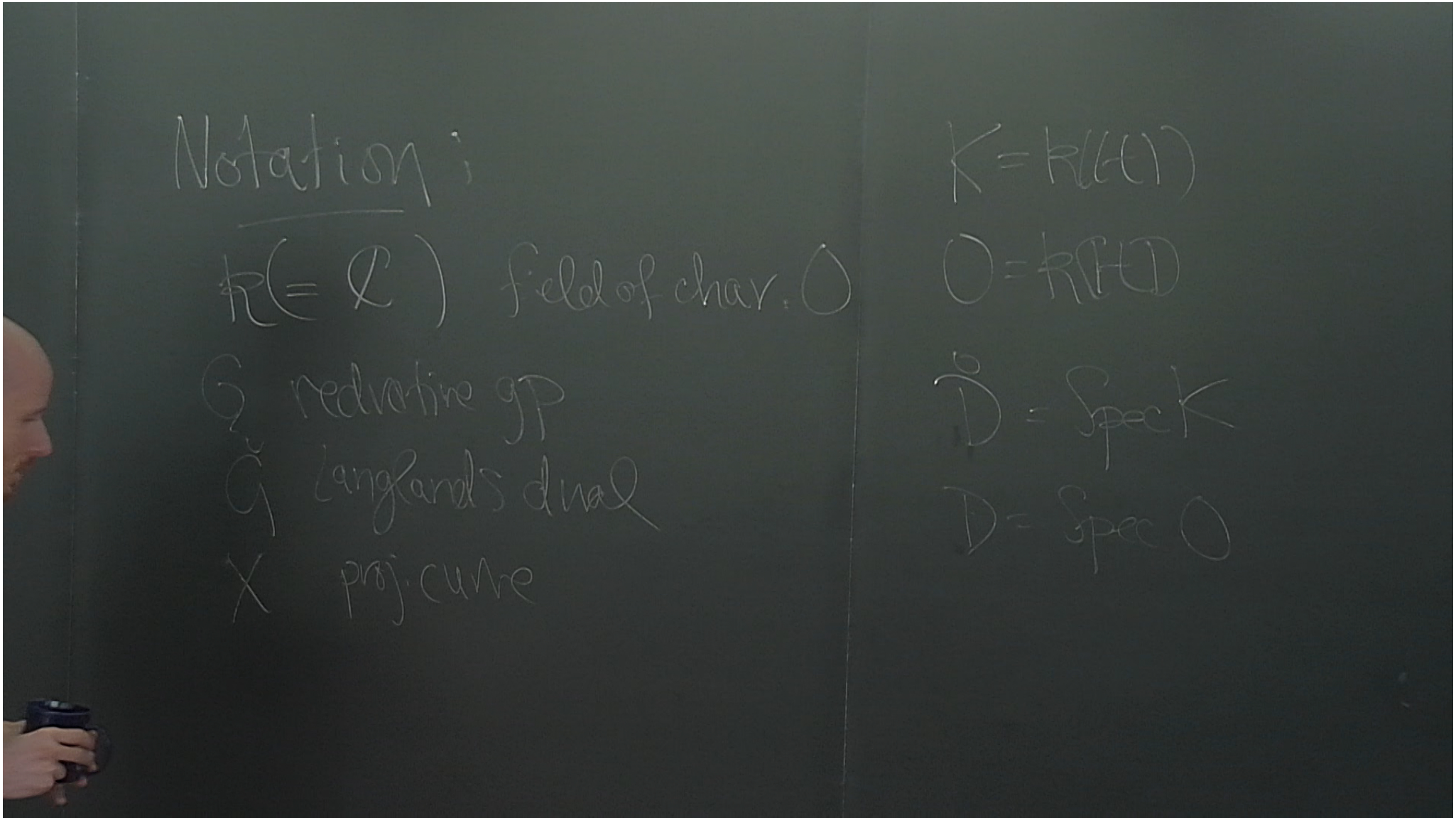
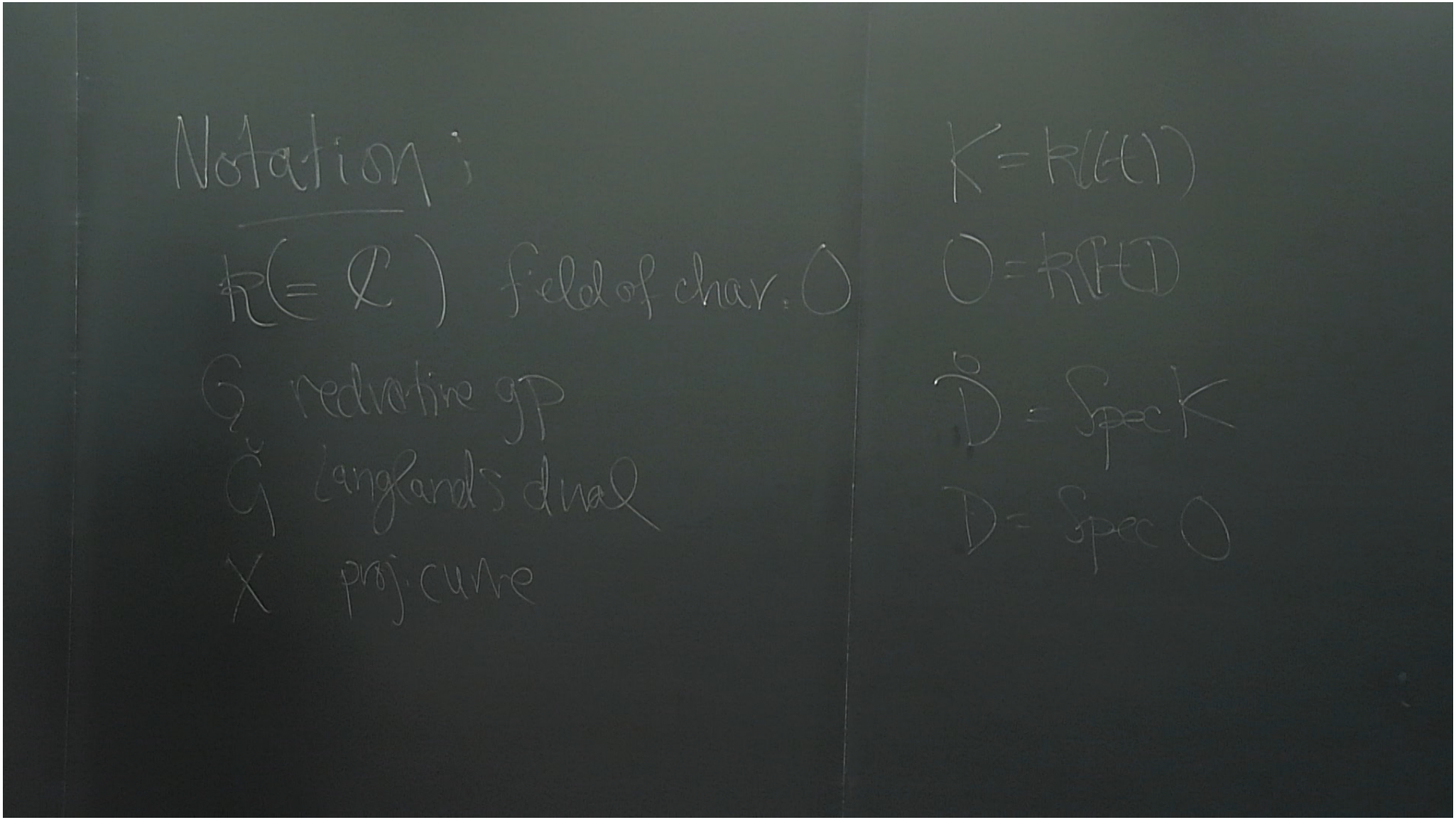
Title: Introduction to local geometric Langlands

