Title: On recent development of representation theory of W-algebras and related topics

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Abstract:

① coset construction

and q-geom. Langlands

② 4d Higgs branches

and Vertex algs

① $\kappa = k + h^\vee$

$W^\kappa(\mathfrak{g})$ principal W-alg

$\|$ at level $k = \kappa - h^\vee$

$DS(V^\kappa(\mathfrak{g}))$ $\quad V^\kappa(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}c)} \mathbb{C}_k$

$\searrow$

$W_\kappa(\mathfrak{g})$ unique simple quotient $\quad L_\kappa(\mathfrak{g})$

① $\kappa = k + h^\vee$

$W^\kappa(\mathfrak{g})$ principal W-alg at level $k = \kappa - h^\vee$

$\|$

$DS(V^\kappa(\mathfrak{g}))$ $\quad V^\kappa(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k$

$W_\kappa(\mathfrak{g})$ unique simple quotient $\quad L_\kappa(\mathfrak{g})$

minimal series W-alg

$\{ W_\kappa(\mathfrak{g}) \mid \kappa = \frac{p}{q},\ p \cdot q = h^\vee,\ (p, q) = 1 \}$

<u>Thm</u> [A.-Creutzig-Linshaw '17] $\mathfrak{g}$ ADE κ

$$W_\kappa(\mathfrak{g}) \cong \left(L_\psi(\mathfrak{g}) \otimes L_1(\mathfrak{g}) \right)^{\mathfrak{g}[t]} \qquad \kappa = \frac{\psi}{\psi + 1}$$

and $W_\kappa(\mathfrak{g})$ & $L_{\psi+1}(\mathfrak{g})$ form a dual pair

$\mathfrak{g}$

$-n-h^\vee$

$U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}[t])} \mathbb{C}_k$

$L_n(\mathfrak{g})$

minimal series W-alg

$\{ W_n(\mathfrak{g}) \mid n = \frac{p}{q},\ p \cdot q = h^\vee,\ (p,q)=1 \}$

<u>Thm</u> [A-Creutzig-Linshaw '17] $\mathfrak{g}$ ADE κ

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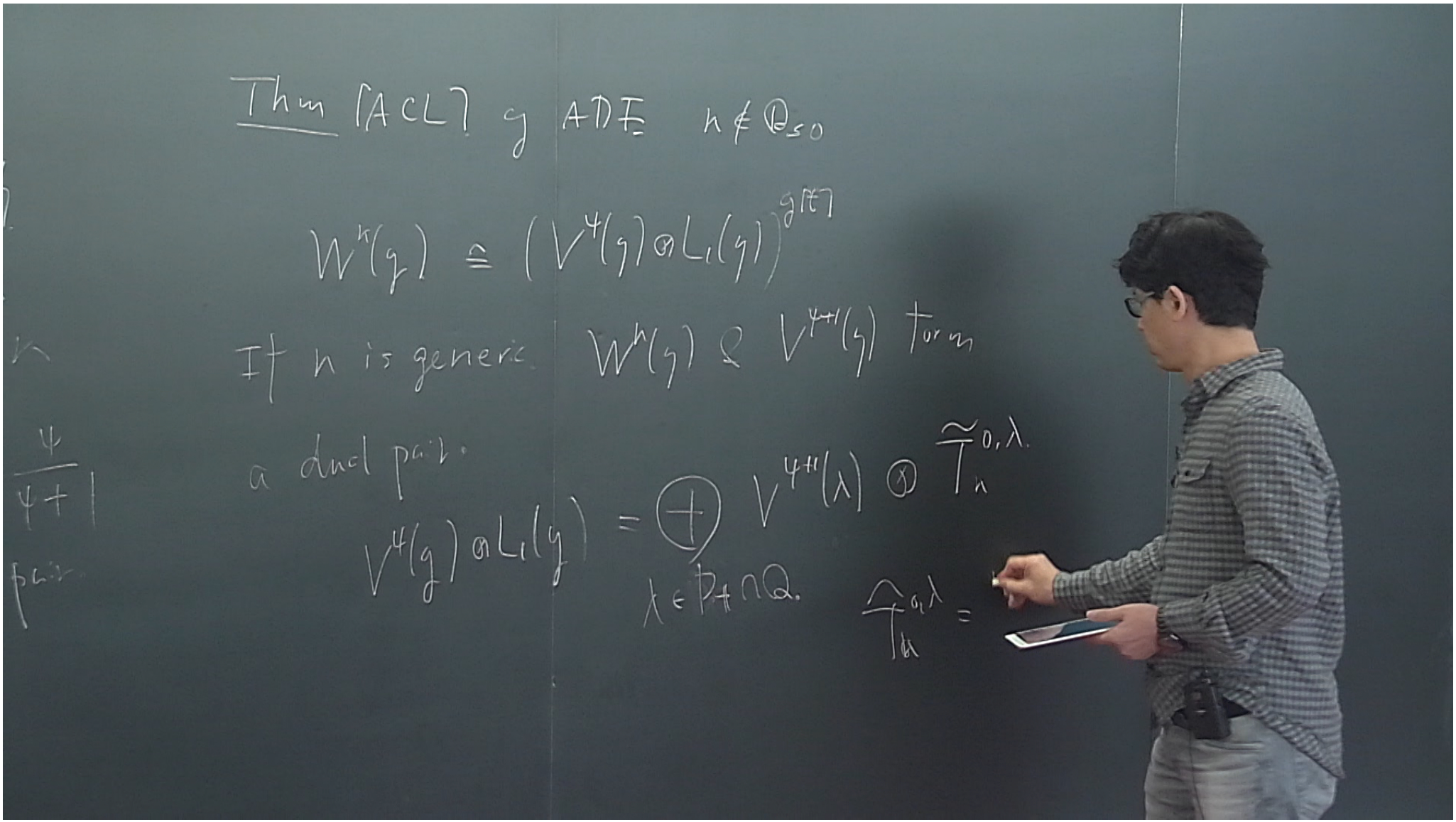
and $W_\kappa(\mathfrak{g})$ & $L_{\psi+1}(\mathfrak{g})$ form a dual pair

