Title: PSI 2017/2018 - Quantum Integrable Models - Lecture 9

Date: Mar 29, 2018 11:30 AM

URL: http://pirsa.org/18030067

Abstract:

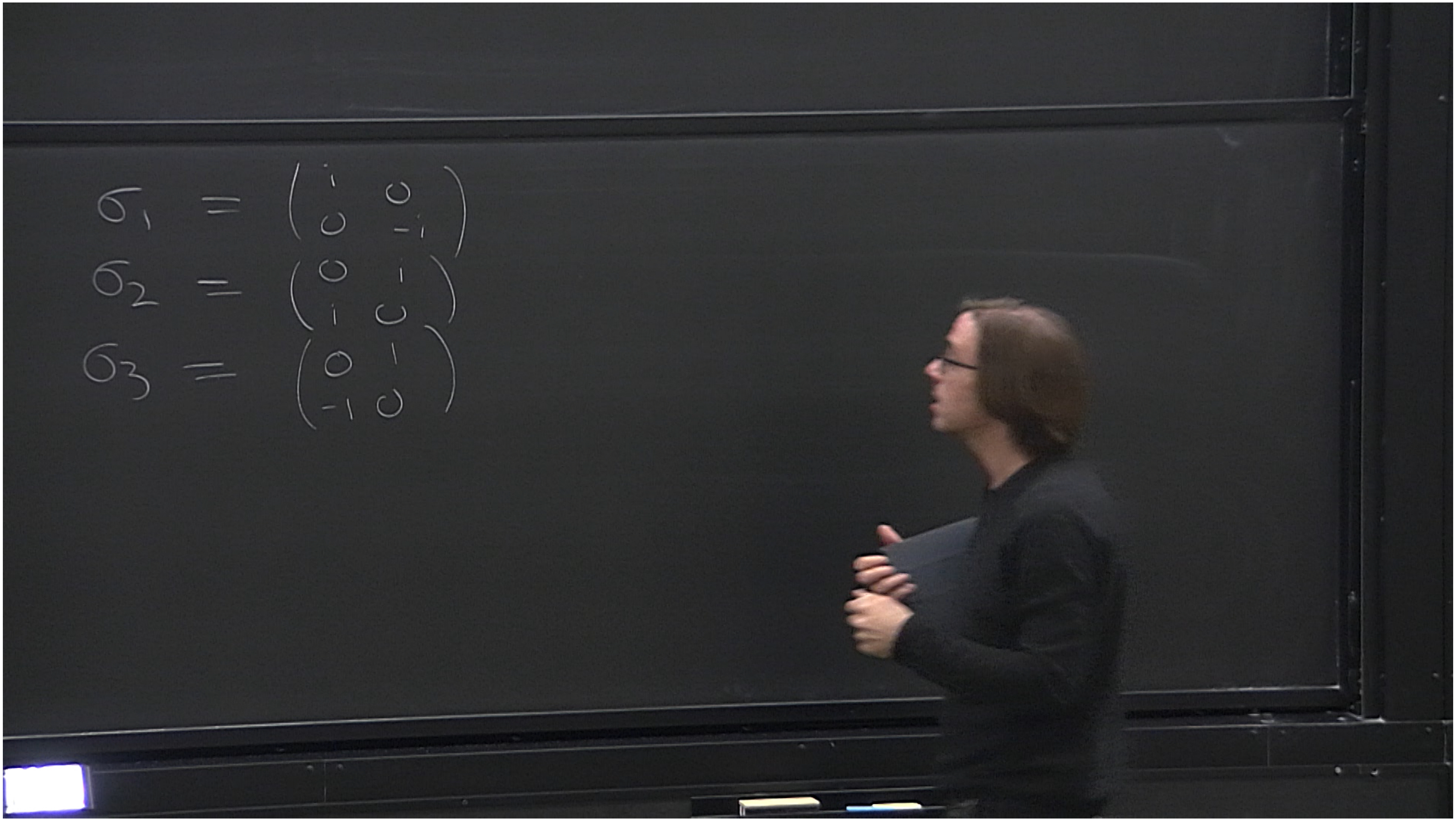
Chiral + A.C. + $J\bar{J}$

Consider free β-γ systems

$$\int \beta^i \partial_{\bar{w}} \gamma_i + \int \bar{\beta}^i \partial_w \bar{\gamma}_i \quad + \text{c.c}$$

i runs from $1, .. 2$

β, γ are in fun. rep of $SU(2)$


$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
$\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
$\sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

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$$J_1 = i\left(\beta^1 \gamma_1 - \beta^2 \gamma_2\right)$$

$$J_2 = i\left(\beta^1 \gamma_2 + \beta^2 \gamma_1\right)$$

$$J_3 = \beta^1 \gamma_2 - \beta^2 \gamma_1$$

$$\bar{J}^1 = -i\left(\bar{\beta}^1 \bar{\gamma}_1 - \bar{\beta}^2 \bar{\gamma}_2\right)$$

$$\bar{J}^2 = -i\left(\bar{\beta}^1 \bar{\gamma}_2 + \bar{\beta}^1 \bar{\gamma}_2\right)$$

$$\bar{J}^3 = \bar{\beta}^1 \bar{\gamma}_2 - \bar{\beta}^2 \bar{\gamma}_1$$

$$\sum J_a \bar{J}_a = J_1 \bar{J}_1 + J_2 \bar{J}_2 + J_3 \bar{J}_3$$

$$= \left(\beta^1 \overline{\beta^1} + \beta^2 \overline{\beta^2} \right) \left(\gamma_1 \bar{\gamma}_1 + \gamma_2 \bar{\gamma}_2 \right)$$

$$\sum J_a \bar{J}_a = J_1\bar{J}_1 + J_2\bar{J}_2 + J_3\bar{J}_3$$

$$= (\beta^1\bar{\beta}^1 + \beta^2\bar{\beta}^2)(\gamma_1\bar{\gamma}_1 + \gamma_2\bar{\gamma}_2)$$

