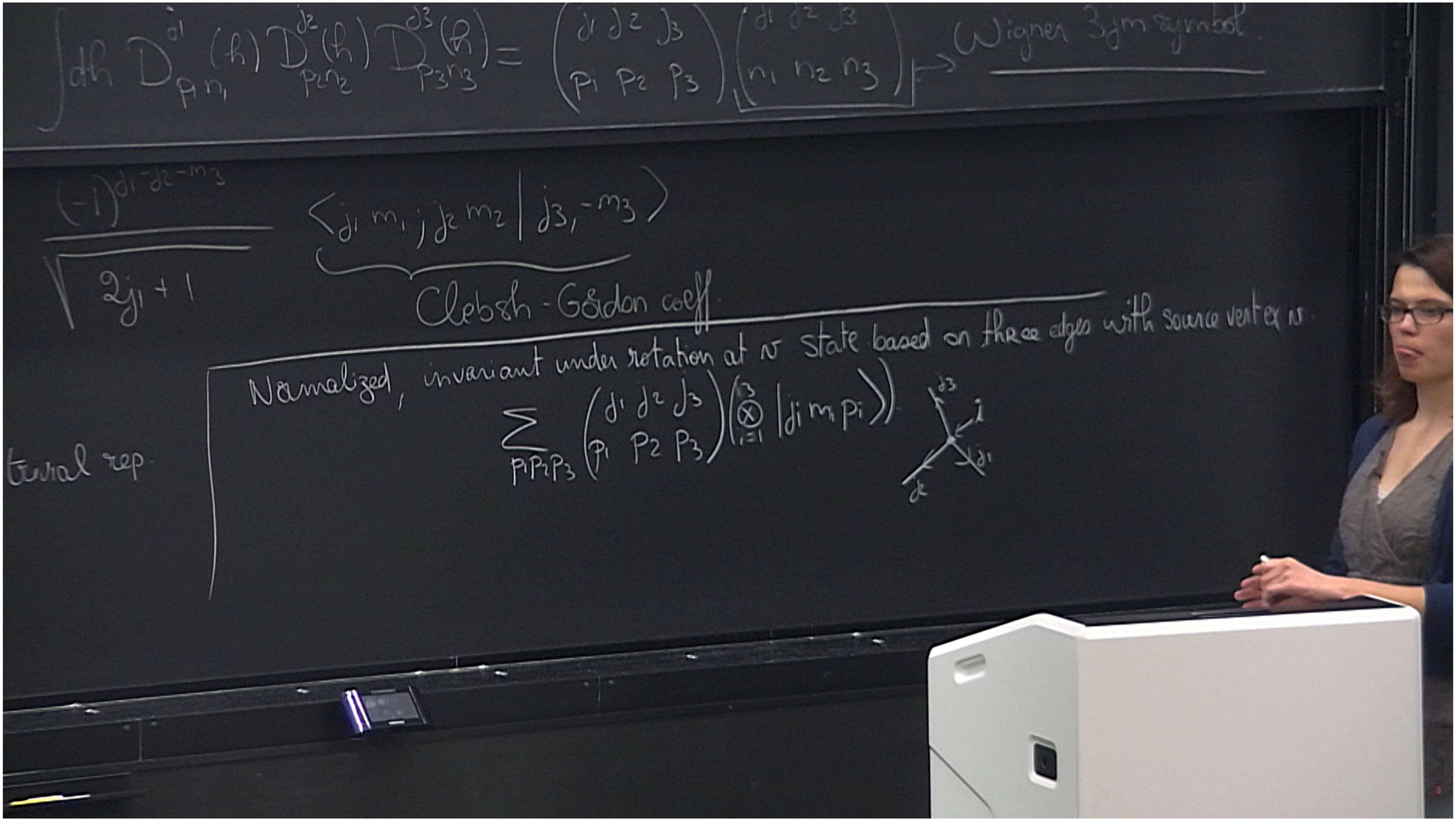


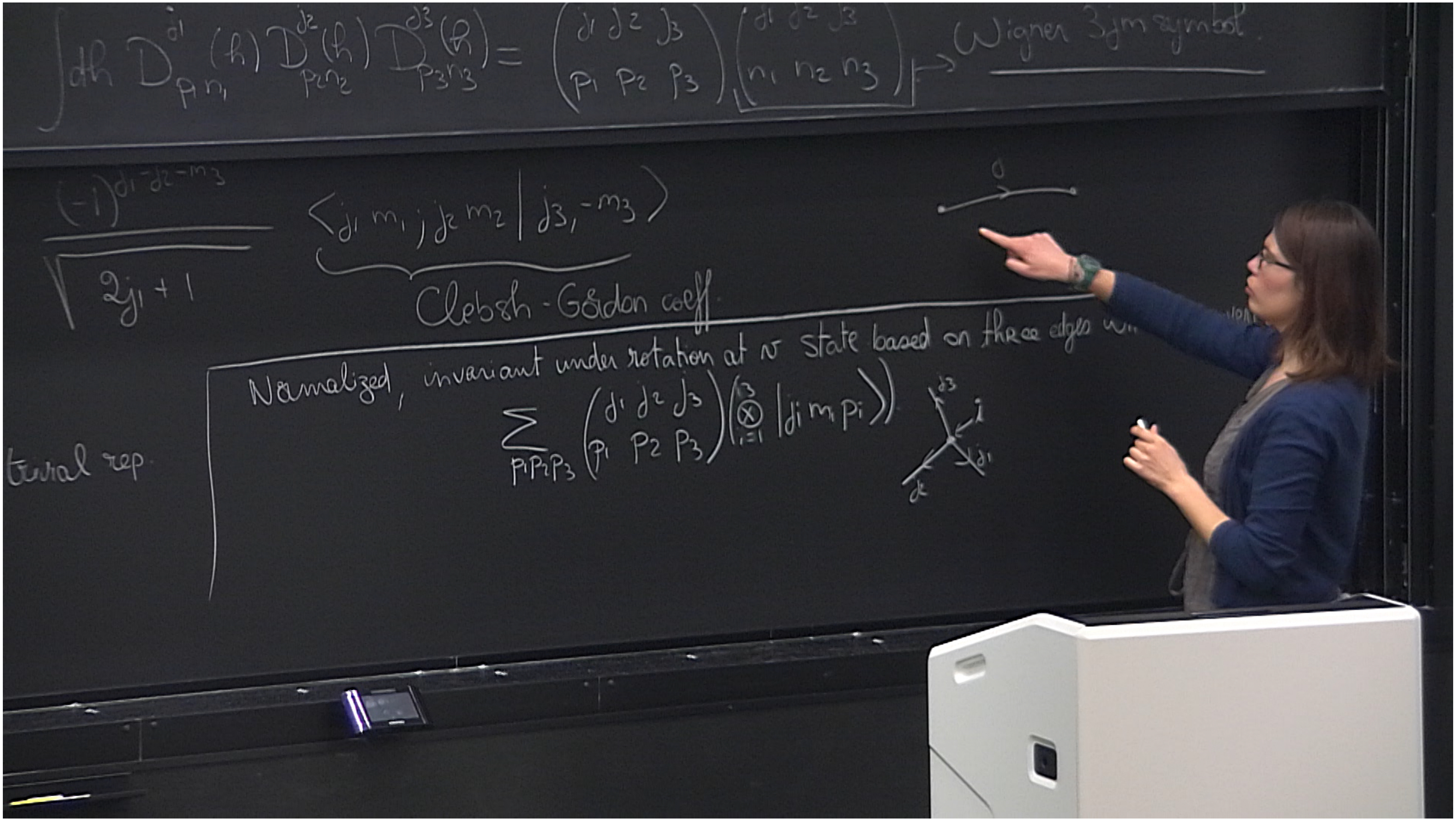
SU(2) recoupling theory:
$\int dh \, D^{j_1}_{p_1 n_1}(h) D^{j_2}_{p_2 n_2}(h) D^{j_3}_{p_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$
Wigner 3jm
$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_1+1}} \langle j_1 m_1 ; j_2 m_2 | j_3, -m_3 \rangle$
Clebsch-Gordan coeff.


