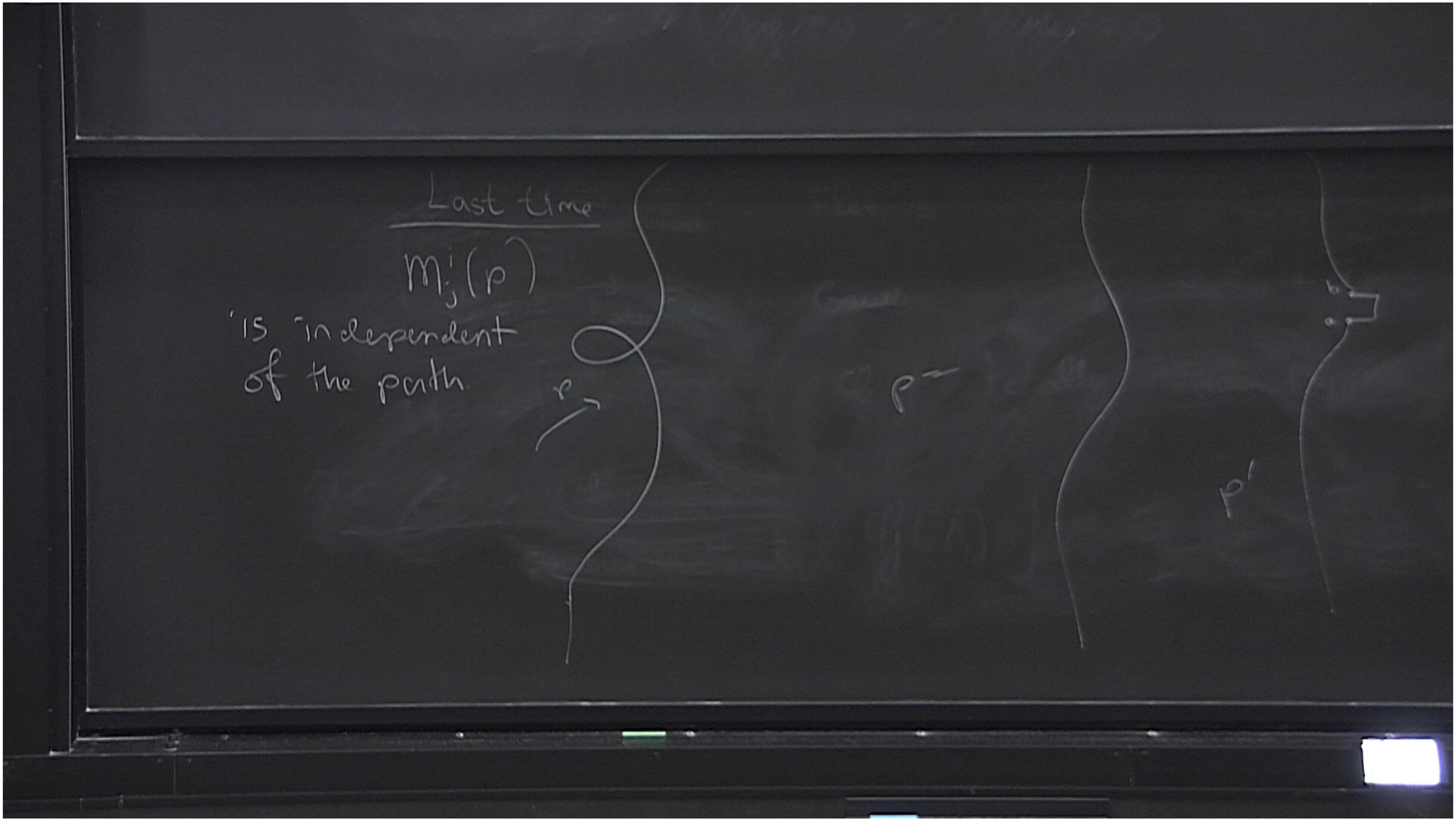
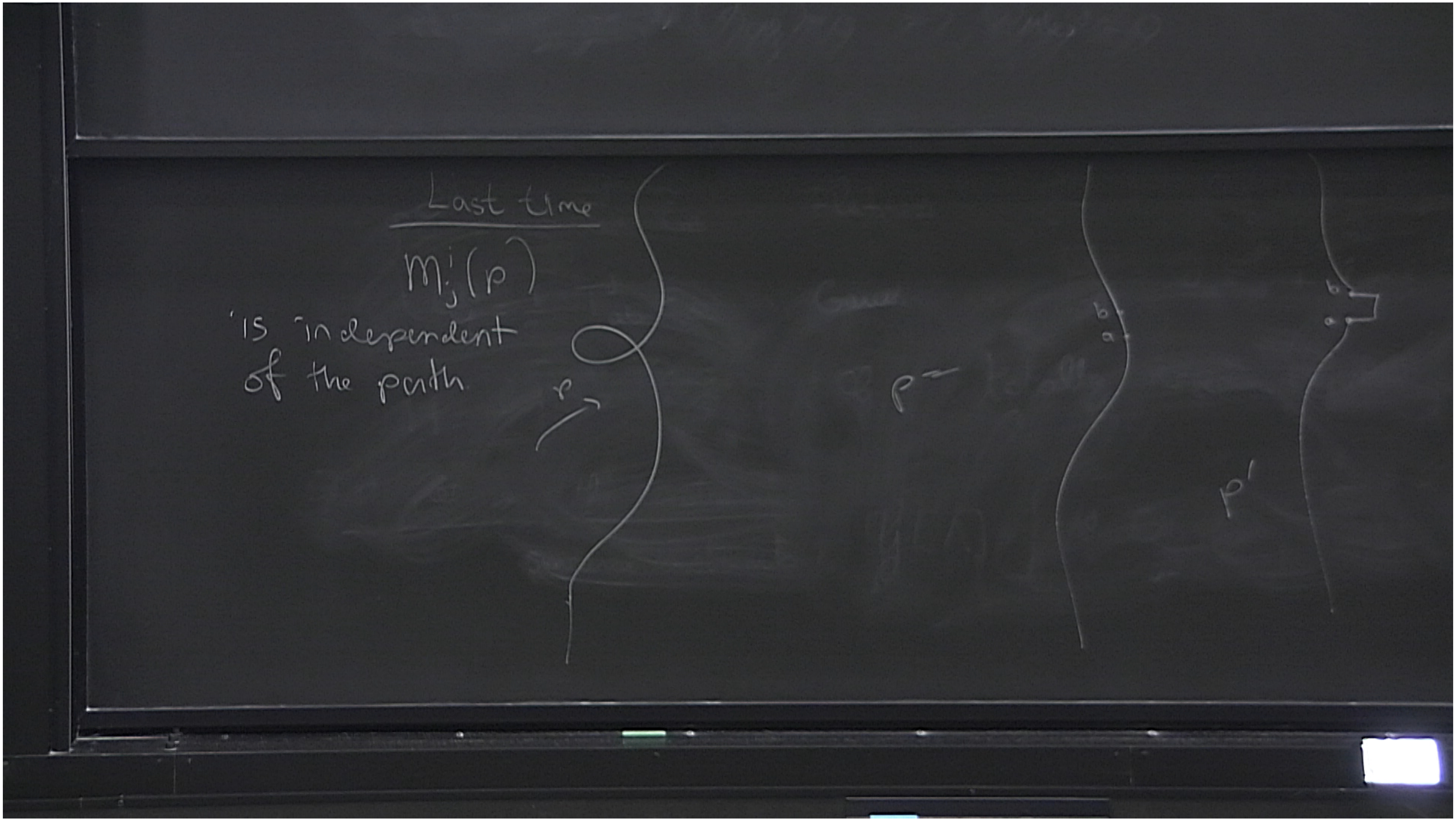
Title: PSI 2017/2018 - Quantum Integrable Models - Lecture 6