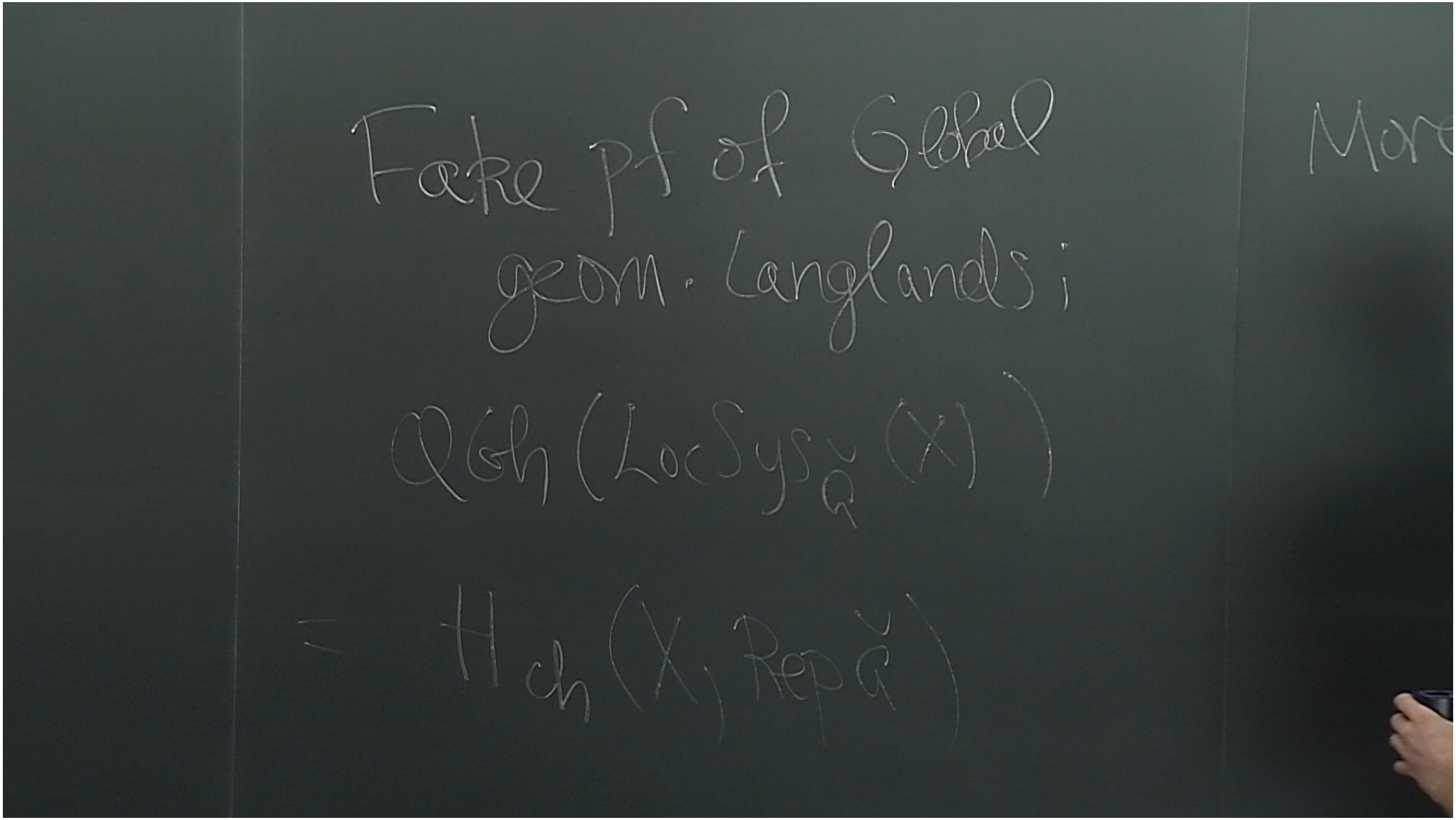
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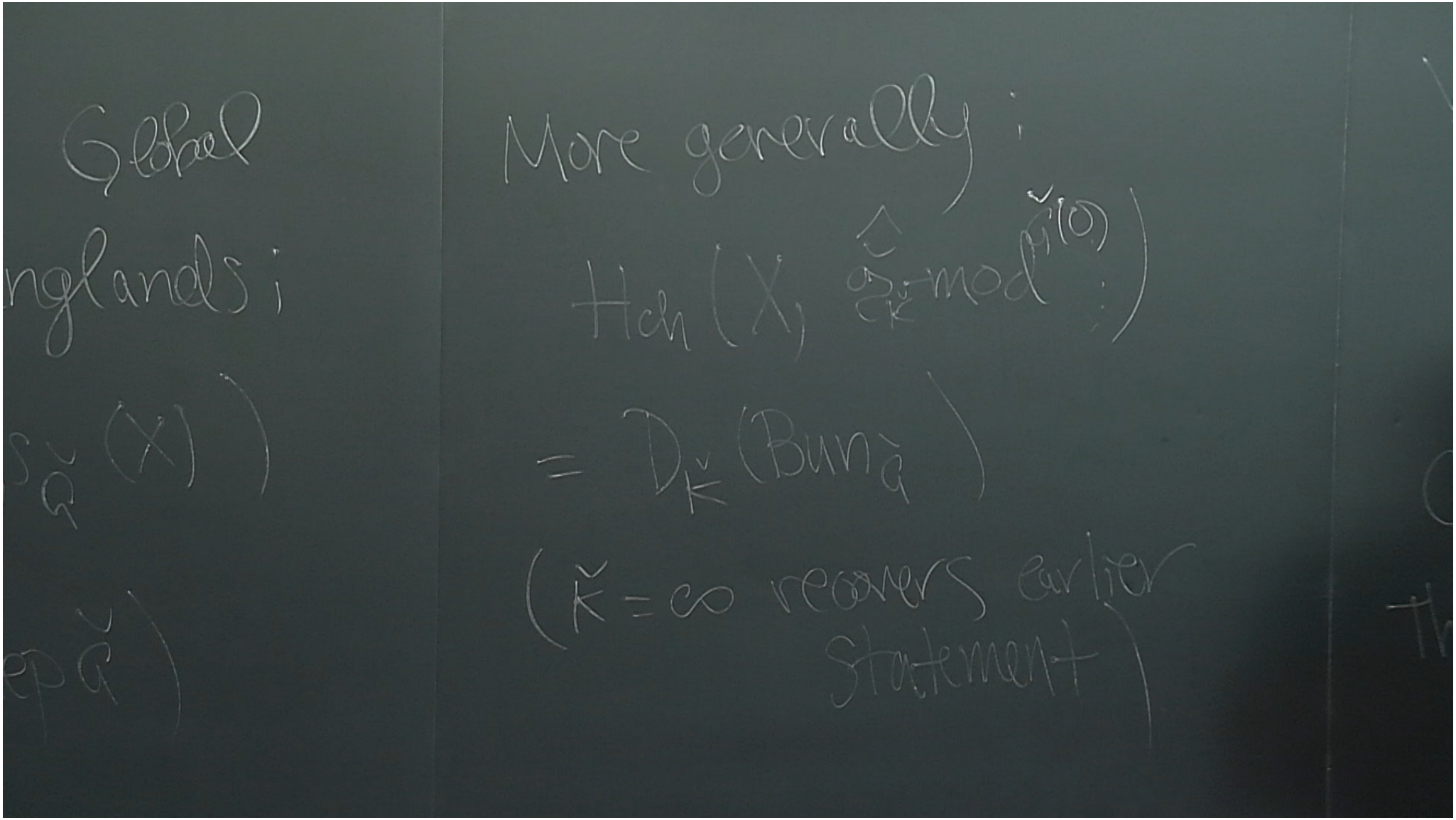
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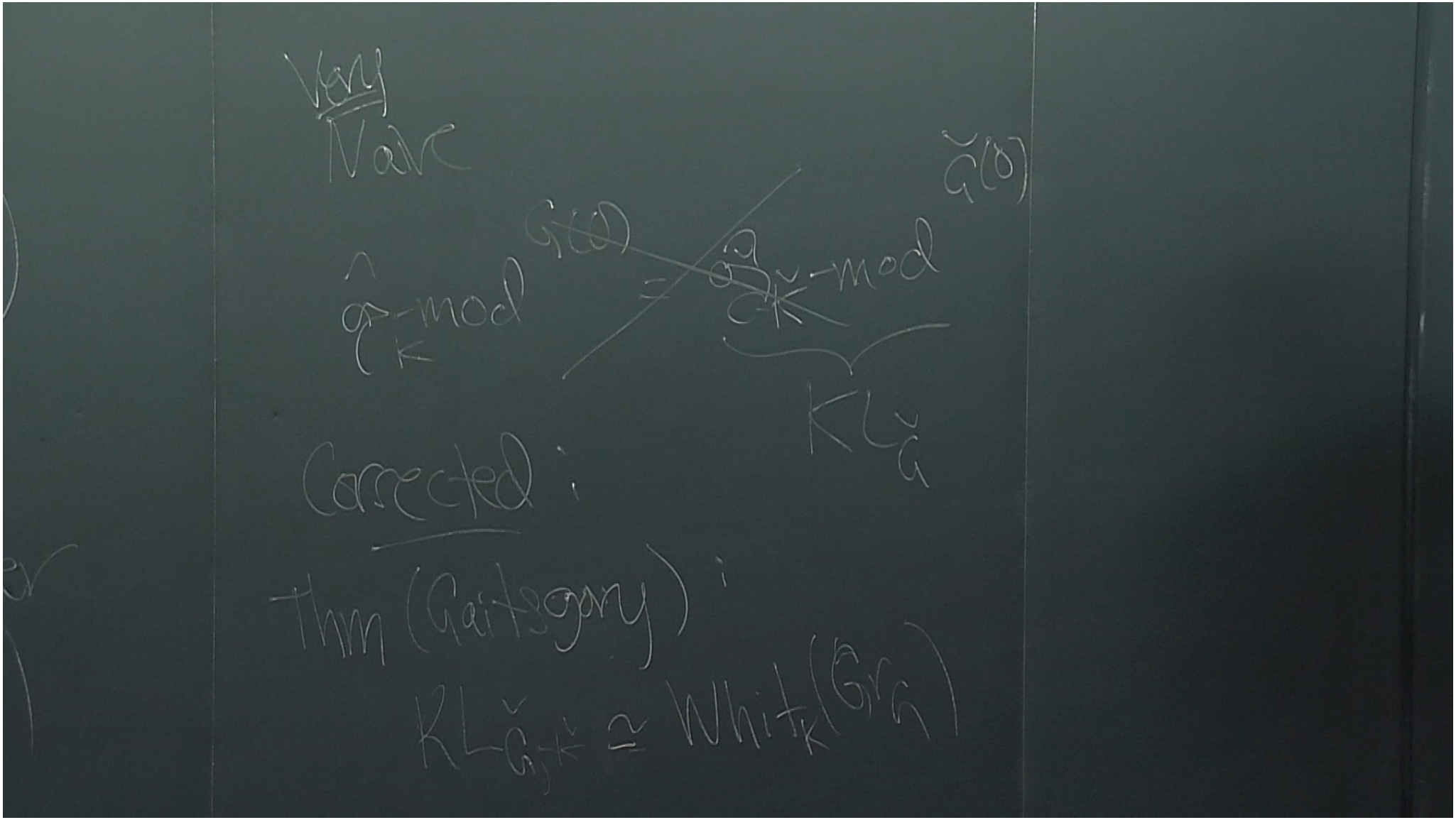
Abstract:

