

Title: Hyperlinear profile and entanglement

Date: Mar 19, 2018 02:00 PM

URL: <http://pirsa.org/18030118>

Abstract:

An approximate representation of a finitely-presented group is an assignment of unitary matrices to the generators, such that the defining relations are close to the identity in the normalized Hilbert-Schmidt norm. A group is said to be hyperlinear if every non-trivial element can be bounded away from the identity in approximate representations of the group. Determining whether all groups are hyperlinear is a major open problem, as a non-hyperlinear group would provide a counterexample to the famous Connes embedding problem.

Given the difficulty of the Connes embedding problem, it makes sense to look at an easier problem: how fast does the dimension of approximate representations grow (as a function of how close the defining relations are to the identity) when we require a given set of elements to be bounded away from the identity. These growth rates are called the hyperlinear profile of the group.

In this talk, I will explain our best lower bounds on hyperlinear profile, as well as the connection to entanglement requirements for non-local games (joint work with Thomas Vidick). Time permitting, I will also mention some other approaches to looking for non-hyperlinear groups, including the recent work of De Chiffre, Glebsky, Lubotzky, and Thom on a group which is not approximable in the unnormalized Hilbert-Schmidt norm.

Hyperlinear profile
& entanglement.

Elementary question.

$$G = \langle S : R \rangle = \langle x_1, \dots, x_n : r_1, \dots, r_m \rangle. \quad \text{M}$$

G

$$G = \langle S : R \rangle$$

G

$$G = \langle S; R \rangle = \langle x_1, \dots, x_n : r_1, \dots, r_m \rangle.$$

$$w \in \text{Free}(S). \quad \delta, \varepsilon > 0.$$

$$\text{hlp}(w; \delta, \varepsilon) = \min \left\{ d : \begin{array}{l} \text{then on unitaries } u_1, \dots, u_n \\ \text{s.t. } \| r_i(u_1, \dots, u_n) - \mathbb{1} \|_f \leq \varepsilon \\ \quad \quad \quad \| w(u_1, \dots, u_n) - \mathbb{1} \|_f \geq \delta. \end{array} \right.$$

↑
hyperlines
profile

G

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↑
hyperlinear
profile.

$$f = \sqrt{\frac{1}{2} \text{tr}(A^T A)}$$

matrices.

$$u_n \leq \varepsilon.$$

$$\delta \}$$

Qn. How fast does $\text{hlp}(w; \delta, \varepsilon)$ grow as a function of ε when $\varepsilon \rightarrow 0$.

When can we see G in
its "finite-dim'l models"?

Def'n. A group G is residually
finite dim'l if for any $w \in G$,
 $w \neq e$, then is a hom $\varphi: G \rightarrow U(n)$
s.t. $\varphi(w) \neq I$.

$$(x_1, \dots, x_n : r_1, \dots, r_m).$$

$$> 0.$$

Then an unitarie $u_1, \dots, u_n$
s.t. $\| \varphi(u_1, \dots, u_n) - I \|$

$$\neq \| w(u_1, \dots, u_n) - I \|$$

$$U\{\infty\}.$$

we see G in
-dim'l models?

group G is residually
iff for any $w \in G$

is a hom $\rho: G \rightarrow U(\mathbb{C}^d)$
 $\rho(w) \neq I$.

Then, an lots of examples
of non-RFD groups.

There is a $w \in G$ $w \neq e$
but $\rho(w) = I$ for any
 $\rho: G \rightarrow U(\mathbb{C}^d)$.

Ex $BS(2,3) = \langle x, y : xy^2x^{-1} = y^3 \rangle$.

$(r_1, \dots, r_m)$.

an unitaire $u_1, \dots, u_n$
 $\| \rho_i(u_1, \dots, u_n) - I \|_F$
 $\| \rho(u_1, \dots, u_n) - I \|_F >$
 $U \{ +\infty \}$.

s of examples
groups.

$w \in G$ $w \neq e$.
 $\gamma = 1$ for any
 $U(\mathbb{Q}^d)$.

$\langle x, y : xy^2x^{-1} = y^3 \rangle$.

What if instead we ask
about "approx. fin. dim'l
models"?

Def'n. An ε -rep'n of G .
is a hom. $\alpha : \text{Free}(S) \rightarrow U(\mathbb{Q}^n)$

Qn. How
good is



s of examples
groups.

$w \in G$ $w \neq e$.
 $\rho(w) = 1$ for any
 $\rho \in \text{Hom}(G, \mathbb{C}^d)$.

$\langle x, y : xy^2x^{-1} = y^3 \rangle$.

What if instead we ask
about "approx. lin. dim'd
models"?