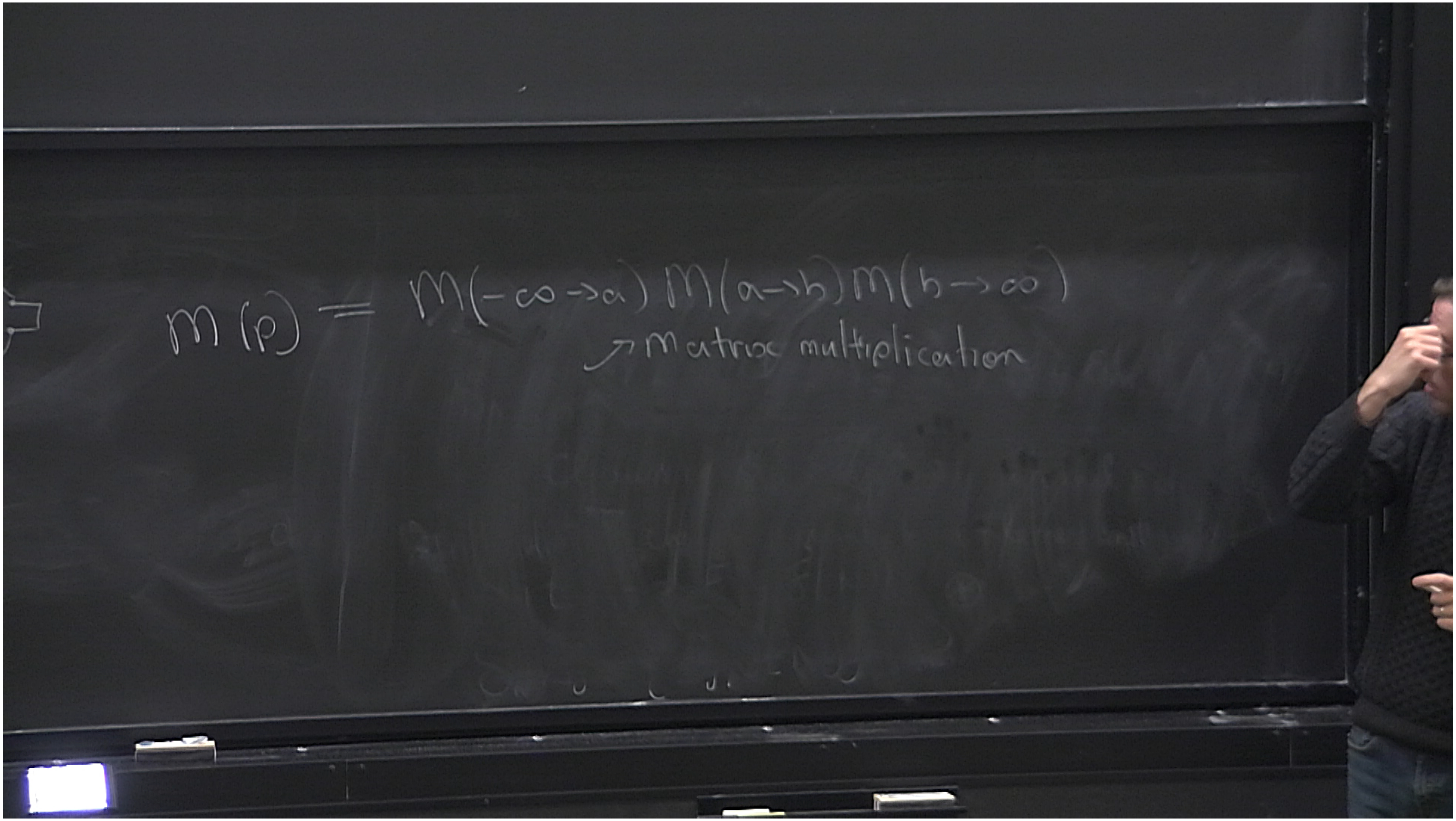
Date: Mar 26, 2018 11:30 AM

URL: http://pirsa.org/18030064

Abstract:


Last time
$M^i_j(p)$
is independent
of the path
p
p
p'


Last time
$m^i_j(\rho)$
is independent of the path
ρ
ρ
b
a
b
a
ρ'


$M(p) = M(-\infty \to a)\, M(a \to b)\, M(b \to \infty)$
matrix multiplication

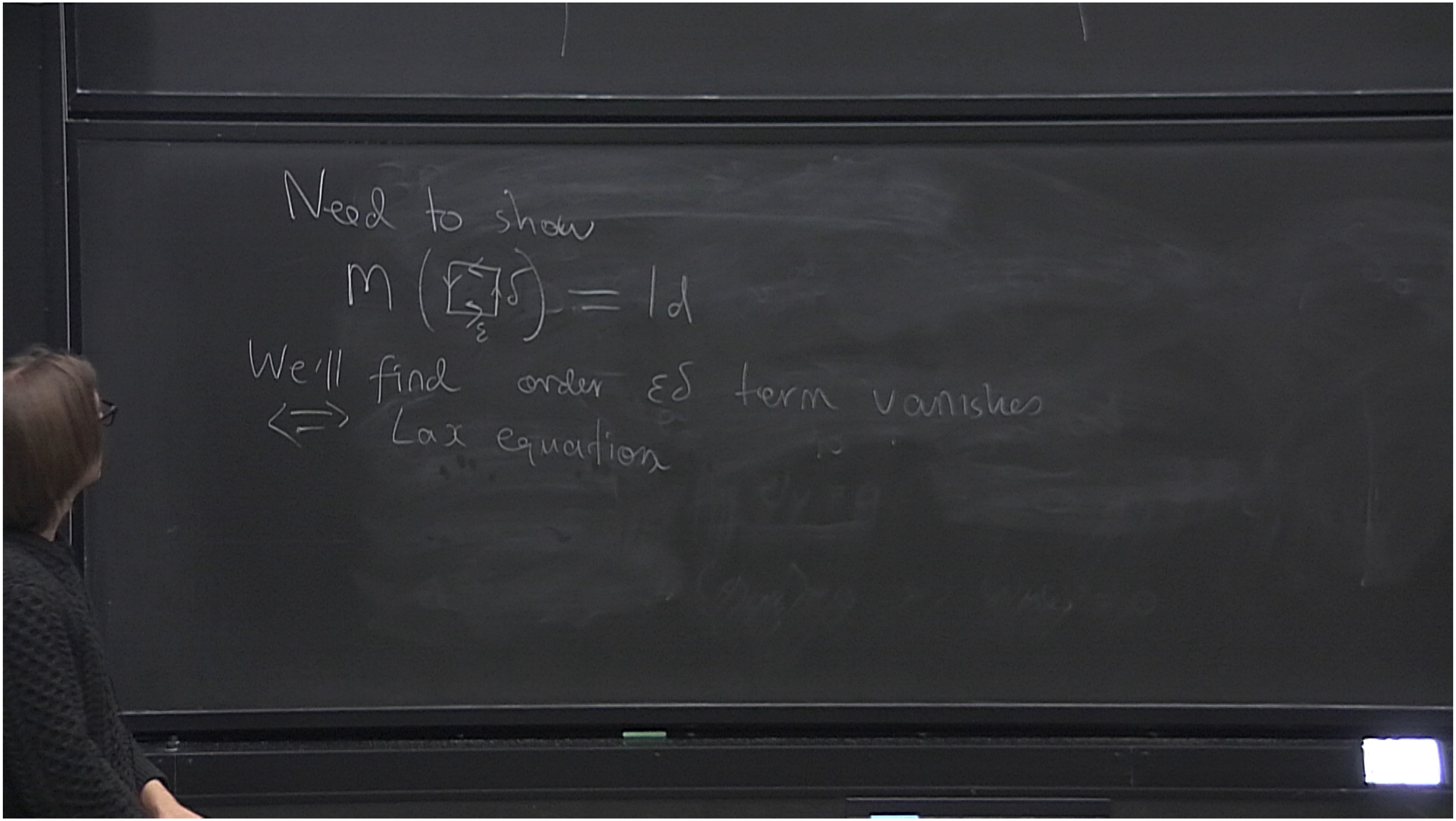
$$m(p) = m(-\infty \to a)\, m(a \to b)\, m(b \to \infty)$$

