Title: Dynamical symmetries and test fields in rotating black hole spacetimes

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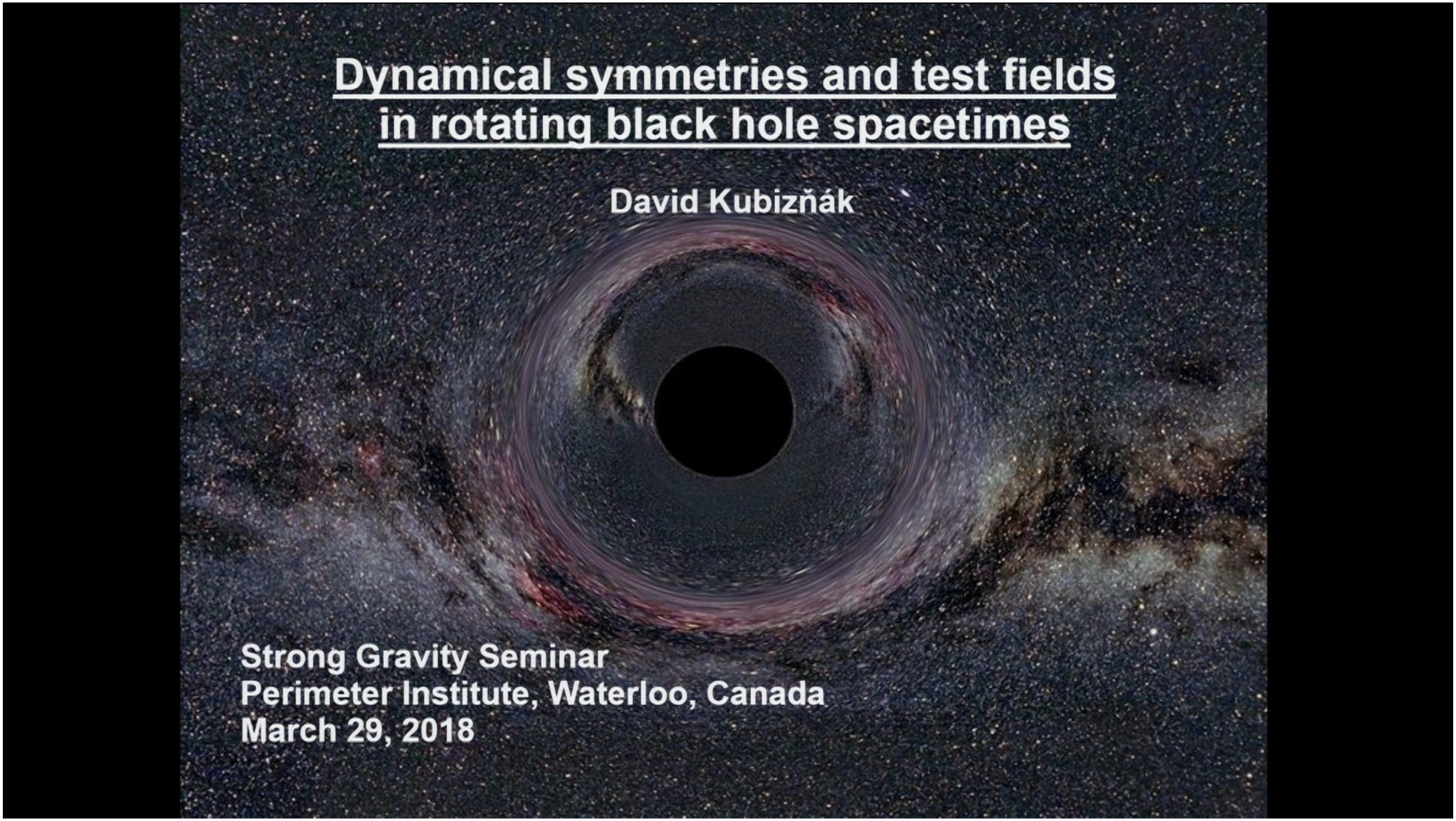
Abstract: <p>Starting from the well known Laplace-Runge-Lenz vector of the Kepler problem, I will introduce a notion of dynamical (hidden)
 symmetries. These are genuine phase space symmetries that stand in contrast to the more familiar symmetries of the configuration space
 discussed in truncated versions of Noether's theorem. Proceeding to a relativistic description, I will demonstrate that such symmetries -- encoded
in the so called Killing-Yano tensors -- play a crucial role in the study of rotating black holes described by the Kerr geometry. Even more remarkably, I will show that one such special symmetry is enough to guarantee complete integrability of particle and light motion in general rotating black hole spacetimes in an arbitrary
number of spacetime dimensions. Recent developments on the separability of test fields in these spacetimes will also be discussed.</p>