<u>Thm</u> [ACL] $\mathfrak{g}$ ADE $n \notin \mathbb{Q}_{\leq 0}$

$$W^n(\mathfrak{g}) \cong \left(V^\psi(\mathfrak{g}) \otimes L_1(\mathfrak{g})\right)^{\mathfrak{g}[t]}$$


Thm [ACL] g ADE n ∉ Q≤0
W^n(g) ≅ (V^ψ(g) ⊗ L_1(g))^{g[t]}
If n is generic W^n(g) & V^{ψ+1}(g) form
a dual pair.
V^ψ(g) ⊗ L_1(g) = ⊕ V^{ψ+1}(λ) ⊗ T̃^{0,λ}_n
λ ∈ P_+ ∩ Q.
T̂^{0,λ}_n =

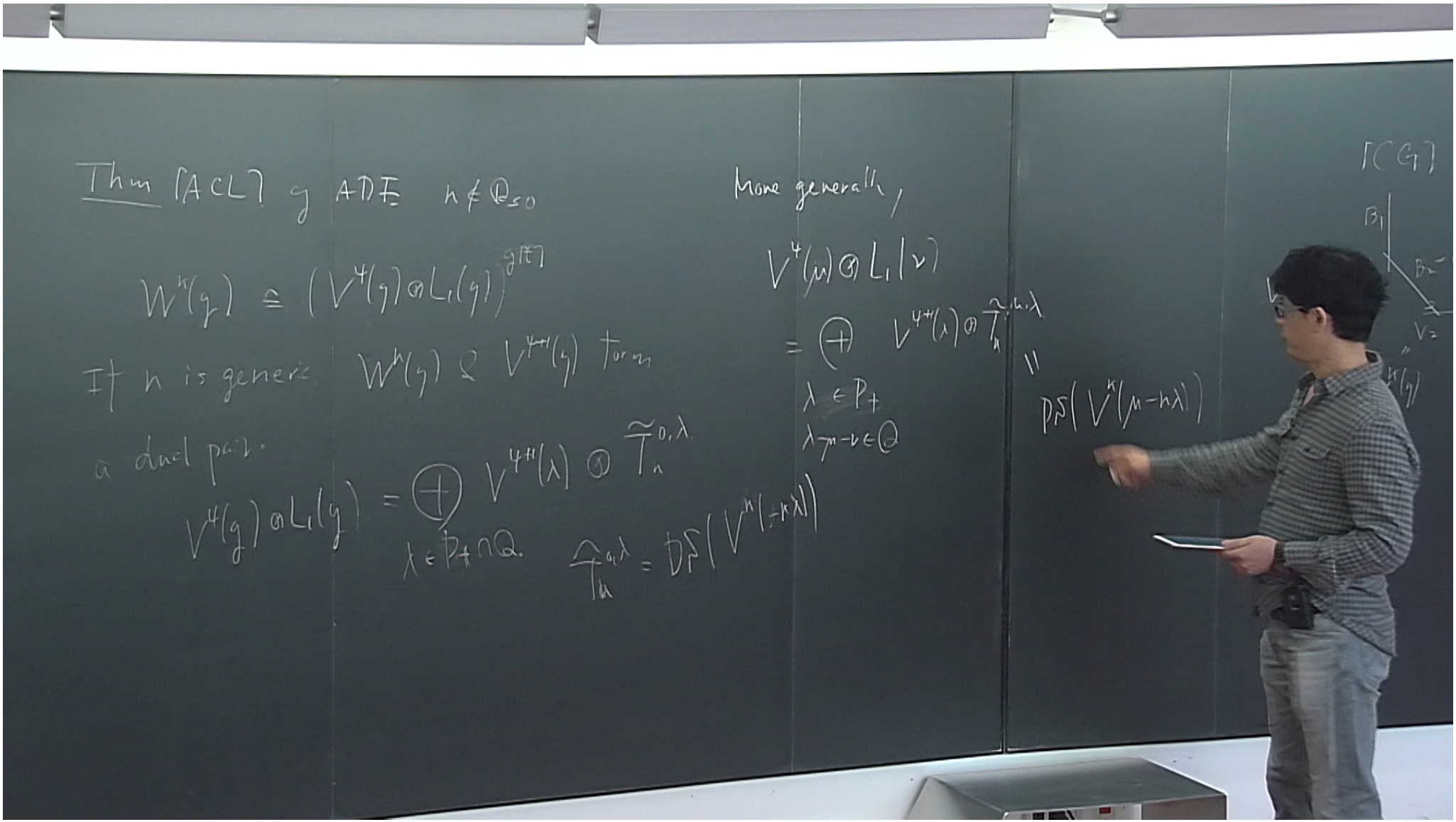
<u>Thm</u> [ACL] $\mathfrak{g}$ ADE $n \notin \mathbb{Q}_{\leq 0}$

$$W^n(\mathfrak{g}) \cong \left(V^\psi(\mathfrak{g}) \otimes L_1(\mathfrak{g})\right)^{\mathfrak{g}[t]}$$

If n is generic $W^n(\mathfrak{g})$ & $V^{\psi+1}(\mathfrak{g})$ form a dual pair.