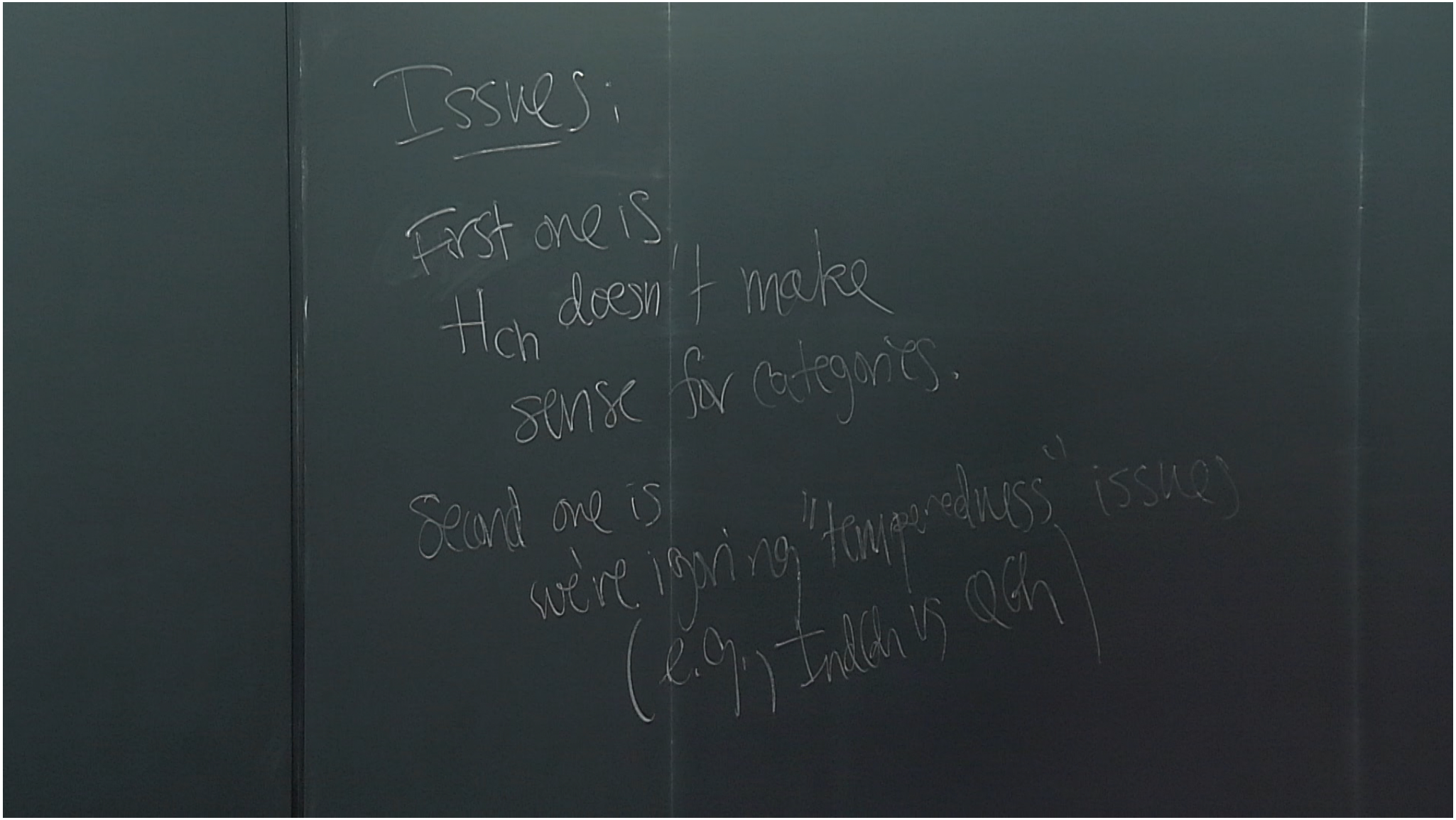

Notation:
k(=ℓ) field of char. 0
G reductive gp
Ǧ Langlands dual
X proj. curve
K=k((t))
O=k[[t]]
D̊ = Spec K
D = Spec O

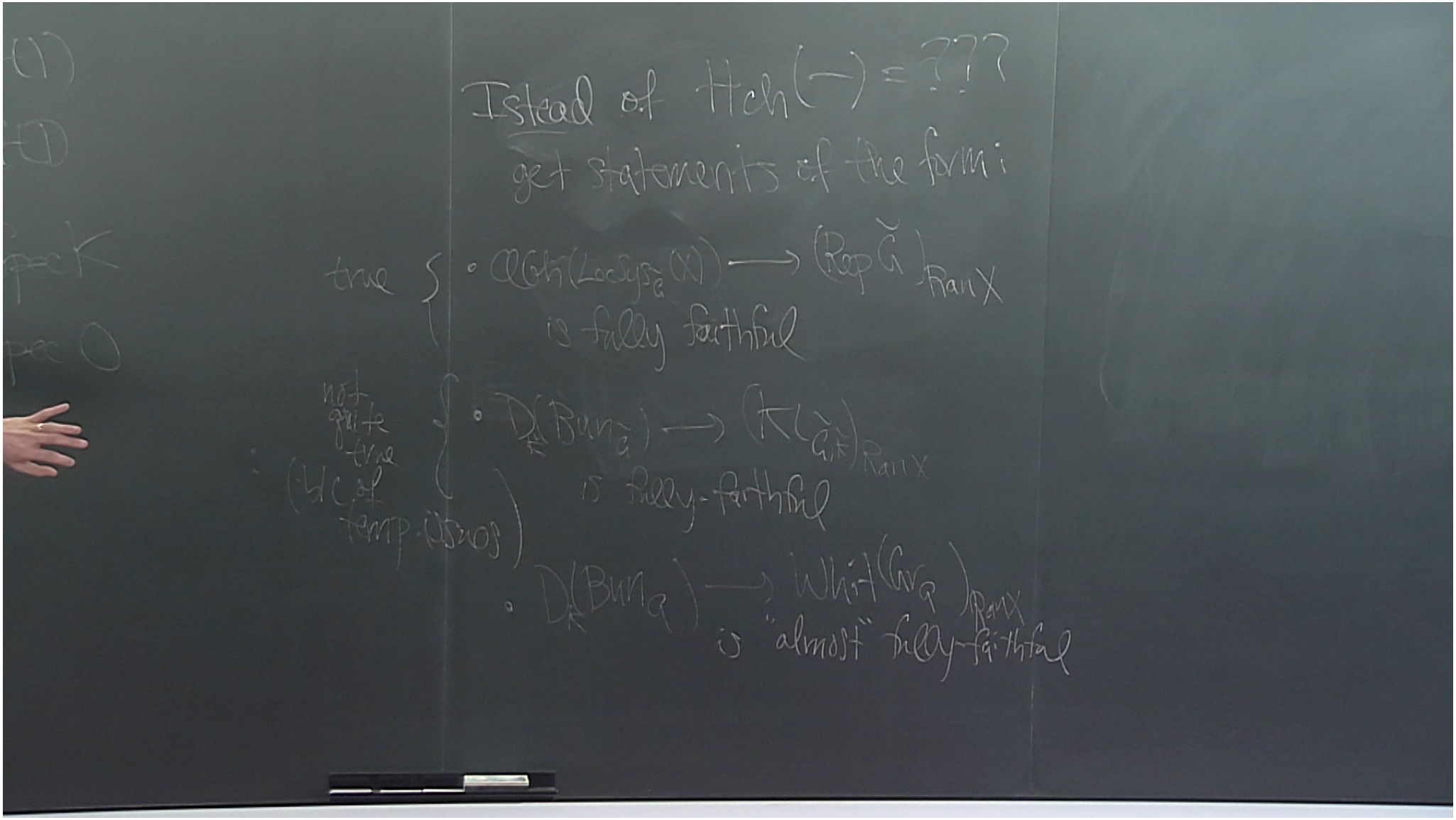

Notation:
$k (= \ell)$ field of char. 0
$G$ reductive gp
$\check{G}$ Langlands dual
$X$ proj. curve
$K = k((t))$
$O = k[[t]]$
$\mathring{D} = \operatorname{Spec} K$
$D = \operatorname{Spec} O$

