Title: Decomposable Specht modules

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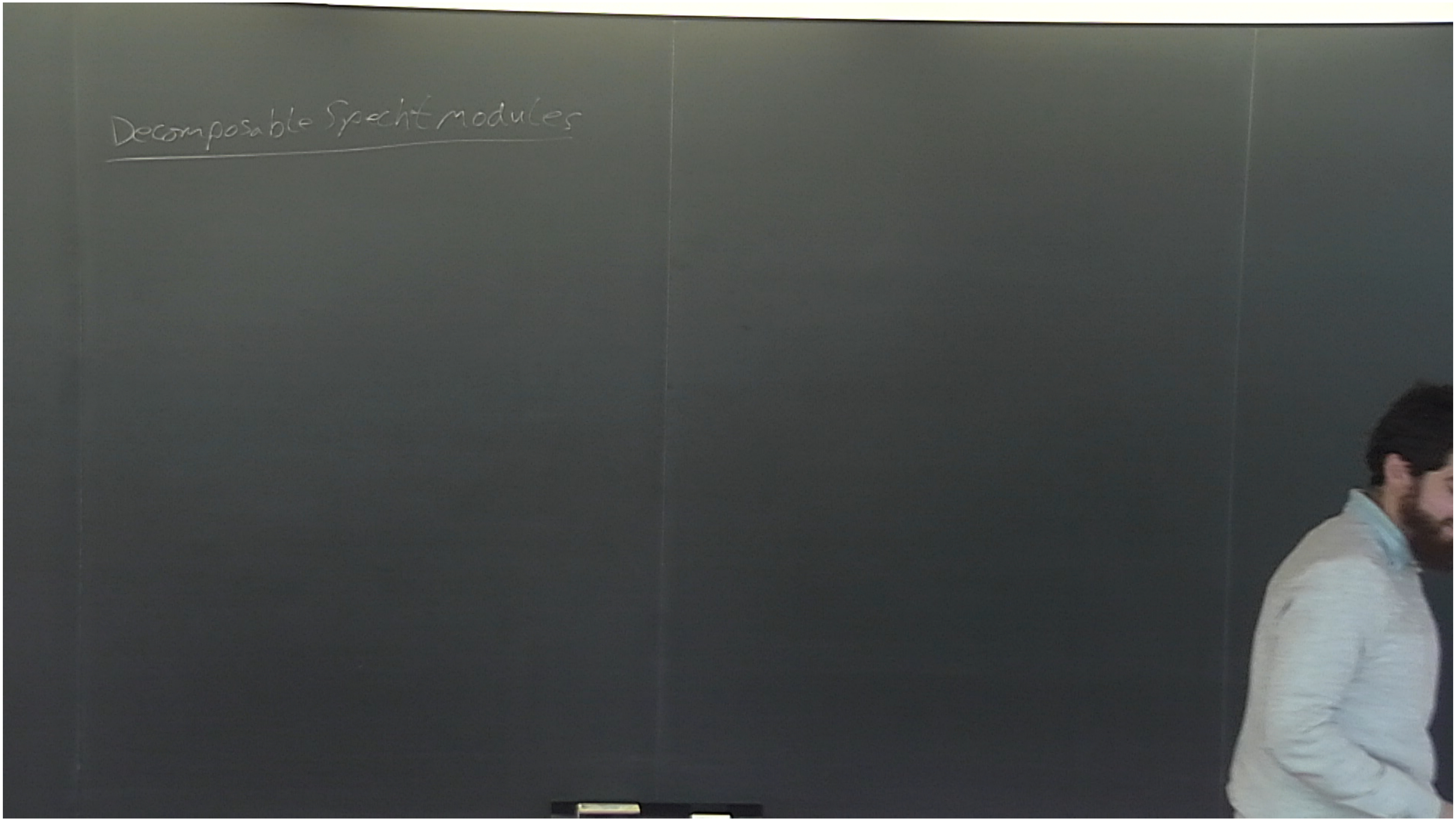
Abstract: <p>I will give a brief survey of the study of decomposable Specht modules for the symmetric group and its Hecke algebra, which includes results of Murphy, Dodge and Fayers, and myself. I will then report on an ongoing project with Louise Sutton, in which we are studying decomposable Specht modules for the Hecke algebra of type B indexed by `bihooksâ€™.</p>


Decomposable Specht modules

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Level 1

$\mathbb{F}$ a field of char $p \geq 0$.