SU(2) recoupling theory:
$\int dh\, D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$
→ Wigner 3jm sy
$\underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}}_{\text{intertwiner } i^{m_1 m_2 m_3}} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_1 + 1}} \underbrace{\langle j_1 m_1, j_2 m_2 \mid j_3, -m_3 \rangle}_{\text{Clebsch-Gordon coeff.}}$
from $j_1 \otimes j_2 \otimes j_3$ to trivial rep.

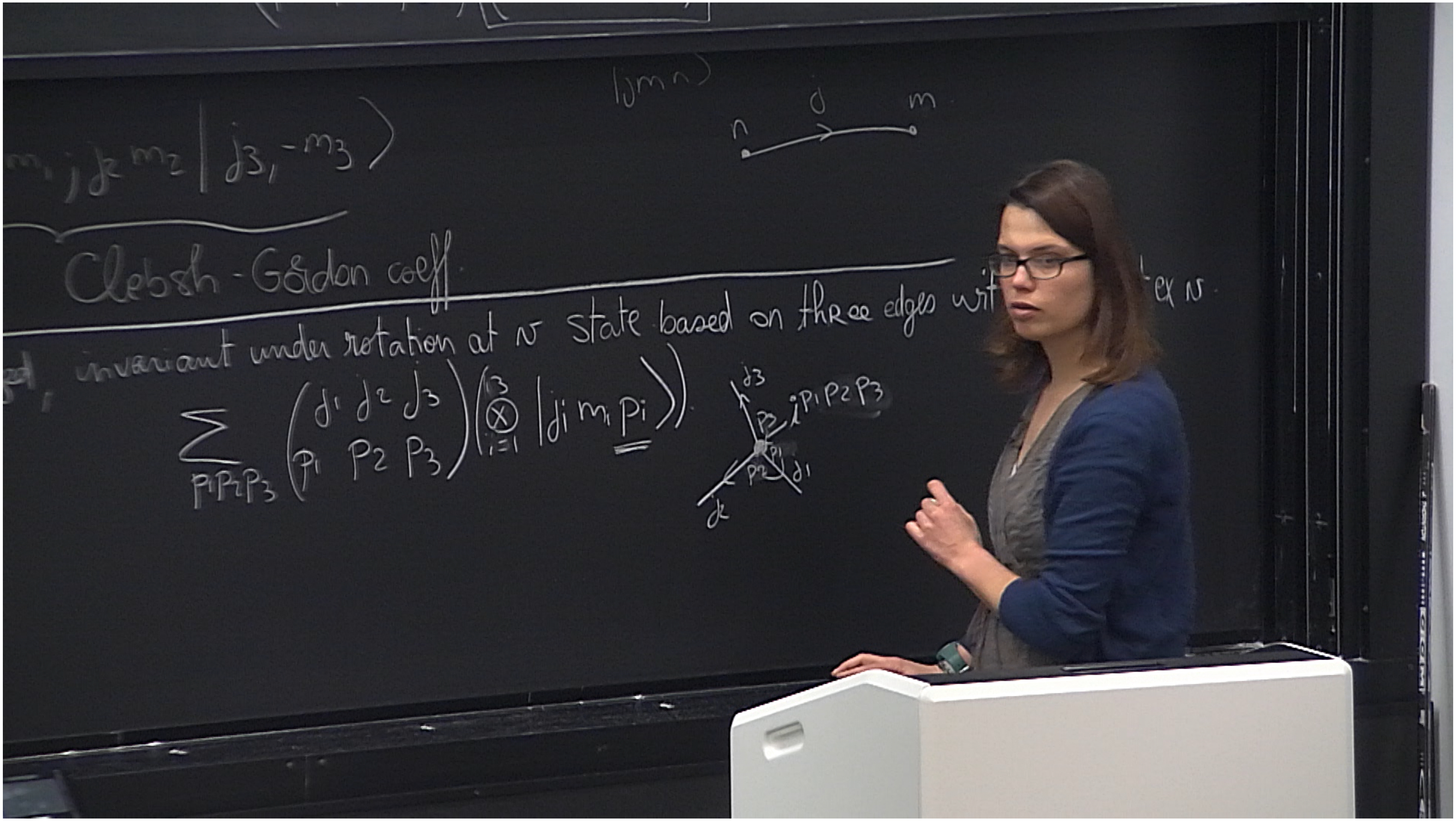

$\int dh\, D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$
Wigner 3jm symbol.
$\frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_1+1}} \underbrace{\langle j_1 m_1, j_2 m_2 | j_3, -m_3 \rangle}_{\text{Clebsch-Gordon coeff.}}$
Normalized, invariant under rotation at v state based on three edges with
$\sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1}^{3} |j_i m_i p_i\rangle \right)$.
trivial rep.


$\int dh\, D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$
Wigner 3jm symbol
$\frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_1+1}} \underbrace{\langle j_1 m_1 ; j_2 m_2 | j_3, -m_3 \rangle}$
Clebsch - Gordon coeff.
Normalized, invariant under rotation at v state based on three edges with source vertex v.
$\sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1}^{3} |j_i m_i p_i\rangle \right)$
trivial rep.