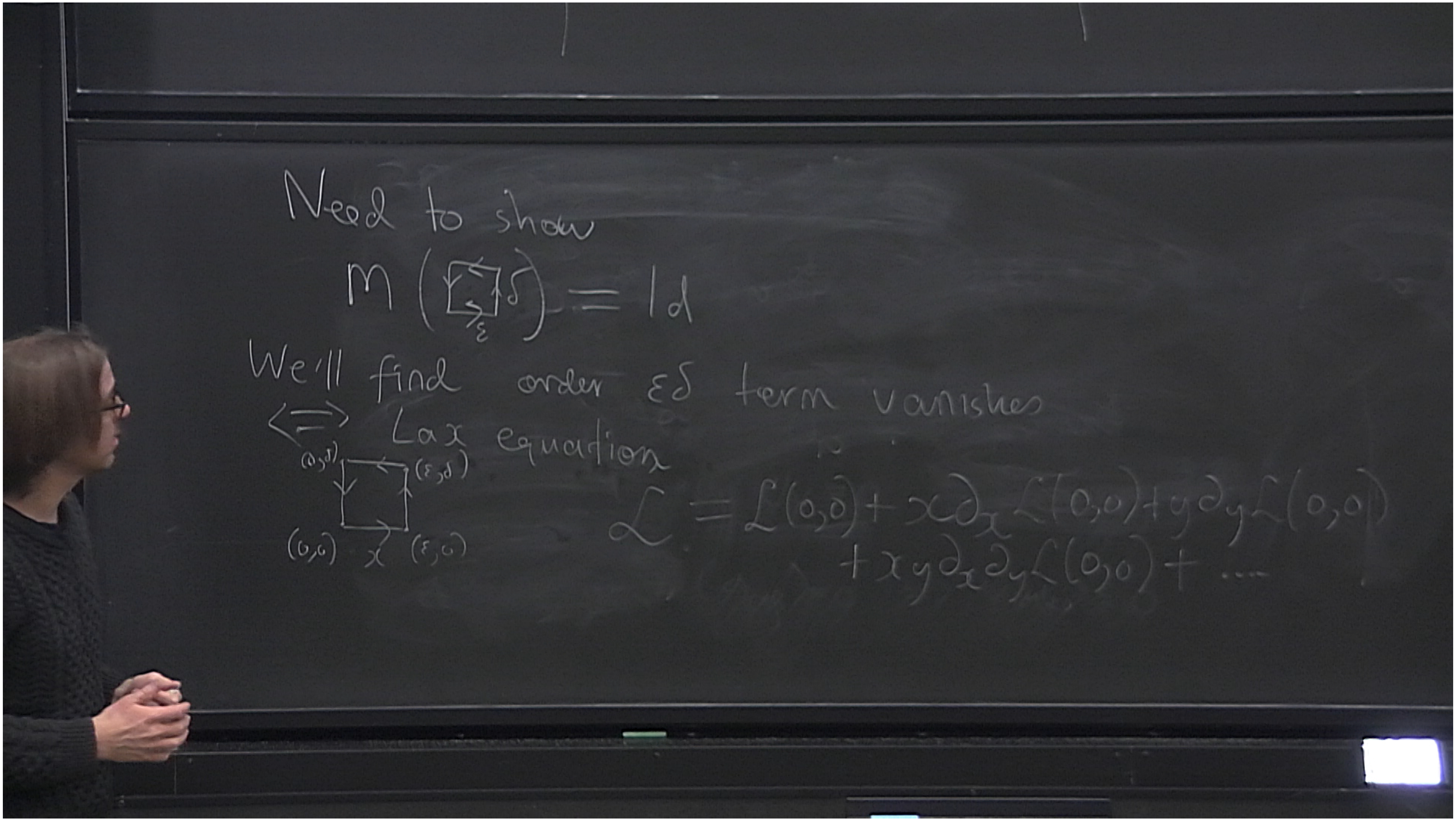
↗ matrix multiplication

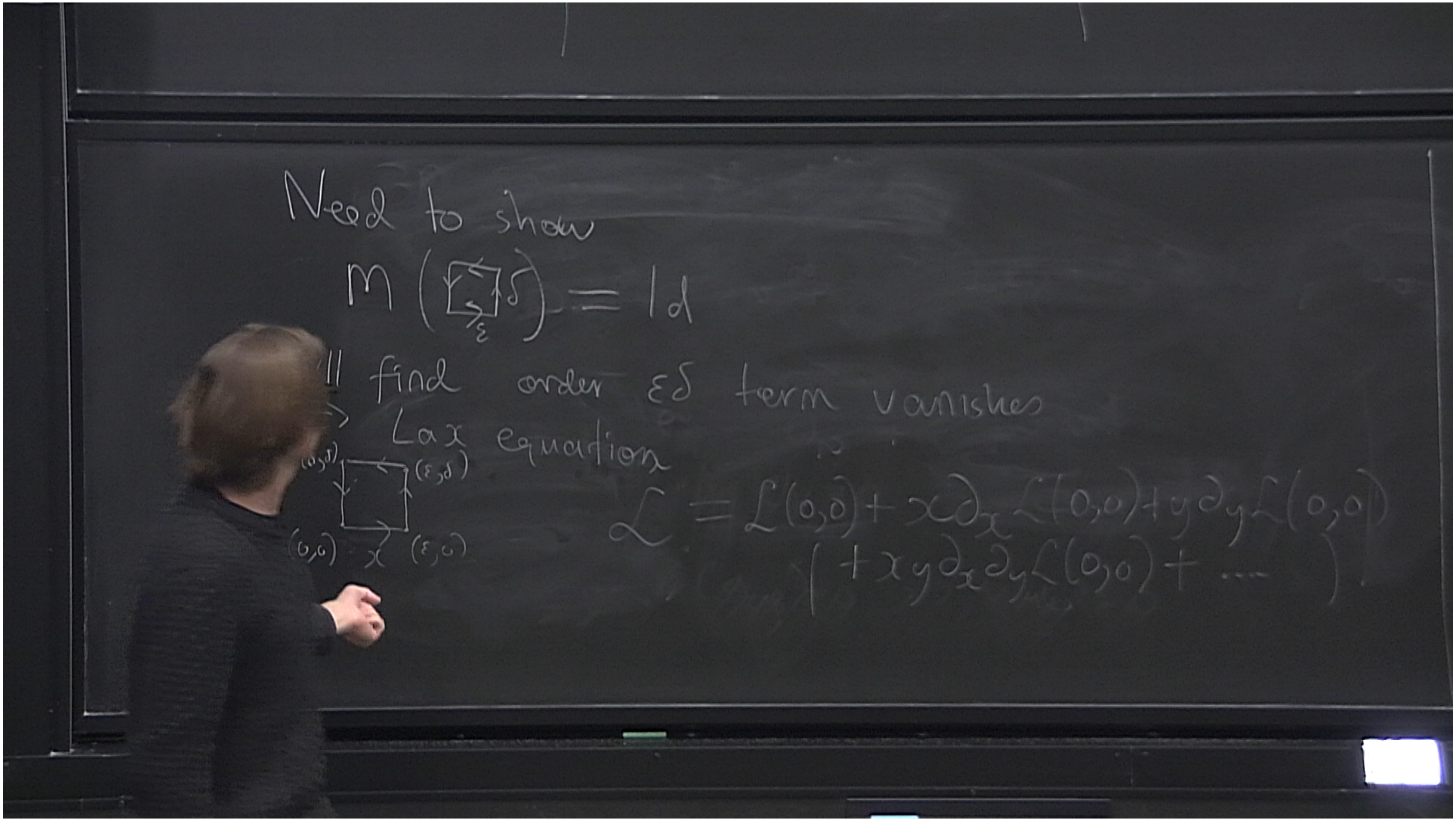
$$m(p') = m(-\infty \to a)\, m\left({}^{b}_{a}\sqsupset\right)\, m(b \to \infty)$$

Need to check:

$$m\left({}^{b}_{a}\sqsupset\right) = m\left({}^{b}_{a}\Big|\right), \text{ or } m\left({}^{b}_{a}\Big|\right)^{-1} m\left({}^{b}_{a}\sqsupset\right) = \mathrm{Id}$$


Need to show
m( ) = Id
We'll find order εδ term vanishes
⟺ Lax equation


Need to show
$m(\square) = \mathrm{Id}$
ε δ
We'll find order $\varepsilon\delta$ term vanishes
$\Longleftrightarrow$ Lax equation
$(0,\delta)$ (ε,δ)
$(0,0)$ x $(\varepsilon,0)$
$\mathcal{L} = \mathcal{L}(0,0) + x\partial_x\mathcal{L}(0,0) + y\partial_y\mathcal{L}(0,0)$
$+ xy\partial_x\partial_y\mathcal{L}(0,0) + \cdots$

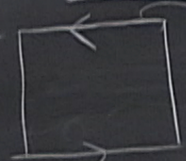

Need to show
m( ) = Id
ll find order εδ term vanishes
⇒ Lax equation
(0,δ)
(ε,δ)
(0,0)
(ε,0)
x
δ
ε
L = L(0,0) + x∂x L(0,0) + y∂y L(0,0)
+ xy ∂x∂y L(0,0) + ...

Path ordered integral

Horizontal integrals (1 point)

Path ordered integral

Horizontal integrals (1 point)

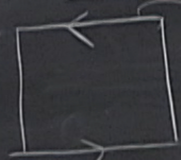


$\to -\varepsilon \mathcal{L}(0,0) - \varepsilon\delta\, \partial_y \mathcal{L}(0,0)$

$\to \varepsilon \mathcal{L}(0,0)$

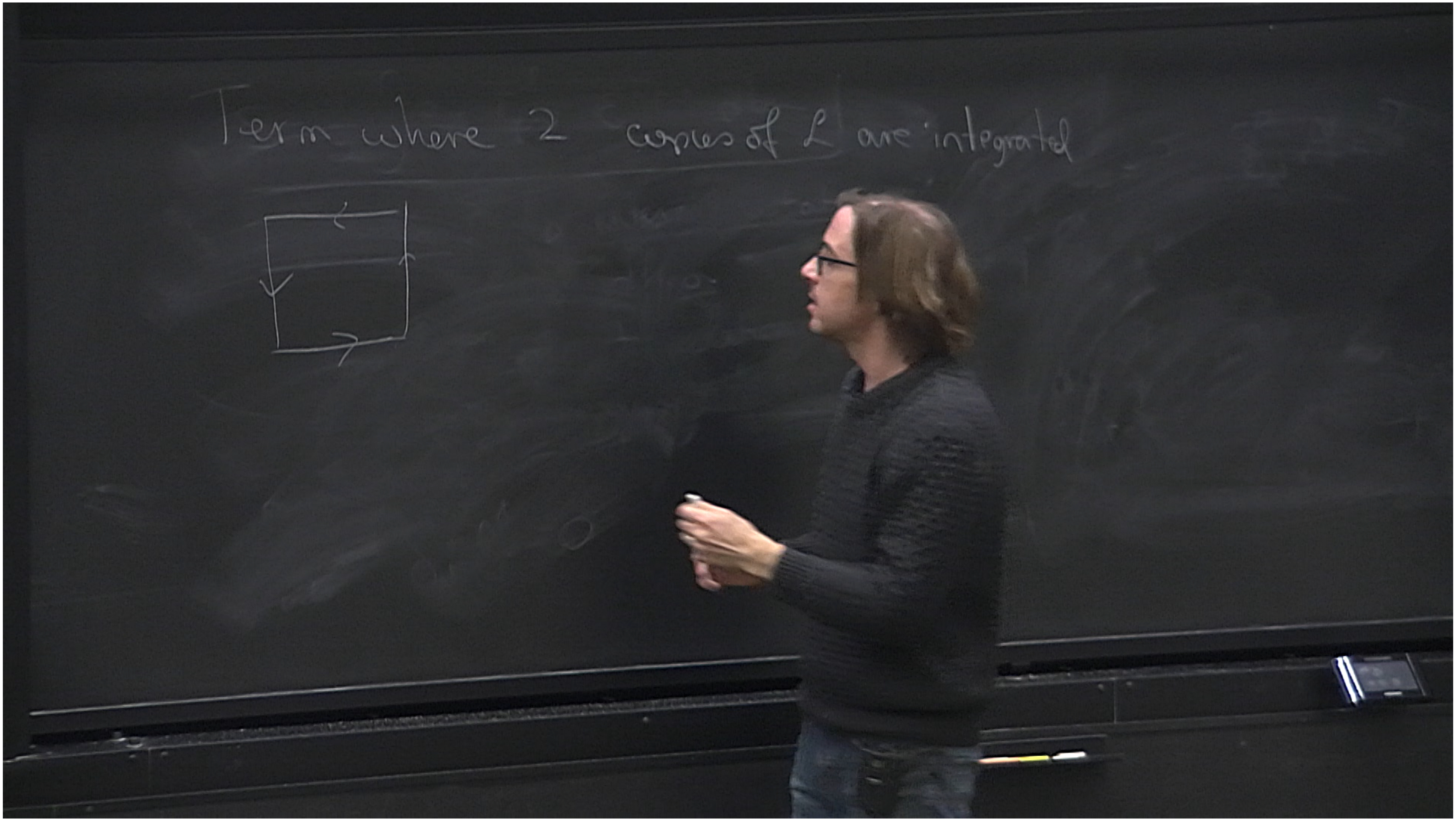
Path ordered integral

Horizontal integrals (1 point)