Def'n. An ε -rep'n of G .
is a hom. $\rho : \text{Free}(S) \rightarrow \mathcal{U}(\mathbb{C}^n)$
st. $\|\rho(r) - 1\|_F \leq \varepsilon$ for all $r \in R$.

Def'n. $w \in \text{Free}(S)$ is non-triv in approx
repl's of G if there is a $\delta > 0$, s.t.
for all $\varepsilon > 0$, there is an ε -rep'n q
w/ $\|q(w) - \mathbb{1}\|_f \geq \delta$.

Notes : If w is non-triv in approx rep'n then
 w is non-triv in G .

• δ is arbitrary. $\in (0, \sqrt{2})$. (amplification)

Def'n. G is hyperlinear
 if $\exists w \in \text{Free}(S)$
 which is non-triv in G
 is non-triv in approx reps.
 of G .

RFD gp is hyperlinear.

Then, on lots of examples
 of non-RFD groups.

There is a $w \in G$ $w \neq e$
 but $\alpha(w) = 1$ for any
 $\alpha: G \rightarrow U(\mathbb{C}^d)$.

Ex $BS(2,3) = \langle x, y : x y^2 x^{-1} = y^3 \rangle$.

profile

element

Def'n. G is hyperlinear

if $\exists w \in \text{Free}(S)$

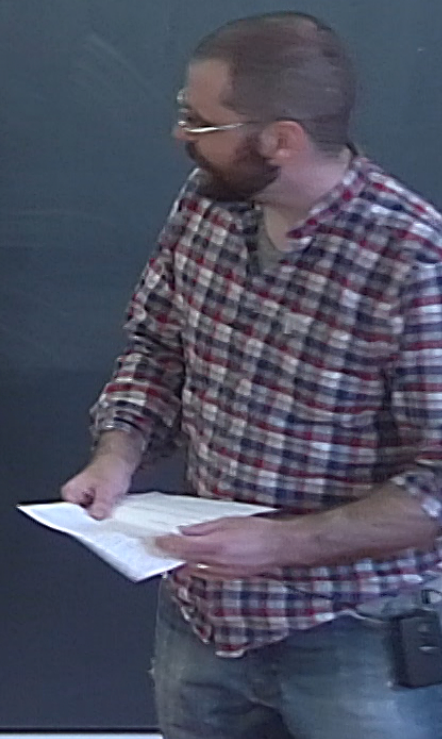
which is non-triv in G

is non-triv in approx rep's.
of G .

RFD_{gp} is hyperlinear.

$BS(2,3)$ is hyperlinear.

amenable gp are hyperlinear.



hyperlinea
 $\in \text{Free}(S)$.
 -driv in G
 approx rep'ns.

linear.

ulinear.
 hyperlinear.

We don't know if there is
 a non-hyperlinear gp.

$$\text{hlp}(w; \delta, \varepsilon) = \min \{ d :$$

there is an ε -rep'n $\alpha : \text{Free}(S) \rightarrow U(d)$
 $w/ \|\alpha(w) - 1\|_2 \geq \delta \} \cup \{ +\infty \}.$

What if instead we ask
 about "approx. lin. dim'l
 mode

$S = \mathbb{R} >$

rep'n
 rec'd

$\leq$

We don't know if there is
a non-hyperlinear gp.

$$\text{hlp}(w; \delta, \epsilon) = \min \{ d : \text{there is an } \epsilon\text{-rep'n } \alpha: \text{Free}(S) \rightarrow U(d) \text{ w/ } \|\alpha(w) - 1\|_2 \geq \delta \} \cup \{+\infty\}.$$

then there is an ϵ -rep'n $\alpha: \text{Free}(S) \rightarrow U(d)$
w/ $\|\alpha(w) - 1\|_2 \geq \delta$.

G is non-hyperlinear $\Leftrightarrow$ there is $w \in \text{Free}(S)$, non-triv
in G
s.t. $\text{hlp}(w; \delta, \epsilon) = +\infty$ as $\epsilon \rightarrow 0$.

If G is RFD, $\text{hlp}(w; \delta, \epsilon)$ is bounded
for all w as $\epsilon \rightarrow 0$.