β-γ + $\bar{\beta}$-$\bar{\gamma}$ system, with a $J\bar{J}$ deformation is 2d σ-model on $\mathbb{C}^2$ with metric $g^{ij}\, du_i\, d\bar{u}_j$ where if g_{ij} satisfies

$$\bar{J}_3 = \beta^1\gamma_2 - \beta^2\gamma_1 \qquad \bar{J}^3 = \bar{\beta}^1\bar{\gamma}_2 - \bar{\beta}^2\bar{\gamma}_1$$

$g_{ij}g^{jk} = \delta_i^k$, then

$$g_{ij} = \delta_{ij}\left(u_1\bar{u}_1 + u_2\bar{u}_2\right)$$

$$\sum J_a \bar{J}_a = J_1\bar{J}_1 + J_2\bar{J}_2 + J_3\bar{J}_3$$

$$= (\beta^1\bar{\beta}^1 + \beta^2\bar{\beta}^2)(\gamma_1\bar{\gamma}_1 + \gamma_2\bar{\gamma}_2)$$

$\beta-\gamma + \bar{\beta}-\bar{\gamma}$ system, with a $J\bar{J}$ deformation is 2d σ-model on $\mathbb{C}^2$ with metric

$g^{ij} du_i d\bar{u}_j$, where if g_{ij} satisfies

u_1, u_2 coords on $\mathbb{C}^2$

$$\bar{J}_3 = \beta^1 \gamma_2 - \beta^2 \gamma_1 \qquad \bar{J}^3 = \bar{\beta}^1 \bar{\gamma}_2 - \bar{\beta}^2 \bar{\gamma}_1$$

$g_{ij} g^{ik} = \delta_i^k$, then

$$g_{ij} = \delta_{ij} \left(u_1 \bar{u}_1 + u_2 \bar{u}_2 \right)$$

Therefore the metric is

$$g = \frac{1}{|u_1|^2 + |u_2|^2} \left(du_1 d\bar{u}_1 + du_2 d\bar{u}_2 \right)$$

$u_1 = x_1 + \sqrt{-1}\, x_2$

$u_2 = x_3 + \sqrt{-1}\, x_4$

$$g = \frac{1}{r^2} \left(\sum dx_i^2 \right)$$

$$r^2 = \sum x_i^2$$

$\mathbb{R}^4 \setminus 0$, with this metric is

$$g = d^2(S^3) + \frac{dr^2}{r^2}$$

$$s = \log r$$

$$d^2(S^3) + ds^2$$

$$(\mathbb{R}^4 \setminus 0, g) \cong S^3 \times \mathbb{R}$$

...h this metric is

$+ \frac{dr^2}{r^2}$

ds^2

$S^3 \times \mathbb{R}$

Conclusion

σ-model on S^3, times a free scalar

is of the form $\left(\text{chiral} \times \text{a.c.} + J\bar{J}\right)$

and is therefore integrable

Why are these models integrable?

Consider a chiral and antichiral theory
with an action of G, a simple group
t_a a basis of $\mathfrak{g}$ $\langle t_a, t_b \rangle = \delta_{ab}$
f_{abc} structure constants

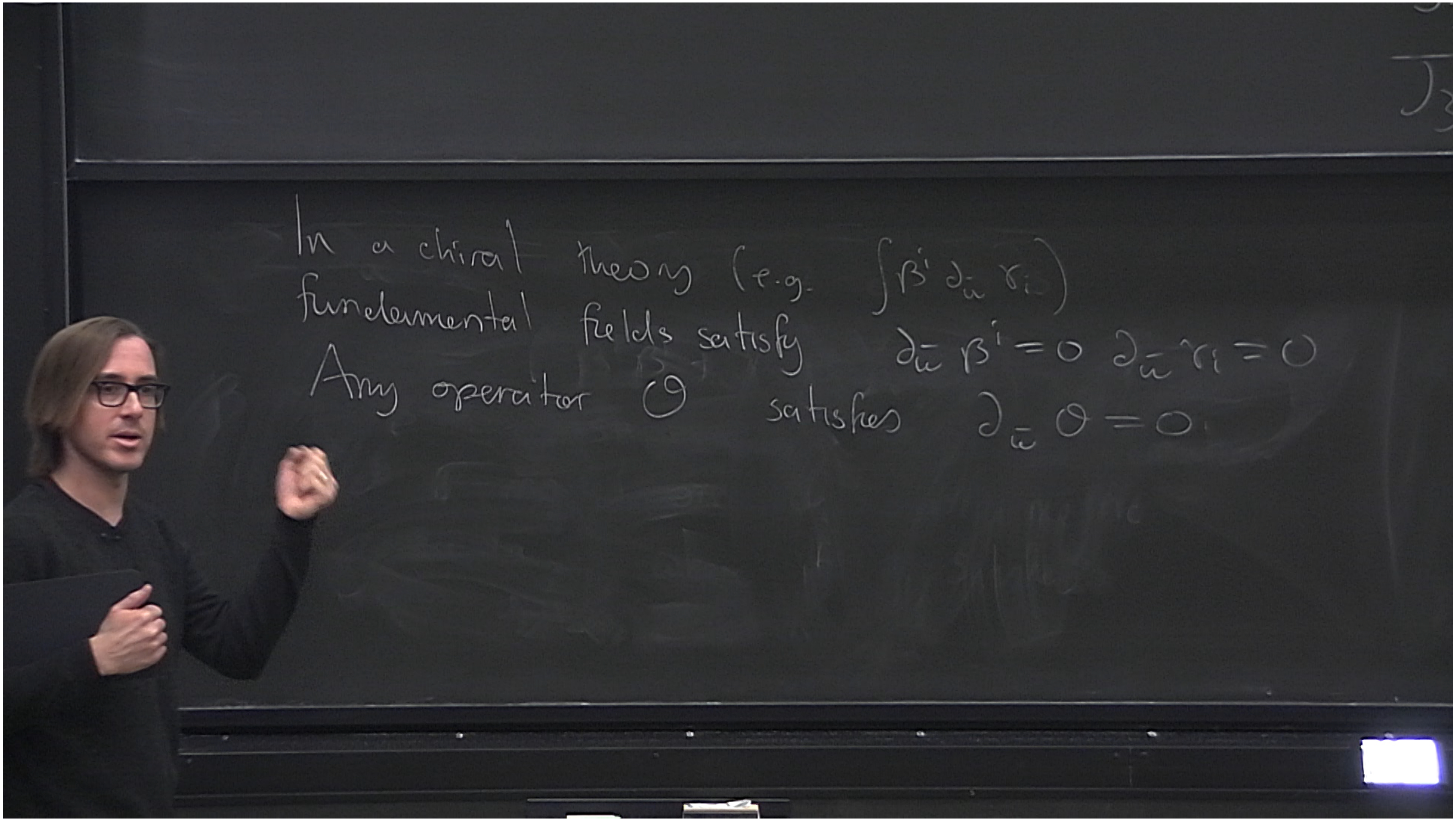
Why are these models integrable?