Dynamical symmetries and test fields
in rotating black hole spacetimes
David Kubizňák
Strong Gravity Seminar
Perimeter Institute, Waterloo, Canada
March 29, 2018

Plan of the talk

I. Introduction: Hidden symmetries

II. Miraculous properties of Kerr geometry

III. Principal Killing-Yano tensor
   I. Families of Killing-Yano tensors
   II. Kerr-NUT-AdS spacetimes
   III. Killing towers of symmetries

IV. Particles and fields: Integrability and Separation of variables
   I. Complete integrability of geodesic motion
   II. Scalar, Dirac, and Maxwell perturbations

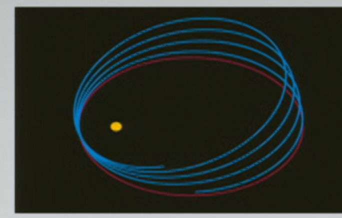
V. Summary

Friends:
M. Cariglia, P. Connell , V.P. Frolov, G.W. Gibbons, T. Houri, P. Krtous, H.K. Kunduri, D.N. Page, J.E. Santos, M. Vasudevan, C.M. Warnick, Y. Yasui

Laplace-Runge-Lenz vector

Central force:

$$E\ ,\ \vec{L}$$

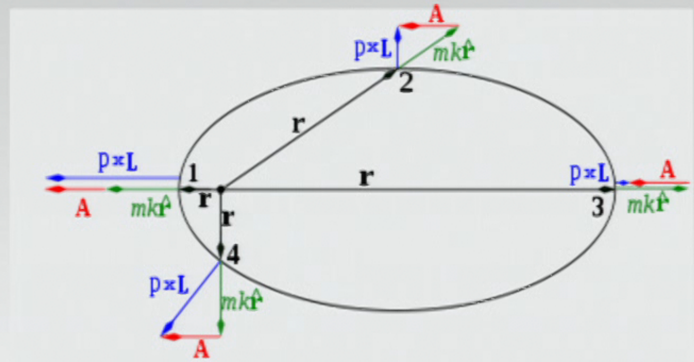


Kepler problem:

$$\vec{F} = -\frac{k}{r^2}\hat{r}$$

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$$

Laplace-Runge-Lenz vector



Wikipedia

Laplace-Runge-Lenz vector

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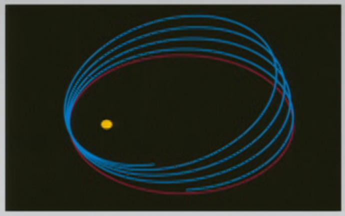

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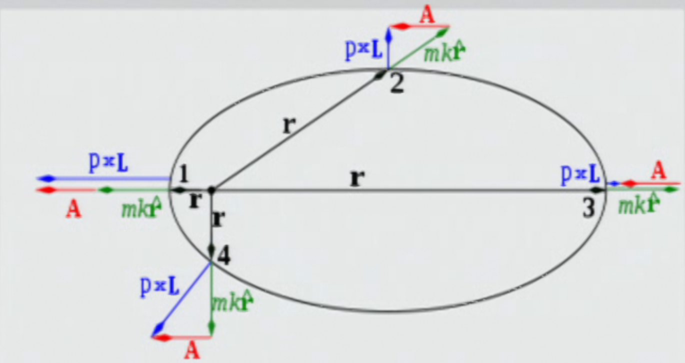


