Title: PSI 2017/2018 - Quantum Integrable Models - Lecture 8

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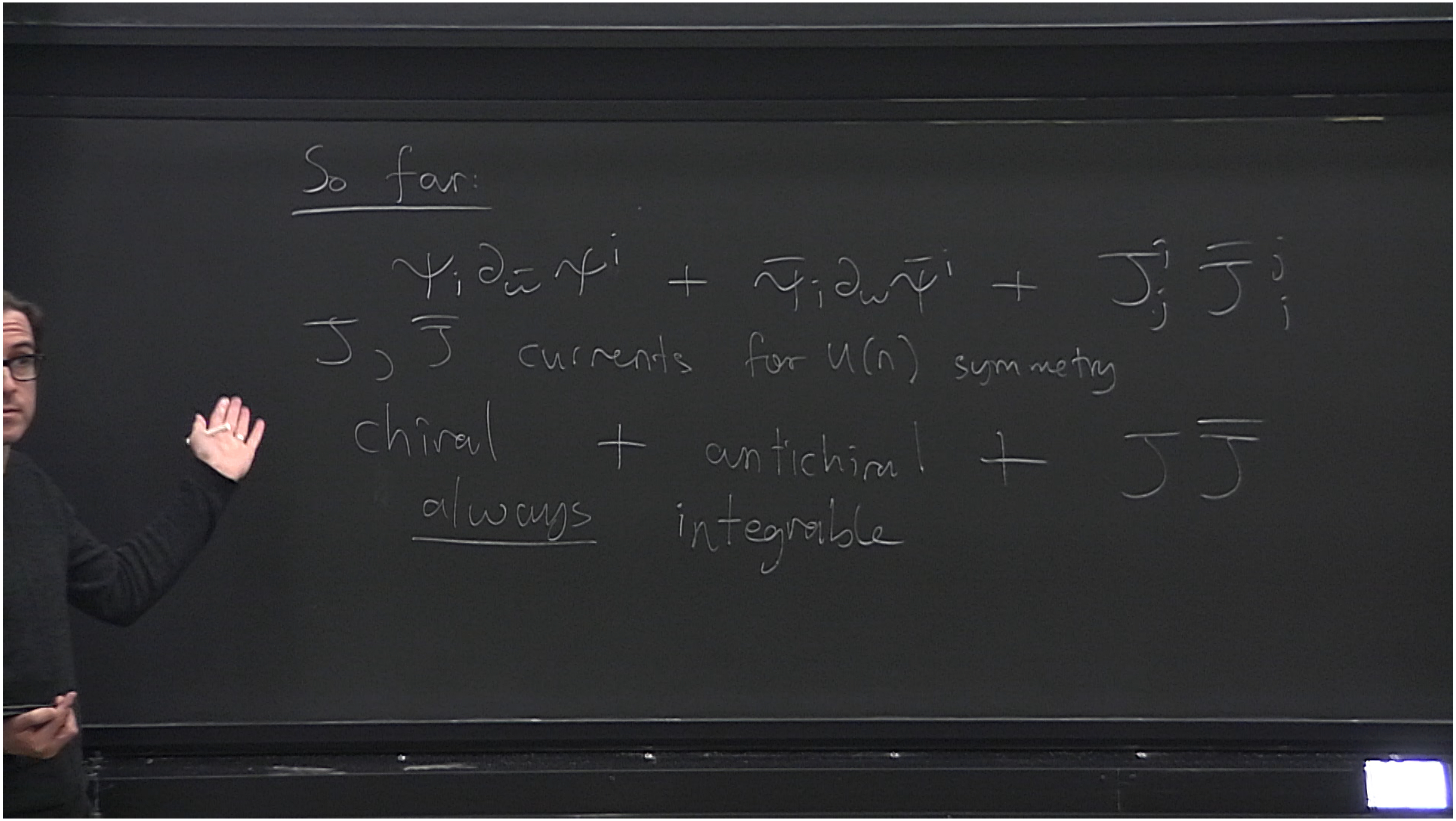
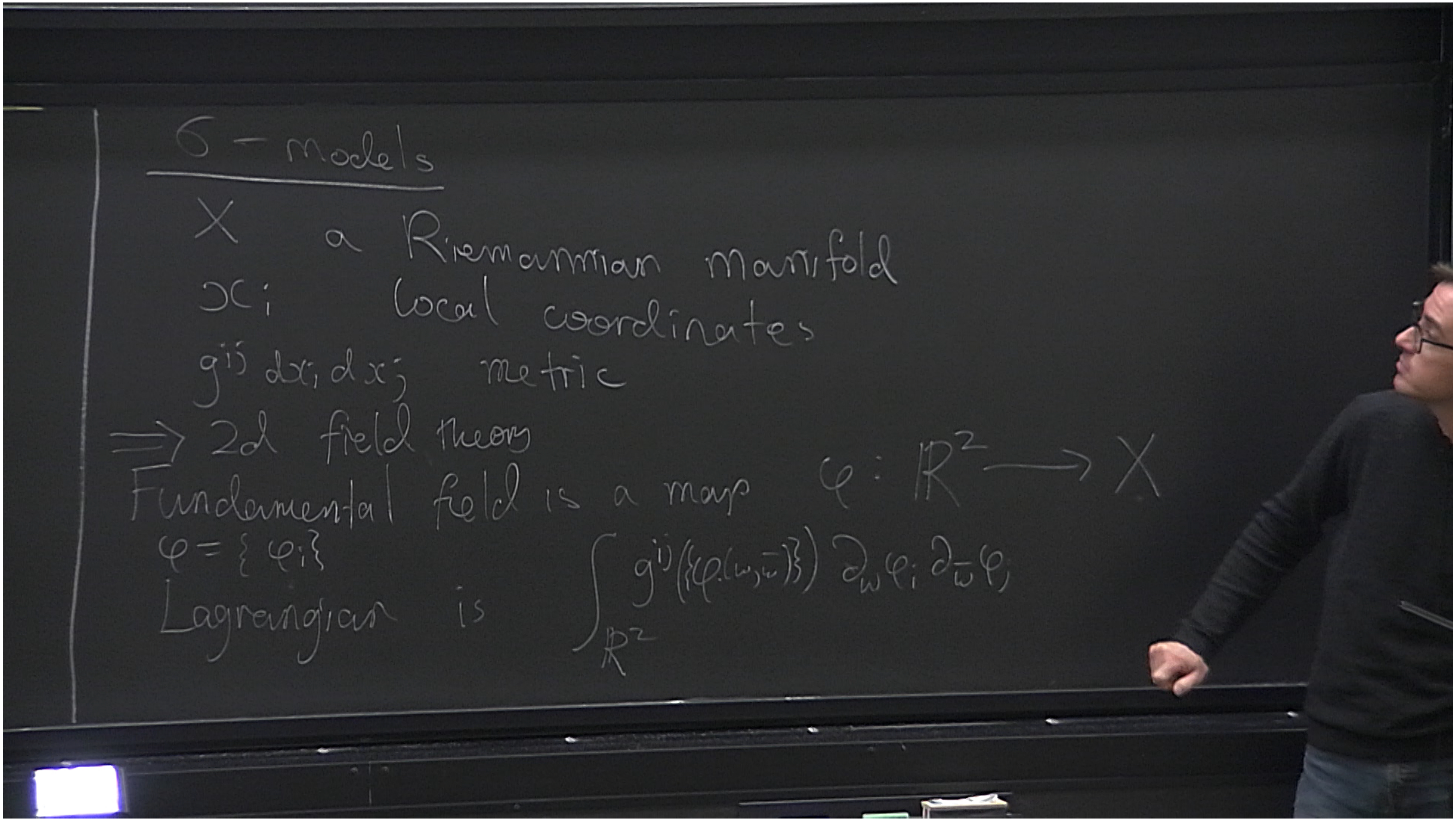
Abstract:


So far:
$\psi_i \partial_{\bar{w}} \psi^i + \bar{\psi}_i \partial_w \bar{\psi}^i + J^i_j \bar{J}^j_i$
$J, \bar{J}$ currents for $U(n)$ symmetry
chiral + antichiral + $J\bar{J}$
always integrable


σ - models
X a Riemannian manifold
x_i local coordinates
$g^{ij}\, dx_i\, dx_j$ metric
⟹ 2d field theory
Fundamental field is a map $\varphi : \mathbb{R}^2 \longrightarrow X$
$\varphi = \{\varphi_i\}$
Lagrangian is $\int_{\mathbb{R}^2} g^{ij}(\{\varphi_\cdot(w,\bar{w})\})\, \partial_w \varphi_i\, \partial_{\bar{w}} \varphi_j$

Example If $g^{ij} = \delta^{ij} + x_i x_j$

Then the Lagrangian is

$$\int (\delta^{ij} + \varphi_i \varphi_j)(\partial_w \varphi_i)(\partial_{\bar{w}} \varphi_j)$$

$$g^{ij} = \delta^{ij} + \sum F^{ij k_1 \dots k_n} x_{k_1} \dots x_{k_n}$$

replace $x_{k_i} \to \varphi_{k_i}$ in Lagrangian.