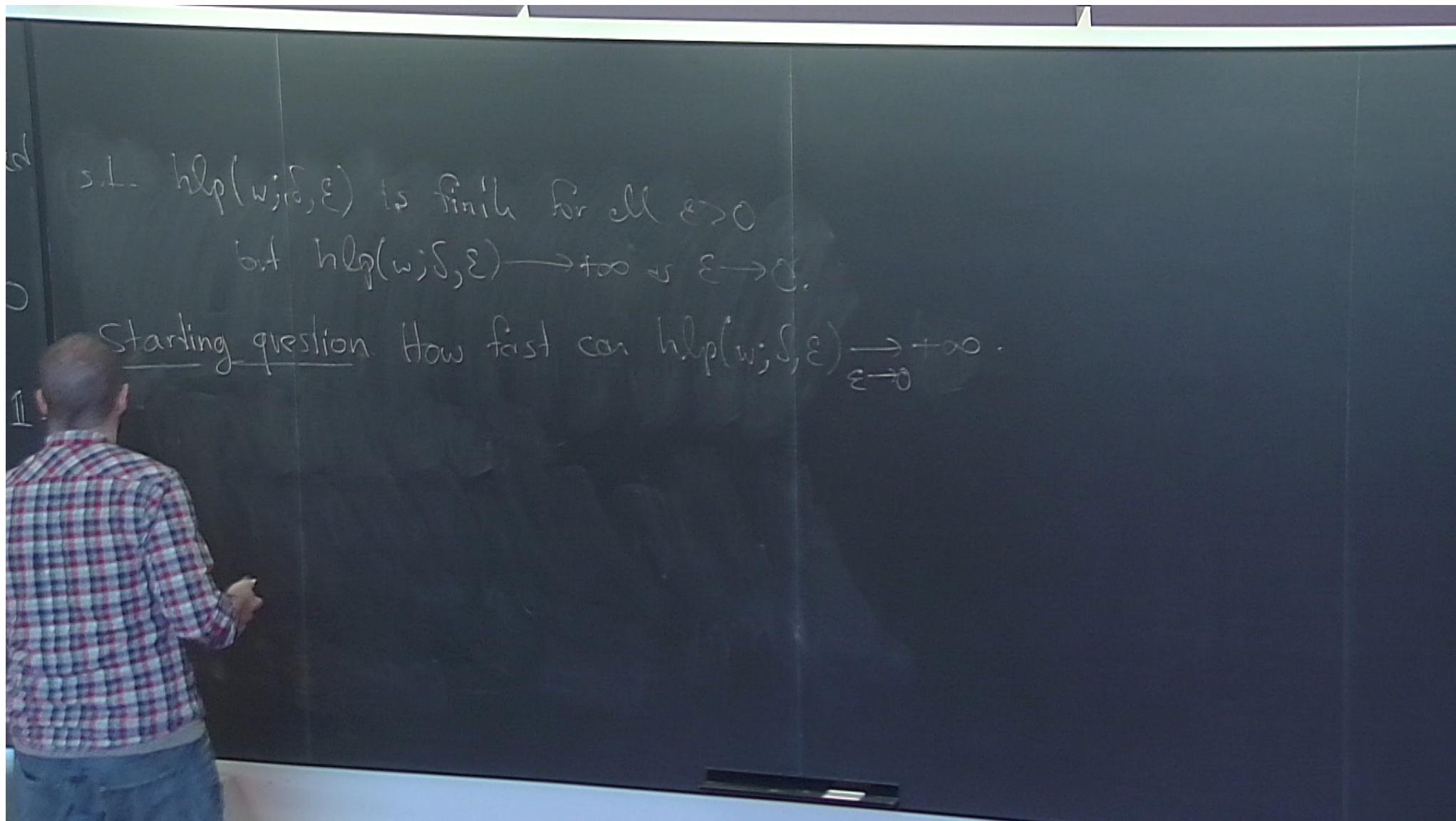
If $\text{hlp}(w; \delta, \epsilon) \leq D$ as $\epsilon \rightarrow 0$

then there is an exact rep'n.

$\alpha: G \rightarrow U(\mathbb{C}^D)$ s.t. $\alpha(w) \neq 1$.

If G is hyperlinear but

non-RFD, then there is
an elt $w \in \text{Free}(S)$



s.t. $\text{hlp}(w; \delta, \varepsilon)$ is finite for all $\varepsilon > 0$
but $\text{hlp}(w; \delta, \varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.

Starting question How fast can $\text{hlp}(w; \delta, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} +\infty$.

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A non-hyperlinear group would give a counterexample to.
the Connes embedding problem.

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A non-hyperlinear group would give a counterexample to the Connes embedding problem.

(Classification of von Neumann algs.)

Does every II_1 factor embed in R^w ?
Answer: no
at hyperfinite
 II_1 factor

s.t. $hlp(w; \delta, \varepsilon)$ is finite for all $\varepsilon > 0$

but $hlp(w; \delta, \varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.