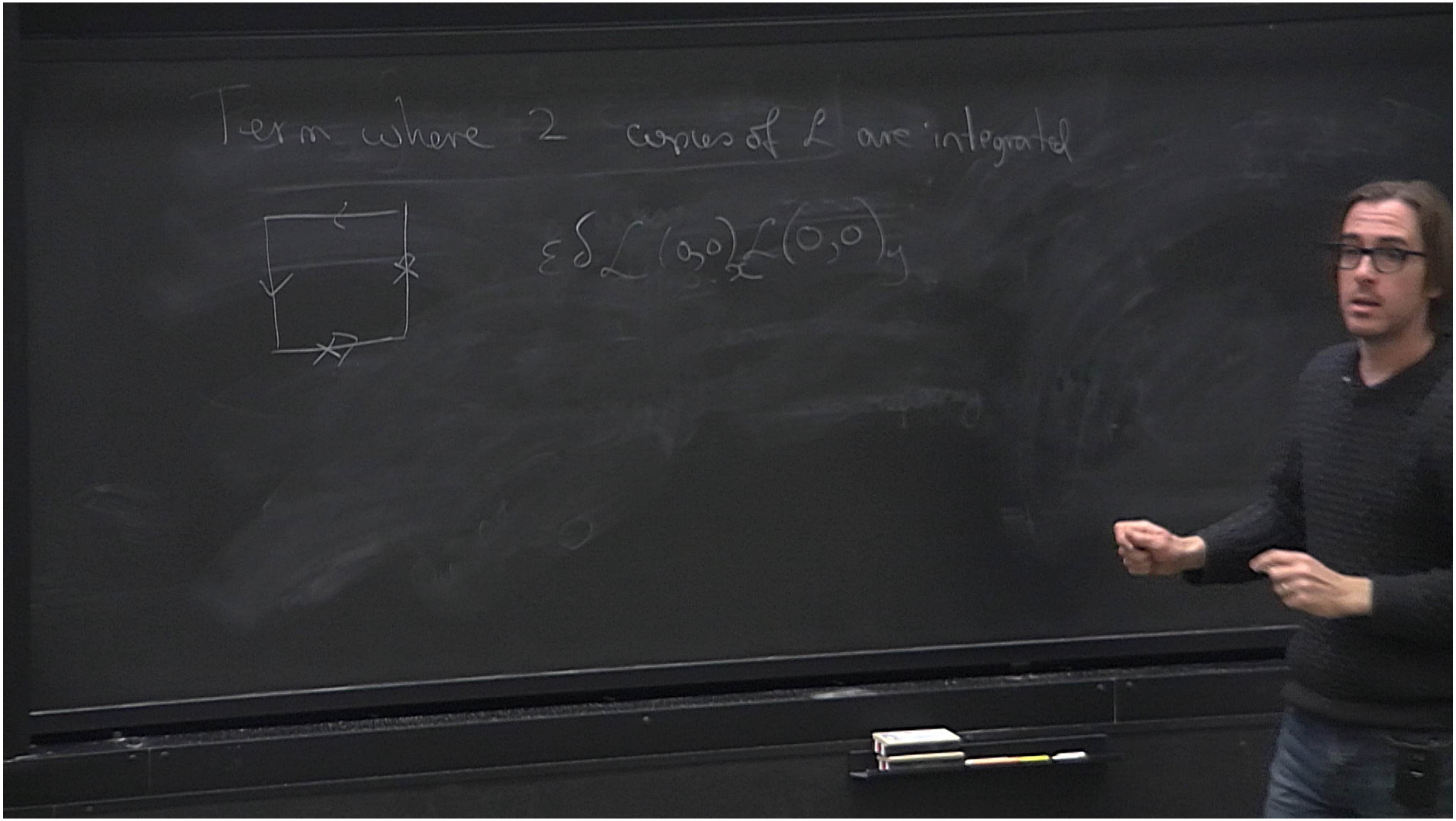


$$\left.\begin{array}{l} -\varepsilon \mathcal{L}(0,0) - \varepsilon\delta\, \partial_y \mathcal{L}(0,0) \\ \varepsilon \mathcal{L}(0,0) \end{array}\right\} = -\varepsilon\delta\, \partial_y \mathcal{L}(0,0)$$

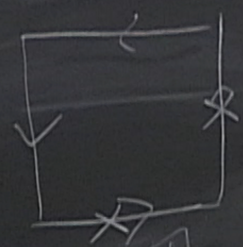
Vertical integrals: $+\varepsilon\delta\, \partial_x \mathcal{L}(0,0)$


Term where 2 copies of $\mathcal{L}$ are integrated


Term where 2 copies of $\mathcal{L}$ are integrated


Term where 2 copies of L are integrated
εδ L(0,0)x L(0,0)y

Term where 2 copies of $\mathcal{L}$ are integrated



$$\varepsilon\delta\left(\mathcal{L}(0,0)_x\,\mathcal{L}(0,0)_y - \mathcal{L}(0,0)_y\,\mathcal{L}(0,0)_x\right)$$