The Specht modules

$\{S^\lambda \mid \lambda \vdash n\}$ over S_n

ptns. of n.

$\|$

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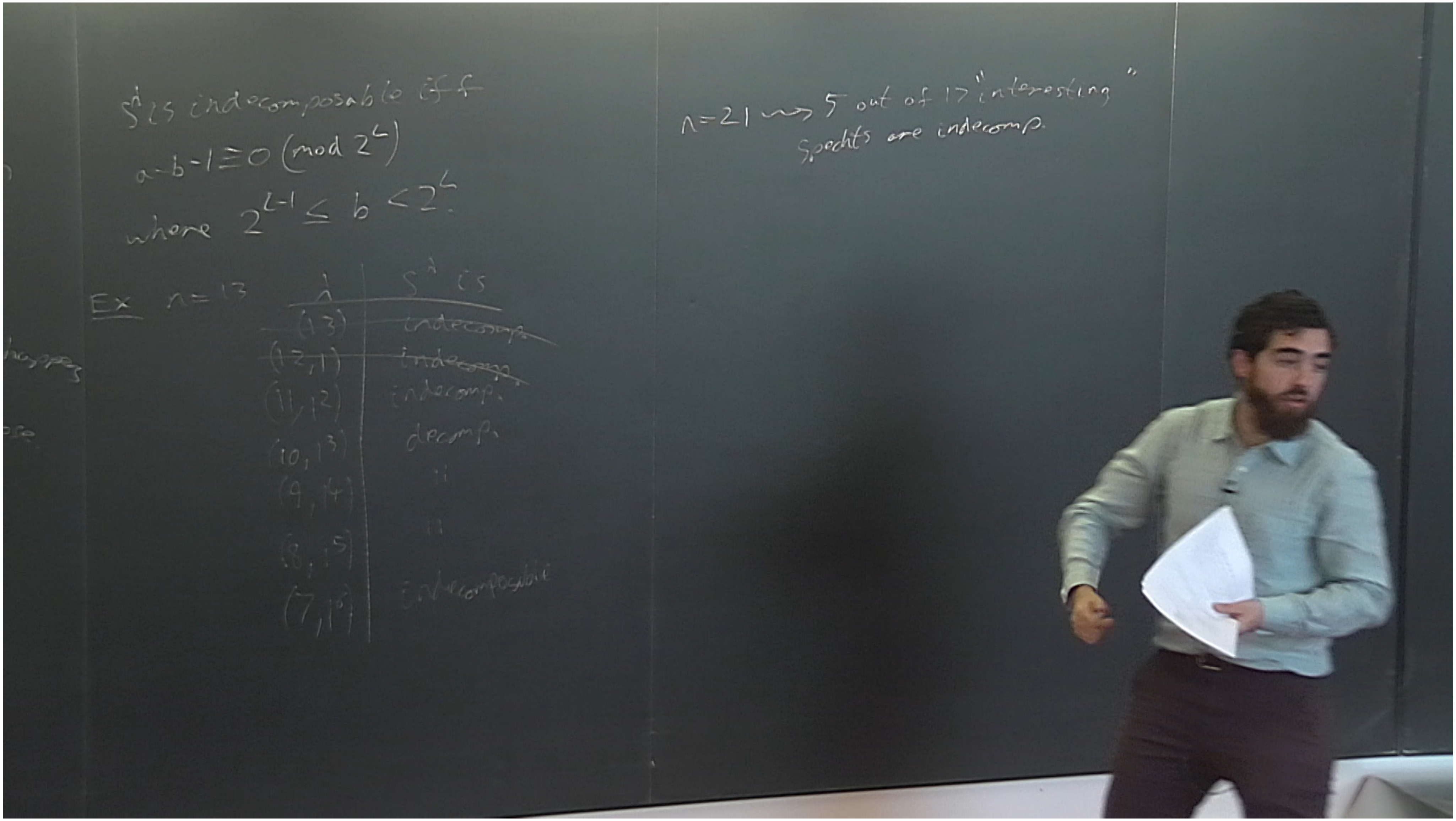
has no repeat.

In general, difficult to see what happens if $p = 2$.

<u>Fact</u> S^{λ} decomposable $\Leftrightarrow$ $S^{\lambda^{T}}$ is

transpose


S^λ is indecomposable iff
$a-b-1 \equiv 0 \pmod{2^L}$
where $2^{L-1} \le b < 2^L$.
Ex $n=13$
λ | S^λ is
~~(13) indecomp~~
~~(12,1) indecomp~~
$(11,1^2)$ indecomp.
$(10,1^3)$ decomp.
$(9,1^4)$ "
$(8,1^5)$ "
$(7,1^6)$ indecomposable
$n=21 \rightsquigarrow 5$ out of 17 "interesting"
Spechts are indecomp.