Wikipedia

- motion maximally superintegrable

$$\vec{A} \cdot \vec{L} = 0 \qquad A^2 = m^2k^2 + 2mEL^2$$

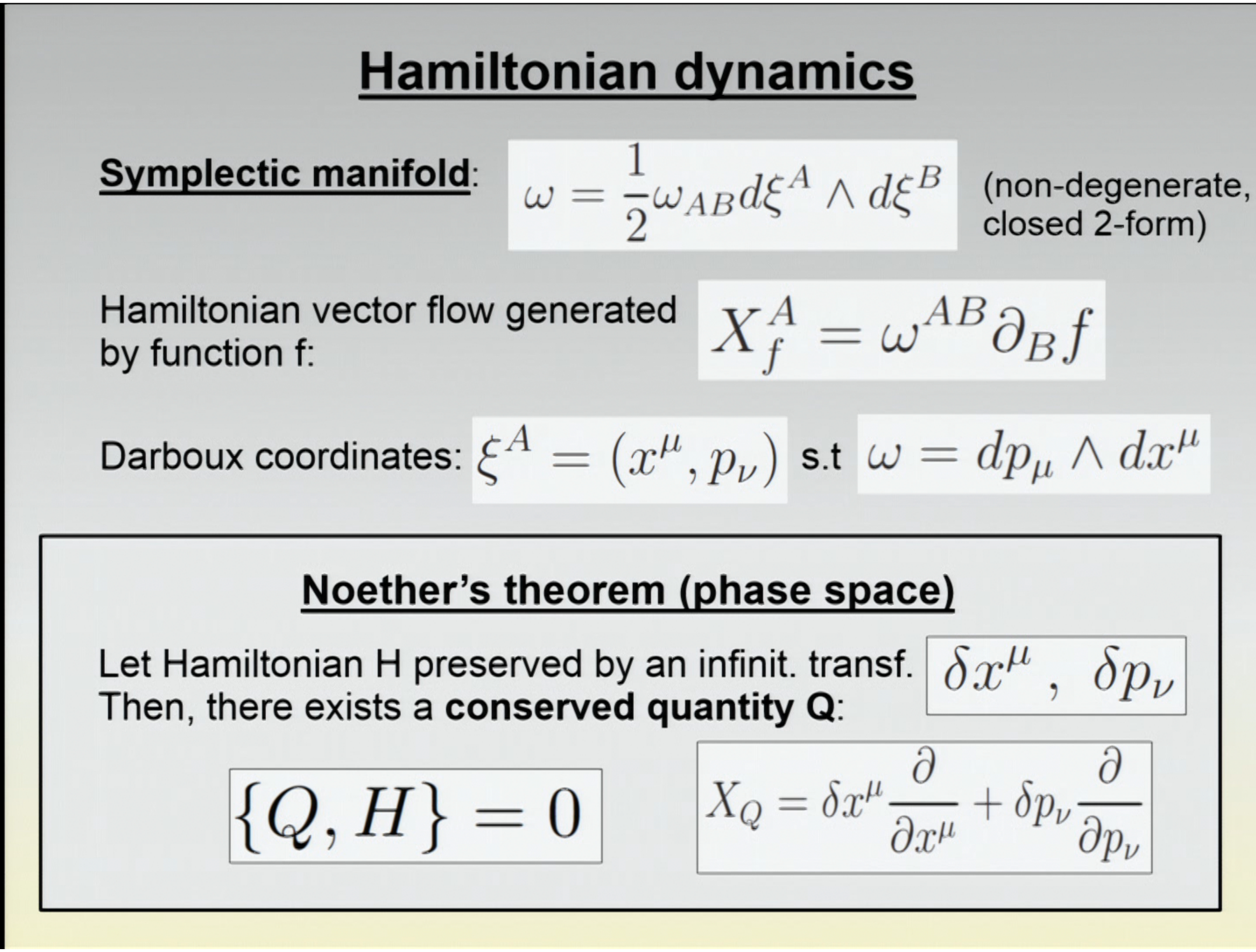
- **dynamical symmetry**

Hamiltonian dynamics

Symplectic manifold: $\omega = \frac{1}{2}\omega_{AB} d\xi^A \wedge d\xi^B$ (non-degenerate, closed 2-form)

Hamiltonian vector flow generated by function f: $X_f^A = \omega^{AB}\partial_B f$

Darboux coordinates: $\xi^A = (x^\mu, p_\nu)$ s.t $\omega = dp_\mu \wedge dx^\mu$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Noether's theorem (phase space)
Let Hamiltonian H preserved by an infinit. transf. $\delta x^\mu \,, \ \delta p_\nu$
Then, there exists a conserved quantity Q:
$\{Q, H\} = 0$
$X_Q = \delta x^\mu \frac{\partial}{\partial x^\mu} + \delta p_\nu \frac{\partial}{\partial p_\nu}$

Dynamical symmetries

Spec: **Phase space** is a cotangent bundle of manifold M, T*(M).