Consider a chiral and antichiral theory
with an action of G, a simple group

t_a a basis of $\mathfrak{g}$ $\quad \langle t_a, t_b \rangle = \delta_{ab}$

f_{abc} structure constants

Recall, J_a appears as coeff. of
a background gauge field $A^a_{\bar{w}}$


In a chiral theory (e.g. $\int \beta^i \partial_{\bar{w}} \gamma_i$)
fundamental fields satisfy $\partial_{\bar{w}} \beta^i = 0$ $\partial_{\bar{w}} \gamma_i = 0$
Any operator $\mathcal{O}$ satisfies $\partial_{\bar{w}} \mathcal{O} = 0$.

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Any operator $\mathcal{O}$ satisfies $\partial_{\bar{w}} \mathcal{O} = 0$.
If we turn on a background gauge field $A_{a\bar{w}}$
then $\partial_{\bar{w}} \mathcal{O} + A_{a\bar{w}} \rho_a(\mathcal{O}) = 0$
ρ_a = transformations of an operator under $\mathfrak{g}$ action

In chiral + a.c. + $J_a \bar{J}_a$ theory

any operator $\mathcal{O}$ built from chiral fields satisfies

$$\partial_{\bar{w}} \mathcal{O} + \bar{J}_a \rho_a(\mathcal{O}) = 0$$

As, from chiral theory P.O.V. $J_a \bar{J}_a$ deformation means have a gauge field $A_{a\bar{w}} = \bar{J}_a$