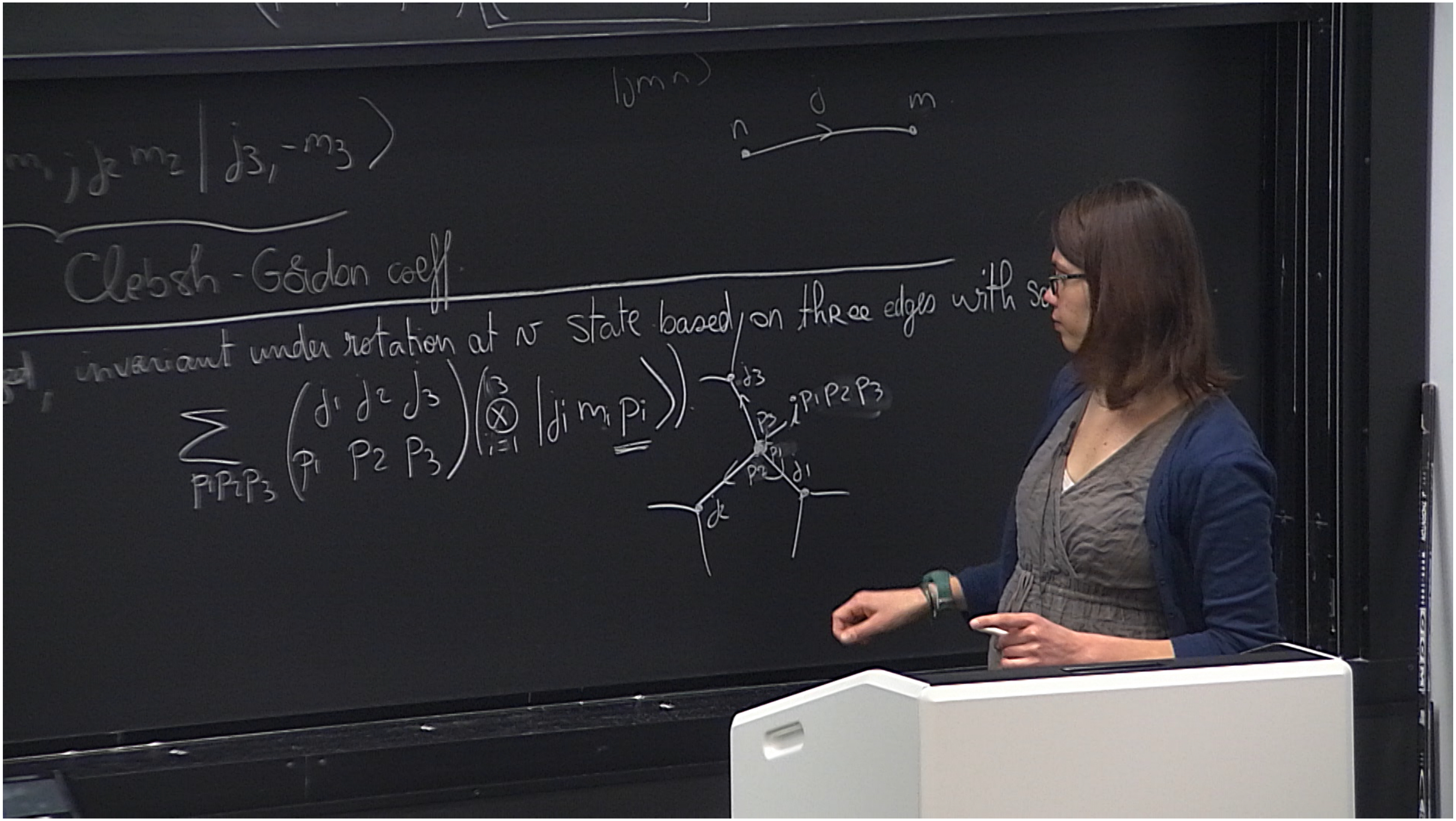

$\int dh\, D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$
Wigner 3jm symbol.
$\frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_1+1}} \underbrace{\langle j_1 m_1 ; j_2 m_2 \mid j_3, -m_3 \rangle}_{\text{Clebsch-Gordon coeff.}}$
trivial rep.
Normalized, invariant under rotation at v state based on three edges with source vertex v.
$\sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1}^{3} |j_i m_i p_i\rangle \right)$.
j_3
j_1
j_2



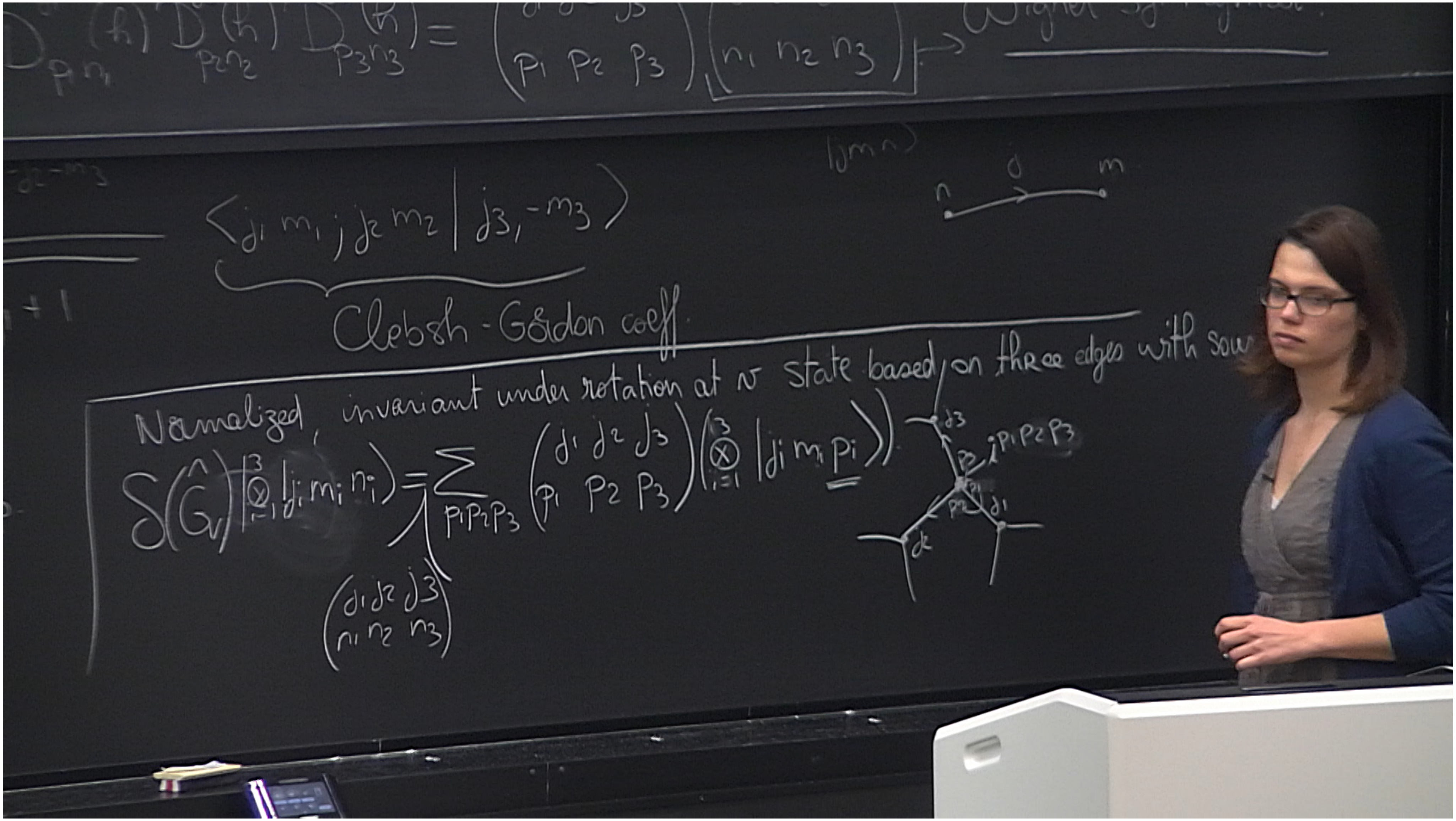
$\int dh\, D^{j_1}_{p_1 n_1}(h)\, D^{j_2}_{p_2 n_2}(h)\, D^{j_3}_{p_3 n_3}(h) = \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$
→ Wigner 3jm symbol.
$\frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_1+1}} \underbrace{\langle j_1 m_1 ; j_2 m_2 \mid j_3, -m_3 \rangle}_{\text{Clebsch-Gordon coeff.}}$
trivial rep.
Normalized, invariant under rotation at v state based on three edges wi...
$\sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1}^{3} |j_i m_i p_i\rangle \right)$
j
j_3
j_2
j_1