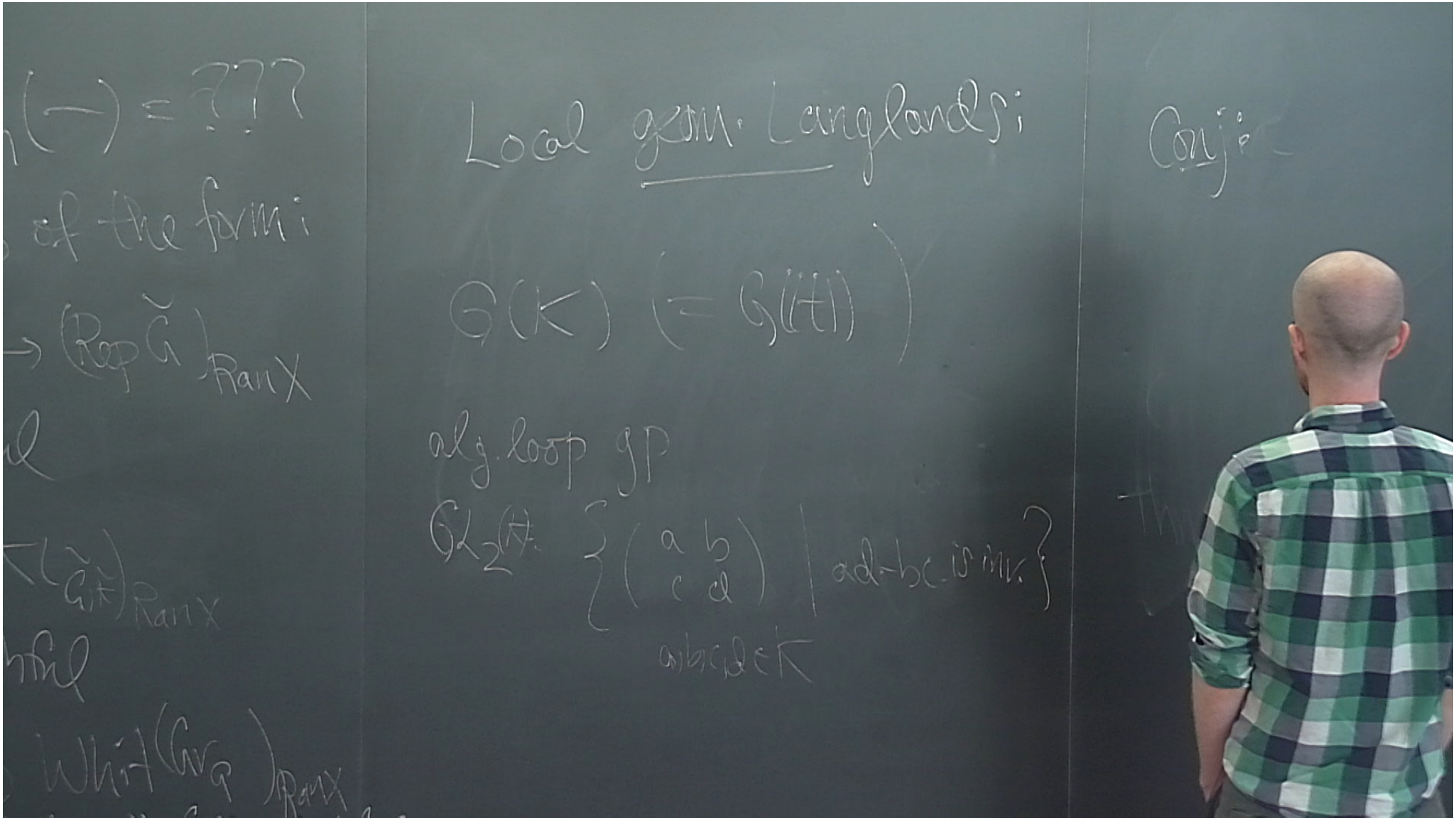

Fake pf of Global
geom. Langlands;
QCh(LocSys$_{\check{G}}$(X))
= Hch(X, Rep $\check{G}$)
More

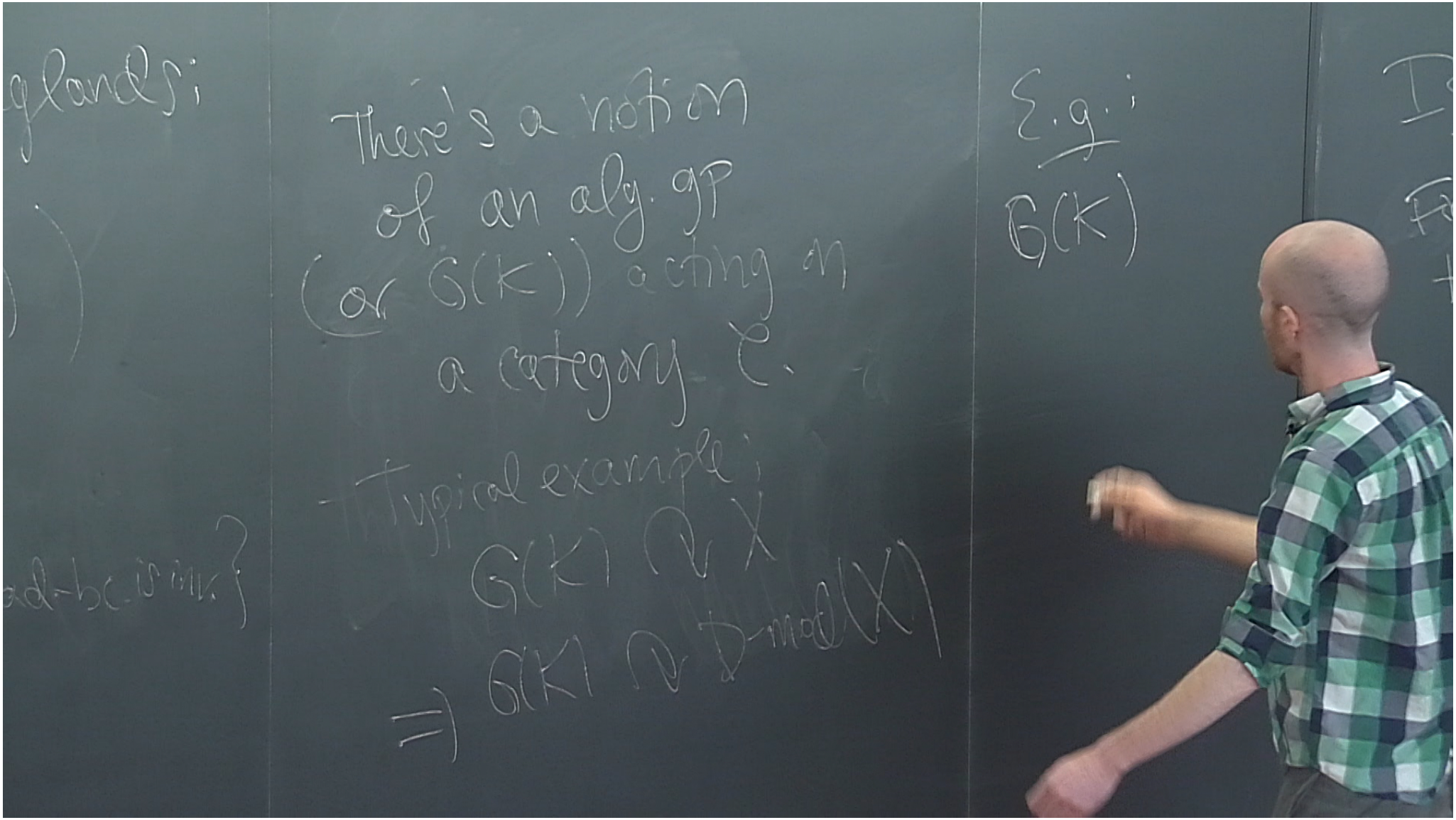

Global
nglands;
More generally:

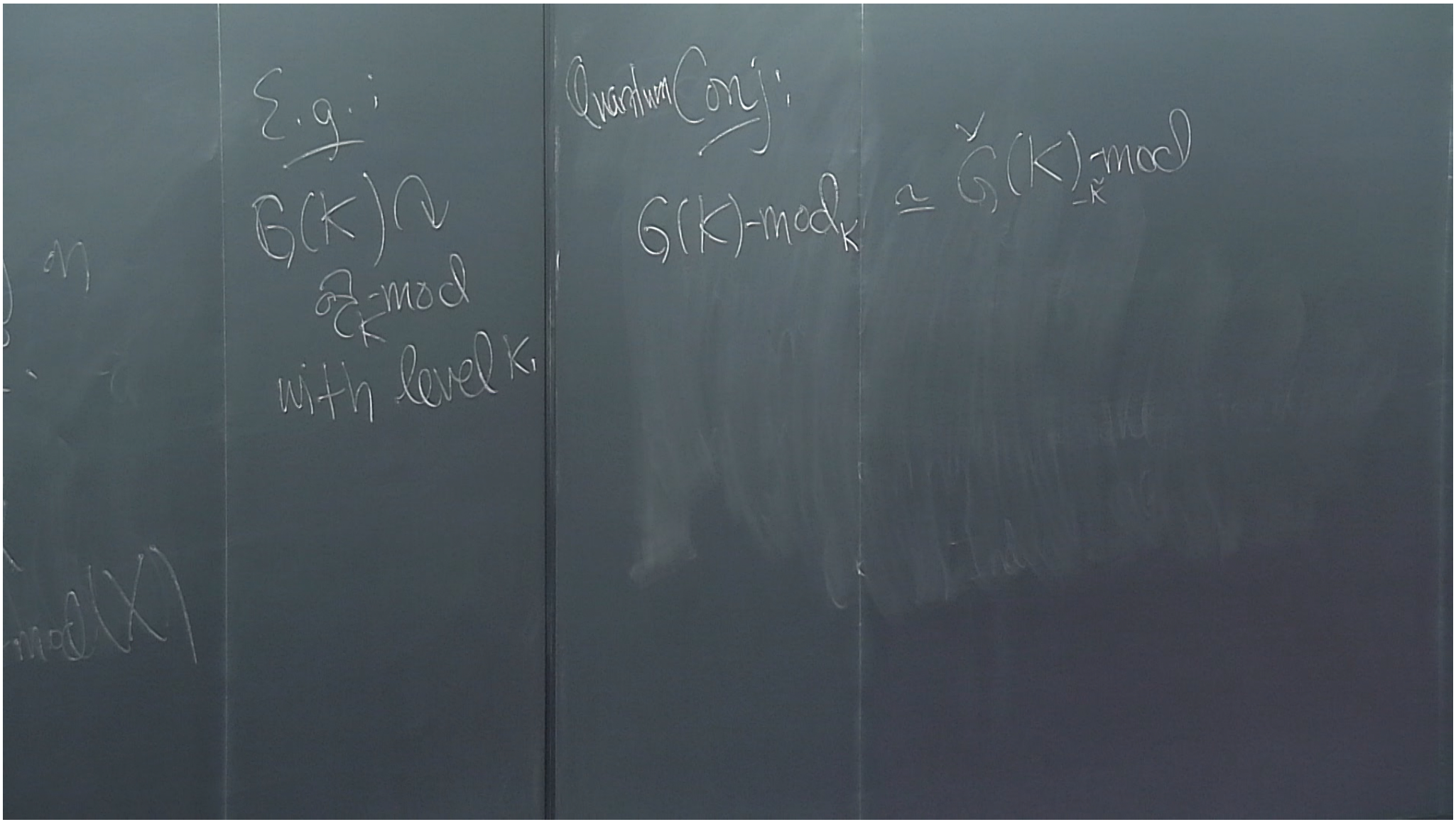

Very
Naive
$\hat{G}(0)$-mod
$\check{G}(0)$
$\check{G}$-mod
$K\check{L}_{\check{G}}$
Corrected:
Thm (Gaitsgory):
$K\check{L}_{\check{G},+} \simeq Whit_K(Gr_{\check{G}})$