$$V^\psi(\mathfrak{g}) \otimes L_1(\mathfrak{g}) = \bigoplus_{\lambda \in P_+ \cap Q} V^{\psi+1}(\lambda) \otimes \widetilde{T}_n^{0,\lambda}$$

$$\widehat{T}_n^{0,\lambda} = DS\left(V^n(-n\lambda)\right)$$


Thm [ACL] g ADE $n \notin \mathbb{Q}_{\le 0}$
$W^n(g) \cong (V^4(g) \otimes L_1(g))^{g[t]}$
If n is generic $W^n(g)$ & $V^{4+1}(g)$ form a dual pair.
$V^4(g) \otimes L_1(g) = \bigoplus_{\lambda \in P_+ \cap Q} V^{4+1}(\lambda) \otimes \widetilde{T}_n^{0,\lambda}$
$\widetilde{T}_n^{0,\lambda} = DS(V^n(-n\lambda))$
More generally,
$V^4(\mu) \otimes L_1(\nu)$
$= \bigoplus_{\substack{\lambda \in P_+ \\ \lambda, \mu-\nu \in Q}} V^{4+1}(\lambda) \otimes \widetilde{T}_n^{\mu,\lambda}$
$\widetilde{T}_n^{\mu,\lambda} = DS(V^n(\mu - n\lambda))$
[CG]
B_1
B_2
V_2
$V^n(g)$

① coset construction

and q-geom. Langlands

② 4d Higgs branches

and Vertex algs

$\mathfrak{g}$ any

FF duality

$$\hat{T}^{\lambda,\mu}_{\lambda} \cong \hat{T}^{\check{\mu},\check{\lambda}}_{\check{\lambda}}$$

$$\sigma_\mu DS(V(\lambda)) = DS_{\psi_\mu}(V^\mu(\lambda))$$

$$= T^{\lambda,\mu}$$

spectral flow

series U

$| \kappa =$

Creutzig-

$(\mathfrak{g}) \cong ($

Gen. $\mathcal{A}$

FF duality

$$T^{\lambda,\mu} \cong T^{\mu,\lambda}$$

$$DS(V^n(\lambda)) = DS_{\psi_\mu}(V^n(\lambda))$$

$$= T^{\lambda,\mu}$$

Thm (Conj [G], [CG])

$$T^{\lambda,\mu} \cong \hat{T}^{\lambda,\mu}$$

$$\cong \qquad\qquad \|$$

$$T^{\mu,\lambda} = \hat{T}^{\mu,\lambda}$$

Thm [ACL] g A

$$W^n(g) \cong (V^4$$

If n is generic V

a dual pair.

$$V^4(g) \otimes L_1(g)$$

Thm (Conj [G], [CG])

$$T_{\lambda}^{\lambda,\mu} \cong \hat{T}^{\lambda,\mu}$$

$$\| \qquad\qquad \|$$

$$T_{\lambda}^{\mu,\lambda} = \hat{T}^{\mu,\lambda}$$

$$\Theta^{[\mu-n\lambda+k\lambda]}_{(\hat{\gamma})} \simeq \Theta(g)^{[\mu]}$$

$$0 \to V^n(\lambda-n\mu) \to W^n(\lambda-n\mu) \to \bigoplus_i W^n(s_i\cdot\lambda-n\mu) \to \cdots$$

$$\tilde{T}_n^{0,\lambda}$$

$$\tilde{T}_n^{0,\lambda} = \mathrm{DS}(V^n(-n\lambda))$$

More generally,

$$V^\nu(\mu) \otimes L_1 \mid$$

$$= \bigoplus V$$

$\lambda \in P_+$

$\lambda - \mu - \nu \in Q$

Thm (Conj [G], [CG])

$$T^{\lambda,\mu} \cong \hat{T}^{\lambda,\mu}$$

$$\wr\| \qquad\qquad \|$$

$$T^{\mu,\lambda} = \hat{T}^{\mu,\lambda}$$

$\because$ $\Theta^{[\mu-n\lambda+k\lambda]}(\hat{T}) \simeq \Theta(g)^{[\mu]}$

$$0 \to V^n(\lambda-n\mu) \to W^n(\lambda-n\mu) \to \bigoplus_i W^n(s_i\cdot\lambda-n\mu) \to \cdots$$

$$\overset{DR(?)}{\rightsquigarrow}$$

$$0 \to \hat{T}^{\lambda,\mu} \to \Pi_{\lambda-n\mu} \to \bigoplus_i \Pi_{s_i\cdot\lambda-n\mu}$$

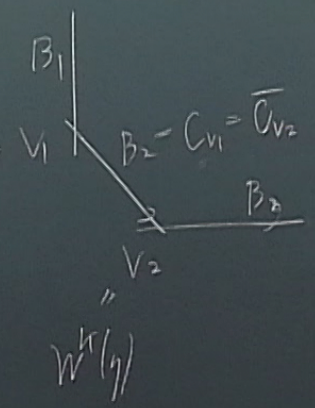
$$0 \to V^h(\lambda) \to W^h(\lambda) \to \bigoplus W^h(s\cdot\lambda)$$

$$\xrightarrow{D\delta(z')^t} \quad 0 \to D\delta(V^h(\lambda)) \to \Pi_\lambda \to \Pi_{s\cdot\lambda}$$

$$\xrightarrow{\sigma_\mu} \quad 0 \to T^{\lambda,\mu} \to \Pi_{\lambda+\mu} \to \bigoplus \Pi_{s\cdot\lambda+\mu}$$