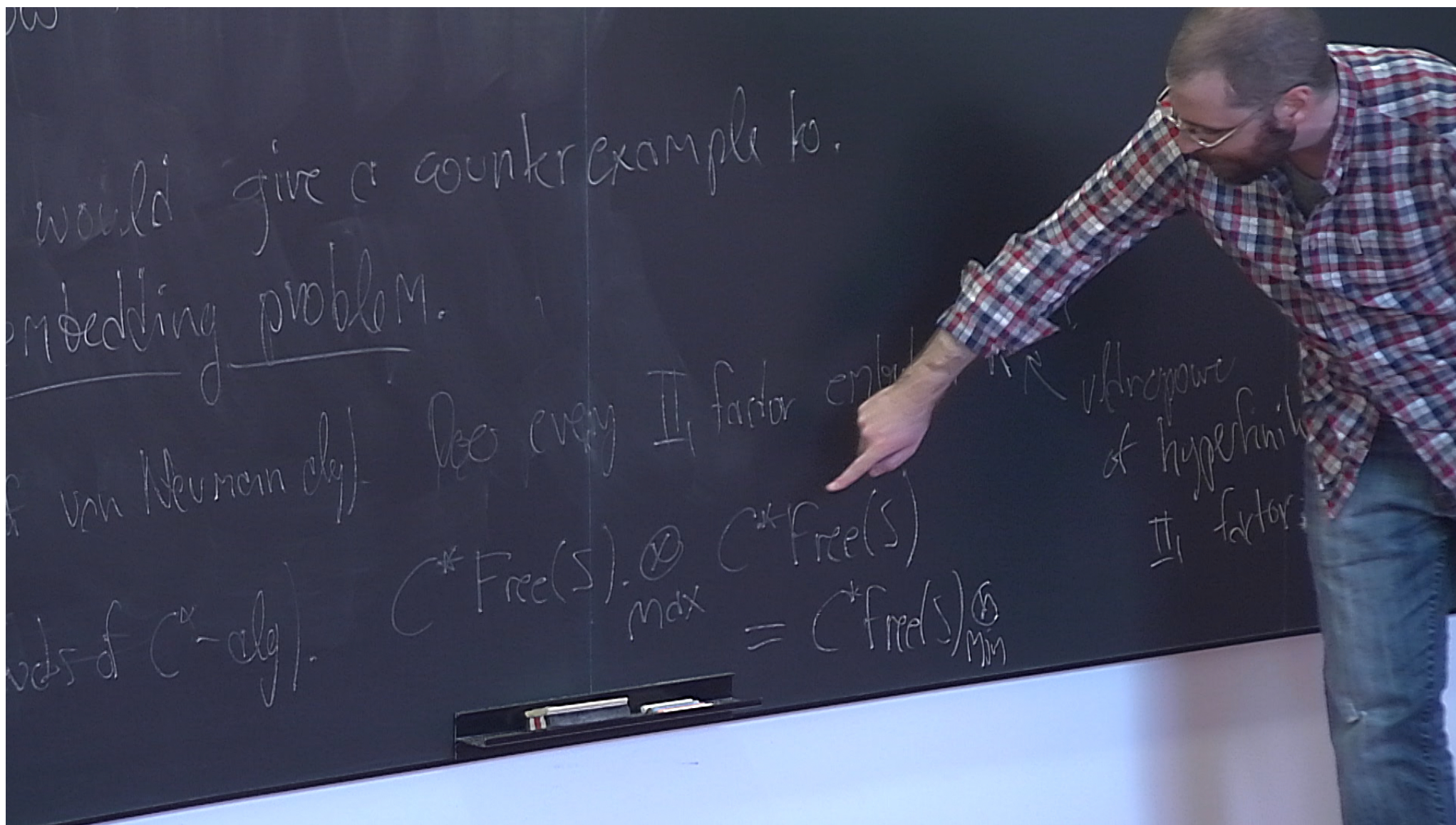
Starting question How fast can $hlp(w; \delta, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} +\infty$.

A non-hyperlinear group would give a counterexample to.
the Connes embedding problem.

(Classification of von Neumann algs).

Kirchberg. (Tensor products of C^* -algs).

Does every II_1 factor embed in R^ω ?
ultrapower
of hyperfinite
 II_1 factor.



would give a counterexample to.
embedding problem.

of von Neumann alg).
 vels of C^* -alg).

Does every II_1 factor embed in $R^{\text{w}}?$
 ultrapower
 of hyperfinite

$$C^* \text{Free}(S) \otimes_{\max} C^* \text{Free}(S) = C^* \text{Free}(S) \otimes_{\min} C^* \text{Free}(S) \Bigg| \text{II}_1 \text{ factor}$$

(Quantum mechanics) - Bell inequality - non-local game -

Axioms for physical systems

How to describe

- state of the system

- measurement

$$\text{map } n \rightarrow \mathbb{N} \text{ to } \mathbb{R}^n \text{ via } \alpha(s) \rightarrow U(\alpha(s))$$

$$\|U\|_{\infty} \geq \delta \text{ for } U \in \{+\infty\}$$

hyperlinear $\Leftrightarrow$ for is $W \in \text{Free}$ in G s.t. $\text{hlp}(W/\delta, \epsilon)$

(Quantum mechanics) - Bell inequality - non-local games - probability.