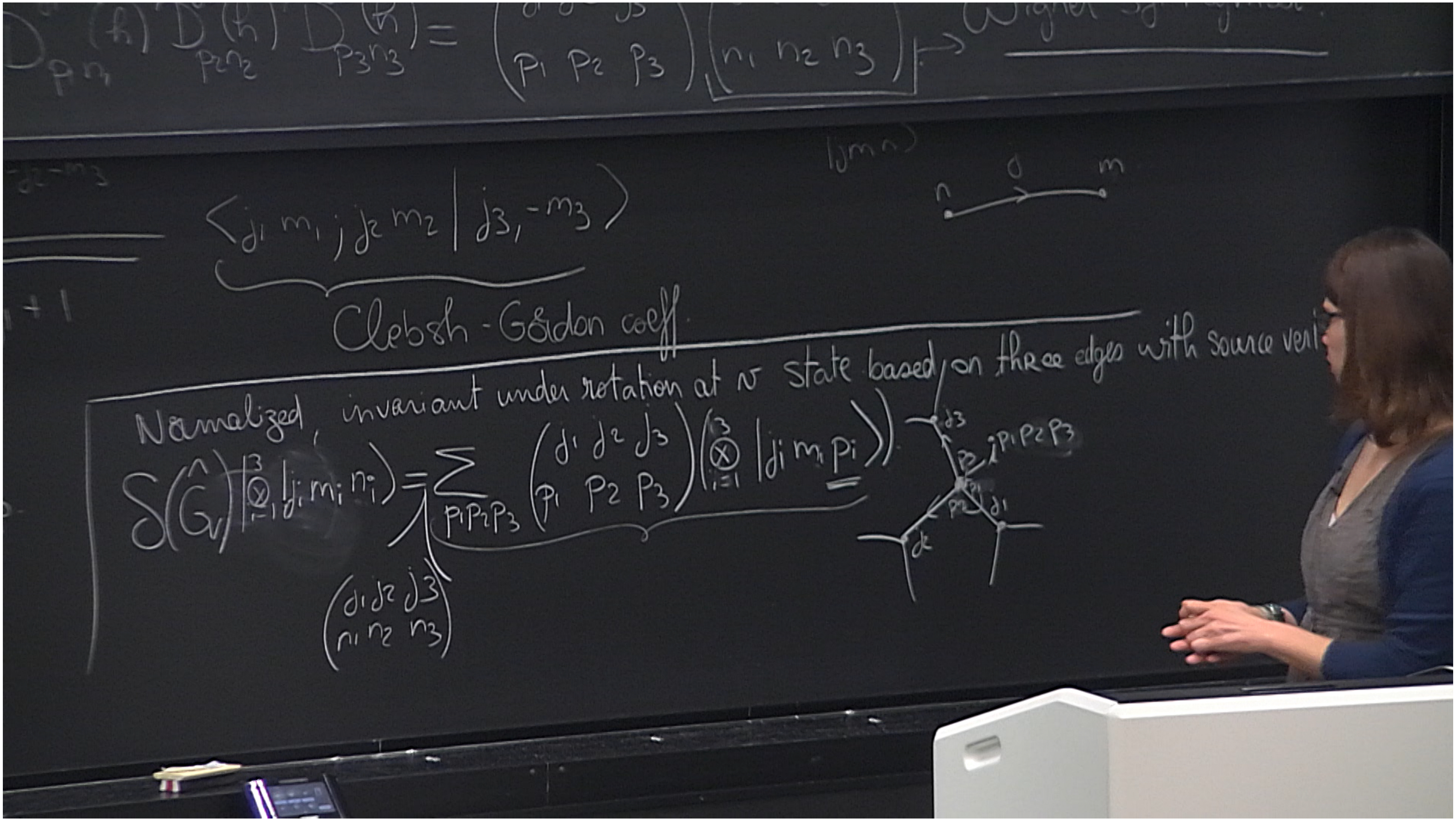


$|jmn\rangle$
$n \quad j \quad m$
$m_1, j_2 m_2 | j_3, -m_3 \rangle$
Clebsh-Gordon coeff.
invariant under rotation at v state based on three edges
ex v.
$\sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1}^{3} |j_i m_i p_i\rangle \right)$
$i_{p_1 p_2 p_3}$