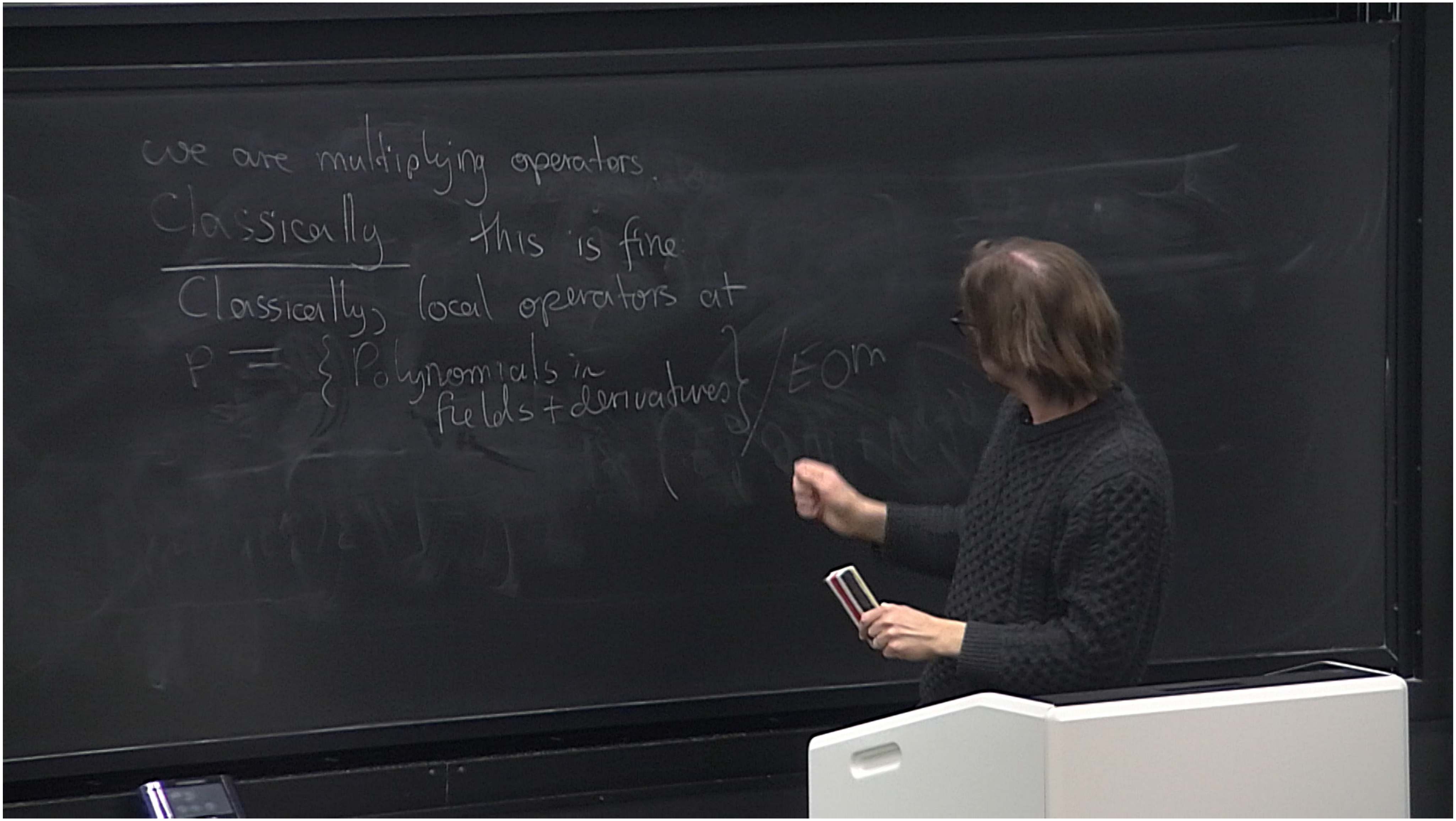
These cancel by Lax equation

Operator Product Expansion

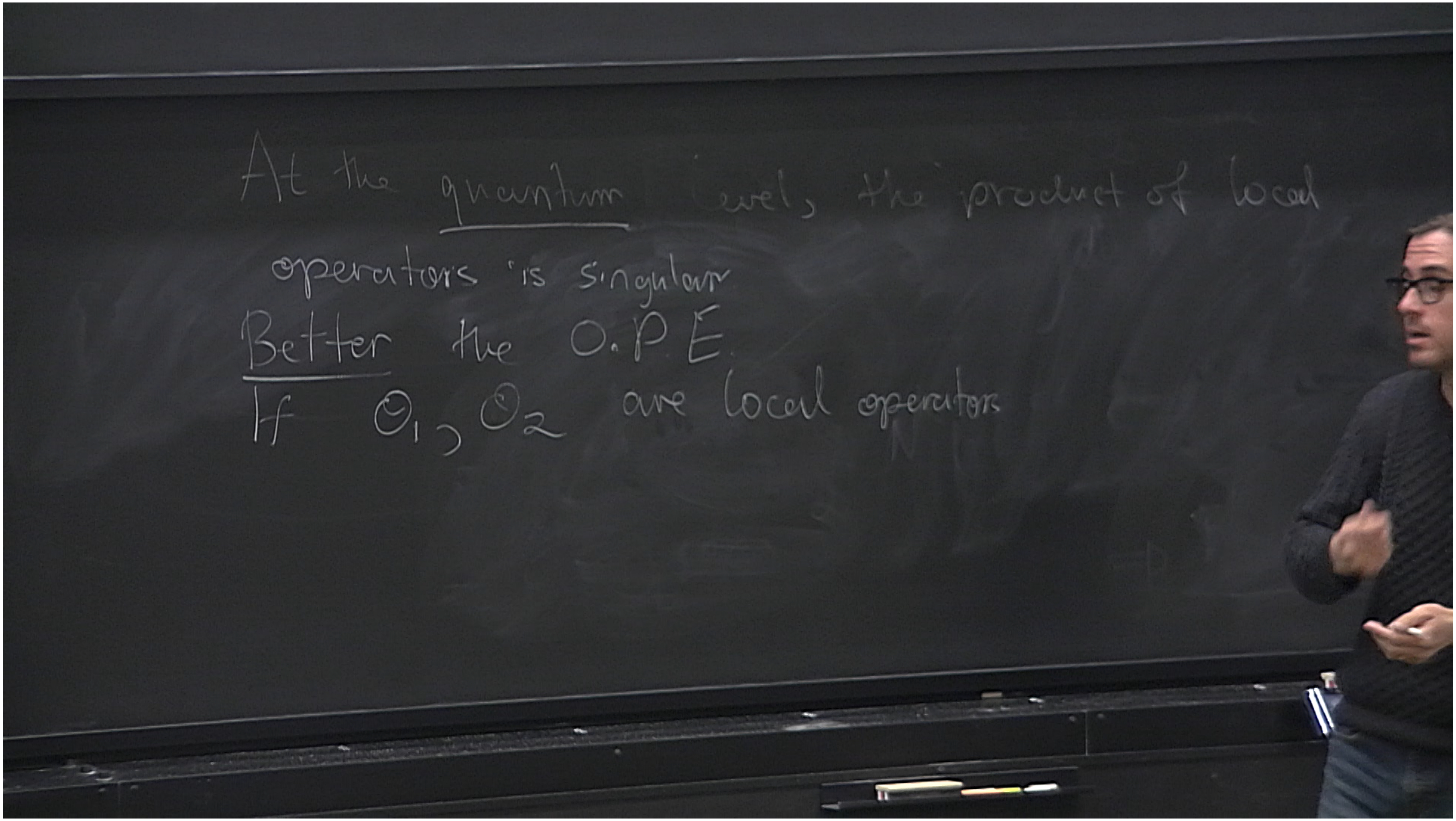
$$\partial_x \mathcal{L}_y - \partial_y \mathcal{L}_x + \mathcal{L}_x \mathcal{L}_y - \mathcal{L}_y \mathcal{L}_x = 0$$

$\mathcal{L}_y \, \mathcal{L}_x =$ N×N matrices whose entries are operators in a 2d QFT

When we write $\mathcal{L}_x \mathcal{L}_y$, or $\mathcal{L}^i_j(z)_x \mathcal{L}^j_k(z)_y$,



we are multiplying operators.
Classically this is fine.
Classically, local operators at
$p = \{\text{Polynomials in fields + derivatives}\}/\text{EOM}$


At the quantum level, the product of local
operators is singular
Better the O.P.E.
If $\mathcal{O}_1, \mathcal{O}_2$ are local operators

At the quantum level, the product of local operators is singular

Better the O.P.E.

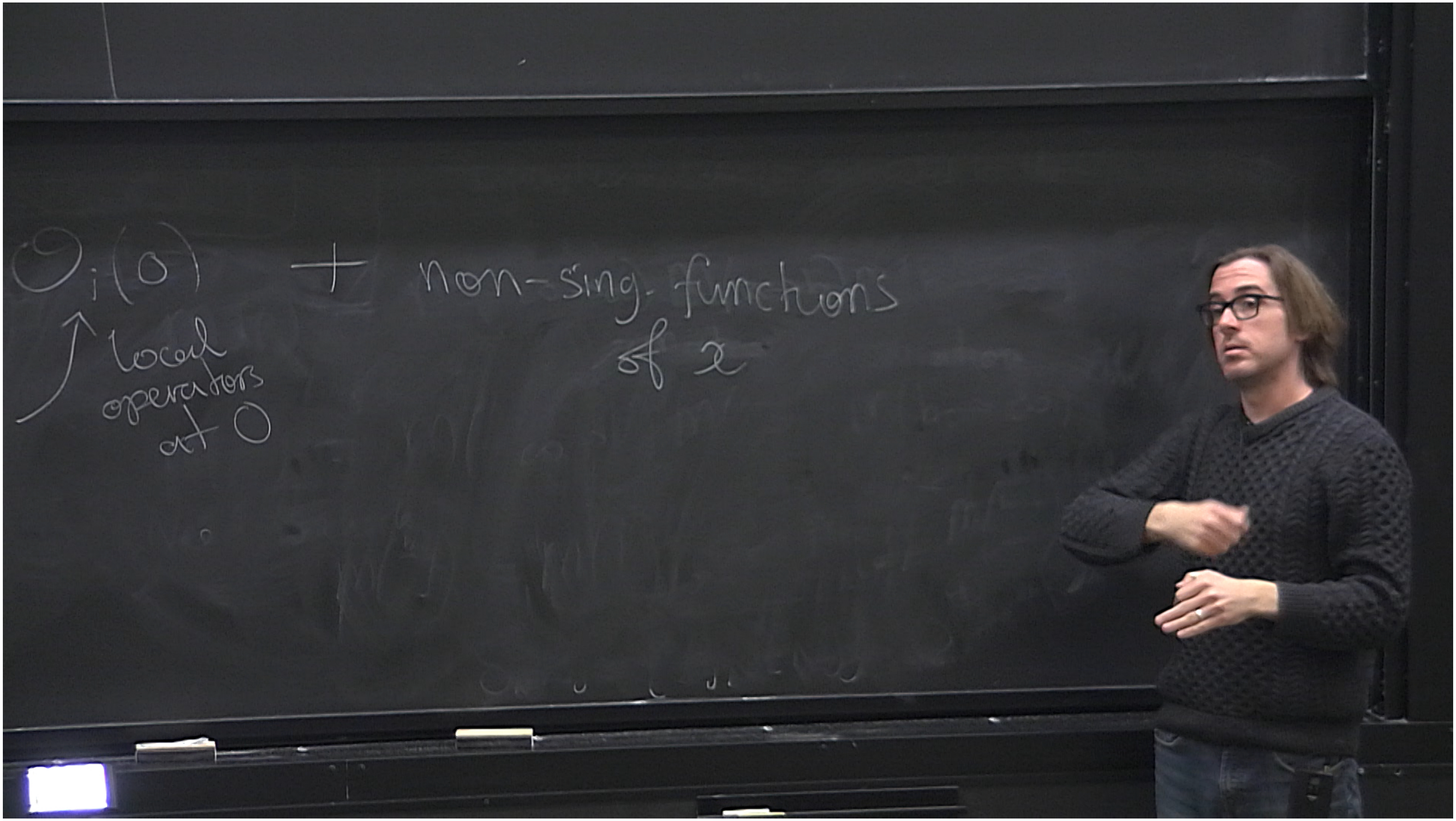
If $\mathcal{O}_1, \mathcal{O}_2$ are local operators