Issues:
First one is
Hch doesn't make
sense for categories.
Second one is
we're ignoring "temperedness" issue
(e.g., Ind(ch is Ech)


Istead of Hch(–) = ???
get statements of the form:
true ⇒ • Hch(Locsys_X(X)) ⟶ (Rep G̃)_RanX
is fully faithful
not quite true (Hch of temp. sheaves)
• D(Bun_G) ⟶ (Π(L_{G,κ}))_RanX
is fully-faithful
• D(Bun_G) ⟶ Whit(Gr_G)_RanX
is "almost" fully-faithful


Local geom. Langlands:
G(K) (= G((t)))
alg. loop gp
GL_2(K) { (a b / c d) | ad-bc is inv. }
a,b,c,d ∈ K
of the form:
Conj:


glands;
There's a notion
of an alg. gp
(or G(K)) acting on
a category C.
Typical example:
G(K) ↷ X
⇒ G(K) ↷ D-mod(X)
E.g.:
G(K)


E.g.:
G(K) ↷
$\mathfrak{a}_K$-mod
with level K.
Quantum Conj.:
G(K)-mod$_K$ ≃ $\check{G}(K)_{-K}$-mod

Conj: ($\kappa=0$ version)

$G(K)\text{-mod} \simeq Shv(Cat/LocSys_{\check{G}}(\mathbb{D}))$

$G(K) \curvearrowright \mathcal{C}$ w/ level $\kappa$

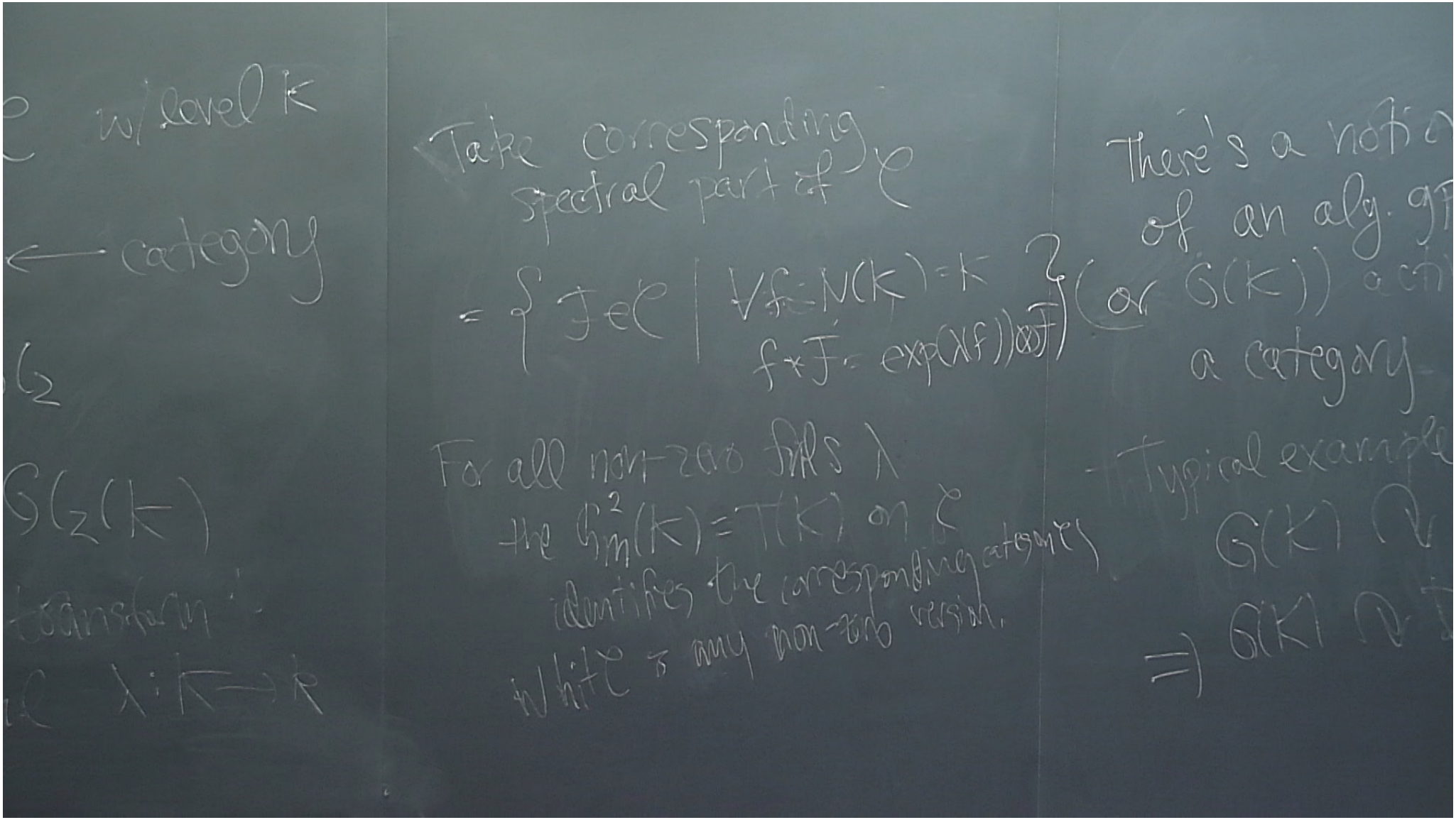
What $\mathcal{C}$ ← category

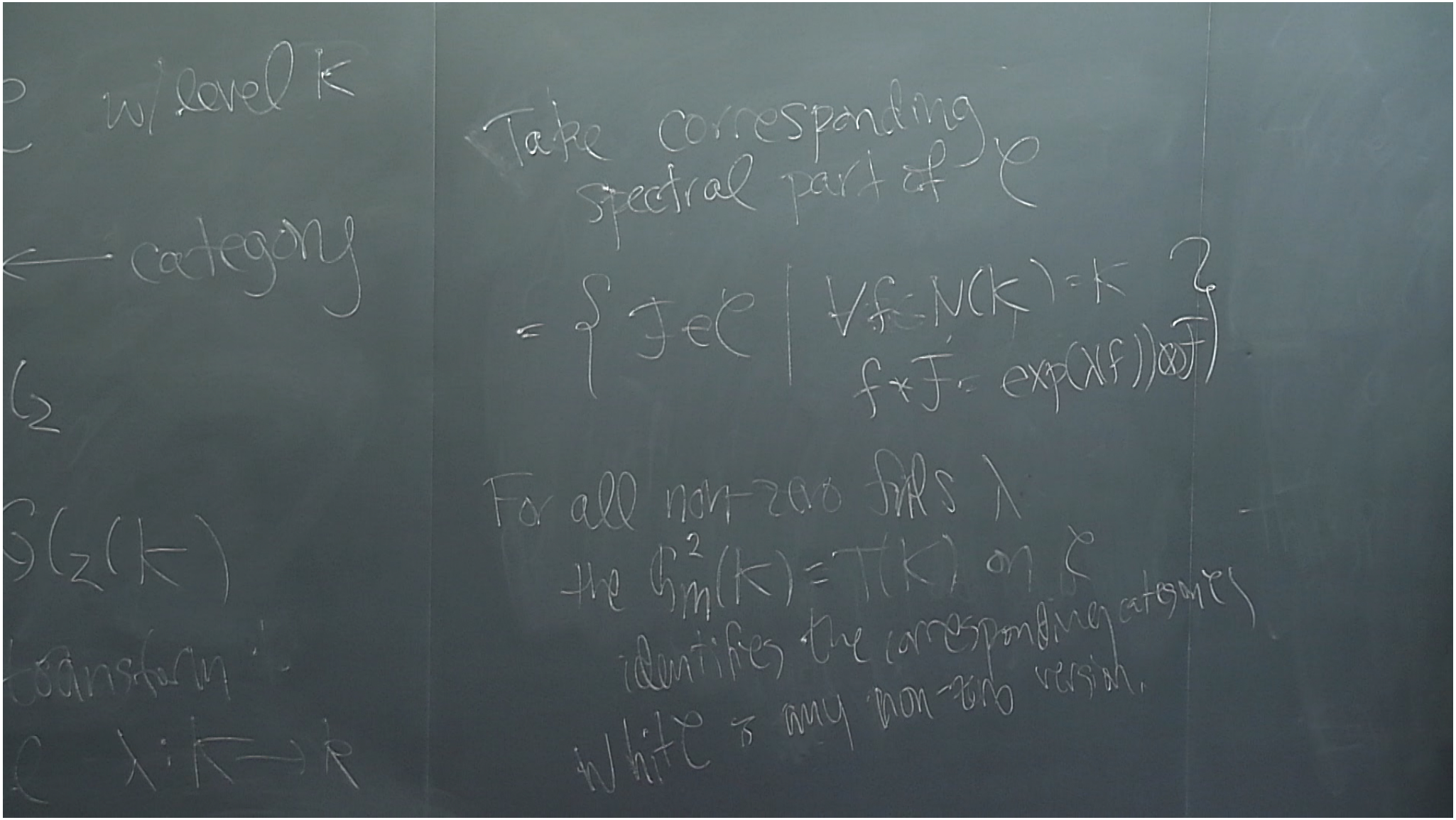
E.g.: $G = GL_2$