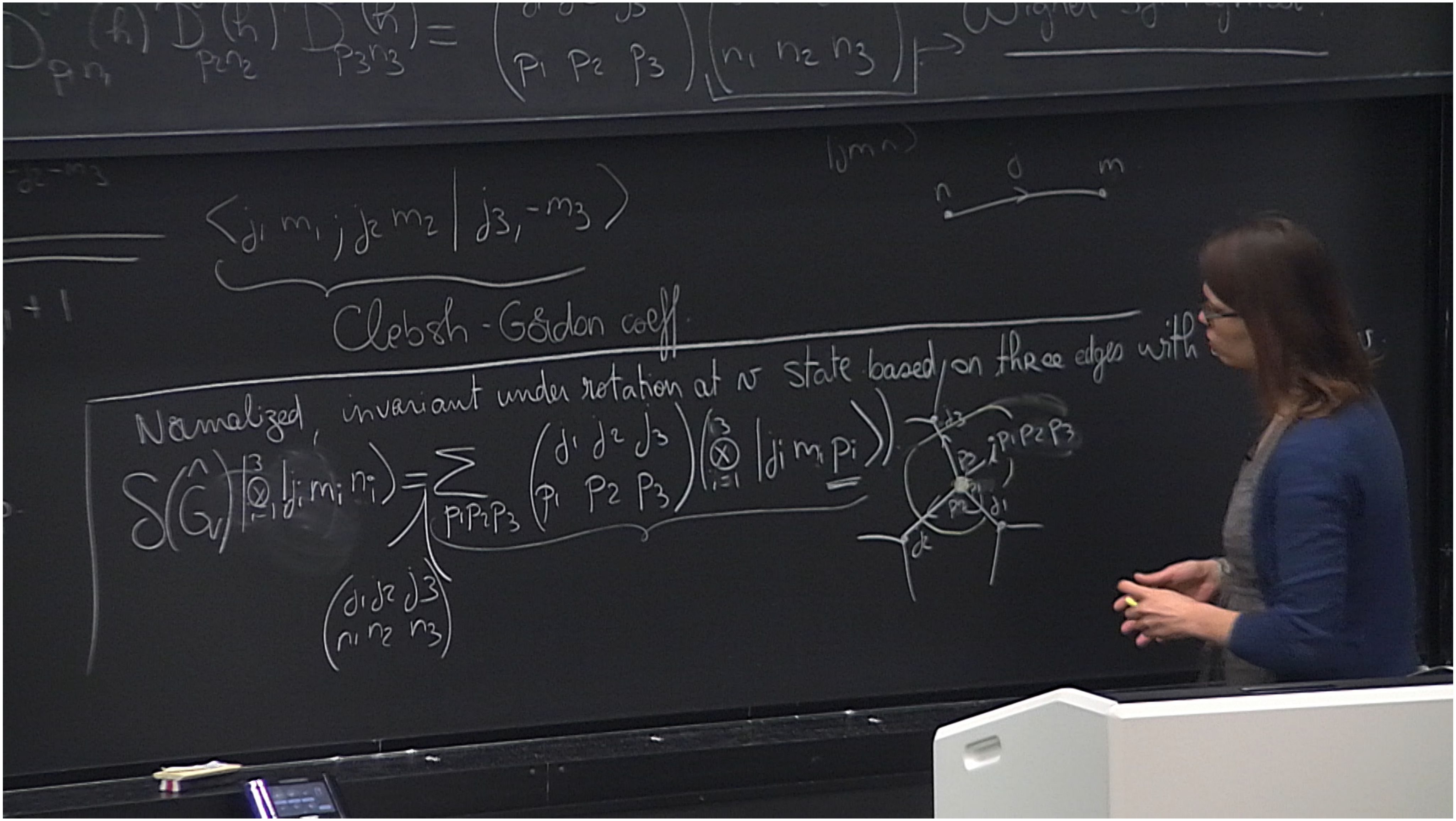


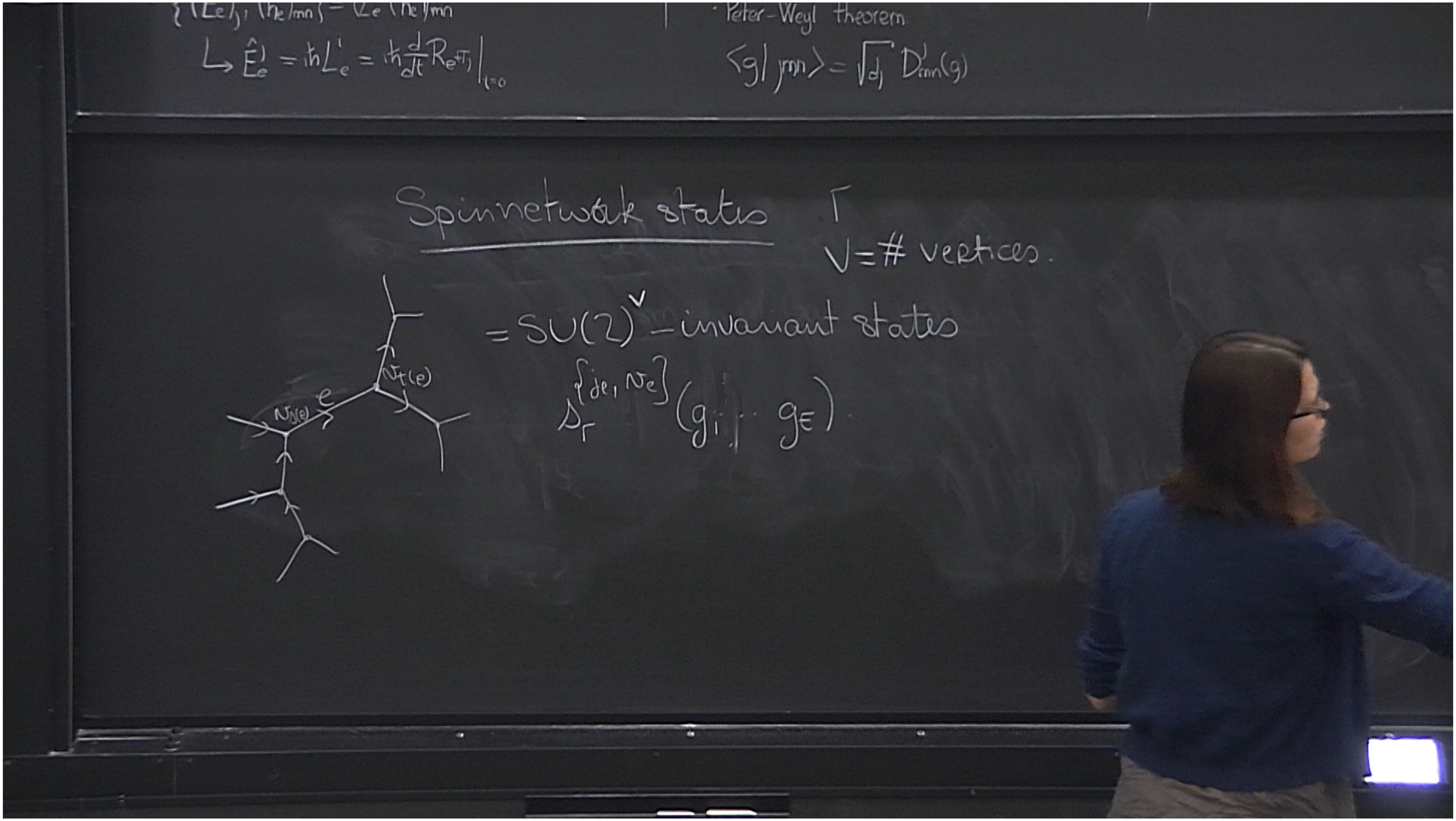
$|j m n\rangle$
n j m
$m_1, j_2 m_2 \mid j_3, -m_3 \rangle$
Clebsh - Gordon coeff
invariant under rotation at v state based on three edges with
$\sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1}^{3} |j_i m_i p_i \rangle \right)$
j_3
p_3
$p_1 p_2 p_3$
p_1
p_2
j_1
j_2


$D_{p_1 n_1}(h) D_{p_2 n_2}(h) D_{p_3 n_3}(h) = \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}$ → Wigner 3j symbol.
$|j m n\rangle$
$\langle j_1 m_1, j_2 m_2 | j_3, -m_3 \rangle$
Clebsh - Gordon coeff.
Normalized, invariant under rotation at v state based on three edges with sou
$\delta(\hat{G}_v) | \bigotimes_{i=1}^{3} |j_i m_i n_i\rangle) = \sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \left(\bigotimes_{i=1}^{3} |j_i m_i p_i\rangle \right)$
$\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$