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Theorem (Dodge–Fayers 2012)

Suppose $\lambda=(a,3,1^b)$ w/ $a\geq 4, b\geq 2$.

Then S^λ is decomp iff at least one of the following holds:

- $a+b\equiv 0$ or $2 \pmod 8$, $a\geq 6, b\geq 4$;
- $a+b\equiv 2 \pmod 4$ & $\binom{a+b-3}{a-3}$ odd;
- $a+b\equiv 0 \pmod 4$ $\binom{a+b-9}{a-5}$ odd.

Natural generalis^n

$\rightsquigarrow$ consider Hecke alg of S_n:

$\mathbb{F}$-alg H_n w/ generators $T_1,\dots,T_{n-1}$ & relations

$(T_i-q)(T_i+1)=0$

$T_iT_j=T_jT_i$ $|i-j|>1$

$T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$

... of 17 "interesting"
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... (...gers 2012)
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Natural generalisn

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$(T_i - q)(T_i + 1) = 0$

$T_i T_j = T_j T_i \quad |i-j| > 1$

$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$

for $q \in \mathbb{F}$ a prim. e^{th} root of 1.

Over H_n, have Specht modules $\{S^\lambda \mid \lambda \vdash n\} = \{\text{simples}\}$ for $e = 0$

Theorem (Dipper–James 1991)
if $e \neq 2$ or λ is 2-regular, S^λ is indecomp.

Theorem (S. 2014)
Suppose $p \neq 2$ & $\lambda = (a, 1^b)$. Then S^λ is indecomposable iff
n is even, or
$b = 2$ or 3 w/ $p \mid \lceil \frac{a}{2} \rceil$.

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KLR algebras

Further generalise Hecke alg.
$\rightsquigarrow$ deform $\mathbb{Z}/\ell\mathbb{Z} \wr S_n$
to get cyclotomic Hecke alg.s.

Thm (Brundan–Kleshchev 2009)
The (integral) cyclotomic Hecke alg. with $e \geq 2$ is isomorphic to a level ℓ cyclotomic KLR algebra R_n^Λ of type $A_{e-1}^{(1)}$ if $e < \infty$ or A_∞ if $e = \infty$. $\Lambda = \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell}$.

R_n^Λ is $\mathbb{F}$-alg. w/ generators
$\{e(i) \mid i \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, \ldots, y_n\}$
$\cup \{\psi_1, \ldots, \psi_{n-1}\}$
& many rels.
$\mathbb{Z}$-graded!

$)$,

$)^b$). Then

$\left[\frac{a}{2}\right]$.

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idempotents orthog. & Σ to 1

$\cup \{\psi_1, \dots, \psi_{n-1}\}$

~ Coxeter gens.

& many rels.

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Decomposable Specht modules

Specht modules for R^Λ_n indexed by l-multipartitions of n.

$\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ s.t. $\sum |\lambda^{(i)}| = n$.

Have tableau combinatorics:

T^λ is column initial λ-tableau;