Π $\quad$ D

$\Gamma(G)$



$\rightsquigarrow$

$V^{h+1}(\eta)$

$\bigoplus M$

$M_i \in \mathrm{Irr}(C_{V_i})$

① coset construction

and $\mathfrak{g}$ - geom. Langlands

② 4d Higgs branches

and Vertex algs

②

Rastelli et al

$\{4d\ N=2\ SCFTs\} \xrightarrow{\Phi} \{VoAs\}$

s.t. Schur index of $\mathcal{T}$ = ch $\Phi(\mathcal{T})$

$\{VOAs\}$

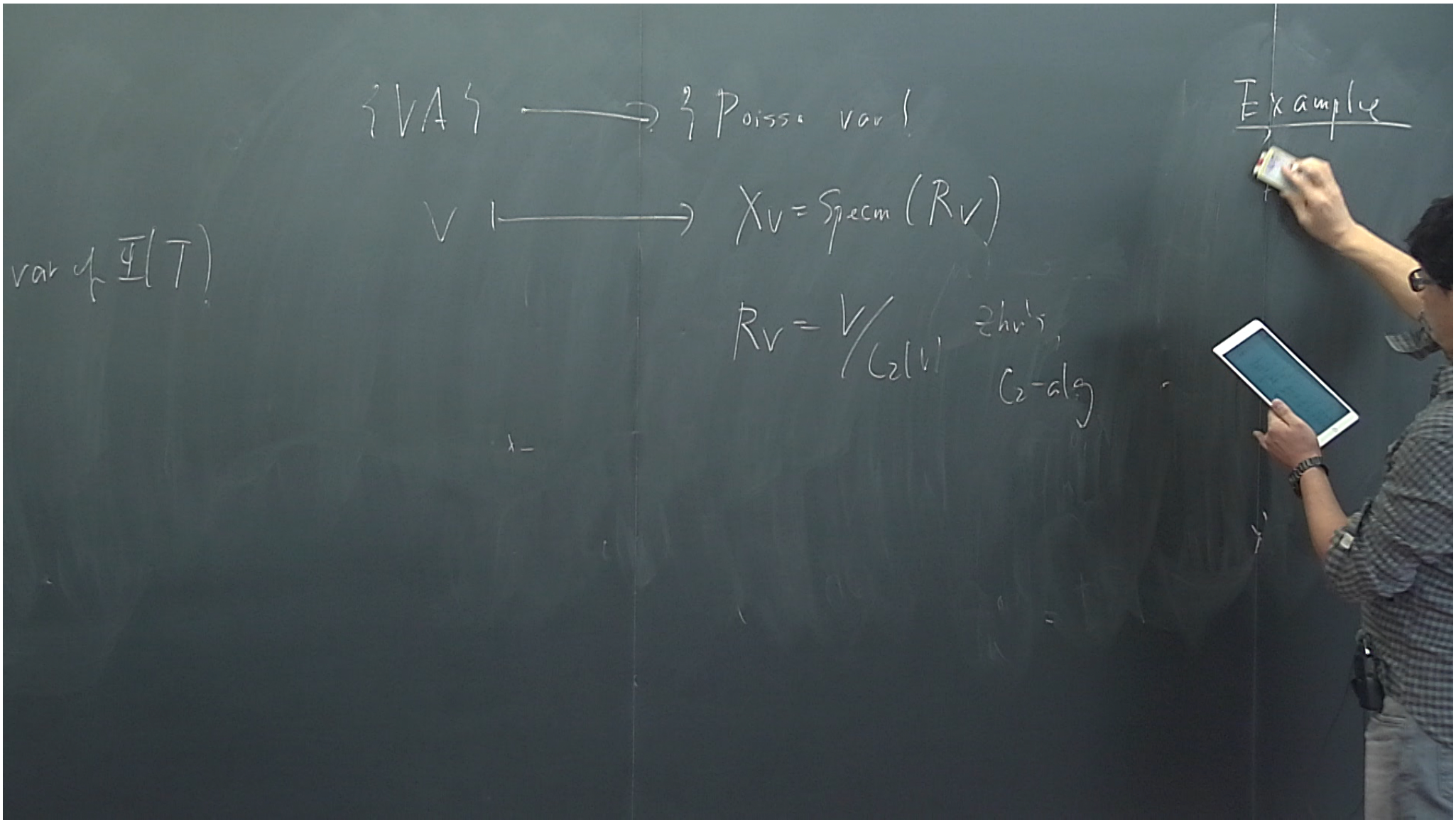
$= \mathrm{ch}\, \Phi(T)$

<u>Conj</u> (Beem-Rastelli)

$\mathrm{Higgs}(T) \cong X_{\Phi(T)}$ ass. var of $\Phi(T)$

$\{VA\}$

V


{VA} ⟶ {Poisson var}
V ⟼ X_V = Specm(R_V)
R_V = V/C_2(V)
C_2-alg
Example
var of Φ(T)

$R_V)$

Zhu's

C_2-alg

Example

1) $V = V^n(\mathfrak{g})$

$R_V = V^n(\mathfrak{g}) / \mathfrak{g}(t^{-1})t^{-2}V^n(\mathfrak{g})$ $\xleftarrow{\sim}$ $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$

$(x_1 t^{-1}) \cdots (x_n t^{-1})(1) \longleftarrow x_1 \cdots x_n$

$X_{V^n(\mathfrak{g})} \cong \mathfrak{g}^*$

2) $V = L_k(\mathfrak{g})$

$R_{L_k(\mathfrak{g})} = L_k(\mathfrak{g}) \big/ \mathfrak{g}(t^{-1})t^{-2}L_k(\mathfrak{g})$

$\mathcal{S}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$

$\chi_{L_k(\mathfrak{g})} \subset \chi_{V^H(\mathfrak{g})} = \mathfrak{g}^*$

$\chi_1 \cdots \chi_s$

$\chi_{V^H(\mathfrak{g})} \simeq \mathfrak{g}^*$

Rastelli et al

$\{N=2 \text{ SCFTs}\} \xrightarrow{\Phi} \{\text{VOAs}\}$

Schur index of $\mathcal{T}$ = ch $\Phi(\mathcal{T})$

- $X_{L_k(\mathfrak{g})} = \{0\} \Leftrightarrow L_k(\mathfrak{g})$ is integrable