Wigner 3j-symbol
|j m n⟩
n j m
⟨j1 m1, j2 m2 | j3, -m3⟩
Clebsch-Gordon coeff.
Normalized, invariant under rotation at v state based on three edges with source vert
δ(Ĝv) ⊗ |ji mi ni⟩ = Σ (j1 j2 j3 / p1 p2 p3) (⊗ |ji mi pi⟩)
p1p2p3
(j1 j2 j3 / n1 n2 n3)
j3 p3 p1 p2 p3 j1 j2


Wigner 3j symbol
$\langle j_1 m_1, j_2 m_2 | j_3, -m_3 \rangle$
Clebsh-Gordon coeff
Normalized, invariant under rotation at v state based on three edges with
$\delta(\hat{G}_v) | \bigotimes_{i=1}^{3} |j_i m_i n_i\rangle) = \sum_{p_1 p_2 p_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ p_1 & p_2 & p_3 \end{pmatrix} (\bigotimes_{i=1}^{3} |j_i m_i p_i\rangle)$
$\begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$



$\hookrightarrow \hat{E}^i_e = i\hbar L^i_e = i\hbar \frac{d}{dt} R_{e^{t\tau_i}}\big|_{t=0}$
Peter-Weyl theorem
$\langle g | jmn \rangle = \sqrt{d_j}\, D^j_{mn}(g)$
Spinnetwork states
Γ
V = # vertices.
$= SU(2)^V$ - invariant states
$s_\Gamma^{\{j_e, \nu_e\}}(g_1 \cdots g_E)$
$\nu_{t(e)}$
$\nu_{s(e)}$
e

$\{(L_e)_j (h_e)_{mn}\} = \ldots (h_e)_{mn}$

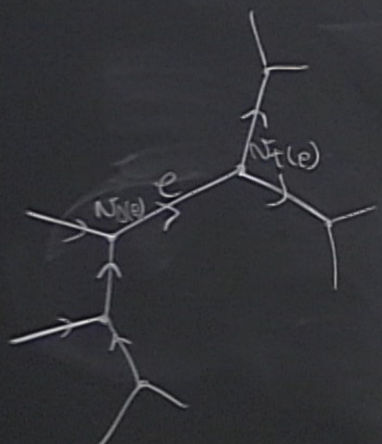
$\hookrightarrow \hat{E}^j_e = i\hbar L^j_e = i\hbar \frac{d}{dt} R_{e^{tT_j}} \Big|_{t=0}$

· Peter-Weyl theorem

$\langle g | j m n \rangle = \sqrt{d_j}\, D^j_{mn}(g)$

Spinnetwork states Γ

$V = \#$ vertices.



$= SU(2)^V$ - invariant states

$$s_\Gamma^{\{j_e, \nu_e\}}(g_1 \cdots g_E) = \sum \prod_{e=1}^{E} \langle g_e | j_e m_e n_e \rangle \Big(\prod_{\substack{\nu/\nu = \nu_{s(e)} \\ \nu/\nu = \nu_{t(e')}}} i_\nu^{m_e n_{e'}} \Big)$$

$dg_1 \dots dg_E$

$\cdot$ ONB $\;|\{\vec{j},\vec{m},\vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \dots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$

$i_v^{m_e \, n_{e'}}$

$= N_{s(e)}$

$= N_{t(e')}$

$\Sigma = 2\text{-sphere}$

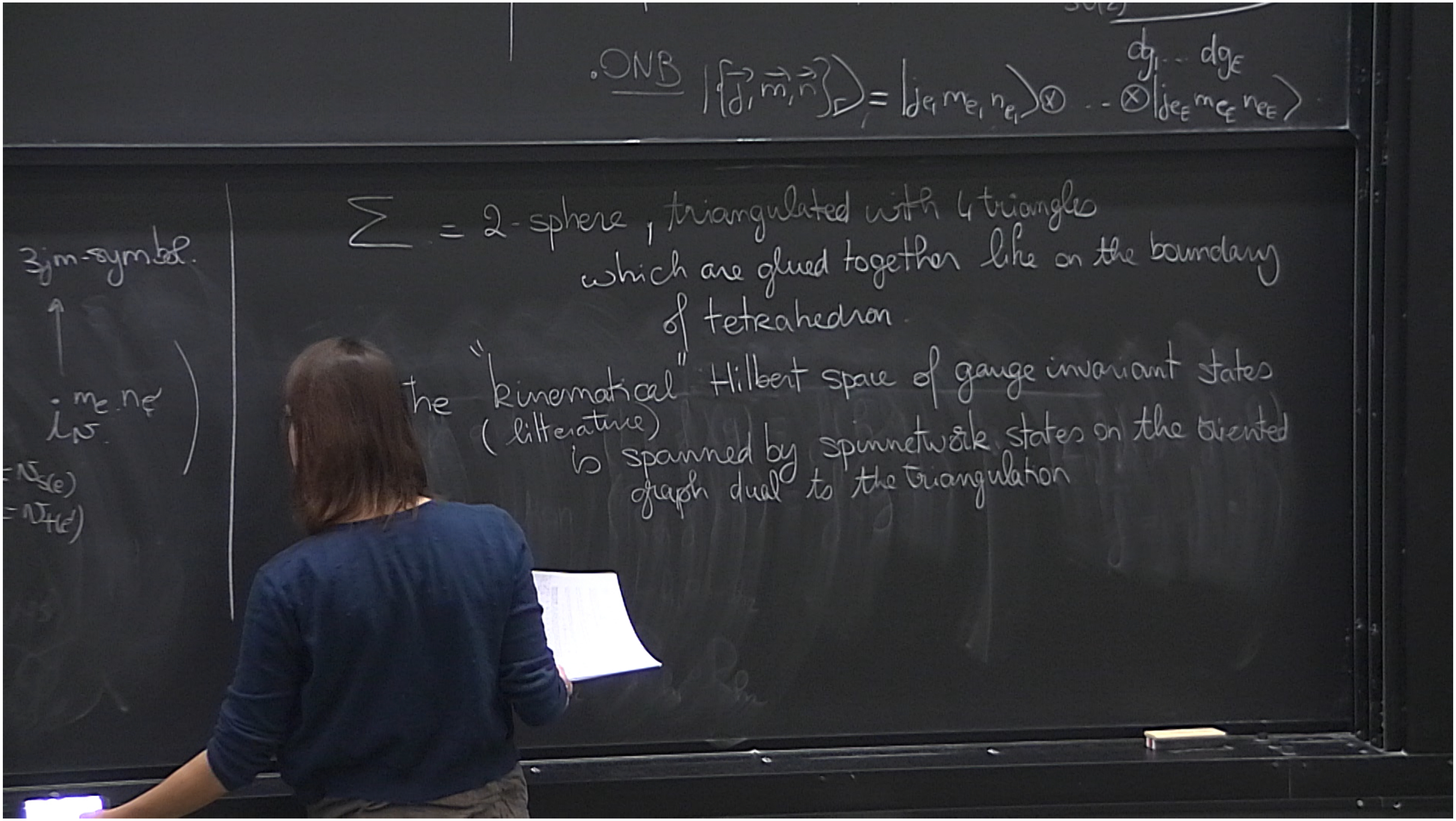
.ONB $|\{\vec{j},\vec{m},\vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1} \ldots\rangle$