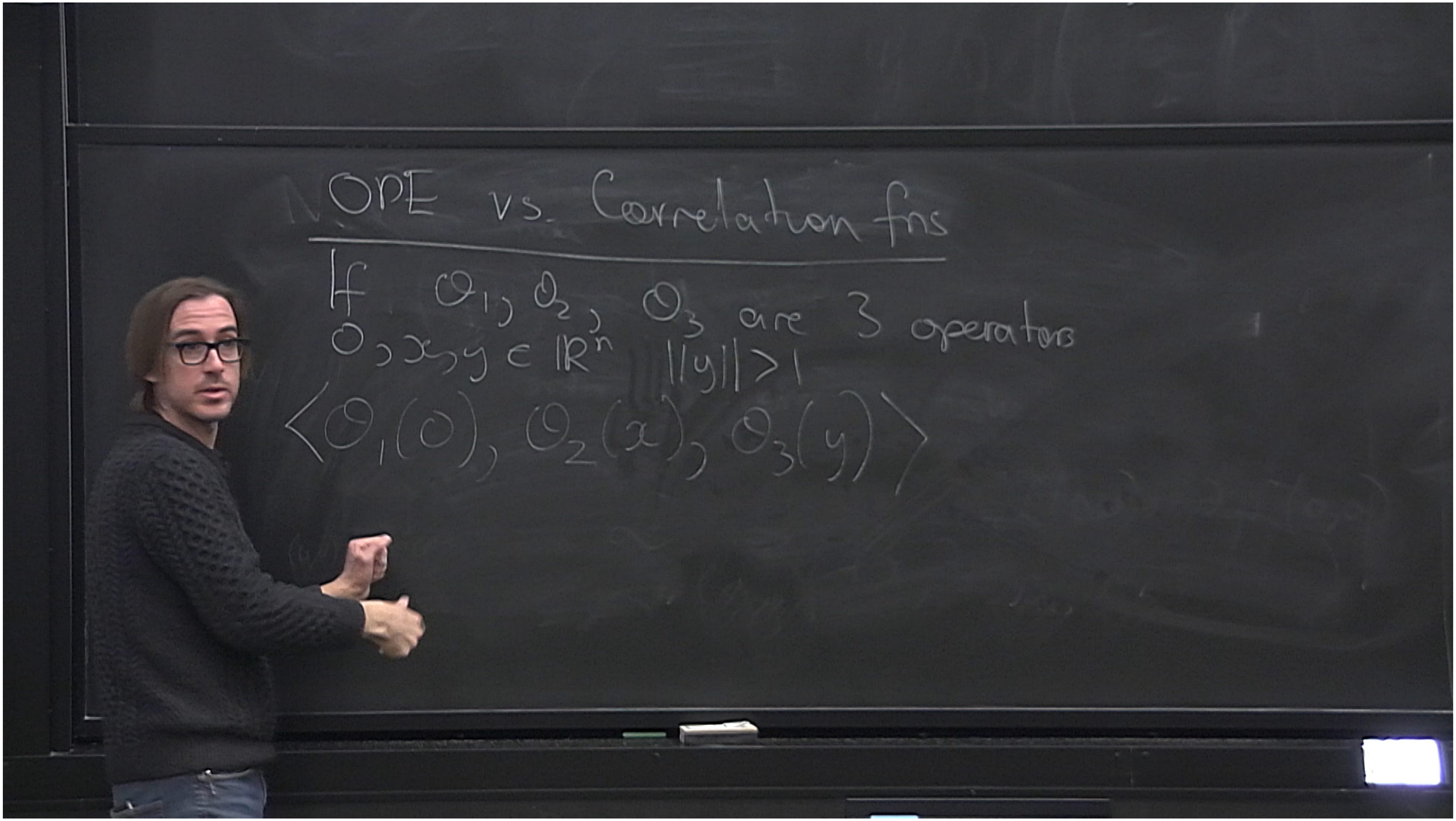
Then the OPE is an expression

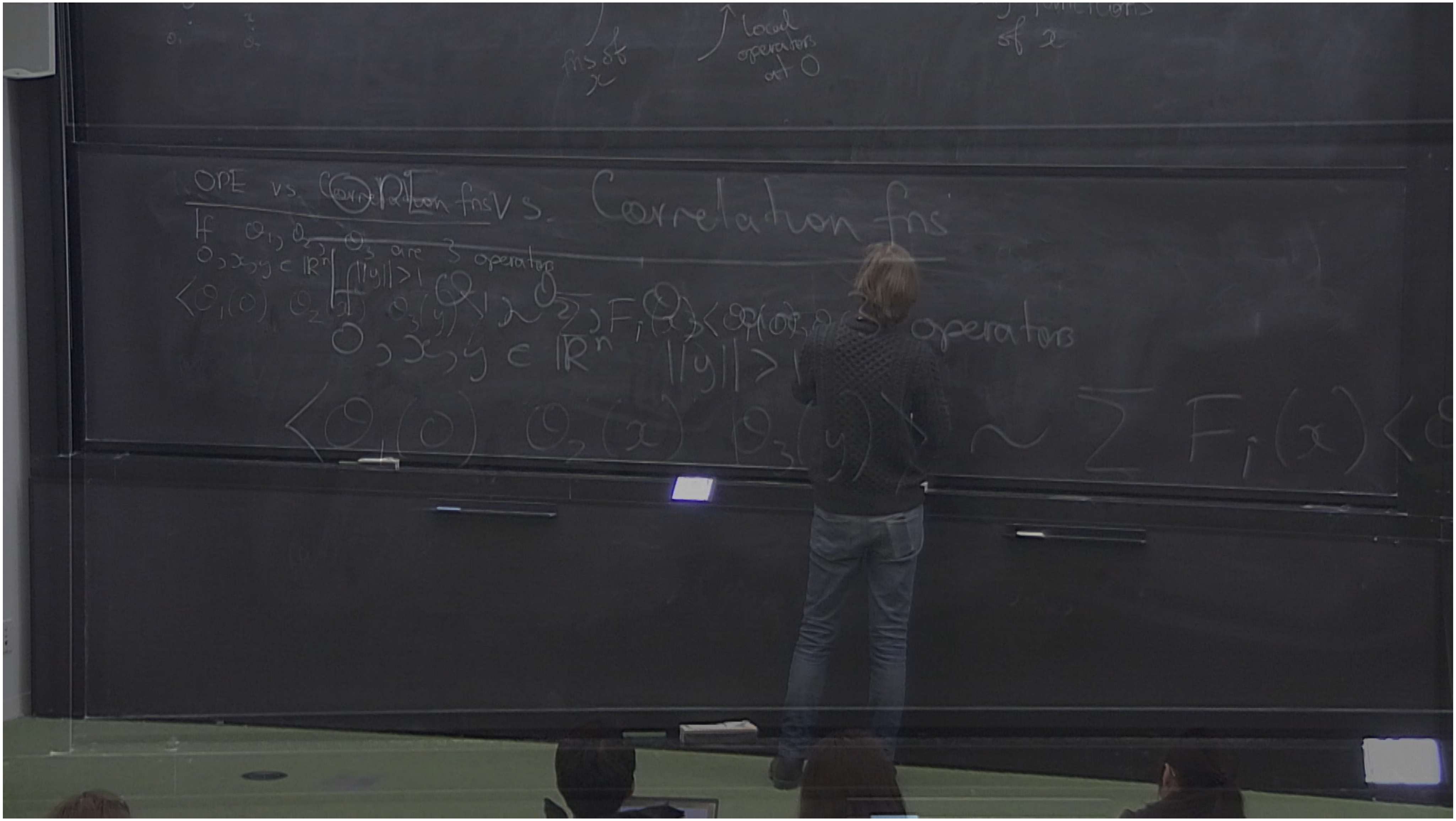
0 · $\mathcal{O}_1$

x · $\mathcal{O}_2$

$$\mathcal{O}_1(0)\cdot\mathcal{O}_2(x) \sim \sum_i \underbrace{F_i(x)}_{\text{fns of } x}\,\underbrace{\mathcal{O}_i(0)}_{\text{local operators at } 0}$$


$\mathcal{O}_i(0)$ + non-sing. functions
of x
local
operators
at 0


OPE vs. Correlation fns
If $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ are 3 operators
$0, x, y \in \mathbb{R}^n$ $\|y\| > 1$
$\langle \mathcal{O}_1(0), \mathcal{O}_2(x), \mathcal{O}_3(y) \rangle$


fns of x
local operators at 0
functions of x
OPE vs. Correlation fns
If $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ are 3 operators
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$\langle \mathcal{O}_1(0)\, \mathcal{O}_2(x)\, \mathcal{O}_3(y) \rangle \sim \sum F_i(x) \langle$

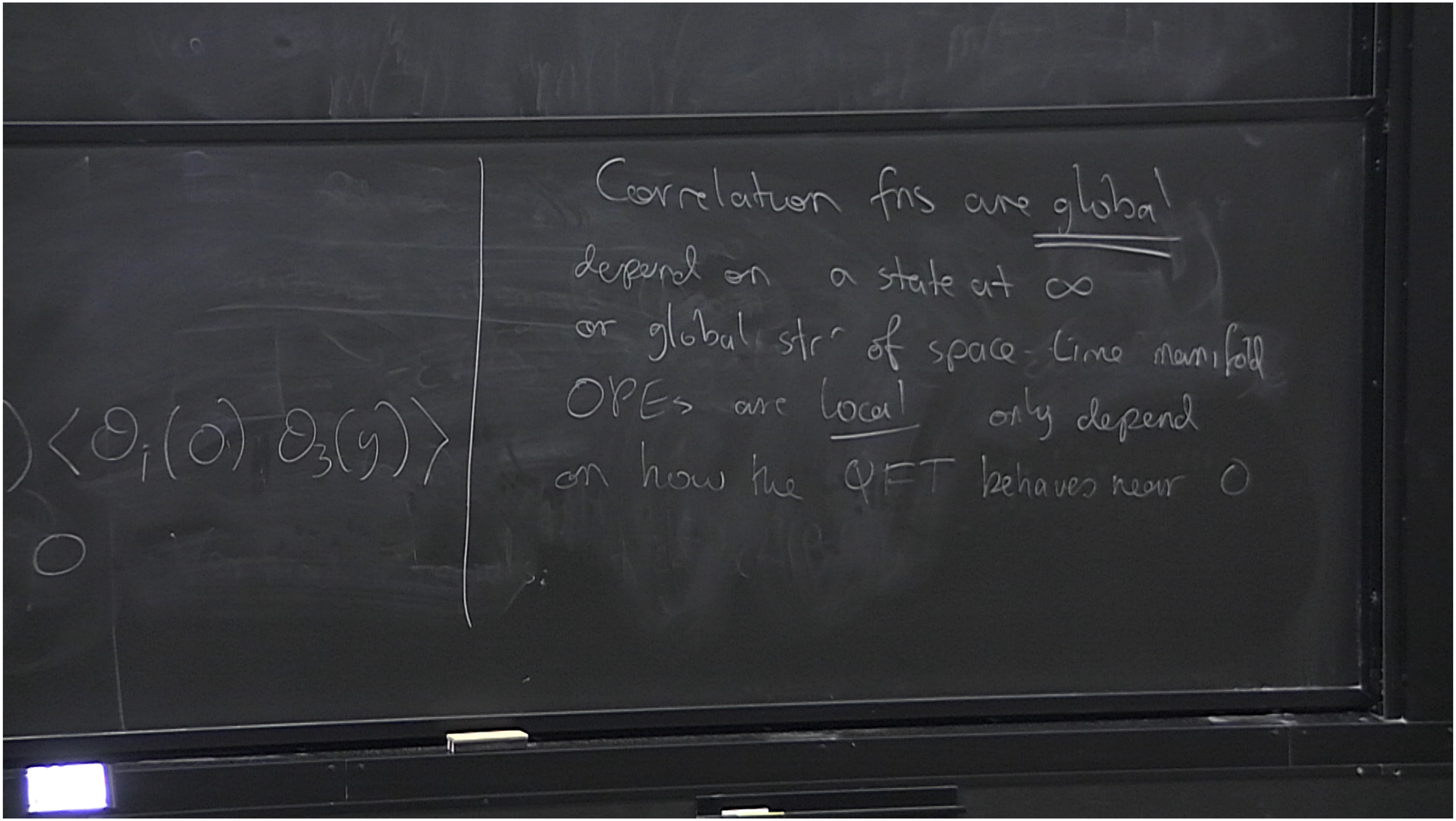
OPE vs. Correlation fns

If $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ are 3 operators

$0, x, y \in \mathbb{R}^n \quad \|y\| > 1$

$$\langle \mathcal{O}_1(0)\, \mathcal{O}_2(x)\, \mathcal{O}_3(y) \rangle \sim \sum F_i(x) \langle \mathcal{O}_i(0)\, \mathcal{O}_3(y) \rangle$$

as $x \to 0$



⟨O₁(0) O₃(y)⟩
O
Correlation fns are global
depend on a state at ∞
or global str^ of space-time manifold
OPEs are local only depend
on how the QFT behaves near 0