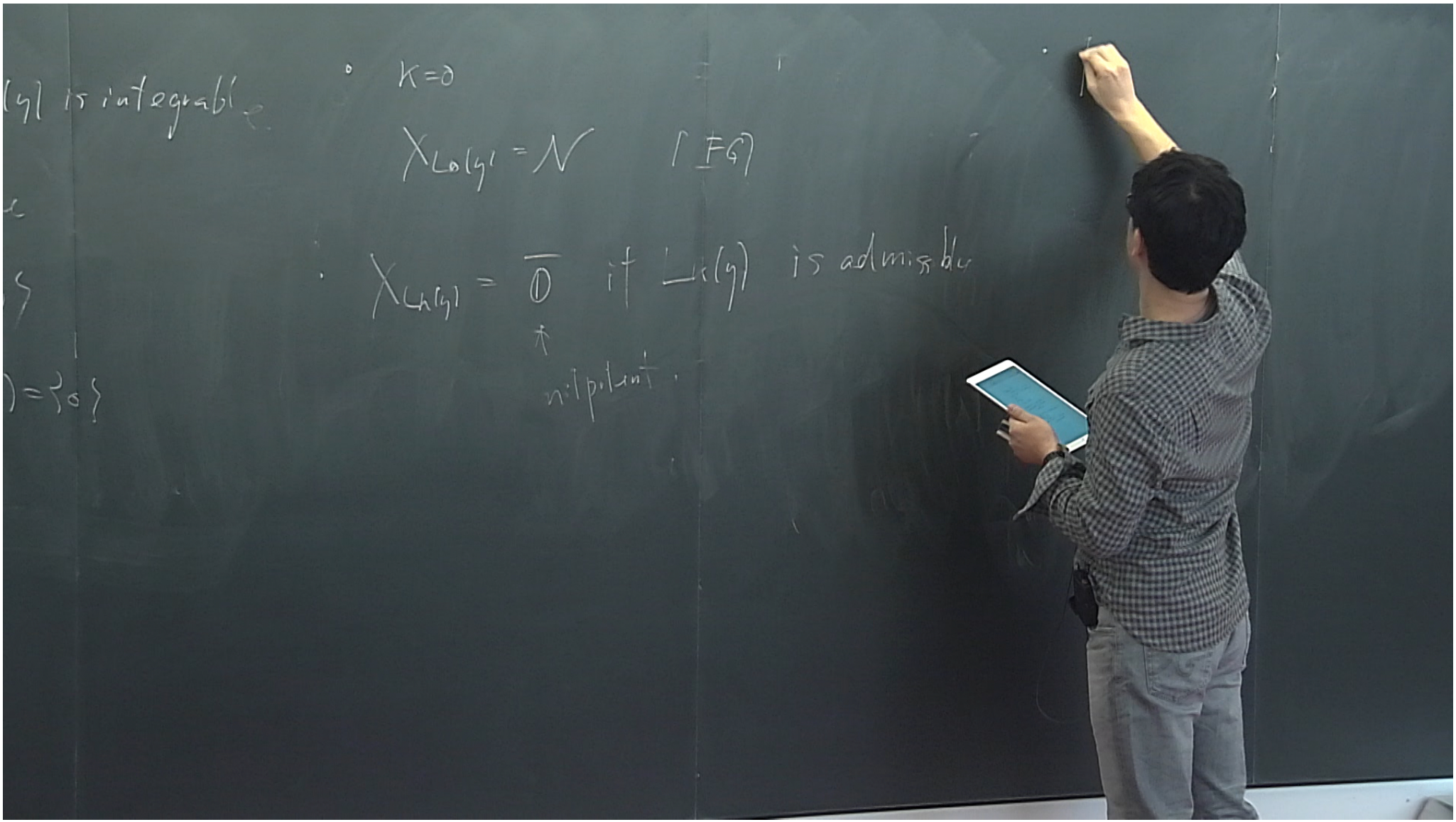
V is lisse (C_2-cofinite)