Axioms for physical systems

How to describe:

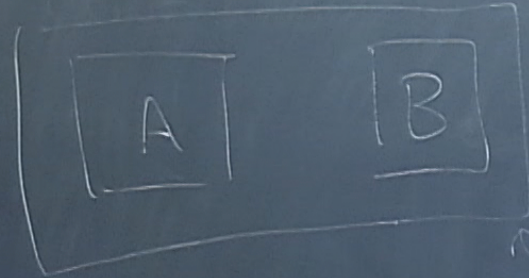
- state of the system.
- measurement.
- subsystems

non-linear $\Rightarrow$ there is $W \in F$ in G s.t. $h(W/\delta, \epsilon)$

ics). Bell inequality.
 non-local.
 game.

physical systems.

be.
 system.



Imagine - two labs.

each capable of making one of b measurements.
 each measurement returns one of b outputs.

joint system
 of physically separated

If G is RFD, $h(p|w; \delta)$
 for all $w \in \mathcal{W}$

If $h(p|w; \delta, \epsilon) \leq \epsilon$
 then there is an encoding
 $c: G \rightarrow \mathcal{U}(\mathbb{C}^D)$

If G is
 non-P
 in $\mathcal{E}$

Quantum mechanics) - Bell inequality
probability - non-local game -

Axioms for physical systems

How to describe

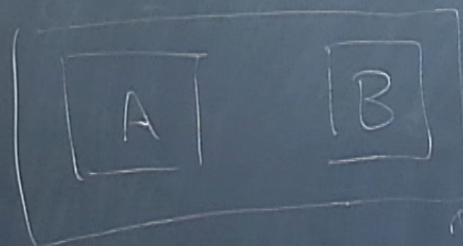
- state of the system
- measurement
- subsystems

Given such a system

Imagine - two labs

each capable of making one of b measurements
each measurement returns one of b outputs

$p(a,b|x,y)$ = prob. of outcome (a,b) on input (x,y) .



joint system
of physically separated

If G is RFD,
for all

If $h(p|w)$

then then

cc: G

$\{p(a,b|x,y)\} \in \mathbb{R}^{n^2 m^2}$ correlation of the system.

What correlations can we get from given axiom systems.

separability

in measurements

on outputs

on input (x,y) .

For instance, if axioms are "reasonable" then they should not allow signalling.

s.t. $h(p; \delta, \epsilon)$ is finite for but $h(p; \delta, \epsilon) \rightarrow \infty$

Starting question. How fast

A non-hyperlinear group would give the Connes embedding problem

(Classification of von Neumann algebras) Kirchberg. (Tensor products of C^* -algebras)

$\{p(a,b|x,y)\} \in \mathbb{R}^{n^2 m^2}$ correlation of the system.

What correlations can we get from given axiom systems.

separably!
 on measurements
 on outputs
 on input (x,y) .

For instance, if axioms on.
 "reasonable" then they.
 should not allow signalling.

$$p(a|x) = \sum_b p(a,b|x,y)$$

s.t. $\text{hlp}(w;\delta,\epsilon)$ is finite for
 but $\text{hlp}(w;\delta,\epsilon) \rightarrow$

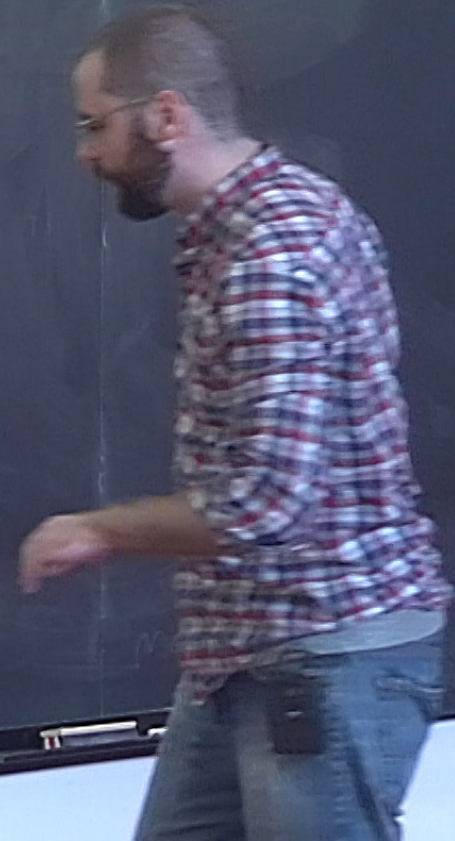
Starting from How fast

A non-hyper...
 (Classical)
 Kirchberg. (Tensor)

non
system.
from.

is independent of y . for all a, x .
& similarly for $p(b|y)$.

on.
key.
ow signalling.
 $p(a, b|x, y)$.



non
system.
from.