Then there exists a **canonical projection:**

$$\pi \ : \ T^*(\mathcal{M}) \to \mathcal{M} \quad \Rightarrow$$

Can distinguish isometries from dynamical symmetries:

$$\pi^*(X_Q) = \begin{cases} \text{vector field on } M & \underline{\textit{isometry}} \\ \text{not well defined on } M & \underline{\textit{dynamical symmetry}} \end{cases}$$

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Laplace-Runge-Lenz:

$$X_{A^i} = (2x^i p^k - \delta^i_k x \cdot p - p^i x^k)\frac{\partial}{\partial x^k} - \left(\delta^i_k p^2 - p^i p_k - mk\delta^i_k \frac{1}{r} + mk\frac{x^i x^k}{r^3}\right)\frac{\partial}{\partial p^k}$$

$$\pi^*(X_{A^i}) = \left(2x^i p^k - \delta^i_k x \cdot p - p^i x^k\right)\frac{\partial}{\partial x^k}$$

dynamical symmetry

Symmmetries in GR

Particle motion $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ geodesics:

$$p^\mu \nabla_\mu p^\nu = 0$$

a) Linear in momentum conserved quantities:

$$C_K = K^\mu p_\mu \iff \nabla_{(\mu} K_{\nu)} = 0$$

...Killing vector equation

Proof: $\dot{C}_K = p^\nu \nabla_\nu (K^\mu p_\mu) = p^\nu p^\mu \nabla_{(\nu} K_{\mu)} + K^\mu \underbrace{p^\nu \nabla_\nu p_\mu}_{0} = 0$

Hamiltonian vector field: $X_{C_K} = K^\mu \frac{\partial}{\partial x^\mu} - \frac{\partial K^\lambda}{\partial x^\mu} p_\lambda \frac{\partial}{\partial p_\mu}$

$$\pi_*(X_{C_K}) = K^\mu \frac{\partial}{\partial x^\mu} = K$$

...isometry

b) Higher-order conserved quantities

$$C_K = K^{\mu_1 \dots \mu_p} p_{\mu_1} \dots p_{\mu_p}$$ ⇔

$$\nabla_{(\mu} K_{\nu_1 \dots \nu_p)} = 0$$

...Killing tensor equation

Walker & Penrose, Comm. Math. Phys. 18 , 265 (1970). (Stackel 1895).

$$\pi_*(X_{C_K}) = p K^{\mu_1 \dots \mu_{p-1}\nu} p_{\mu_1} \dots p_{\mu_{p-1}} \frac{\partial}{\partial x^\nu}$$

...dynamical symmetry

Hidden symmetries

Explicit symmetries

…Killing vectors (isometries)

Hidden symmetries

…Killing tensors (dynamical symmetries)

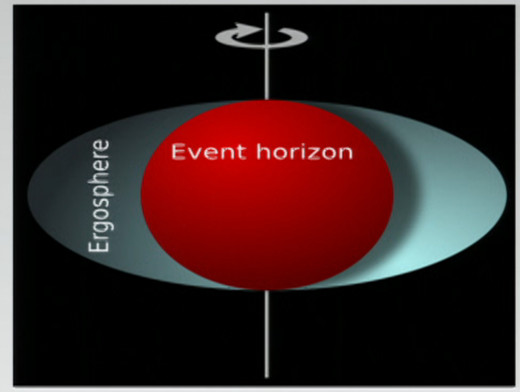
…Killing-Yano tensors (even more "fundamental" – they square to Killing tensors)

$$K_{\mu\nu} = f_{\mu\alpha} f_{\nu}{}^{\alpha}$$

Although we derived these as symmetries of the particle motion, they have far-reaching consequences for the properties of the spacetime and the dynamics of fields in it.

Kerr geometry

- **Unique vacuum** solution of Einstein equations describing a rotating black hole



Roy Patrick Kerr

- Discovered in 1963 by Kerr (4 years before Wheeler coins the term "black hole").
- Possesses two parameters: mass and rotation (no hair theorem)
- Provided cosmic censorship, Kerr solution is a final configuration of gravitational collapse – generic in our Universe.

- **Field equations decouple and separate**

  Scalar field, Dirac, electromagnetic, and gravitational perturbations **decouple and separate variables** (Carter 1968, Teukolsky 1972, Chandrasekhar & Page 1976, Wald 1978)

Enables to study:
- black hole shadow
- plasma accretion
- black hole stability
- Hawking evaporation
- …

- **Kerr-Schild form**: the metric can be written as a linear in mass deformation of the flat space

$$\boldsymbol{g} = \mathring{\boldsymbol{g}} + \frac{2Mr}{\Sigma}\,\boldsymbol{l}\,\boldsymbol{l}$$

- **Special algebraic type** of the Weyl tensor

Principal tensor

"All" the above properties can be attributed to the existence of a single object called:

Principal tensor = a (non-degenerate) closed conformal Killing-Yano 2-form

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a$$

For example: Carter's constant corresponds to the "square" of principal tensor

$$K_{ab} = h_{ac}h_b{}^c + \frac{1}{2}g_{ab}h^2$$

Special algebraic type: follows from integrability conditions of the above object

What about black holes in higher dimensions?