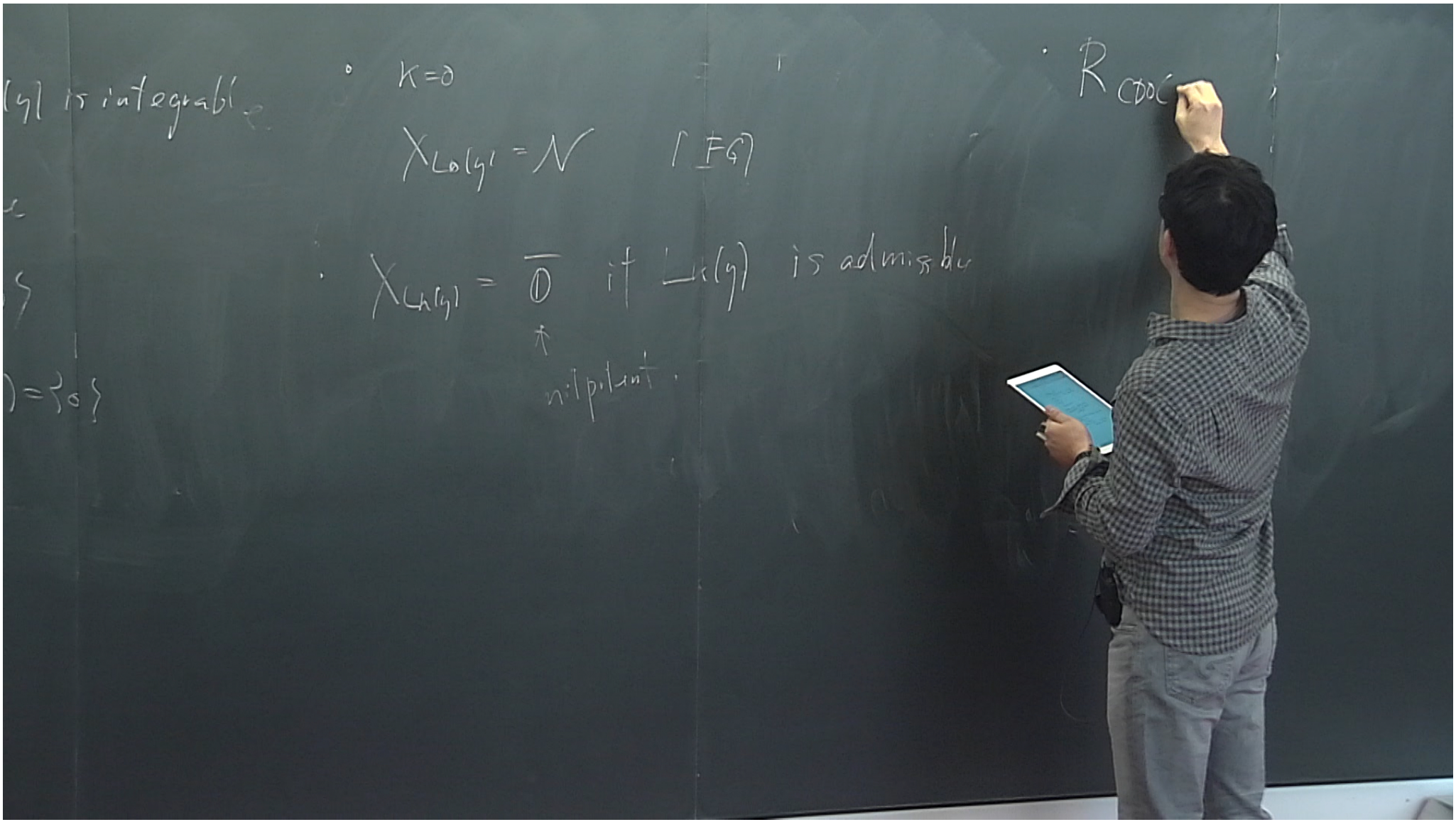
$\Leftrightarrow X_V = \{0\}$

$\Leftrightarrow \mathrm{Specm}(\mathrm{gr}\, V) = \{0\}$

$\{VA$

V


(y) is integrable
K=0
X_{L_0(y)} = N [FG]
X_{L_k(y)} = 0 if L_k(y) is admissible
nilpotent


(y) is integrabl
K=0
X_L0(y) = N [FG]
X_L0(y) = 0 if L_K(y) is admissible
nilpotent
R CDOC

$\cdot\ X_{CDO(G)_{n,-n}} = T^*G$

$\cdot\ V$ VA object $KL_n(G)$

$V^n(y)$

$\Rightarrow X_{DS(V)} \cong X_V \times_{g^*} \mathcal{S}$

$\mathcal{S} = e + g^+$ Kostant-slodowy slice

$X_{W^n(y)} = g^{+} \times_{g^+} \mathcal{S} = \mathcal{S}$

$X_{DS(L_n(y))} = X_{L_n(y)} \cap \mathcal{S}$

$$X_{W^n(\mathfrak{g})} = \mathfrak{g}^* \times_{\mathfrak{g}^*} \mathfrak{S} = \mathfrak{S}$$

$$X_{DS(L_n(\mathfrak{g}))} = X_{L_n(\mathfrak{g})} \cap \mathfrak{S}$$

$$X_{DS(CDG(\mathfrak{g})_{n-1})} = G \times \mathfrak{S} = G \times_N (n + n^\perp)$$

$$X_{DS(DS(CDG(\mathfrak{g})_{n-1}))} = (G \times \mathfrak{S}) \times_{\mathfrak{g}^*} \mathfrak{S} = \mathrm{Specm}\, H_*^{G(O)}(G \times \mathfrak{S})$$

① Coset construction
and q-geom Langlands

② 4d Higgs branches
and Vertex algs

V is quasi-lisse

$\Leftrightarrow$ X_V has f.m symplectic leaves

$\Rightarrow$ $\mathrm{Loc}(V)_E$ is f.d.

$\cong R_L$

R_L, R_S

$\Rightarrow$ modular inv of $\mathrm{ch}V$

$$\{4d\ \mathcal{N}=2\ \text{SCFTs}\}$$

$$\cup$$

$$\{\text{the theory of Class } \mathcal{S}\}$$

$$\Big\downarrow \text{ MT conj ([BFM])}$$

$$\{\text{symplec var}\}$$

$$\begin{tikzcd}
\{4d\ \mathcal{N}=2\ \text{SCFTs}\} \arrow[d, phantom, "\cup"] & & \\
\{\text{the theory of Class } \mathcal{S}\} \arrow[rr, "\Phi"] \arrow[d, "\text{MT conj (IBFM)}"] & & \{\text{chiral algs of class } \mathcal{S}\} \arrow[dll] \\
\{\text{symplec var}\} & &
\end{tikzcd}$$

$\Downarrow$ co

objects

morphism

$\mathbb{V}$ category of VA.