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

ρ_a - Transformations of an operator under g action

Similarly, if $\bar{\Theta}$ is an operator built a.c. fields then

$$\partial_w \bar{\Theta} + J_a \bar{\rho}_a(\bar{\Theta}) = 0$$

Applying to $\Theta = J_a$

$\bar{\Theta} = \bar{J}_a$

we find

$\partial_{\bar{w}}$

$$\partial_{\bar{w}} J_a + \bar{J}_b f_{cba} J_c = 0$$

$$\partial_{w} \bar{J}_a + J_b f_{cba} \bar{J}_c = 0$$

r g action

$$\partial_{\bar w} J_a + \bar J_b f_{cba} J_c = 0$$

$$\partial_w \bar J_a + J_b f_{cba} \bar J_c = 0$$

Claim $\mathcal{L}(z)_{a w} = \frac{1}{z} J_a$

$$\mathcal{L}(z)_{a \bar w} = \frac{1}{1-z} \bar J_a$$

satisfy the Lax equations

$$\partial_{\bar w} \mathcal{L}(z)$$

$$\partial_{\bar{w}} J_a + \bar{J}_b f_{cba} J_c = 0$$

$$\partial_w \bar{J}_a + J_b f_{cba} \bar{J}_c = 0$$

claim $\mathcal{L}(z)_{aw} = \frac{1}{z} J_a$

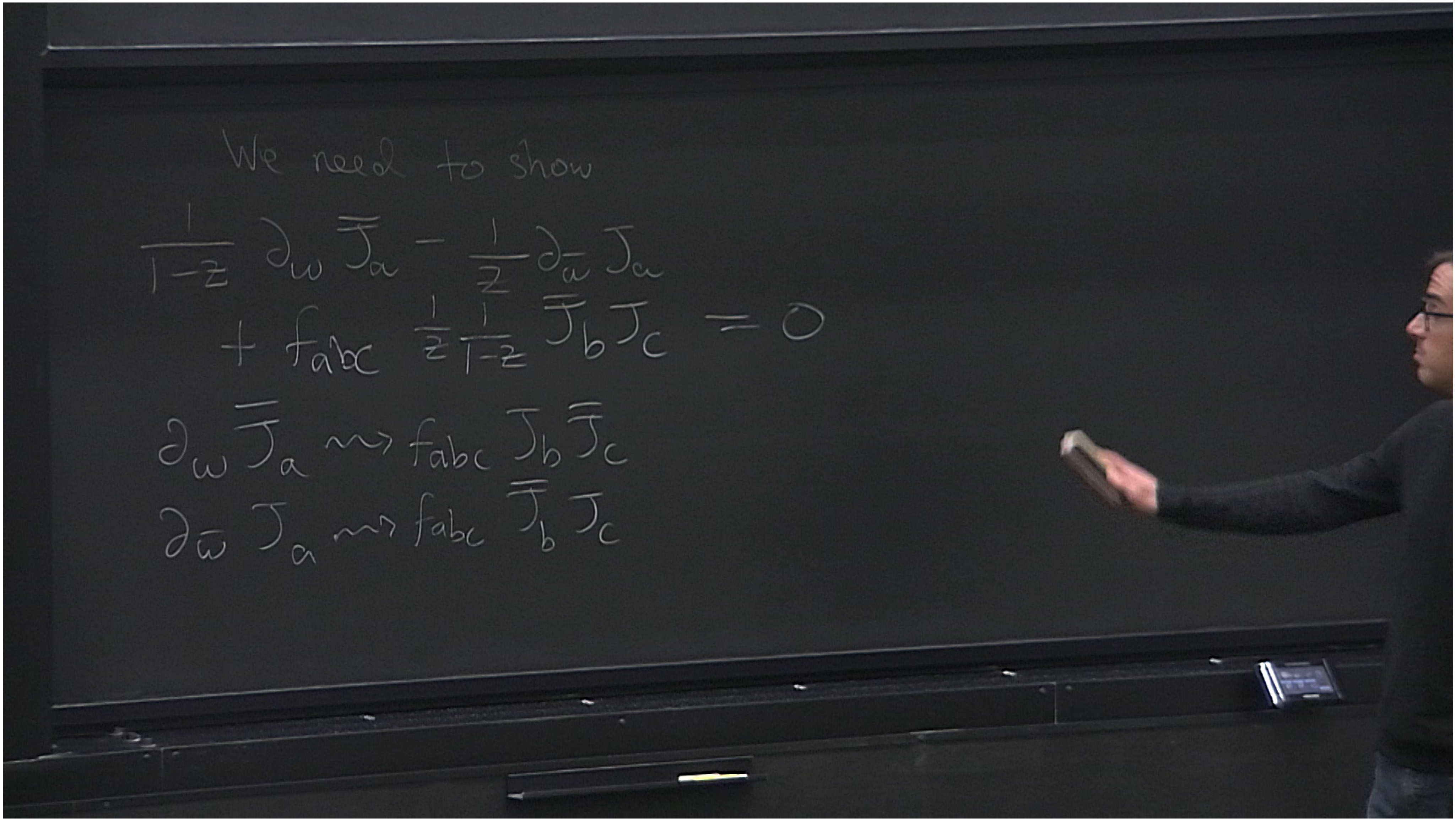
$\mathcal{L}(z)_{a\bar{w}} = \frac{1}{1-z} \bar{J}_a$

satisfy the Lax equations

$$\partial_w \mathcal{L}(z)_{a\bar{w}} - \partial_{\bar{w}} \mathcal{L}(z)_{aw} + f_{abc} \mathcal{L}(z)_{bw} \mathcal{L}(z)_{c\bar{w}} = 0$$

We need to show

$$\frac{1}{1-z}\partial_w \bar{J}_a - \frac{1}{z}\partial_{\bar{w}} J_a + f_{abc}\frac{1}{z}\frac{1}{1-z}\bar{J}_b J_c = 0$$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$\partial_w \bar{J}_a \rightsquigarrow f_{abc} J_b \bar{J}_c$
$\partial_{\bar{w}} J_a \rightsquigarrow f_{abc} \bar{J}_b J_c$