(motivated by string theory, brane world scenarios, GR)

- **Myers-Perry** generalization of the Kerr metric (1986)

rotates in [(D-1)/2]
orthogonal planes

Robert C. Myers

Malcolm J. Perry

Families of Killing-Yano tensors

for a general differential p-form

$$\nabla\omega = (\text{exterior} + \text{divergence} + \text{harmonic})\ \text{parts}$$

Conformal Killing-Yano (CKY) tensor

$$\nabla_X k = \frac{1}{p+1} X \lrcorner\, dk - \frac{1}{D-p+1} X^\flat \wedge \delta k\,.$$

Killing-Yano (KY) tensor: divergence part is missing

closed CKY tensor: exterior part is missing

Under Hodge duality divergence part transforms into exterior part and vice versa.

$$*(\text{closed CKY}) = \text{KY}$$

Principal Killing-Yano tensor

= (non-degenerate) closed CKY 2-form

$$\nabla_X \boldsymbol{h} = \boldsymbol{X}^\flat \wedge \boldsymbol{\xi}\,.$$

$$\nabla_X h_{ab} = 2X_{[a}\,\xi_{b]}$$

It follows

$$\boldsymbol{dh} = 0$$

$$\xi_b = \frac{1}{D-1}\nabla_a h^a{}_b$$

non-degenerate: full matrix rank, eigenvalues are functionally independent (can be used as coordinates)

Kerr-NUT-(A)dS spacetimes

In all dimensions admit the principal Killing-Yano tensor

V.P. Frolov, DK, PRL 98, 011101 (2007); DK, V.P. Frolov, Class. Quant. Grav. 24 , F1-F6 (20017).

The metric

In **canonical coordinates** $\{x_\mu, \psi_j\}$ the metric reads:

$$\boldsymbol{g} = \delta_{ab}\boldsymbol{\omega}^{\hat{a}}\boldsymbol{\omega}^{\hat{b}} = \sum_{\mu=1}^{n}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon\boldsymbol{\omega}^{\hat{0}}\boldsymbol{\omega}^{\hat{0}},$$

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{\omega}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{\mu}}.$$

$$D = 2n + \varepsilon.$$

Darboux basis: Euclidean & non-degenerate h

where

$$\boldsymbol{\omega}^{\hat{\mu}} = \frac{\boldsymbol{d}x_\mu}{\sqrt{Q_\mu}}, \quad \tilde{\boldsymbol{\omega}}^{\hat{\mu}} = \sqrt{Q_\mu}\sum_{j=0}^{n-1} A_\mu^{(j)}\boldsymbol{d}\psi_j, \quad \boldsymbol{\omega}^{\hat{0}} = \sqrt{\frac{-c}{A^{(n)}}}\sum_{j=0}^{n} A^{(j)}\boldsymbol{d}\psi_j.$$

$$A^{(j)} = \sum_{\nu_1<\dots<\nu_j} x_{\nu_1}^2 \dots x_{\nu_j}^2, \quad A_\mu^{(j)} = \sum_{\substack{\nu_1<\dots<\nu_j \\ \nu_i\neq\mu}} x_{\nu_1}^2 \dots x_{\nu_j}^2,$$

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2).$$

$$X_\mu = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2}.$$

Towers of hidden symmetries

Lemma ([Krtouš *et al.*, 2007b]). *Let* $\boldsymbol{k}^{(1)}$ *and* $\boldsymbol{k}^{(2)}$ *be two closed CKY tensors. Then their exterior product* $\boldsymbol{k} \equiv \boldsymbol{k}^{(1)} \wedge \boldsymbol{k}^{(2)}$ *is also a closed CKY tensor.*

closed CKY tensors:

$$\boldsymbol{h}^{(j)} \equiv \boldsymbol{h}^{\wedge j} = \underbrace{\boldsymbol{h} \wedge \ldots \wedge \boldsymbol{h}}_{\text{total of } j \text{ factors}} .$$

Killing-Yano tensors:

$$\boldsymbol{f}^{(j)} \equiv * \boldsymbol{h}^{(j)} .$$

$$\nabla_{(\alpha_1} f_{\alpha_2)\alpha_3 \ldots \alpha_{p+1}} = 0 .$$

Killing tensors:

$$K^{(j)}_{ab} \equiv \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}{}_{ac_1 \ldots c_{D-2j-1}} f^{(j)}{}_{b}{}^{c_1 \ldots c_{D-2j-1}} .$$

$$\boldsymbol{K}^{(j)} = \sum_{\mu=1}^{n} A^{(j)}_{\mu} (\boldsymbol{\omega}^{\hat{\mu}} \boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}} \tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon A^{(j)} \boldsymbol{\omega}^{\hat{0}} \boldsymbol{\omega}^{\hat{0}} .$$

$$\nabla_{(a} K^{(j)}_{bc)} = 0$$

Tower of explicit symmetries

Primary Killing vector:

$$\boldsymbol{\xi} = \boldsymbol{l}_{(0)} = \frac{1}{D-1} \nabla \cdot \boldsymbol{h}$$