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Def'n. A correlation $\{p(a, b|x, y)\}$ is quantum.

on -
key.
signalling.
($a, b|x, y$).

on
system.
from.

is independent of y . for all a, x .
 $\&$ similarly for $p(b|y)$.

Def'n. A correlation $\{p(a,b|x,y)\}$ is quantum.

if there is $|\psi\rangle \in H_A \otimes H_B$ & $\{M_a^x\}_{a=1}^n, \{N_b^y\}_{b=1}^n$

$$\text{s.t. } p(a,b|x,y) = \langle \psi | M_a^x \otimes N_b^y | \psi \rangle$$

$\in H_A \quad H_B$

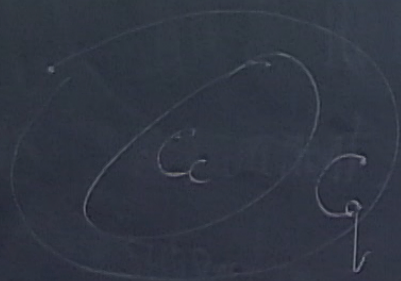
on.
they.
w signalling.
 $p(a,b|x,y)$.

Bell inequalities.

$C_{\text{classical}}(m,n) = \text{classical correlations}$

} convex

$C_{\text{quantum}}(m,n) = \text{quantum.}$
 "



Bell inequalities.

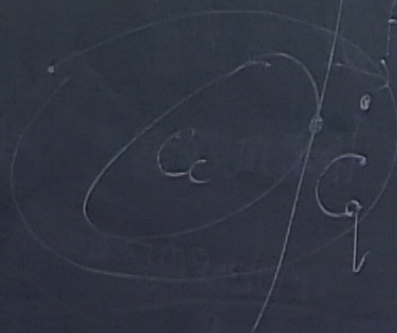
$C_{\text{classical}}(m,n) = \text{classical correlations}$

} convex

$C_{\text{quantum}}(m,n) = \text{quantum.}$

"
 $f(p) = \alpha$

$f(p) > \alpha$

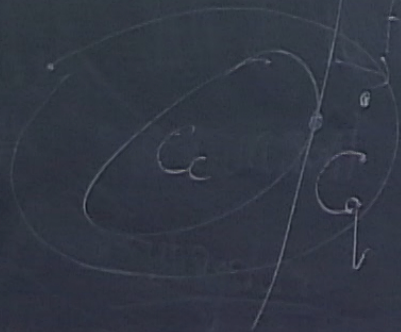


Bell inequalities.

$C_{\text{classical}}(m,n) = \text{classical correlations}$

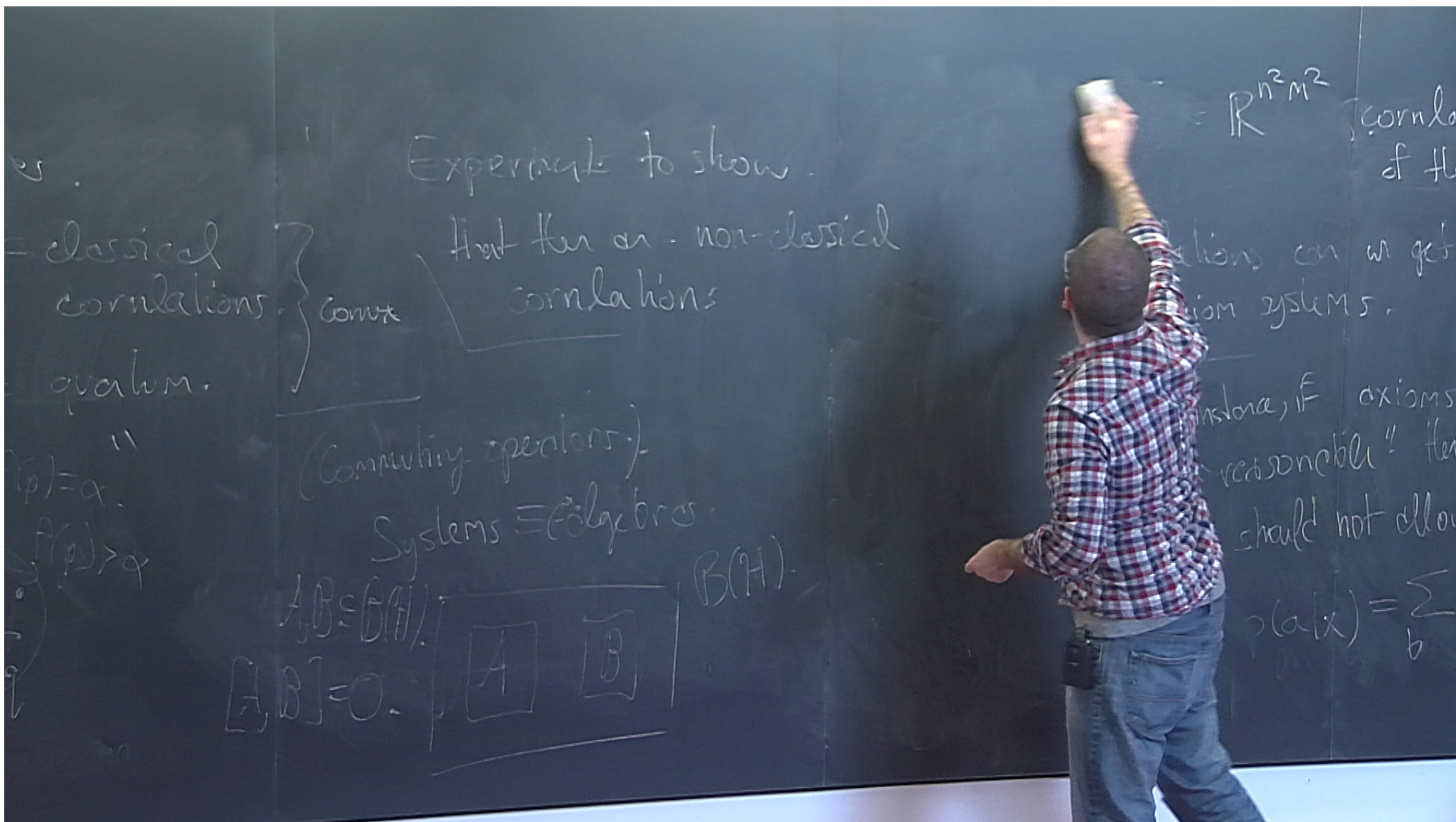
$C_{\text{quantum}}(m,n) = \text{quantum.}$