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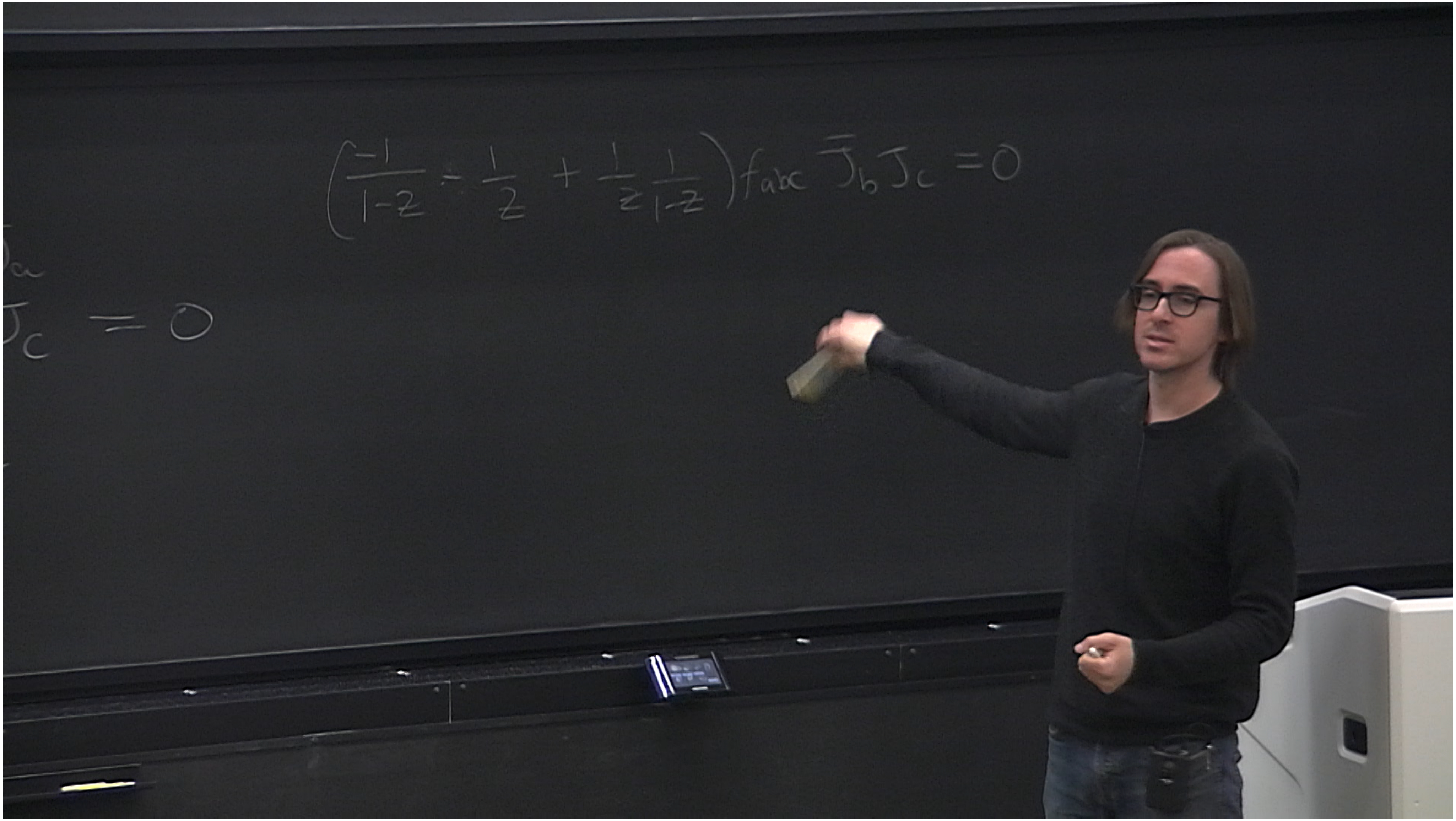
$$\partial_w \bar{J}_a \rightsquigarrow -f_{abc}\bar{J}_b J_c$$

$$\partial_{\bar{w}} J_a \rightsquigarrow f_{abc}\bar{J}_b J_c$$

$$\left(\frac{-1}{1-z} - \frac{1}{z} + \frac{1}{z}\frac{1}{1-z}\right) f_{abc}\, \bar{J}$$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$$\left(\frac{-1}{1-z} - \frac{1}{z} + \frac{1}{z}\frac{1}{1-z}\right) f_{abc} \bar{J}_b J_c = 0$$

More generally

Suppose we have $r_{ab}(z) \in g \otimes g$
which satisfies

1) $r_{ab}(z) = r_{ba}(-z)$

2) Classical YBE

$$f_{bce}\, r_{ab}(z_1-z_2)\, r_{cd}(z_2-z_3) + \text{cyclic} = 0$$

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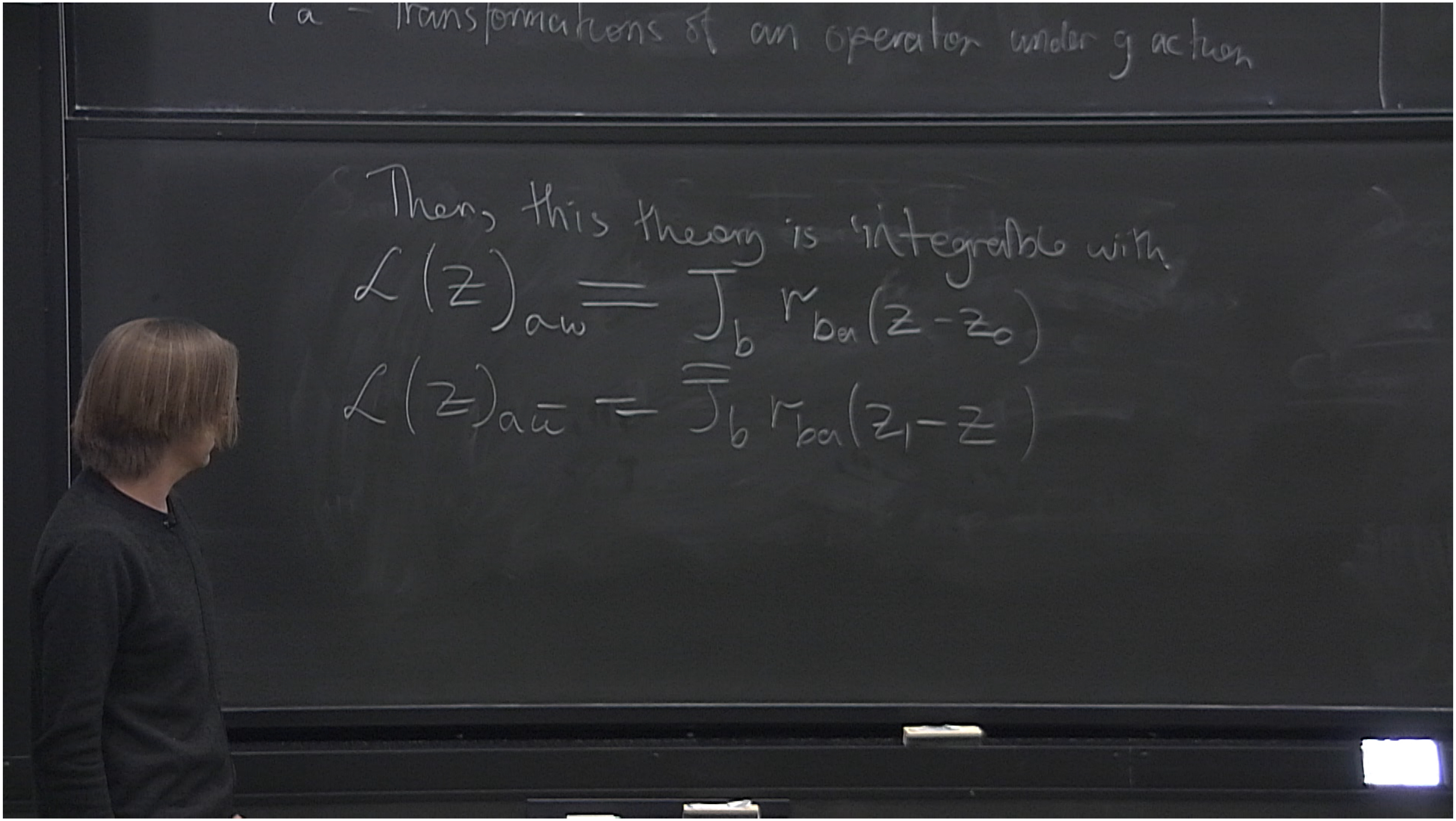
Then, we can define a field theory by deforming chiral x a.c. by