Secondary Killing vectors:

$$\boldsymbol{l}_{(j)} = \boldsymbol{K}_{(j)} \cdot \boldsymbol{\xi}$$

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Since all symmetries generated from a single object h, they all **mutually** (Schouten-Nijenhuis) **commute**:

$$[\boldsymbol{l}_{(i)}, \boldsymbol{K}_{(j)}] = 0\,, \quad [\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}] = 0\,.$$

$$[K^{(j)}, K^{(l)}]_{abc} \equiv K^{(j)}_{e(a} \nabla^e K^{(l)}_{bc)} - K^{(l)}_{e(a} \nabla^e K^{(j)}_{bc)} = 0\,.$$

Complete integrability of geodesic motion

Definition. A motion in M^D is *completely integrable* if there exist D functionally independent integrals of motion which are in *involution*, that is, they mutually Poisson commute of one another [Arnol'd, 1989], [Kozlov, V. V., 1983].

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D=2n+ε constants of motion:

- Killing vectors: $$\Psi_k = \boldsymbol{l}_{(k)} \cdot \boldsymbol{u}$$ ... $n + \epsilon$
- Moreover we have Killing tensors:

$$\kappa_j = K^{(j)}_{\ \ ab} u^a u^b = \boldsymbol{u} \cdot \boldsymbol{K}^{(j)} \cdot \boldsymbol{u}\,.$$ ... n

D.N. Page, DK, M. Vasudevan, P. Krtouš, *Complete Integrability of Geodesic Motion in General Higher-Dimensional Rotating Black-Hole Spacetimes*, PRL 98 (2007) 061102.

Separability of scalar perturbations

$$(\Box - m^2)\phi = 0$$

Early results:

• 5D case: direct generalization of Carter's result in Myers-Perry coordinates

Frolov, Stojkovic, PRD 68 (20013) 064011

• Higher-dimensional attempts restricted to "special rotating black holes with enhanced symmetry (e.g. 2 sets of equal rotation parameters)

E.g. Vasudevan, Stevens, Page, CQG 22 (2005) 339.

Key observations:

- Separation occurs in **canonical coordinates** (not in Myers-Perry) that are completely fixed by the principal tensor.
- **Miraculous indentities** have to be pulled out of the hat.
- A slightly "**more involved**" separation of variables occurs:

Elementary separation:

$$\sum_n f_\nu = 0 \text{ where } f_\nu = f_\nu(x_\nu) \quad \Rightarrow \quad f_\nu = q_\nu = \text{const.}, \ \sum_\nu q_\nu = 0\,.$$

Needed:

$$\sum_\nu \frac{1}{U_\nu} f_\nu = 0 \qquad U_\mu = \prod_{\nu\neq\mu}(x_\nu^2 - x_\mu^2) \qquad \Rightarrow \qquad f_\nu = \sum_{k=0}^{N-2} Q_k(-x_\nu^2)^k$$

Separability of Dirac fields

$$(\gamma^a D_a + m)\Psi = 0$$

- Solution found in R-separable form:

$$\psi = R\,\exp\left(i \sum_k \Psi_k \psi_k\right) \bigotimes_\nu \chi_\nu \qquad R = \prod_{\substack{\kappa,\lambda \\ \kappa<\lambda}} \left(x_\kappa + \iota_{\langle\kappa\lambda\rangle} x_\lambda\right)^{-\frac{1}{2}}$$

Oota, Yasui, PLB 659 (2008) 688.

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Oota, Yasui, PLB 659 (2008) 688.

- Is **intrinsically characterized** by a complete set of operators (this time constructed from closed CKY tensors):

$$K_k \equiv K_{\xi_{(k)}} = X^a \lrcorner\, \xi_{(k)} \nabla_a + \frac{1}{4} d\xi_{(k)}$$

$$M_j = M_{h^{(j)}} \equiv e^a \wedge h^{(j)} \nabla_a - \frac{n-2j}{2(n-2j+1)} \delta h^{(j)}$$

Cariglia, Krtous, DK, PRD 84 (2011) 024008.