integrable

Fundamental field is a map $\varphi : \mathbb{R}^2 \to X$

$\varphi = \{\varphi_i\}$

Lagrangian is $\int_{\mathbb{R}^2} g^{ij}(\{\varphi_i(u,\bar{u})\})\, \partial_u \varphi_i \, \partial_{\bar{u}} \varphi_j$

$g^{ij} = \delta^{ij} + x_i x_j$

...an is

$(\partial_u \varphi_i)(\partial_{\bar{u}} \varphi_j)$

$(ij), k_1 \ldots k_n$ $\quad x_{k_1} \ldots x_{k_n}$

$\to \varphi_{k_j}$

$\varphi_i = x_i \circ \varphi : \mathbb{R}^2 \to \mathbb{R}$

$i = 1 \ldots n$

Suppose we can choose local coords x_i, y_i on X

$u_i = x_i + \sqrt{-1}\, y_i$

$\bar{u}_i = x_i - \sqrt{-1}\, y_i$

Such that the metric is

$g^{ij}\, du_i\, d\bar{u}_j$ where

$\overline{g^{ij}(u,\bar{u})} = g^{ji}(u,\bar{u})$

Lagrangian is $\int_{\mathbb{R}^2}$

$\varphi_i = x_i \circ \varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$

Suppose we can choose local coords x_i, y_i on X $\quad i=1..n$

$u_i = x_i + \sqrt{-1}\, y_i$

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Such that the metric is

$g^{ij}\, du_i\, d\bar{u}_j$ where

$\overline{g^{ij}(u,\bar{u})} = g^{ji}(u,\bar{u})$

X is complex manifold w. Hermitian metric

If in addition, $\partial_{\bar{u}_k} g^{ij} - \partial_{\bar{u}_j} g^{ik} = 0$

X is Kähler

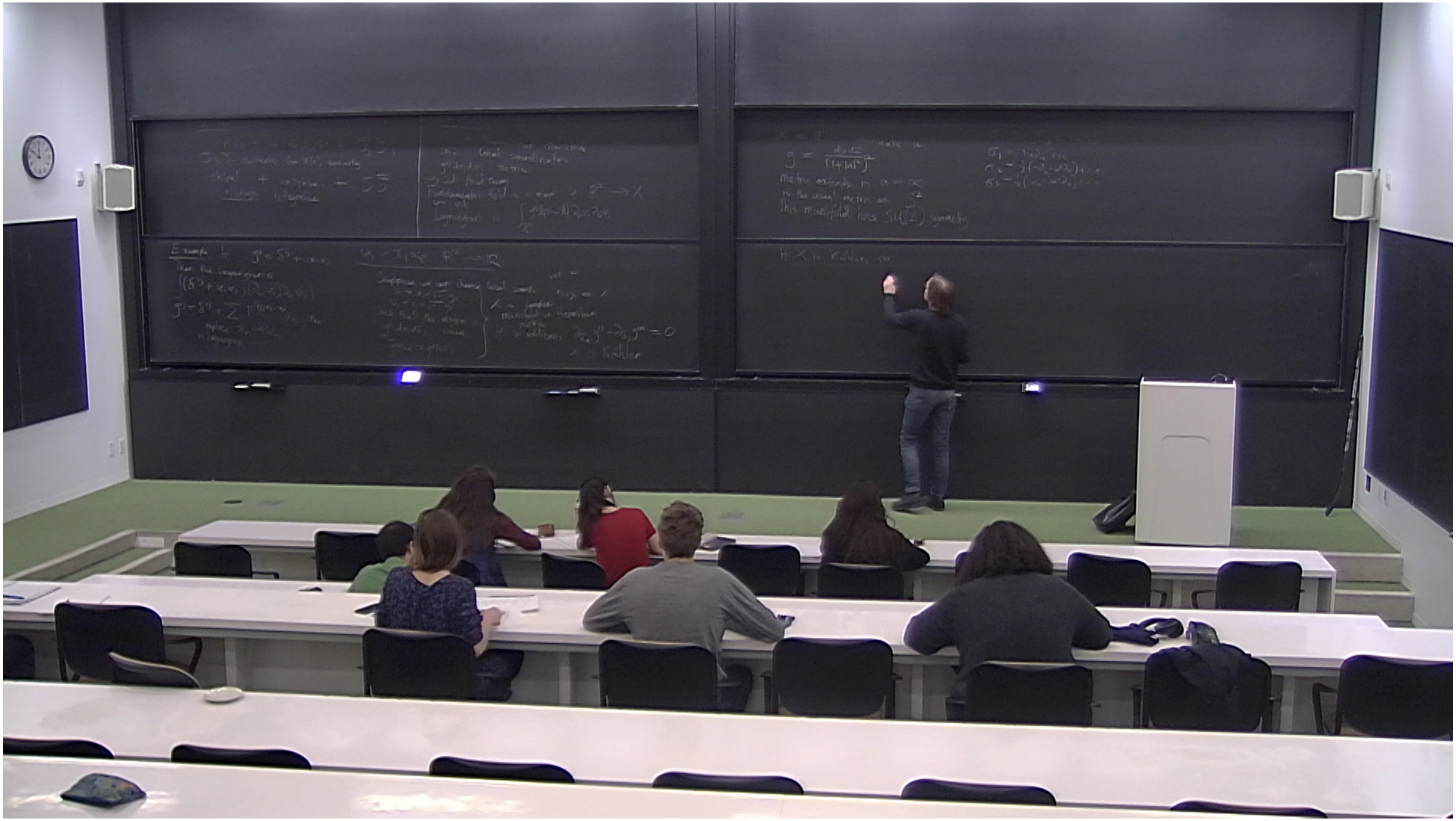
$X = \mathbb{C}$ coordinate u

$$g = \frac{du\,d\bar{u}}{(1+|u|^2)^2}$$

metric extends to $u = \infty$

is the usual metric on S^2

This manifold has $SU(2)$ symmetry



$\sigma_1, \sigma_2, \sigma_3 \in su(2)$

$\sigma_1 = i u \partial_u + c.c.$

$\sigma_2 = \frac{1}{2}\left(-\partial_u - u^2 \partial_u\right) + c.c.$

$\sigma_3 = \frac{1}{2}\left(i\partial_u - i u^2 \partial_u\right) + c.c.$

If X is Kähler, then