$$J_a \bar{J}_b\, r_{ab}(z_0 - z_1)$$


a – Transformations of an operator under g action
Then, this theory is 'integrable' with
$\mathcal{L}(z)_{aw} = J_b \, r_{ba}(z - z_0)$
$\mathcal{L}(z)_{a\bar{w}} = \bar{J}_b \, r_{ba}(z_1 - z)$

t_a - Transformations of an operator under g action

Then, this theory is 'integrable' with

$$\mathcal{L}(z)_{a w} = J_b \, r_{ba}(z - z_0)$$

$$\mathcal{L}(z)_{a \bar{w}} = \bar{J}_b \, r_{ba}(z_1 - z)$$

<u>Idea</u> EOM will say

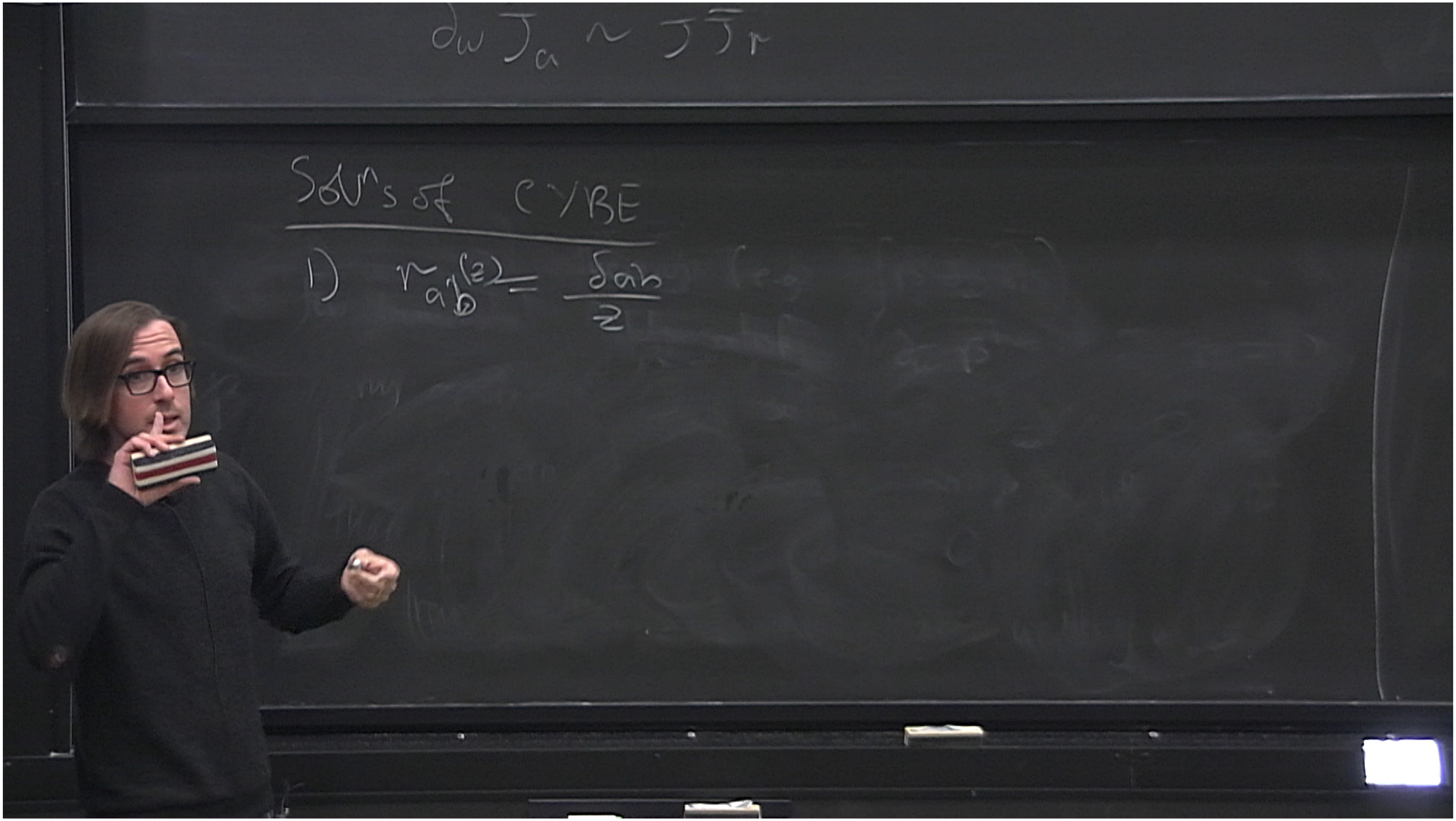
$$\partial_{\bar{w}} J_a \sim J \bar{J} r$$