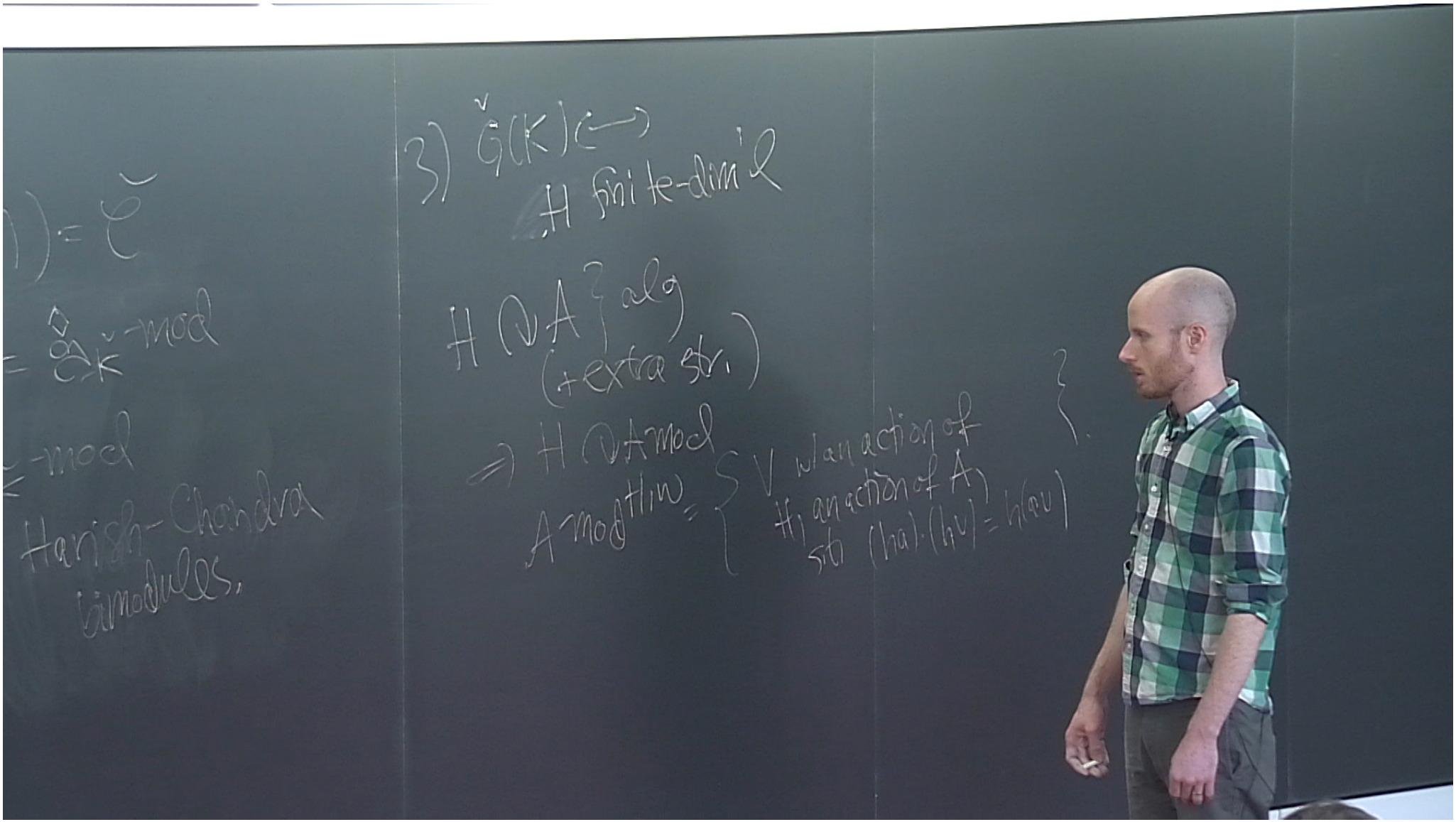
$\kappa = N(K)$ $\left\{\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}\right\} \leq GL_2(K)$

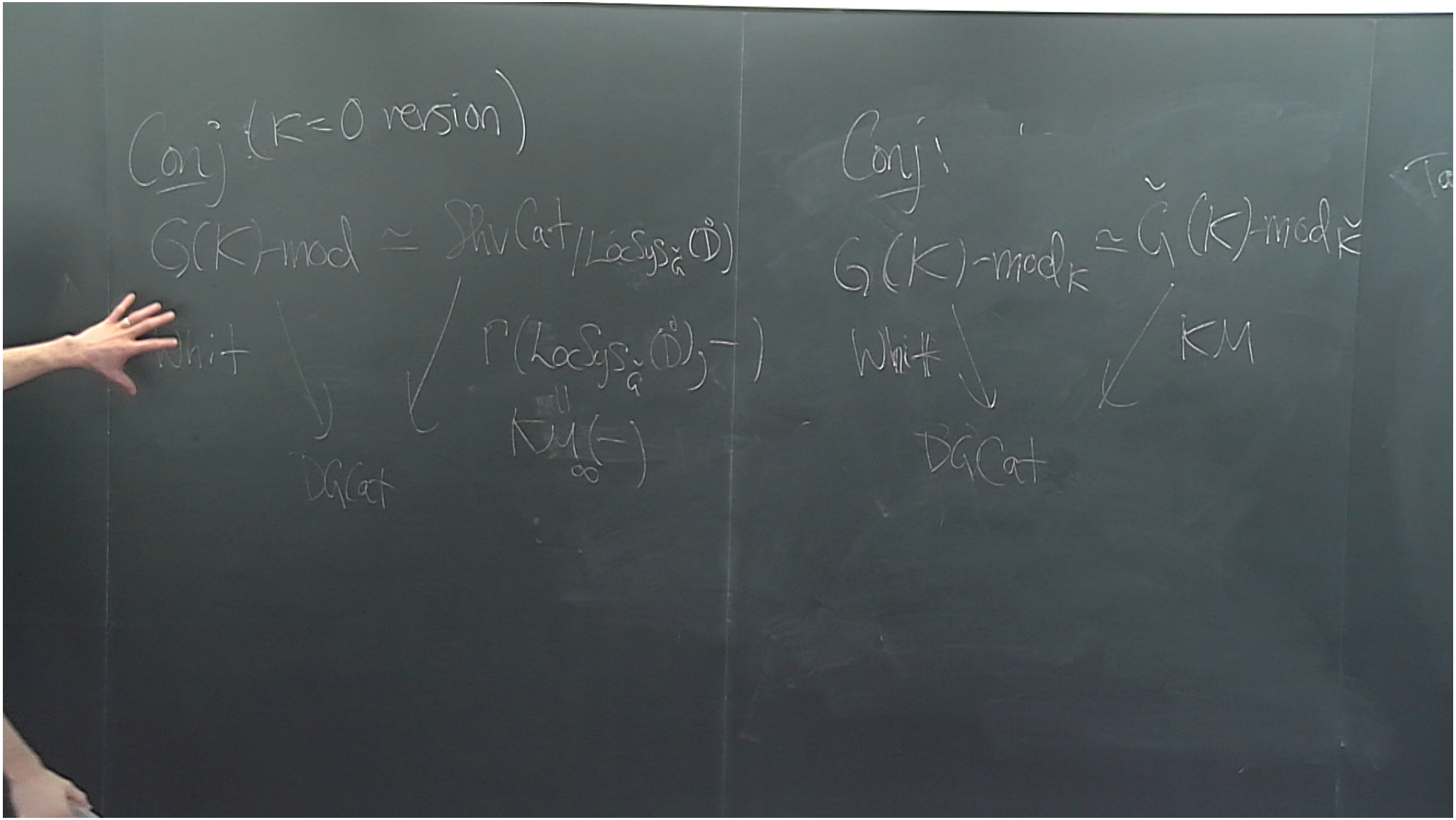
$\kappa \curvearrowright \mathcal{C}$ do Fourier transform:

For any $\mathbb{R}$-l $\lambda: K \to \mathbb{R}$


w/ level K
category
Take corresponding spectral part of
$= \{ f \in \mathcal{C} \mid \forall f : N(K) = K \}$
$f * F = \exp(\lambda f) \otimes F$
(or G(K))
a category
For all non-zero fills λ
the $G_m^2(K) = T(K)$ on $\mathcal{C}$
identifies the corresponding category
with $\mathcal{C}$ as any non-trivial version.
Typical example
$\Rightarrow$ G(K)


w/ level k
category
$G_2$
$S(G_2(k))$
transform
$\lambda : k \to R$
Take corresponding spectral part of $\mathcal{C}$
$= \{ f \in \mathcal{C} \mid \forall f : N(k) = k \quad f * f = \exp(\lambda(f))(\otimes f) \}$
For all non-zero $\lambda$'s the $G_m^2(k) = T(k)$ on $\mathcal{C}$ identifies the corresponding category with $\mathcal{C}$ & any non-zero version.