use fields σ_i, $\bar{\sigma}_i$
corresponding to $u_i, \bar{u}_i$ coordinates.
Lagrangian is

$$\int \partial_{\bar{w}} \sigma_i \, \partial_w \bar{\sigma}_j \, g^{ij}(\sigma, \bar{\sigma})$$

$$\left(+ \text{ top}^{l}\text{. term} \right)$$

$X = \mathbb{C} \subseteq \mathbb{CP}^1$, with Fubini-Study metric

then σ-model has fields

$\sigma, \bar{\sigma}$ with Lagrangian

$$\int \frac{\partial_{\bar{w}} \bar{\sigma}\, \partial_w \sigma}{(1+\sigma\bar{\sigma})^2}$$

Consider Lag. Introduce auxiliary fields $\rho^i, \bar{\rho}^i$ of dim^n 1 and spin

$(1, -1)$ spins

$\rho^i \sim dw \qquad \bar{\rho}^i \sim d\bar{w}$

Consider the Lagrangian

$$\int \partial_{\bar{w}} \sigma_i \, \partial_w \bar{\sigma}_j \, g^{ij}(\sigma, \bar{\sigma}) - \int \rho^i \bar{\rho}^j g_{ij}(\sigma, \bar{\sigma})$$

$$g_{ij} g^{ik} = \delta_j^k$$

ρ^i: mass, but no kinetic term

Equivalent to previous Lagrangian.

Lagrangian is $\int_{R^2}$ g

$$\beta^i = \rho^i - g^{ij}\partial_w \bar{\sigma}_j$$

$$\bar{\beta}^i = \bar{\rho}^i - g^{ji}\partial_{\bar{w}} \sigma_j$$

Lagrangian becomes

$$\int \partial_{\bar{w}}\sigma_i\,\partial_w\bar{\sigma}_j\,g^{ij} - \int \partial_w\bar{\sigma}_l\,\partial_{\bar{w}}\sigma_k\,g^{il}g^{kj}\,g_{ij}$$

cancel

$$-\int \beta^i\bar{\beta}^j g_{ij} - \int \beta^i\partial_{\bar{w}}\sigma_i - \int \bar{\beta}^i\partial_w\bar{\sigma}_i$$

Lagrangian is $\int_{\mathbb{R}^2} g$

$$\beta^i = \rho^i - g^{ij}\partial_w \bar{\sigma}_j$$

$$\bar{\beta}^i = \bar{\rho}^i - g^{ji}\partial_{\bar{w}} \sigma_j$$

Lagrangian becomes

$$\int \partial_{\bar{w}}\sigma_i \,\partial_w \bar{\sigma}_j \, g^{ij} - \int \partial_w \bar{\sigma}_l \,\partial_{\bar{w}} \sigma_k \, g^{il} g^{kj} g_{ij}$$