Ex $\lambda = ((4,3),(3,2,1))$

$T^\lambda =$

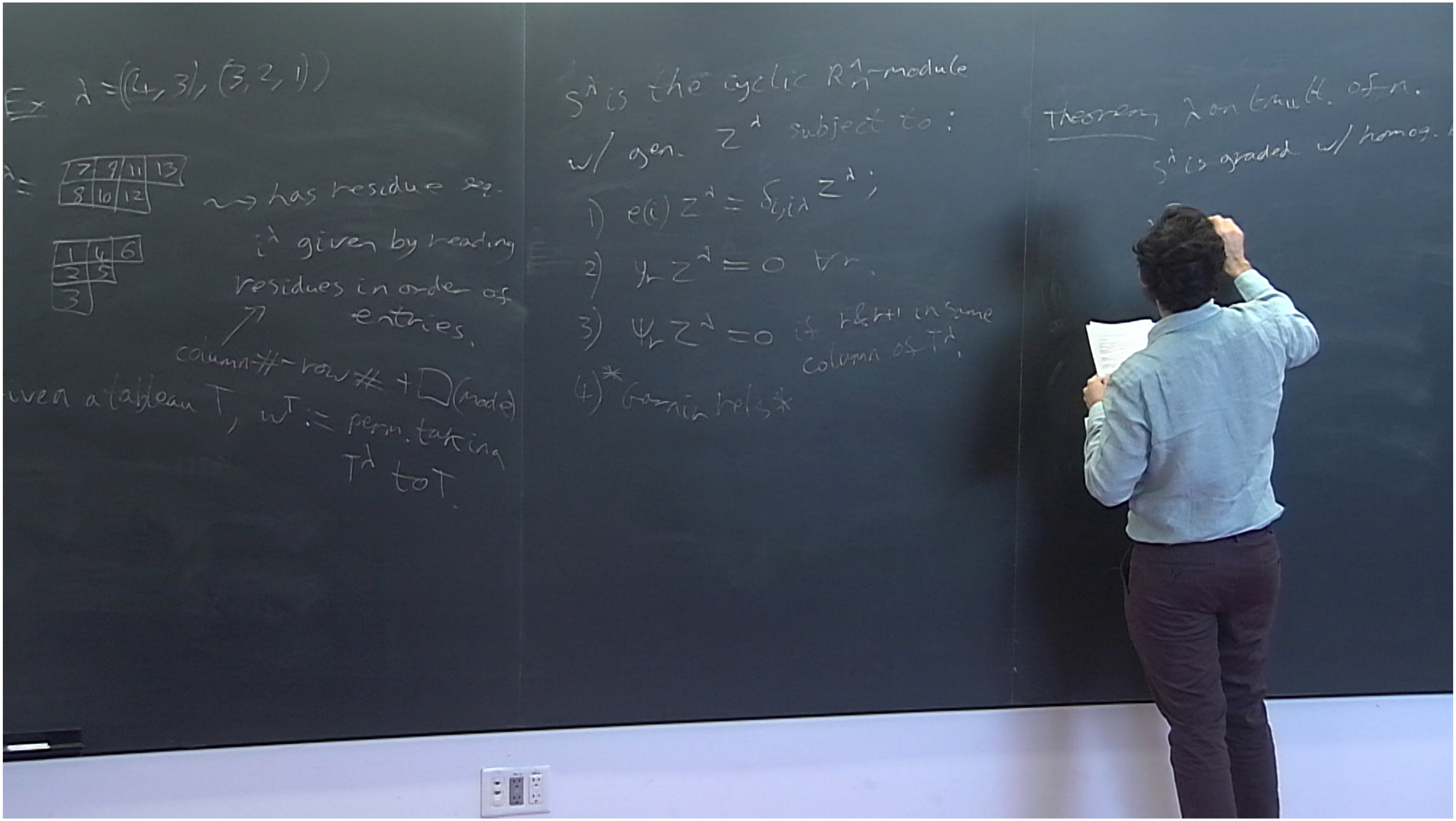
7	9	11	13
8	10	12	

1	4	6
2	5	
3		

⇝ has residue seq. i^λ given by reading residues in order of entries.

column # − row # + $\square$ (mod e)