3) $\check{G}(K) \hookrightarrow$
$H$ finite-dim'l
$H \otimes A$ {alg
(+extra str.)
$\Rightarrow H \otimes A$-mod
$A$-mod$^{fin}$ = { $V$ w/ an action of
$H$, an action of $A$
s.t. $(ha)\cdot(hv) = h(av)$ }
$\check{G}$-mod
Harish-Chandra
bimodules.


Conj: (κ=0 version)
G(K)-mod ≃ Shv(Cat)/LocSys_Ǧ(D°)
Whit
Γ(LocSys_Ǧ(D°), −)
DGCat
Conj:
G(K)-mod_κ ≃ Ǧ(K)-mod_κ̌
Whit
KM
DGCat

Conj: (K=0 version)

$G(K)\text{-mod} \simeq \mathrm{Shv}(\mathrm{Gr}_{/\mathrm{LocSys}_{\check{G}}(\mathring{D})})$

$KW_{\mathrm{Whit}}$ ↘ ↙ $\Gamma(\mathcal{O}_{\mathrm{Gr}_{\check{G}}}(\mathring{D}), -)$

$\mathrm{DGCat}$

$\mathcal{O}_{\mathrm{DGCat}}(\mathring{D}) = \{ d + \begin{pmatrix} 0 & f \\ t & 0 \end{pmatrix} dt,\ f \in K \}$

Local-to-Local princ.

The functors

$\mathrm{Whit} : G(K)\text{-mod} \to \mathrm{DGCat}$

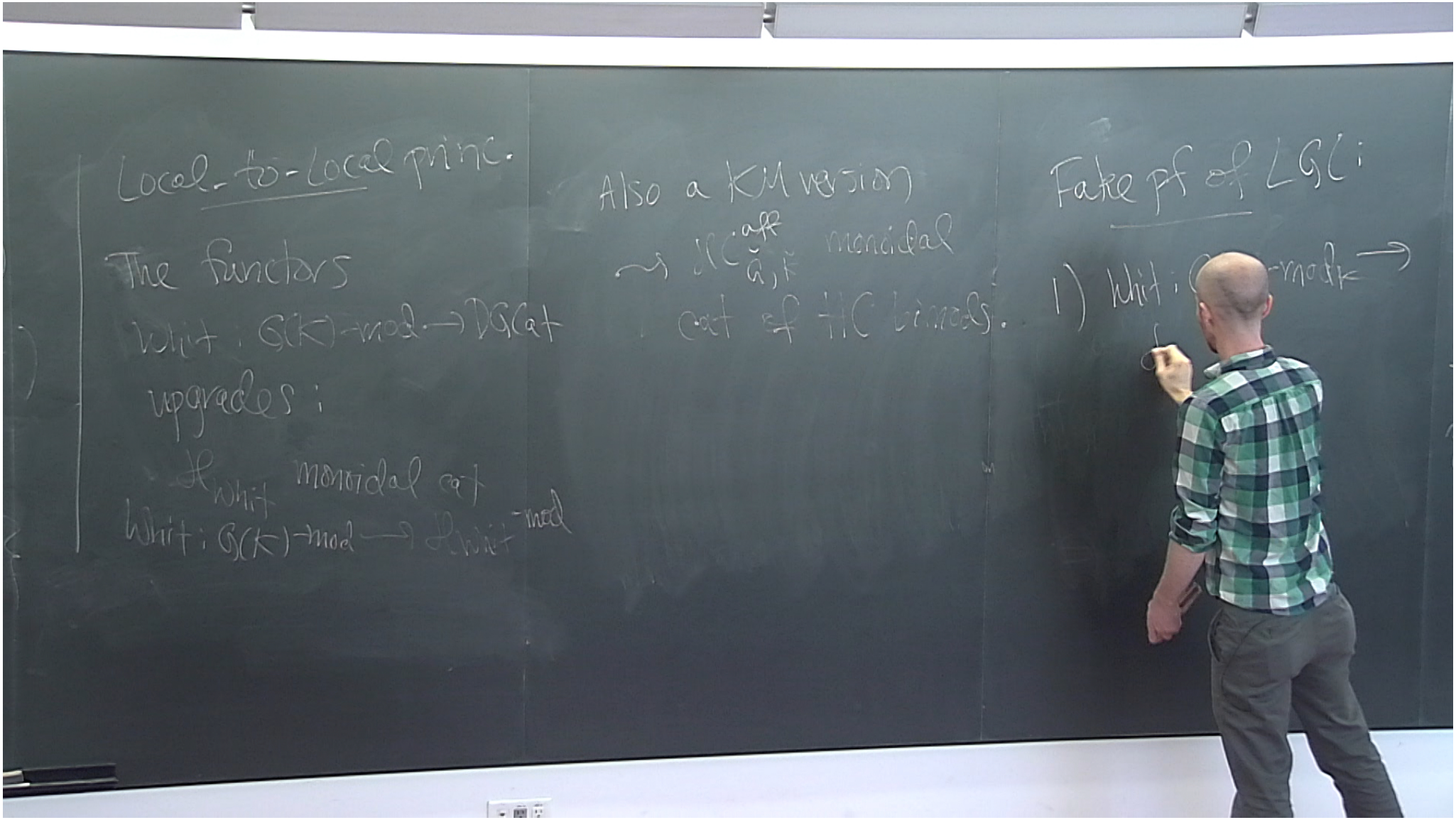
upgrades:

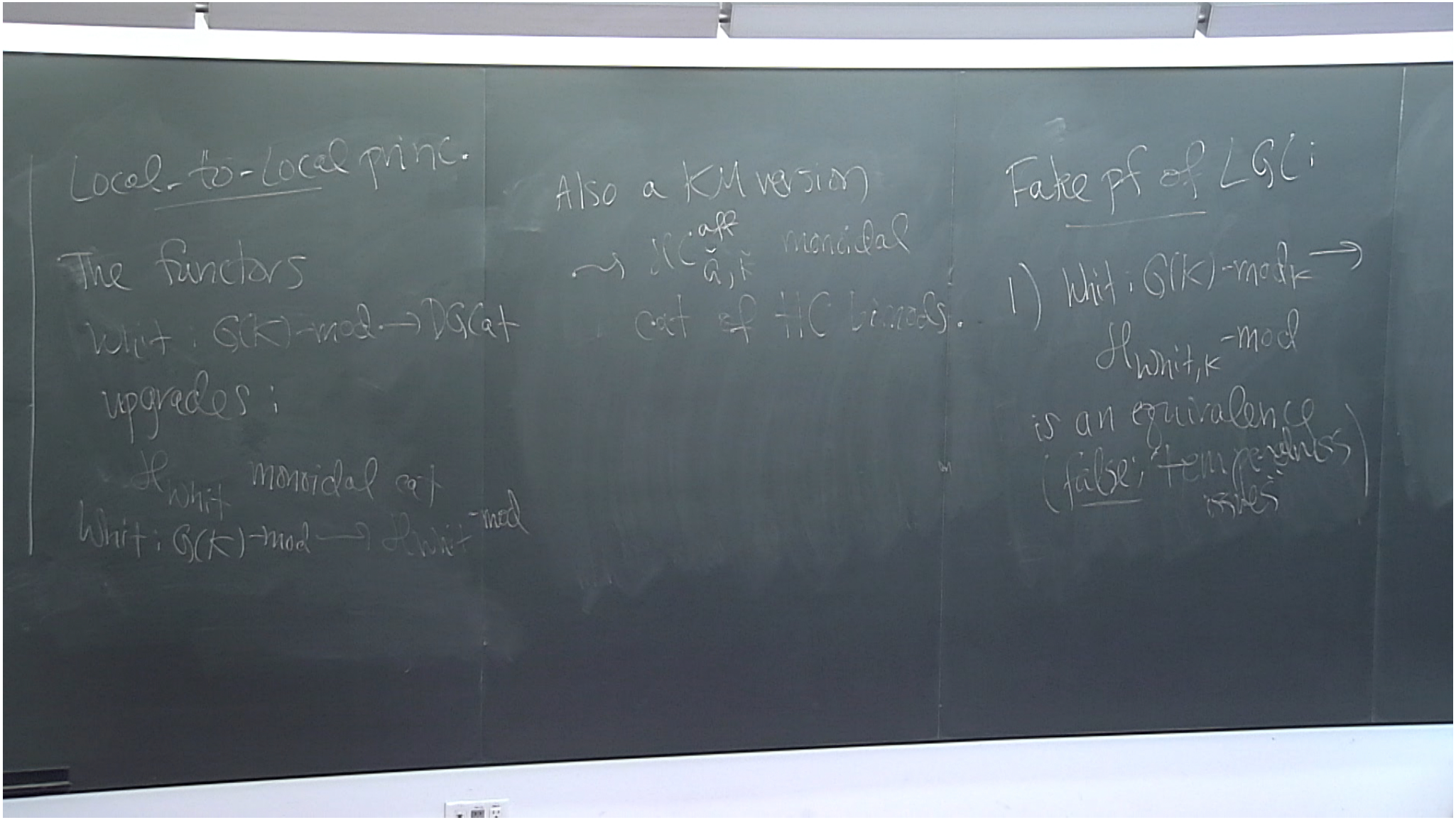
$\mathcal{H}_{\mathrm{Whit}}$ monoidal cat