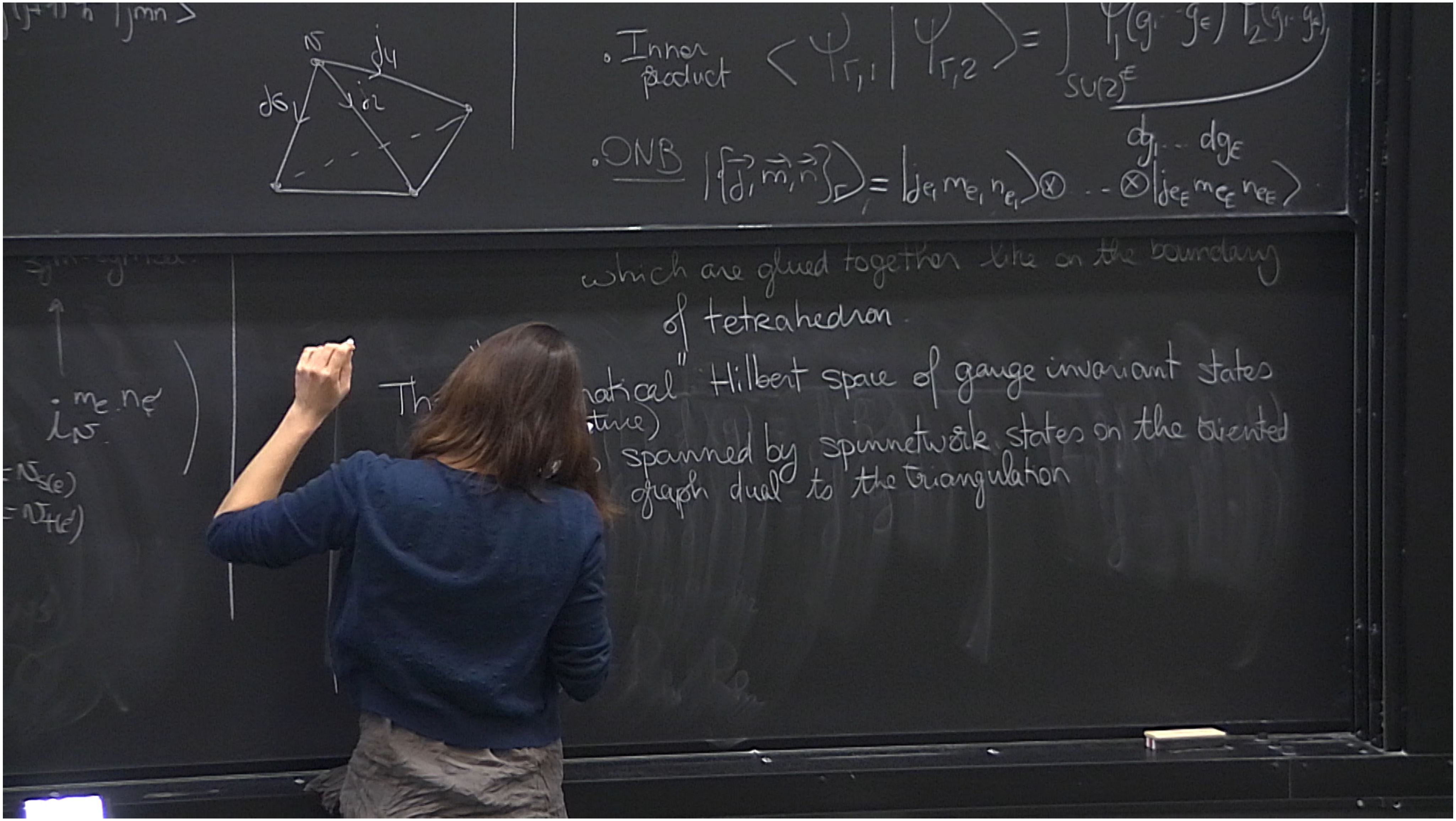
...rtices.

$$\prod_{e=1}^{E} \langle g_e | j_e m_e n_e \rangle \left(\prod_{\substack{v / v = v_{s(e)} \\ v / v = v_{t(e')}}} i_v^{m_e . n_{e'}} \right)$$

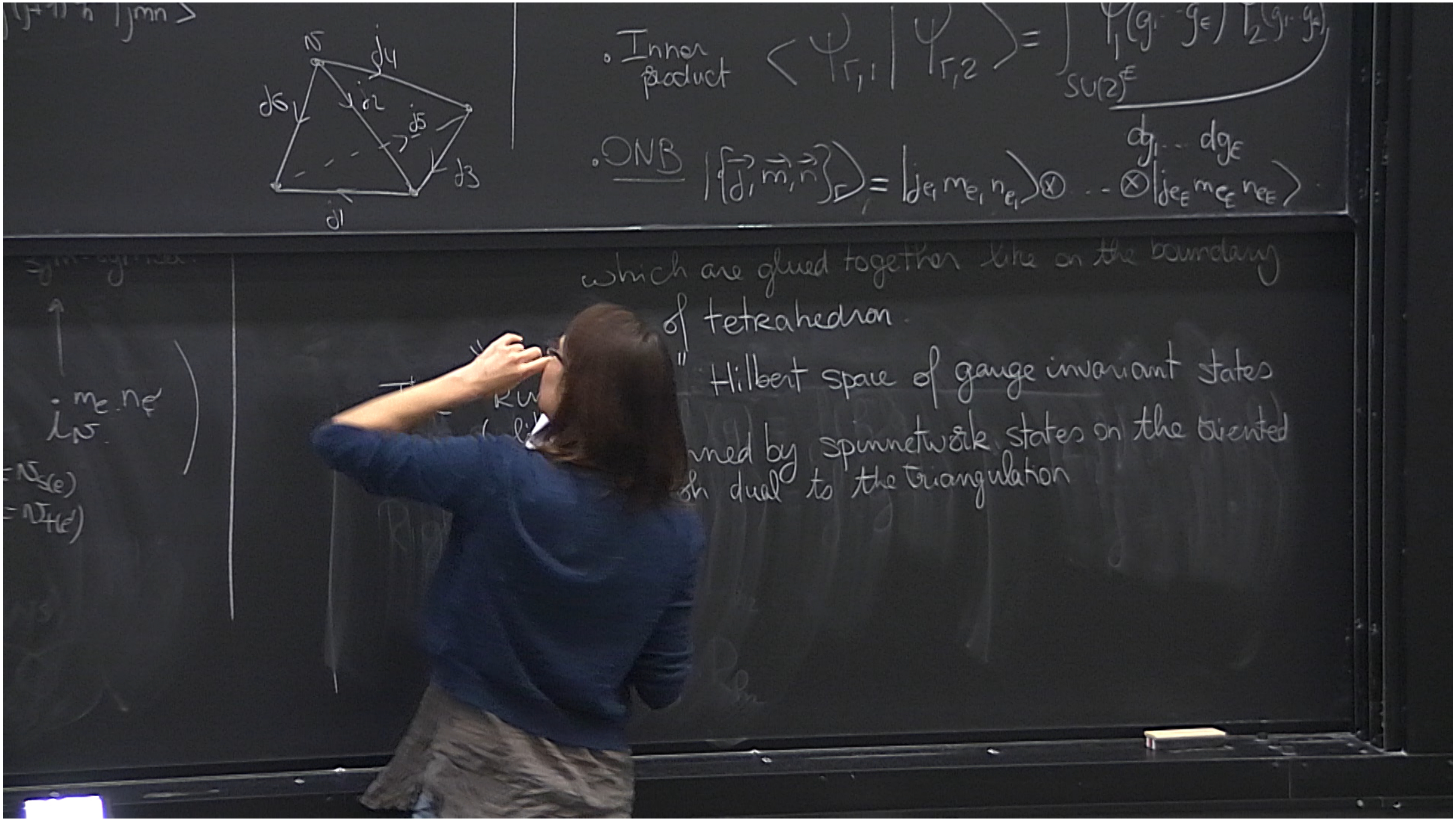
3jm-symbol.