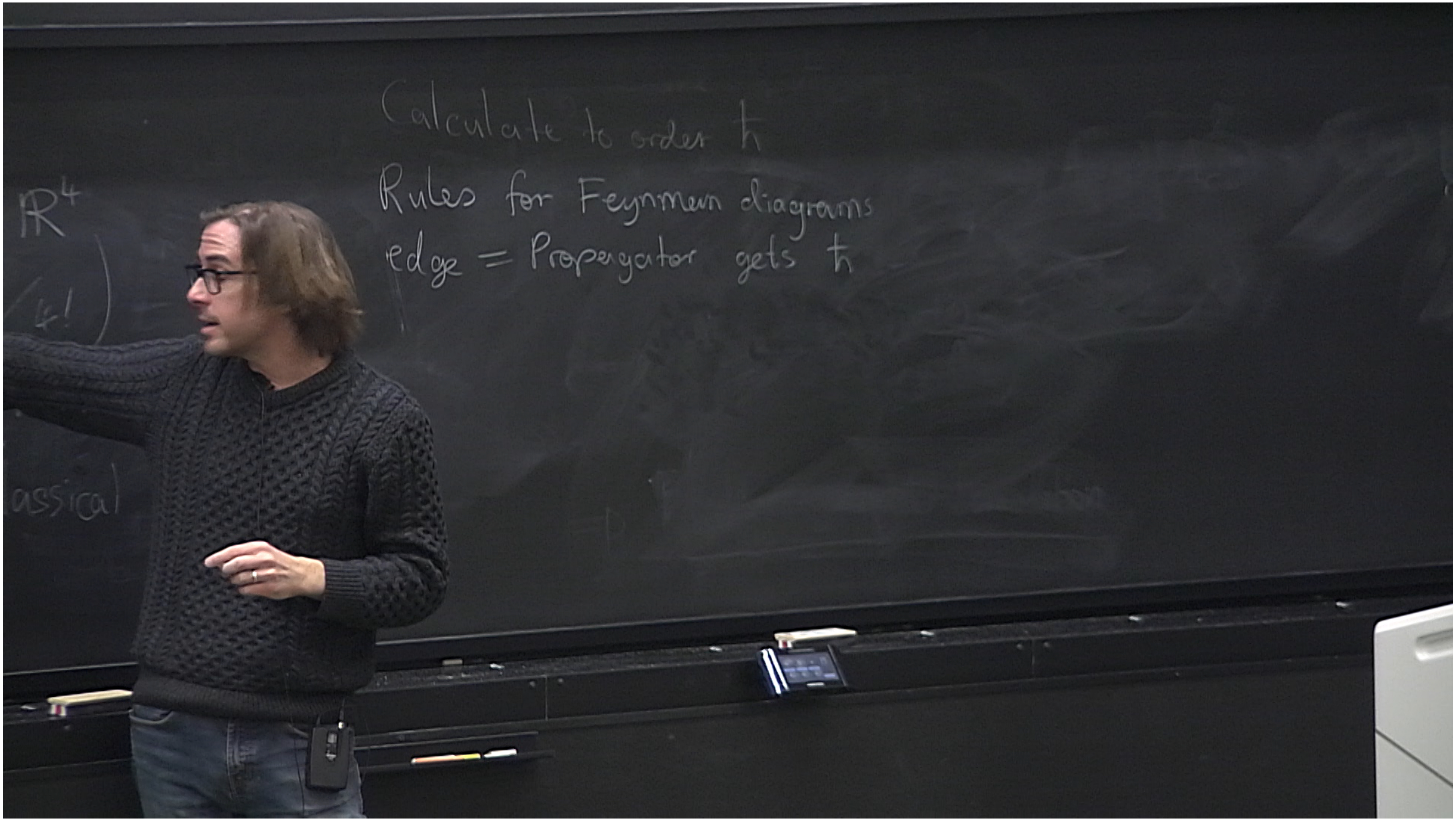
Example

φ^4 theory on $\mathbb{R}^4$

$$\frac{1}{\hbar}\left(-\frac{1}{2}\int \varphi \Delta \varphi + \int \varphi^4/4!\right)$$

$\hbar$ = Loop expansion param.

$\hbar = 0$ Classical


Calculate to order ħ
Rules for Feynman diagrams
edge = Propagator gets ħ

Example

φ^4 theory on $\mathbb{R}^4$

$$\frac{1}{\hbar}\left(-\frac{1}{2}\int \varphi \Delta \varphi + \int \varphi^4/4!\right)$$

$\hbar$ = Loop expansion param.