Separability of Maxwell fields

$$\nabla_a F^{ab} = 0\,, \quad \boldsymbol{dF} = 0$$

Traditional approach: uses **Newmann-Penrose formalism** (separation for field strength). This does not quite work in higher dimensions – only partial success (for near horizon geometries) achieved.

For example:

- Durkee, Reall, PRD83 (2011) 104044.
- Araneda, arXiv:1711.09872.

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Breakthrough achieved by Lunin, by abandoning the Newmann-Penrose paradigm

$$\mathrm{A}^a = B^{ab}\nabla_b Z \qquad Z = \Big(\prod_\nu R_\nu\Big)\,\exp\Big(i\sum_j L_j\psi_j\Big)$$

O. Lunin, *Maxwell's equations in the Myers-Perry geometry*, JHEP 1712 (2017) 138.

The ansatz can be covariantly written in terms of the principal tensor.

$$\mathrm{A}^a = B^{ab}\nabla_b Z$$

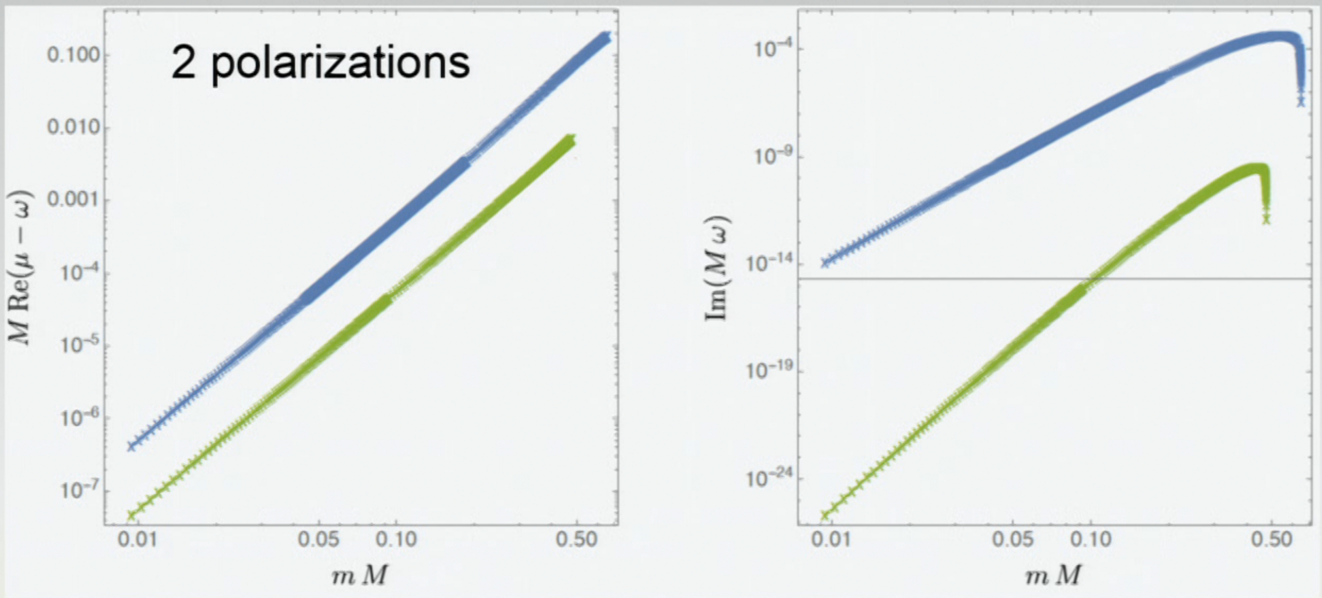
$$(g_{ab} + i\mu h_{ab})B^{bc} = \delta_a^c$$

The Maxwell equations can be written as a composition of operators, which form a complete set of commuting operators acting on Z. The corresponding separation constants are the eigenvalues of these operators and their common eigenfunction is the separated solution.

Frolov, Krtous, DK, arXiv:1802.09491.
Krtous, Frolov, DK, arXiv:1803.02485.