$$\partial_w \bar{J}_a \sim J \bar{J} r$$

$\mathcal{L}_{a w} \mathcal{L}_{b \bar{w}} f_{abc} \sim r \sim$

The Lax eq^{n}

$\Longleftrightarrow$ an equation quadratic in r, which is exactly CYBE.


$\partial_w J_a \sim J \bar{J} r$
Sol^ns of CYBE
1) $r_{ab}(z) = \frac{\delta_{ab}}{z}$

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Sol^{n}s of CYBE

1) $r_{ab}(z) = \dfrac{\delta_{ab}}{z}$ (using so far)

$\beta\gamma + \bar\beta\bar\gamma + J\bar J \rightsquigarrow$ round metric on S^2 or S^3

2) Trigonometric:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then,

$$r(u) = \frac{e \otimes f}{1-e^{u}} - \frac{f \otimes e}{1-e^{-u}} + \frac{1}{4}\left(\frac{1+e^{u}}{1-e^{u}}\right) h \otimes h$$

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If we start w. $\beta-\gamma$ on $\mathbb{C}$ (or $\mathbb{CP}^1$) acted on by $SU(2)$

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If we start w. β-γ on $\mathbb{C}$ (or $\mathbb{CP}^1$) acted on by $SU(2)$ we get a σ-model on a funny squashed S^2

"Sausage Model"

$\partial_w J_a \sim J \bar{J} r$

Sol^{n}s of CYBE   XXX

1) $r_{ab}(z) = \frac{\delta_{ab}}{z}$ (using so far)

$\beta\gamma + \bar{\beta}\bar{\gamma} + J\bar{J} \rightsquigarrow$ round metric on S^2 or S^3

2) Trigonometric: XXZ

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Thm:

$$r(u) = \frac{e\otimes f}{1-e^{u}} - \frac{f\otimes e}{1-e^{-u}} + \frac{1}{4}\left(\frac{1+e^{u}}{1-e^{u}}\right) h\otimes h$$