Given a tableau T, $w^T :=$ perm. taking T^λ to T.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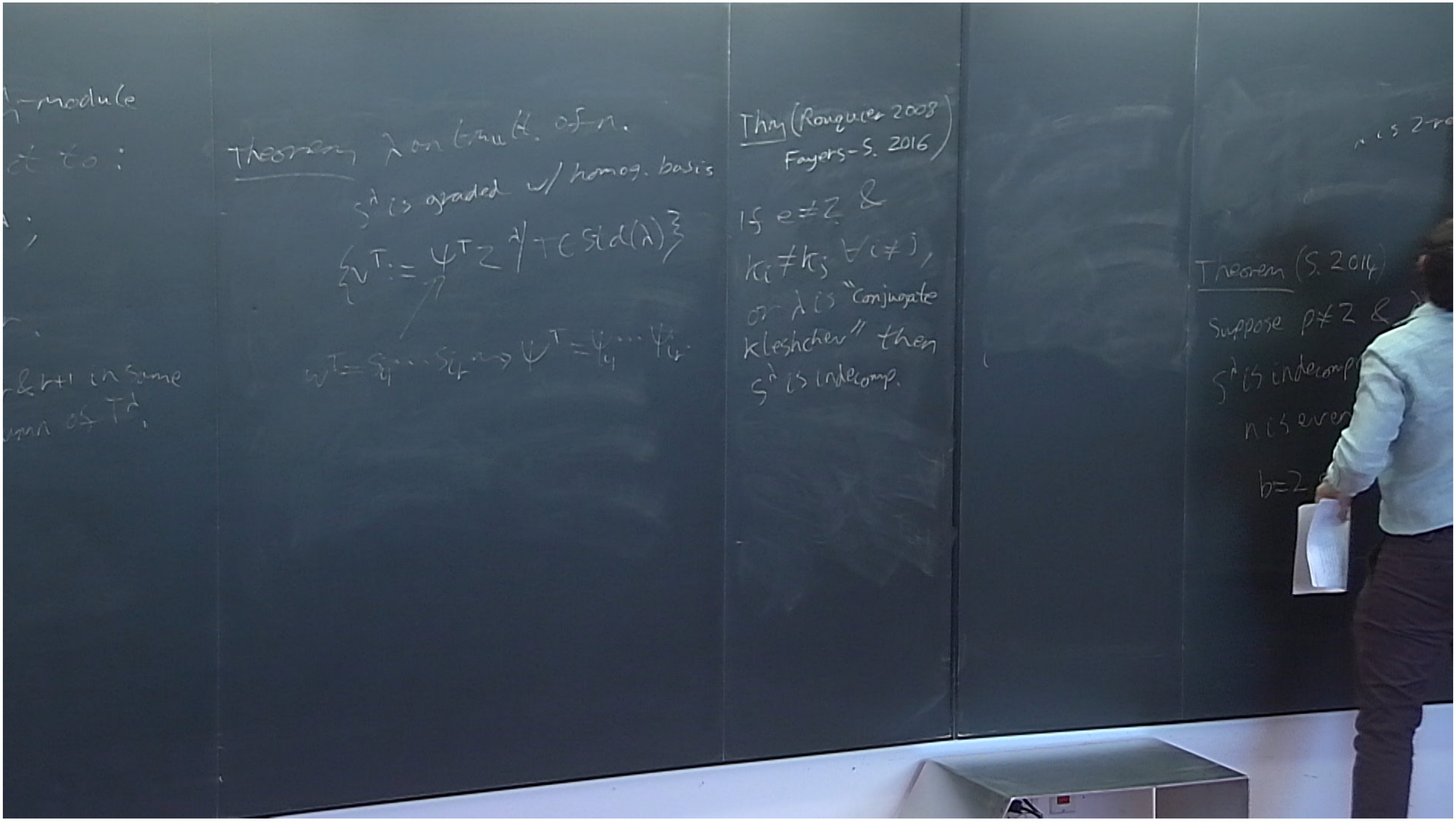
Given a tableau T, w^T := perm. taking
T^λ to T.
S^λ is the cyclic R^Λ_n-module
w/ gen. z^λ subject to:
1) e(i) z^λ = δ_{i,i^λ} z^λ;
2) y_r z^λ = 0 ∀ r.
3) ψ_r z^λ = 0 if r & r+1 in same
column of T^λ.
4)* Garnir rels*
Theorem
S^λ is graded w/ homog.