$\Sigma = 2$-sphere.


dg1 ... dgE
.ONB |{j, m, n}Γ⟩ = |je1 me1 ne1⟩ ⊗ ... ⊗ |jeE meE neE⟩
3jm-symbol
Σ = 2-sphere, triangulated with 4 triangles
which are glued together like on the boundary
of tetrahedron.
the "kinematical" Hilbert space of gauge invariant states
(litterature)
is spanned by spinnetwork states on the oriented
graph dual to the triangulation

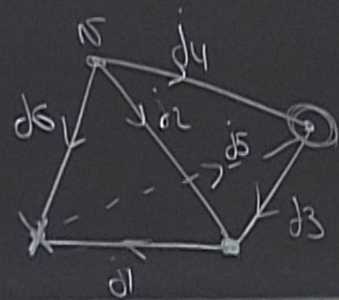


Inner product
SU(2)^E
dg_1 ... dg_E
ONB
which are glued together like on the boundary
of tetrahedron.
Hilbert space of gauge invariant states
spanned by spinnetwork states on the oriented
graph dual to the triangulation



Inner product
ONB
$SU(2)^E$
$dg_1 \dots dg_E$
which are glued together like on the boundary
of tetrahedron.
Hilbert space of gauge invariant states
spanned by spinnetwork states on the oriented
graph dual to the triangulation

$\dots (j+1)\hbar \; |jmn\rangle$



• Inner product $\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle = \int_{SU(2)^E} \Psi_1(g_1 \dots g_E)\, \Psi_2(g_1 \dots g_E)\; dg_1 \dots dg_E$

• ONB $|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \dots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$

$\iota_v^{m_e n_e}$

$= v_{s(e)}$

$= v_{t(e)}$

which are glued together like on the boundary of tetrahedron.

The "kinematical" Hilbert space of gauge invariant states (litterature) is spanned by spinnetwork states on the oriented graph dual to the triangulation

$$s^{j_e}_{tet}(g_1 \dots g_6) = \sum \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ -m_1 & n_5 & -m_6 \end{pmatrix} \begin{pmatrix} j_2 & j_6 & j_4 \\ n_2 & n_6 & n_4 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ n_3 & -m_4 & -m_5 \end{pmatrix}$$

$\rightarrow [\ ,\ ] = 0$

$|h_e \Psi_e\rangle$

$(h_e)_{mn}$

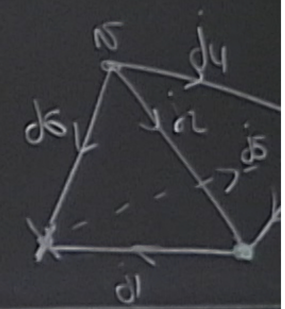
$i\hbar \frac{d}{dt} R_{e}(\tau)\big|_{t=0}$

(3) Orthonormal basis

· Peter-Weyl theorem

$$\langle g | jmn\rangle = \sqrt{d_j}\, D^j_{mn}(g)$$

$$L_e^2 |jmn\rangle = j(j+1)\hbar^2 |jmn\rangle$$



symmetries

V = # vertices.

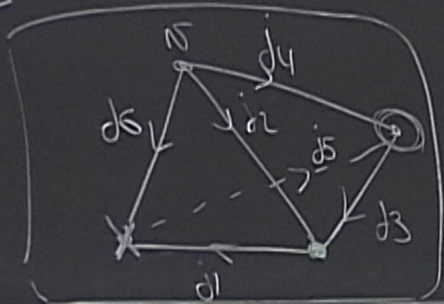
= SU(2)^V – invariant states

$v_{t(e)}$

$$\psi_\Gamma^{\{j_e, v_e\}}(g_1 \dots g_E) = \sum \prod_{e=1}^{E} \langle g_e | j_e m_e n_e\rangle \left(\boxed{\prod_{\substack{v / v = v_{s(e)} \\ v / v = v_{t(e)}}}} i_v^{m_e n_e} \right)$$

The

$$\psi_{tet}^{j_e}(g_1 \dots g_6) =$$



$|jmn\rangle$

- Inner product $\langle \Psi_{\Gamma,1} | \Psi_{\Gamma,2} \rangle = \int_{SU(2)^E} \overline{\Psi_1(g_1 \ldots g_E)}\, \Psi_2(g_1 \ldots g_E)\, dg_1 \ldots dg_E$

- ONB $|\{\vec{j}, \vec{m}, \vec{n}\}_\Gamma\rangle = |j_{e_1} m_{e_1} n_{e_1}\rangle \otimes \ldots \otimes |j_{e_E} m_{e_E} n_{e_E}\rangle$

which are glued together like on the boundary of tetrahedron.

The "kinematical" Hilbert space of gauge invariant states (litterature) is spanned by spinnetwork states on the oriented graph dual to the triangulation

$$\Psi^{j_e}_{tet}(g_1 \ldots g_6) = \sum \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ -m_1 & n_5 & -m_6 \end{pmatrix} \begin{pmatrix} j_2 & j_6 & j_4 \\ n_2 & n_6 & n_4 \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j_5 \\ n_3 & -m_4 & -m_5 \end{pmatrix} \prod_{e=1}^{6} (-1)^{j_e - m_e} \langle g_e | j_e m_e n_e \rangle$$

$i_v^{m_e n_e}$

$= v_{s(e)}$

$= v_{t(e)}$