Separability of Proca equations

$$\nabla_b F^{ab} + m^2 A^a = 0 \Rightarrow \nabla_a A^a = 0$$



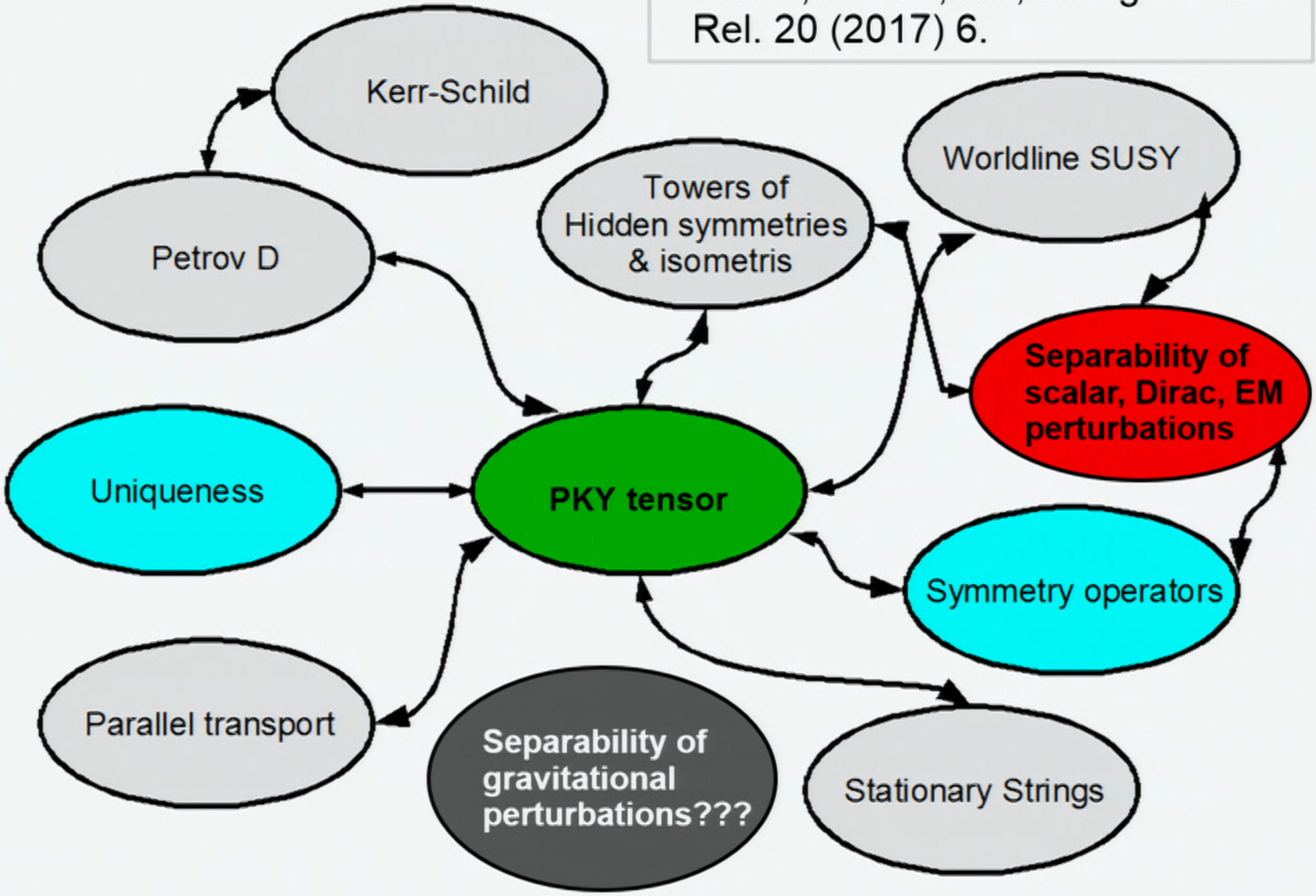
Describes **superradiant instability** due ultralight massive vector particles.

1) Baryakhtar, Lasenby, Teo, PRD96 (2017) 035019.
2) Cardoso, Dias, Hartnett, Middleton, Pani, Santos, JCAP 1803 (2018) 043.

V.P. Frolov, P. Krtous, DK, J.E. Santos, in preparation

Miraculous properties of rotating black holes

Frolov, Krtous, DK, Living Review Rel. 20 (2017) 6.



Summary

1) **Dynamical symmetries** are genuine phase space symmetries that play interesting role in many areas of physics. They are **hidden** in configuration space and “escape” traditional simplified formulations of Noether’s theorem.

2) In GR these are described by **Killing** and **Killing-Yano** tensors. In particular, the **principal Killing-Yano (PKY)** tensor plays a crucial role for various integrability properties of black holes (geodesics, KG, Dirac, type D, Kerr-Schild form,…).

3) Long-standing open question was whether these symmetries can also be exploited for **higher-spin equations –** to separate EM & gravitational perturbations.

4) Very recently this problem was resolved for the **EM** and **Proca perturbations**!

5) This perhaps gives hope that one could also separate the **gravitational perturbations** in these spacetimes.