$\to$ { chiral algs of class $\mathcal{S}$ }

objects — semi-simple groups G

morphism — $\mathrm{Hom}(G_1, G_2) = \{$ VA object in $KL(G_1)_0 \times KL(G_2)_0 \}$

$\hat{g}_1 \circlearrowright V \circlearrowleft \hat{g}_2$

$V_1 \in \mathrm{Hom}(G_1, G_0)$

$V_2 \in \mathrm{Hom}(G_2, G_1)$

$V_1 \circ V_2 \in \mathrm{Hom}(G_1, G_3)$

$\| $

$H^{\frac{10}{2}+0}(\hat{G}_{21}, \mathfrak{g}_{21}, V_1 \otimes V_2)$

$KL(G_0)_0$

$X_{V_1 \circ V_2} = X_{V_1} \circ X_{V_2} /\!/ \alpha(G_2)$

monoidal

on

anglands

anches

Thm (Conj. by Raskell et al)

$\forall G \quad \exists!$ monoidal fun

$$\mathbb{B}_2 \longrightarrow \mathbb{V}$$

s.t. i) $\mathcal{Y}_G(S^1) = G$

ii) $\mathcal{Y}_G(\square) = CDO(G)_{G\times G}$

iii) $\mathcal{Y}_G(\triangleright) = DS(CDO(G)_{G\times G})$

{4d $N=2$ SCFTs}

$\cup$

{the theory of Clas

$\downarrow$ MT con

(AG) $CDO(G)_{G\times G} \circ M = M$

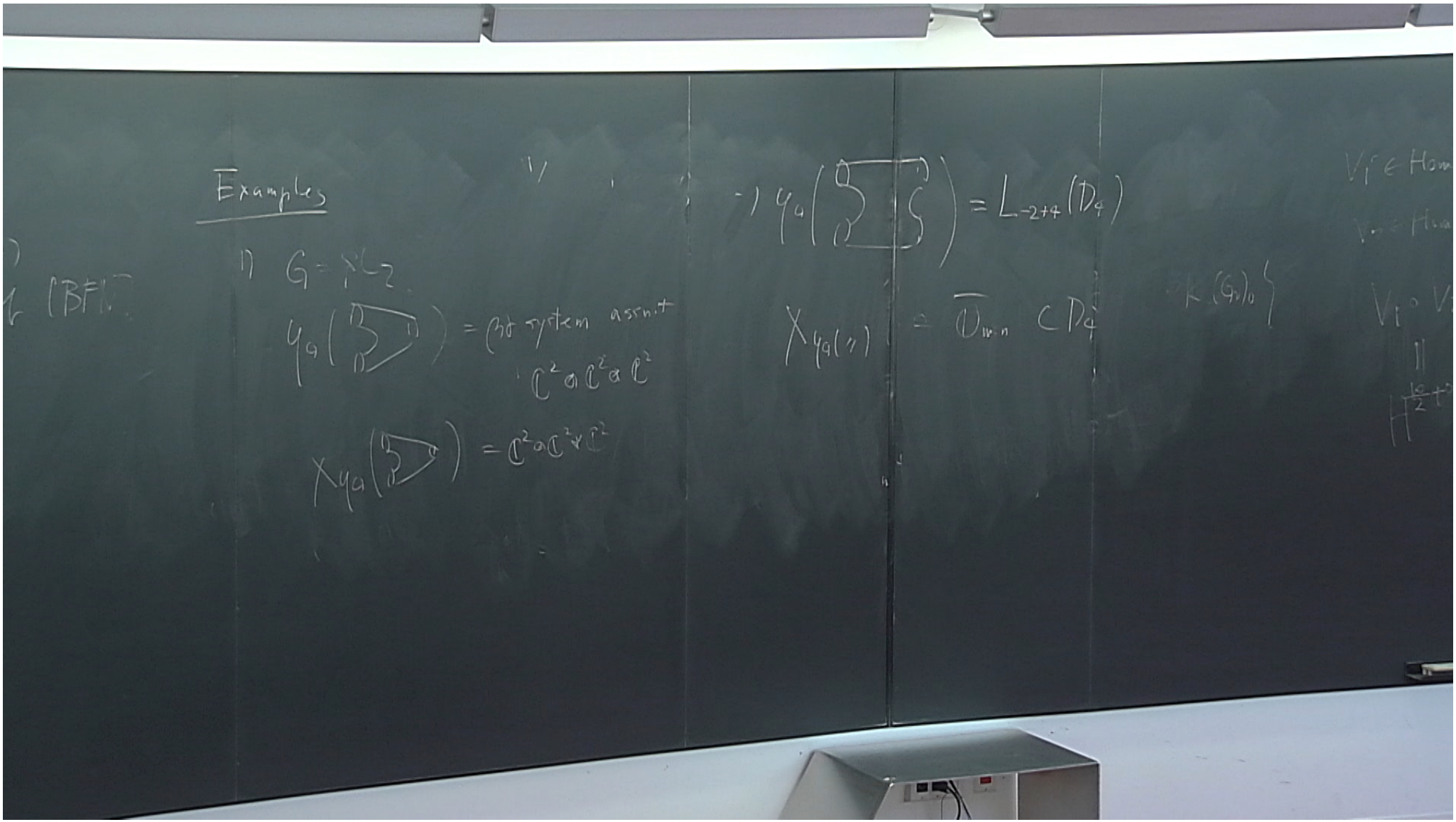
symplec var

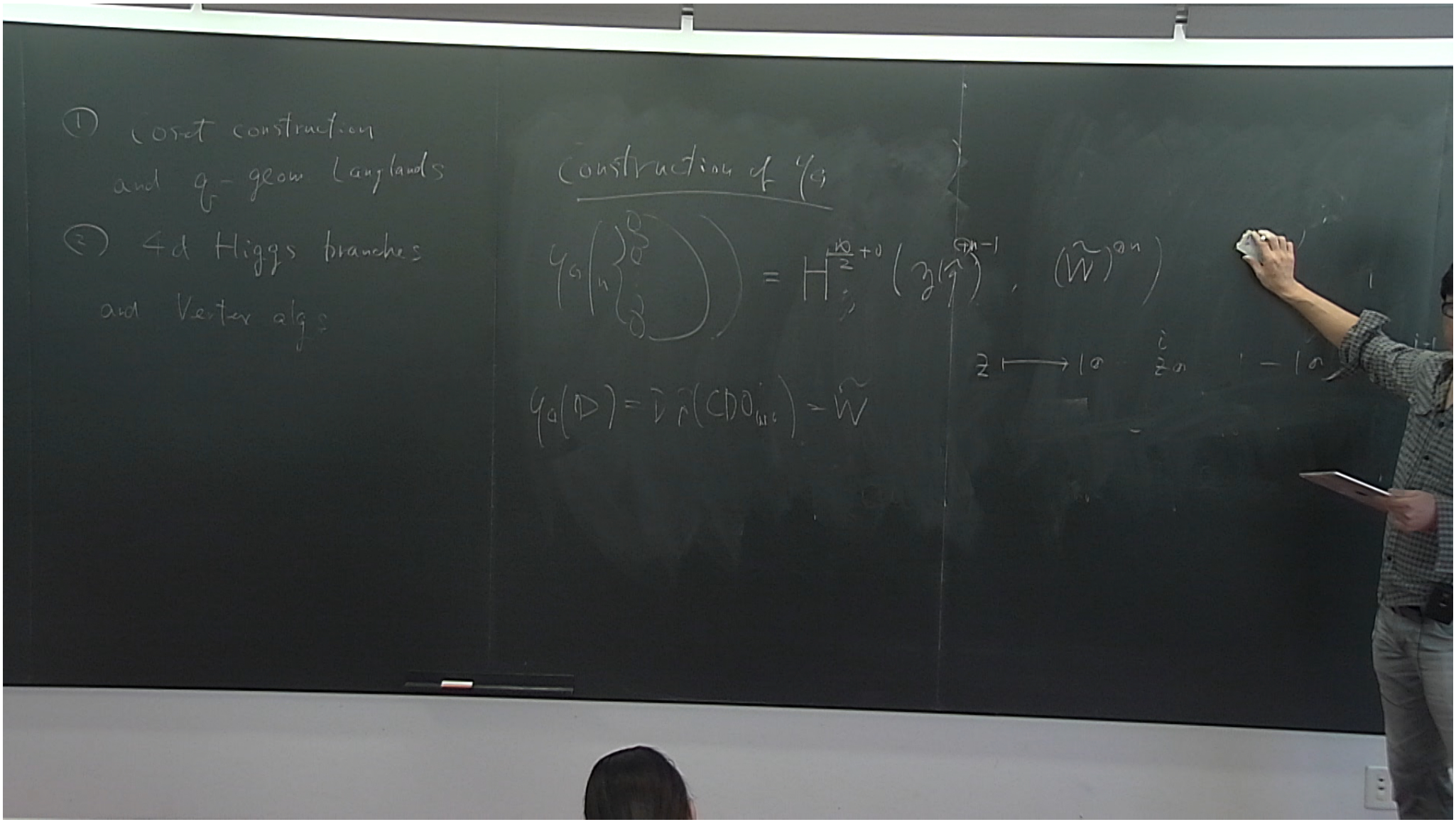
Moreover.

$$B \longmapsto X_{\psi_0(B)}$$

recover the functor of [BF]. $\rightarrow$ { chiral algs of class $\mathcal{B}$ }

$$(A^!G) \quad CDO(G)_{G,0} \circ M = M$$


Examples
G = S_2