$f(p) = \alpha$
"
 $f(p) > \alpha$



Experiments to show.

that there are non-classical correlations



to show.

. non-classical

ions

pro

$B(H)$.

Def'n. A correlation $\{p(a,b|x,y)\}$ is

a commuting operator correlation

if there is $N \geq H$.

$$\{M_a^x, \{N_b^y\} \subseteq H.$$

$$[M_a^x, N_b^y] = 0.$$

$$w/ \quad p(a,b|x,y) = \langle \psi | M_a^x N_b^y | \psi \rangle$$

is independent of y . for all
+ similarly for $p(b|y)$.

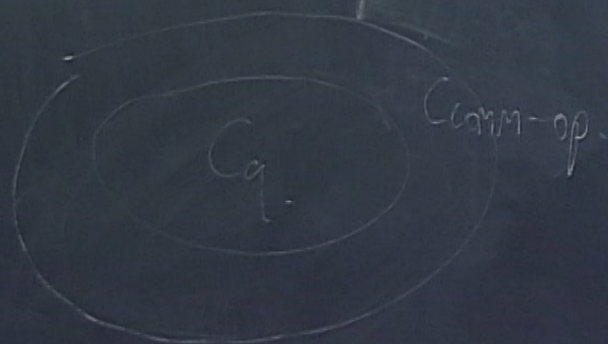
Def'n. A correlation $\{p(a,b|x,y)\}$

if there is $N \geq H \otimes H_B$.

$$s.t. \quad p(a,b|x,y) = \langle \psi | M_a^x \otimes N_b^y | \psi \rangle$$

$C_{\text{comm-op}}(n,n) = \text{community op. correlations}$

$C_{\text{quantum}}(n,n) = \text{quantum.}$
 " " "



Q: Are the Bell ineq?

(No $C_{\text{comm-op}} \neq C_q$).
 $\hookrightarrow C_{\text{comm-op}} = \overline{C_q} = \text{fin-dim'l quant corr.}$

Def'n. A ρ is a comm ρ if it is

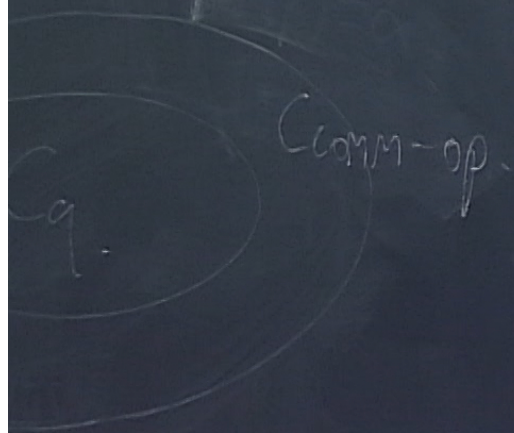
$AB = \overline{BA}$
 $[A, B] = 0$

A	B
---	---

$\rho(a,b|x,y) =$

$(m,n) = \text{commuting op. correlations}$

$(m,n) = \text{quantum.}$
"



Qv Another Bell ineq!