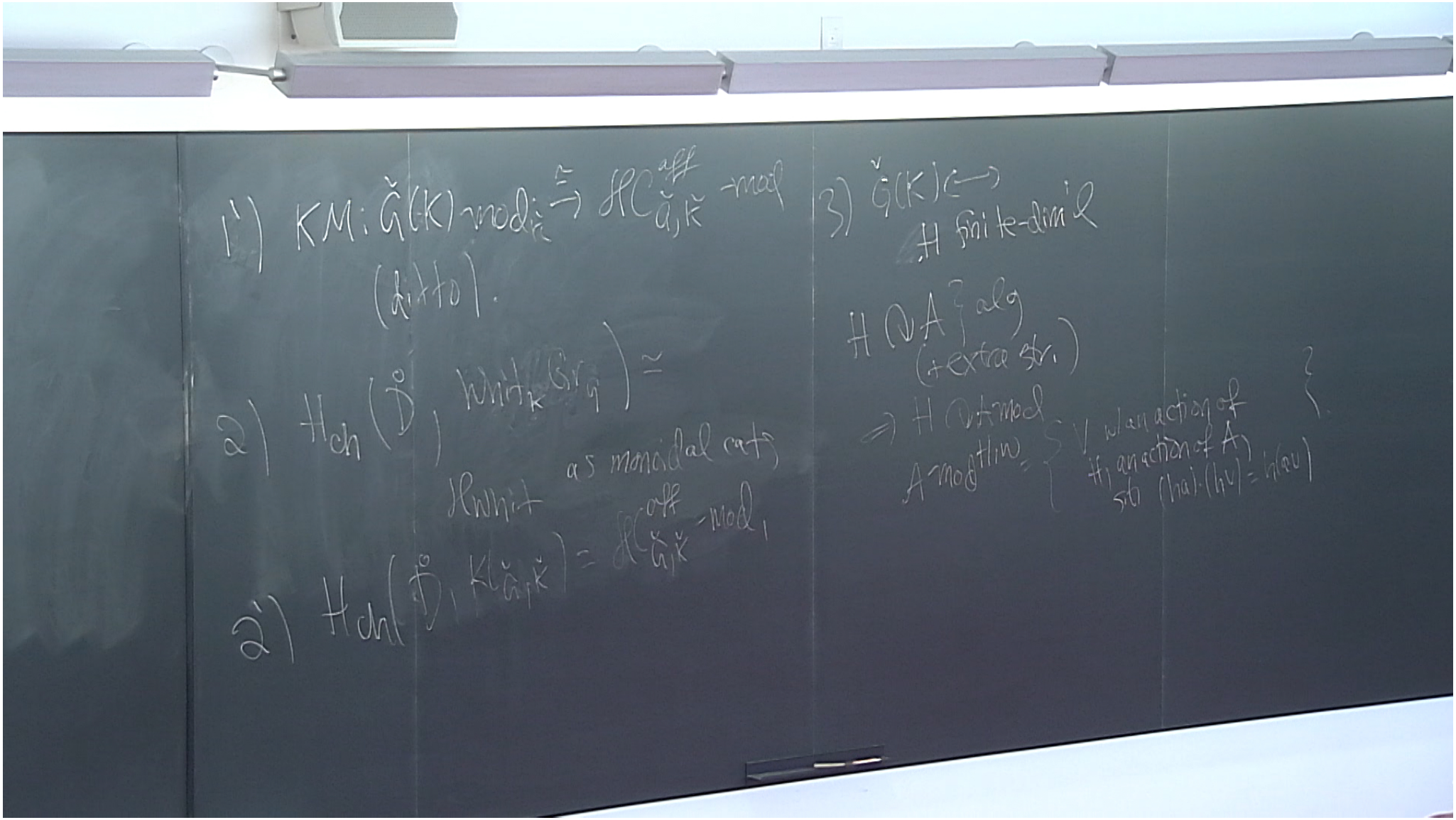
$\mathrm{Whit} : G(K)\text{-mod} \to \mathcal{H}_{\mathrm{Whit}}\text{-mod}$

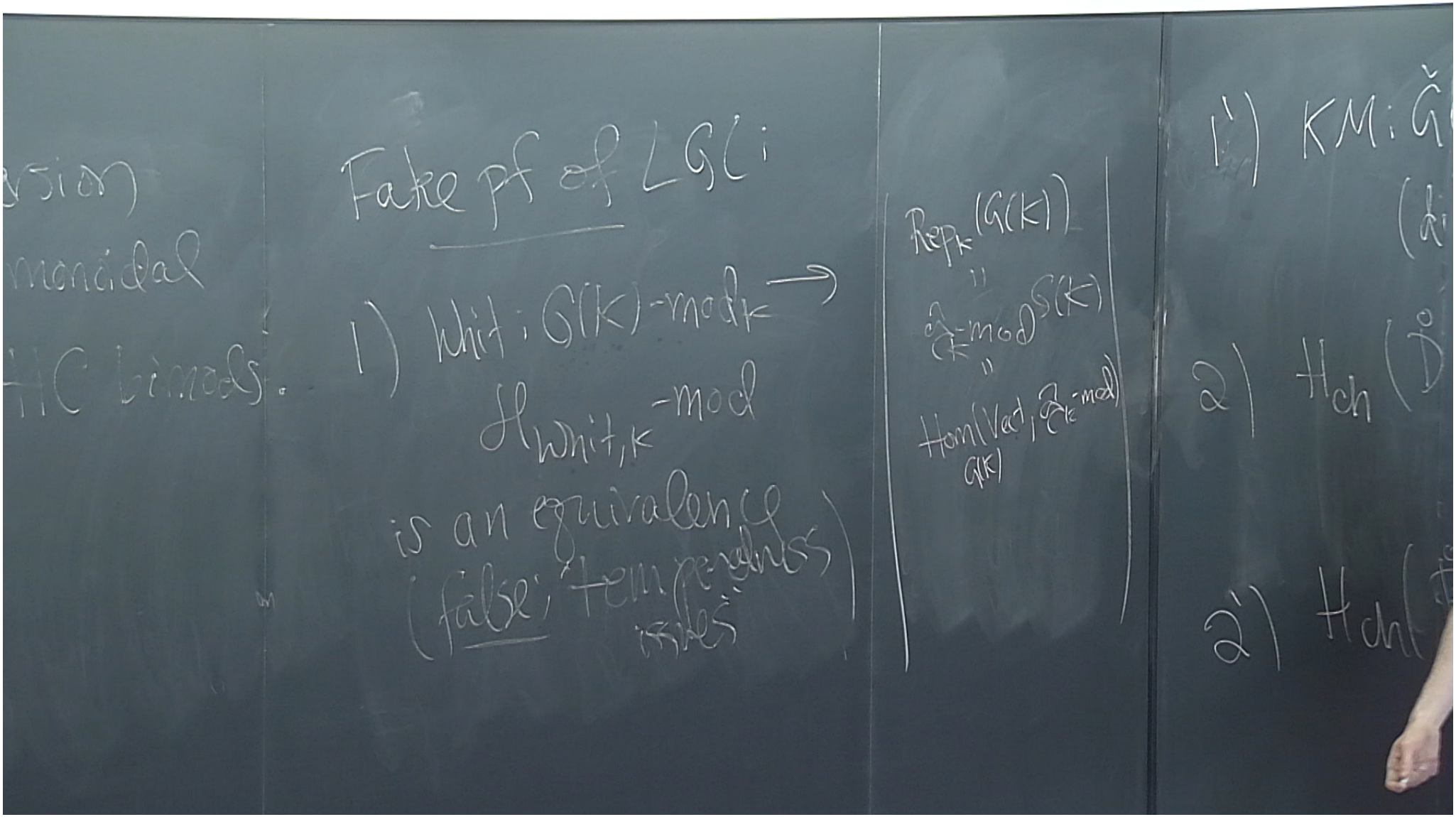
Also a

↝


Local-to-Local princ.
The functors
Whit : G(K)-mod → DGCat
upgrades :
$\mathcal{L}_{Whit}$ monoidal cat
Whit : G(K)-mod → $\mathcal{L}_{Whit}$-mod
Also a KM version
→ $\mathcal{H}(\mathcal{C}^{aff}_{\check{G},\check{I}})$ monoidal
cat of $\mathcal{H}$C bimods.
Fake pf of LGC:
1) Whit :


Local-to-Local princ.
The functors
Whit : G(K)-mod → DGCat
upgrades;
$\mathcal{H}_{Whit}$ monoidal cat
Whit : G(K)-mod → $\mathcal{H}_{Whit}$-mod
Also a KM version
→ $\mathcal{H}(\tilde{G}^{aff}_{\tilde{a},\tilde{r}})$ monoidal
cat of $\mathcal{H}$ bimods.
Fake pf of LGC:
1) Whit : G(K)-mod$_K$ →
$\mathcal{H}_{Whit,K}$-mod
is an equivalence
(false; technicalities
issues)


1') KM: $\check{G}(K)$-mod$_{fd}^{\heartsuit} \xrightarrow{\sim}$ $\mathcal{HC}^{aff}_{\check{a},\check{\kappa}}$-mod
(Lurie)
2) $H_{ch}(\mathring{D}, \mathcal{W}hit_K(Gr_G)) \simeq$
$\mathcal{H}_{Whit}$ as monoidal cats
2') $H_{dR}(\mathring{D}, K(\check{a},\check{\kappa})) = \mathcal{HC}^{aff}_{\check{a},\check{\kappa}}$-mod$_1$
3) $\check{G}(K) \hookrightarrow$
$\Rightarrow H$ finite-dim'l
$H(QA)$ alg
(+extra str.)
$\Rightarrow H$ $A$-mod
$A$-mod$_{Lin}$ = { V / an action of $\check{G}(K)$ on $V$ and an action of $A$ s.t. $(ha)\cdot(hv) = h(av)$ }


Fake pf of LG(i:
1) Whit: G(K)-mod_K →
H_Whit,K -mod
is an equivalence
(false; temperedness issues)
Rep_K(G(K))
"
Z(K)-mod
"
Hom(Vect, Z(K)-mod)
G(K)
1') KM: Ǧ
(di
2) Hch (D
2') Hch(