Inverse metric

cancel

$$\underbrace{-\int \beta^i \bar{\beta}^j g_{ij}}_{\text{deformation}} + \underbrace{\int \beta^i \partial_{\bar{w}}\sigma_i}_{\text{Chiral}} - \underbrace{\int \bar{\beta}^i \partial_w \bar{\sigma}_i}_{\text{Anti-chiral}}$$

– Only depends on X as a cx manifold

$X = \mathbb{C}$ $\quad g = \frac{du\,d\bar{u}}{(1+|u|^2)^2}$

Inverse metric is $g_{u}\,(1+|u|^2)^2$

Our Lagrangian is

$$\int \beta\bar{\beta}\,(1+\sigma\bar{\sigma})^2 + \int \beta\,\partial_{\bar{w}}\sigma + \int \bar{\beta}\,\partial_{w}\bar{\sigma}$$

$\sigma_1, \sigma_2, \sigma_3 \in su(2)$

$\sigma_1 = iu\partial_u + \text{c.c.}$

$\sigma_2 = \frac{1}{2}(-\partial_u - u^2\partial_u) + \text{c.c.}$

$\sigma_3 = \frac{1}{2}(i\partial_u - iu^2\partial_u) + \text{c.c.}$

$X = \mathbb{C}$ $\quad g = \dfrac{du\, d\bar{u}}{(1+|u|^2)^2}$

Inverse metric is $g_u (1+|u|^2)^2$

Our Lagrangian is

$$\int \beta \bar{\beta} (1+\sigma\bar{\sigma})^2 + \int \beta \partial_{\bar{w}} \sigma + \int \bar{\beta} \partial_w \bar{\sigma}$$

Claim $SU(2)$ acts on the free theory

$$\int \beta \partial_{\bar{w}} \sigma + \int \bar{\beta} \partial_w \bar{\sigma},$$ and

interaction term is $J \bar{J}$

$\sigma_1, \sigma_2, \sigma_3 \in su(2)$

$\sigma_1 = i u \partial_u + c.c.$

$\sigma_2 = \frac{1}{2}(-\partial_u - u^2 \partial_u) + c.c.$

$\sigma_3 = \frac{1}{2}(i\partial_u - i u^2 \partial_u) + c.c.$

What are currents for ∂_u, $u\partial_u$, $u^2\partial_u$
in the chiral theory $\int \beta \partial_{\bar{w}} \sigma$?
∂_u: translate σ: $\sigma \longrightarrow \sigma + \varepsilon$
If $A_{\bar{w}}$ is a gauge field gauging this symmetry
then $\int \beta \partial_{\bar{w}} \sigma + \int \beta A_{\bar{w}}$

$J_{\partial_u} = \beta$

σ - Models

X a Riemannian manifold

x_i local coordinates

$g^{ij} dx_i dx_j$ metric

$\Longrightarrow$ 2d field theory

Fundamental field is a map $\varphi : \mathbb{R}^2 \longrightarrow X$

$\varphi = \{\varphi_i\}$

Lagrangian is $\displaystyle\int_{\mathbb{R}^2} g^{ij}(\{\varphi_\cdot(w,\bar{w})\})\, \partial_w \varphi_i\, \partial_{\bar{w}} \varphi_j$

We can calculate $\sum J_{\sigma_i} \bar{J}_{\sigma_i}$ by calculating the inverse metric

$$\sigma_1^u \sigma_1^{\bar{u}} + \sigma_2^u \sigma_2^{\bar{u}} + \sigma_3^u \sigma_3^{\bar{u}}$$

$$= \left(-u\bar{u} + \frac{1}{2}u^2 + \frac{1}{2}\bar{u}^2\right)\partial_u \partial_{\bar{u}}$$

After coordinate change $\bar{u} \to -\bar{u}^{-1}$ this becomes $\frac{1}{2}(1+|u|^2)^2 \partial_u \partial_{\bar{u}}$ ✓

We can calculate $\sum J_{\sigma_i} \bar{J}_{\sigma_i}$ by calculating the inverse metric:

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$$= \left(-u\bar{u} + \tfrac{1}{2}u^2 + \tfrac{1}{2}\bar{u}^2\right)\partial_u \partial_{\bar{u}}$$

After coordinate change $\bar{u} \to -\bar{u}^{-1}$

this becomes $\frac{1}{2}(1+|u|^2)^2 \partial_u \partial_{\bar{u}}$ ✓

Conclusion

σ-model on S^2 is free chiral + a.c. + $J\bar{J}$

$\Longrightarrow$ integrable.