= BF system as v.s
C^2 ⊗ C^2 ⊗ C^2
= C^2 ⊗ C^2 × C^2
= L_{-2+4}(D_4)
= D_{w,n} ⊂ D_4


① coset construction
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Construction of $\mathcal{Y}_G$
$\mathcal{Y}_G\left(\begin{pmatrix} \} & 0 \\ n & \} \\ & 0 \end{pmatrix}\right) = H^{\frac{\kappa_0}{2}+0}\left(\mathcal{Z}(\hat{\mathfrak{g}})^{\otimes n-1}, (\tilde{W})^{\otimes n}\right)$
$z \longmapsto 1 \otimes$
$\mathcal{Y}_G(\mathbb{D}) = \mathrm{Tr}\,\hat{\rho}(\mathbb{D}\,\mathcal{O}_{\mathrm{WZ}}) = \tilde{W}$

Construction of ψ_G

$$\psi_G\left(\begin{pmatrix} \delta & & \\ & \ddots & \\ & & \delta \end{pmatrix}\right) = H^{\frac{\dim}{2}+0}\left(\mathcal{Z}(\hat{\eta})^{\otimes n-1}, (\tilde{W})^{\otimes n}\right)$$

$$\psi_G(\mathbb{D}) = \mathrm{Tr}(CDO_{W_G}) = \tilde{W}$$

$$p_i(z) \cdots q_n(z)$$
$$\| $$
$$\sum p_i' z^{-n-1}$$

$$\mathcal{L} = \bigoplus_{i=1}^{n} \mathbb{C} z_{(n)}^{i} \quad \mathrm{rk}\, \mathcal{L} = \bigoplus_{n=-\infty}^{\infty}$$

$$z \longmapsto (0 \cdots \overset{i}{z} 0 \cdots) - (0 \cdots 0 \overset{i+1}{z} 0 \cdots)$$

Moreover.

$$B \longmapsto X_{\psi_G(B)}$$

recover the Tutte of (BF).

Examples

1) $G =$

$\psi_G($