$\left. \begin{array}{l} (m,n) = \text{commuting op. correlations} \\ (m,n) = \text{quantum.} \end{array} \right\} \text{convex} \rightarrow \text{Noh } C_{\text{comm-op}} \neq C_q$

$C_{\text{comm-op}} = \overline{C_q} = C_{\text{fin-dim. quant corr}}$

This question is equivalent to CIP.

w/ plat

Hyperlinear profile
& entanglement.

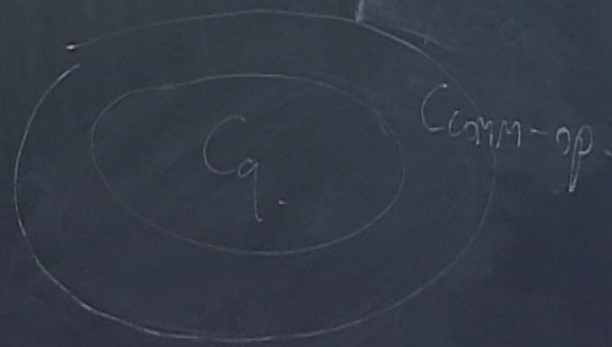
$$\left(\frac{1}{g^2}\right)$$

$$2\sqrt{e}$$

$$C_{\text{comm-op.}}(m,n) = \left. \begin{array}{l} \text{commuting op.} \\ \text{correlations} \end{array} \right\} a$$

$$C_{\text{quantum}}(m,n) = \text{quantum.}$$

"



= commuting op-
correlations

= quantum.

"

Comm-op.

Q Another Bell ineq?

(No) Comm-op $\neq C_q$ (S, Z) - this is $N \geq 0$

$\Rightarrow C_{\text{comm-op}} = \overline{C_q} = C_{\text{fin-dimil quant corr.}}$

This question is equivalent to CEP.

(Ozawa)

(Fritz, JNPR-SW);

Def'n. A correlation $\{p(a,b) \mid a, b \text{ commuting operator corr.}\}$

$\{M_a^x\}, C$

$[M_a^x]$

w/ $p(a,b|x,y) = \langle \psi | M_a^x M_b^y | \psi \rangle$

Q: Answer Bell ineq?

Def'n. A correlation $\{p(a,b|x,y)\}$ is a commuting operator correlation

(Noh $C_{\text{comm-op}} \neq C_q$ $(\sum_i 1, 2)$ - the is $N \in H$ $P_1 + \dots + P_n = I$

$\rightarrow$ is $C_{\text{comm-op}} = \overline{C_q} = C_{\text{fin-dimil quant corr.}}$

$$\{M_a^x, \{N_b^y\} \subseteq H$$

$$[M_a^x, N_b^y] = 0$$

$$p(a,b|x,y) = \langle \psi | M_a^x N_b^y | \psi \rangle$$

is independent
+ similar

Def'n. A co
if the is

st. p(a)

Are the Bell ineq?

Def'n. A correlation $\{p(a,b|x,y)\}$ is a commuting operator correlation

if there is $N \geq 1$ s.t. $P_1 + \dots + P_N = \mathbb{1}$ and $\{M_a^x, N_b^y\} \subset H$ with $[M_a^x, N_b^y] = 0$.

is $C_{\text{comm-op}} = C_q = C_{\text{fin-dimil quant corr.}}$

is equivalent to CEP.

$$p(a,b|x,y) = \langle \psi | M_a^x N_b^y | \psi \rangle$$

is independent of x and similarly for y .

Def'n. A correlation is a commuting operator correlation if there is $N \geq 1$ s.t. $P_1 + \dots + P_N = \mathbb{1}$ and $\{M_a^x, N_b^y\} \subset H$ with $[M_a^x, N_b^y] = 0$.

Qv. Another Bell ineq?

(Noh $C_{\text{comm-op}} \neq C_q$)

$C_{\text{comm-op}} = \overline{C_q} = C_{\text{fin-dimil quant corr.}}$

This question is equivalent to CIP.

(Ozawa).

Fritz, JNPPSW).

Def'n. A correlation $\{p(a,b|x,y)\}$ is a commuting operator correlation

$$\{M_a^x\}, \{N_b^y\} \subset H, \quad [M_a^x, N_b^y] = 0.$$

$$p(a,b|x,y) = \langle \psi | M_a^x N_b^y | \psi \rangle$$

$$C_q \neq \overline{C_q} = C_{qs}$$