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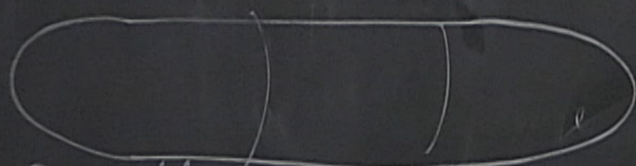
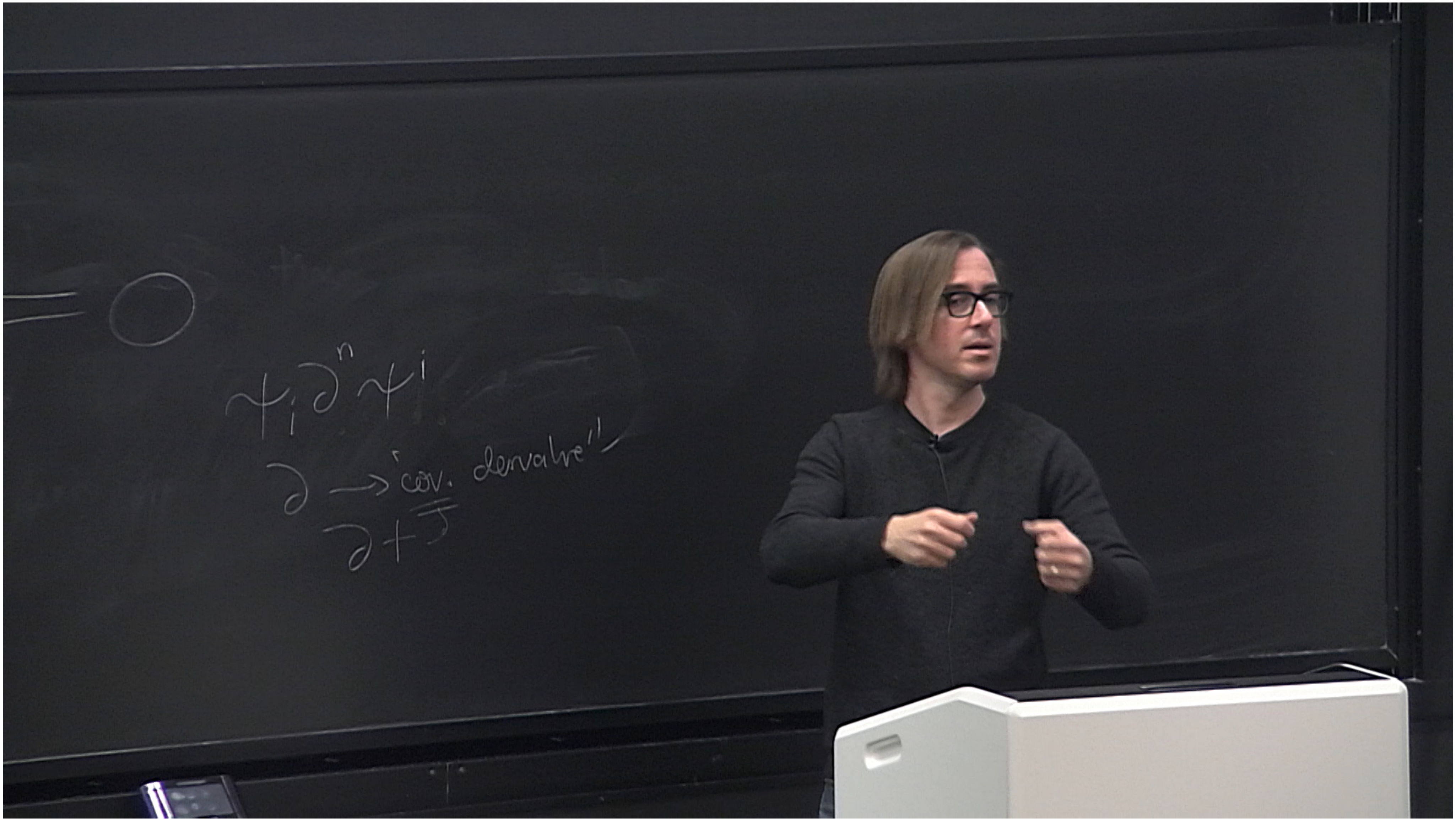
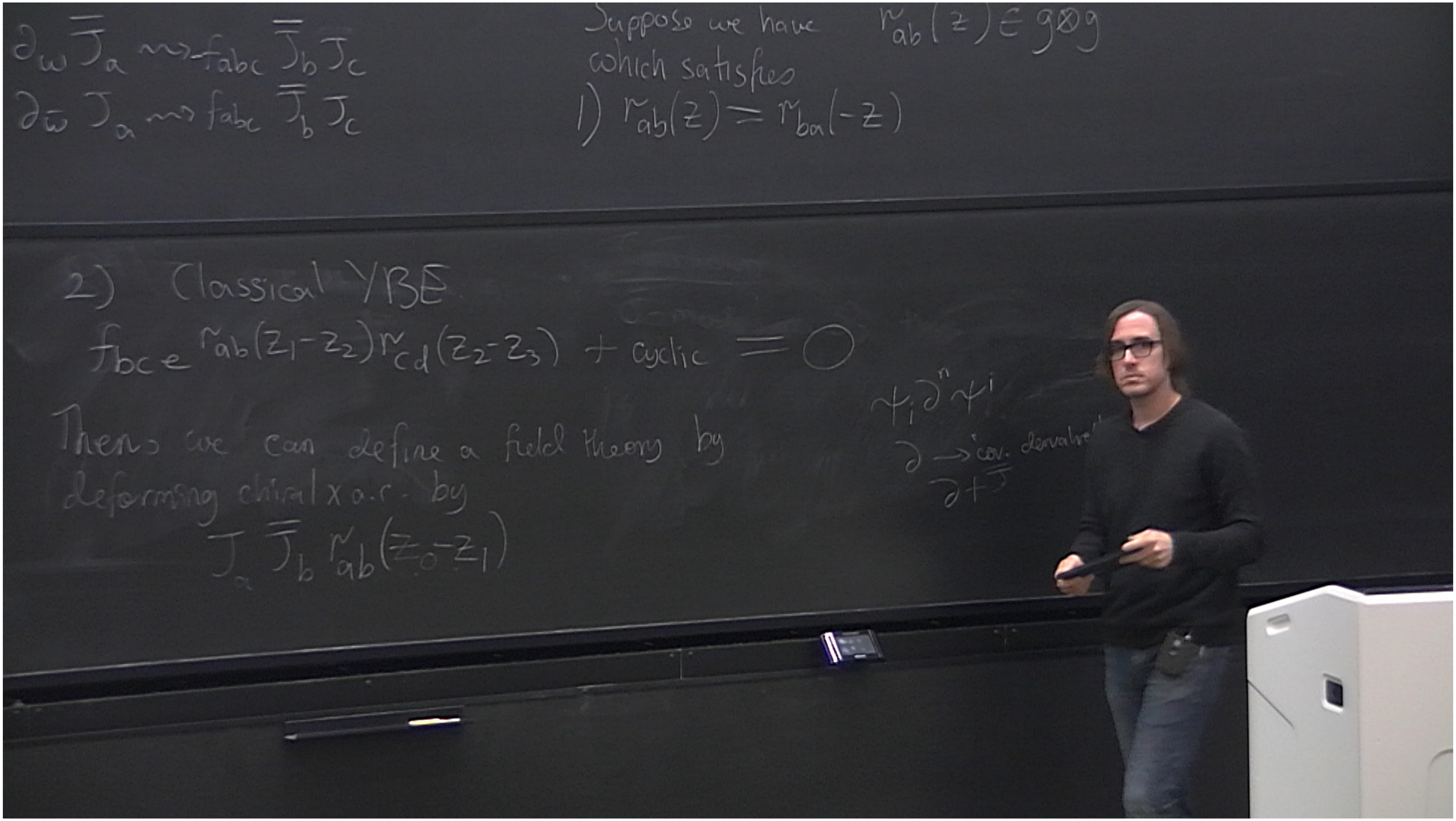


"Sausage Model"

3) Elliptic: XYZ spin chain


$\psi_i \partial^n \psi^i$
$\partial \rightarrow$ "cov. derivative"
$\partial + J$



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