-module
Theorem λ on ... of n.
S^λ is graded w/ homog. basis
{v^T := ψ^T z^λ / T ∈ Std(λ)}
ψ^T = ψ_{i_1} ... ψ_{i_r}
& r+1 in same
column of T^λ.
Thm (Rouquier 2008
Fayers-S. 2016)
If e≠2 &
κ_i ≠ κ_j ∀ i≠j,
or λ is "Conjugate
Kleshchev" then
S^λ is indecomp.
Theorem (S. 2014)
Suppose p≠2 &
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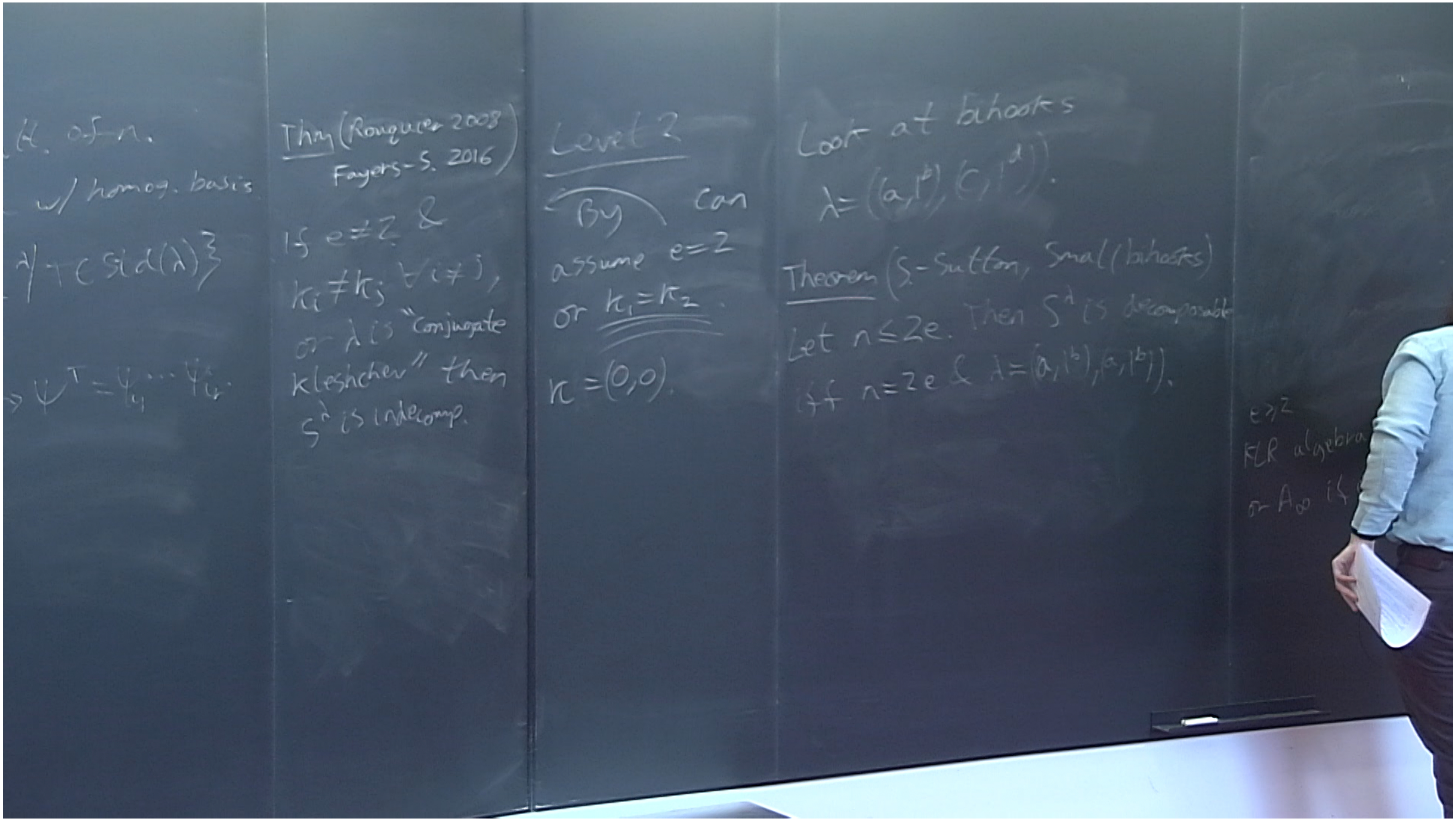

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$\kappa_i \neq \kappa_j \ \forall i \neq j$,
or λ is "conjugate
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Level 2
By con
assume e=2
or $\kappa_1 = \kappa_2$.
$\kappa = (0,0)$.
Look at bihooks
$\lambda = ((a,1^b),(c,1^d))$.
Theorem (S.-Sutton, Small bihooks)
Let n≤2e. Then S^λ is decomposable
iff n=2e & $\lambda = ((a,1^b),(a,1^b))$.
e>2
KLR algebra

...at bihooks

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(S.-Sutton, Small bihooks)

...$\leq 2e$. Then S^λ is decomposable

...$= 2e$ & $\lambda = ((a,1^b),(a,1^b))$.

Theorem (S.-Sutton) $p \neq 2$.

Let $\lambda = ((k),(k))$. Then S^λ is decomposable iff $k \geq e$.

Tricks using induction/restr.:

Thm $S^{((ke),(ke))}$ is decomposable iff $S^{((ke+a,1^b),(ke+a,1^b))}$ is, for $0 \leq a, b \leq e-1$ & $a+b \neq e$.

Technique

Prove $S^{((ke),(ke))}$ is decomp. by finding endomorphism φ s.t. φ has distinct e-values mod p.

at bihooks

$(a,1^b),(c,1^d))$.

(S.-Sutton, Small bihooks)

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