$\hbar = 0$ Classical

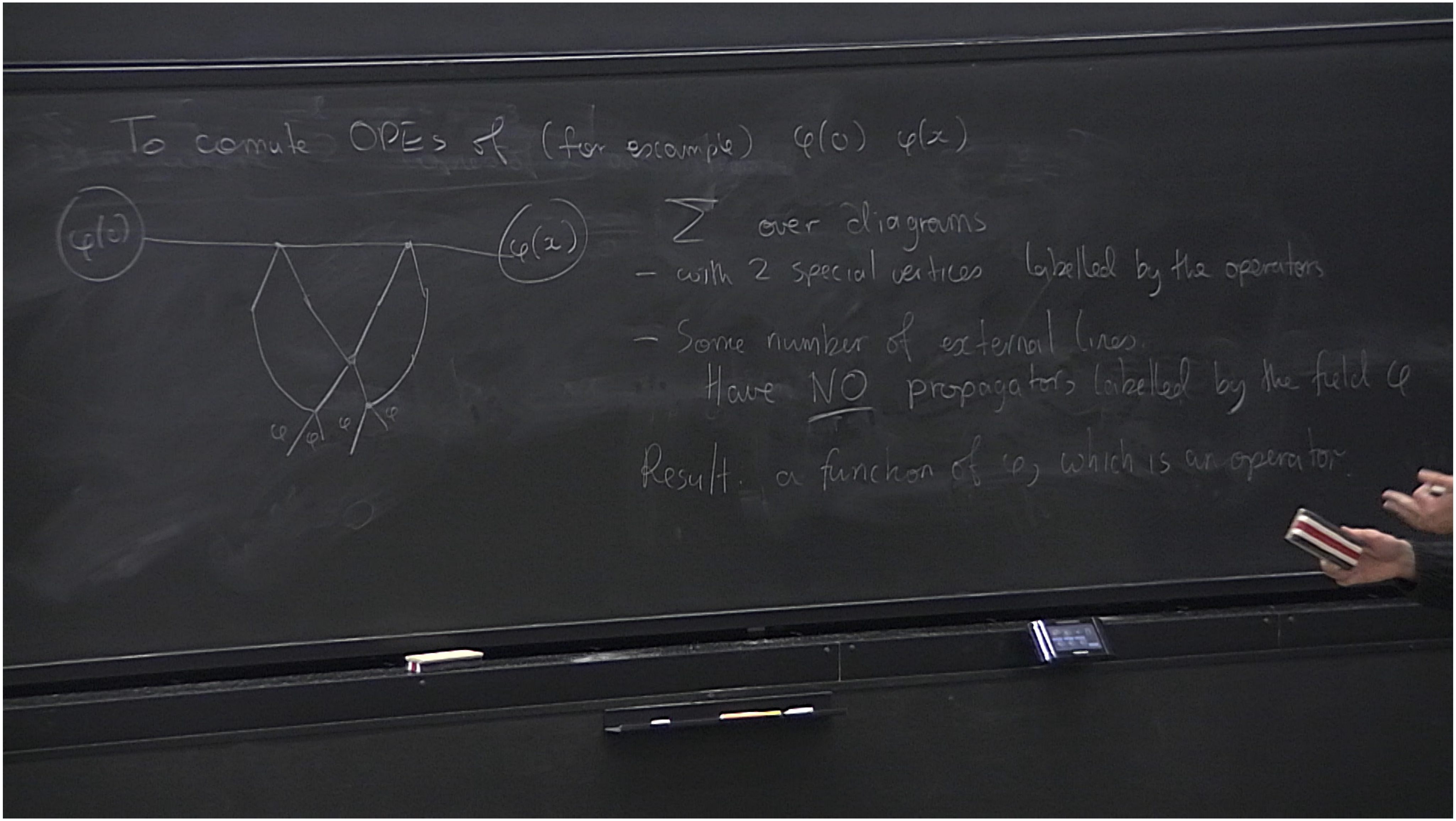
Calculate to order $\hbar$

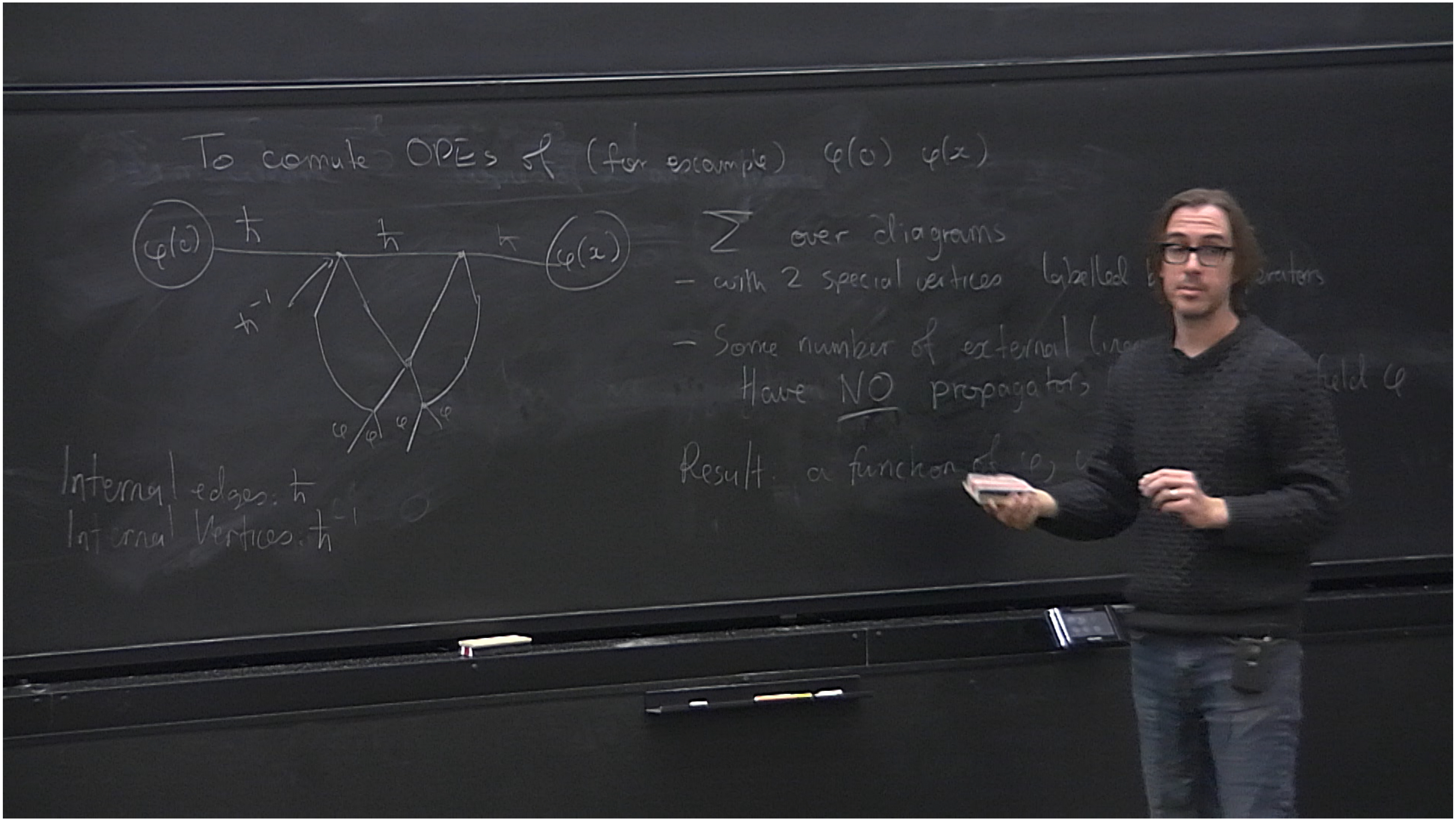
Rules for Feynman diagrams

edge = Propagator gets $\hbar$

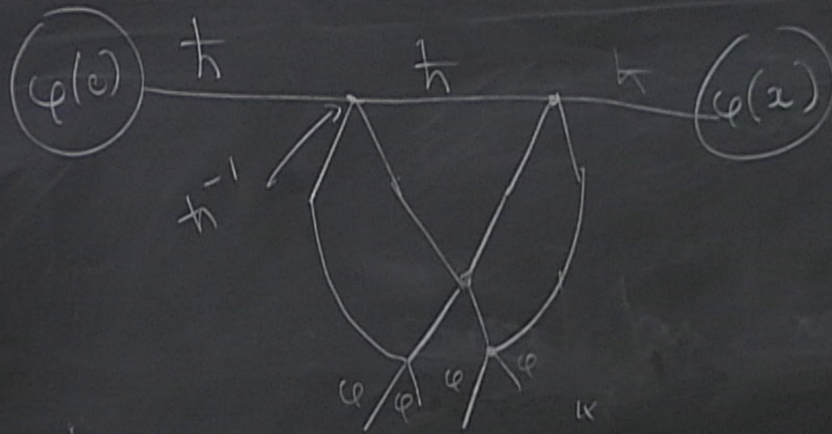
Vertex $\sim \int \varphi^4$ gets a $\hbar^{-1}$

Total weight = # edges − # vertices


To comute OPEs of (for example) φ(0) φ(x)
φ(0)
φ(x)
φ φ φ φ
Σ over diagrams
- with 2 special vertices labelled by the operators
- Some number of external lines.
Have NO propagators labelled by the field φ
Result: a function of φ, which is an operator.


To comute OPEs of (for example) φ(0) φ(x)
φ(0)
φ(x)
Σ over diagrams
- with 2 special vertices labelled
- Some number of external
Have NO propagators
Result: a function of
Internal edges: ħ
Internal Vertices: ħ

To comute OPEs of (for example) $\varphi(0)\ \varphi(x)$



$\sum$ over diagrams

- with 2 special vertices labelled b

- Some number of external lines.

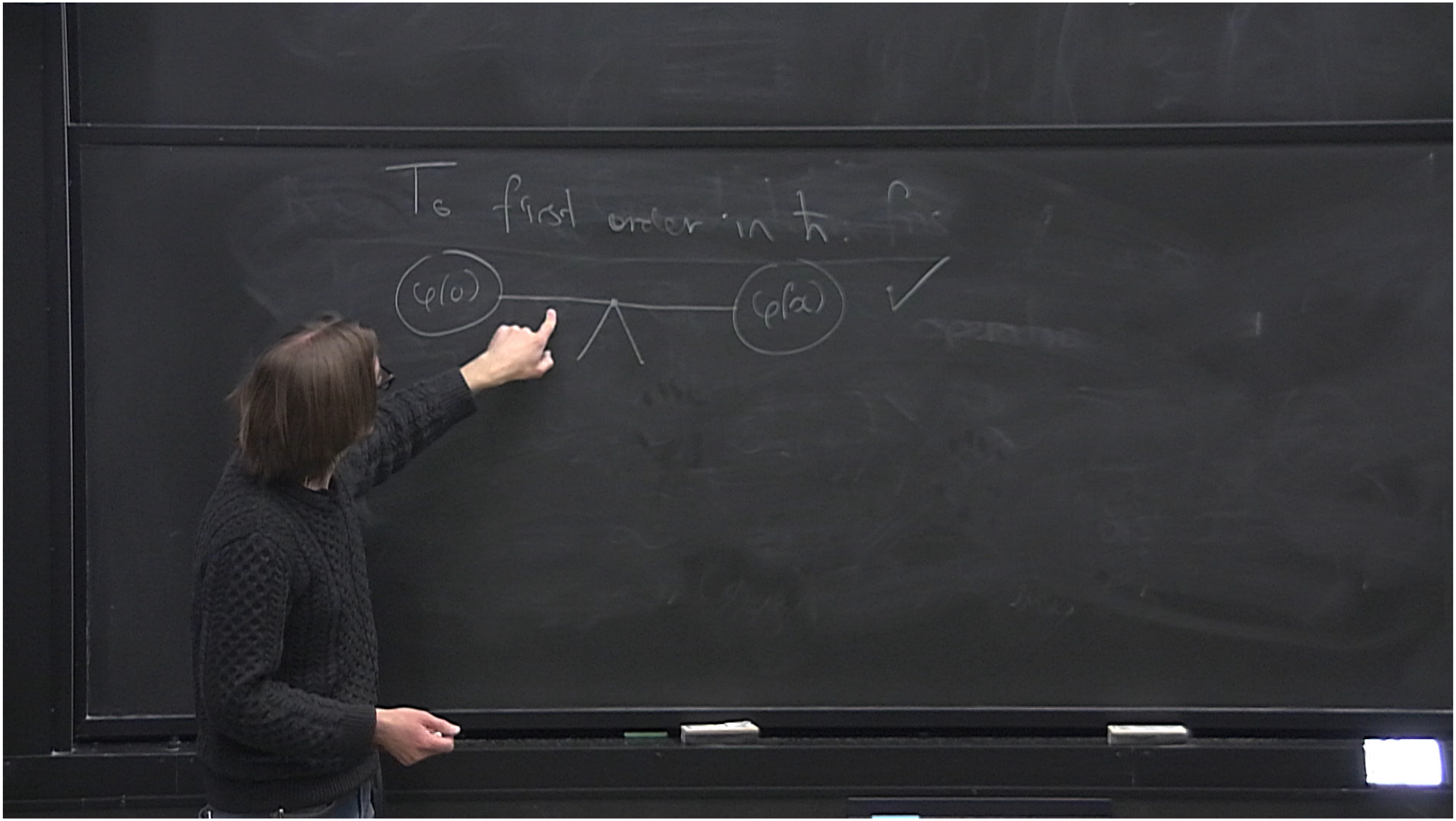
Have NO propagators label

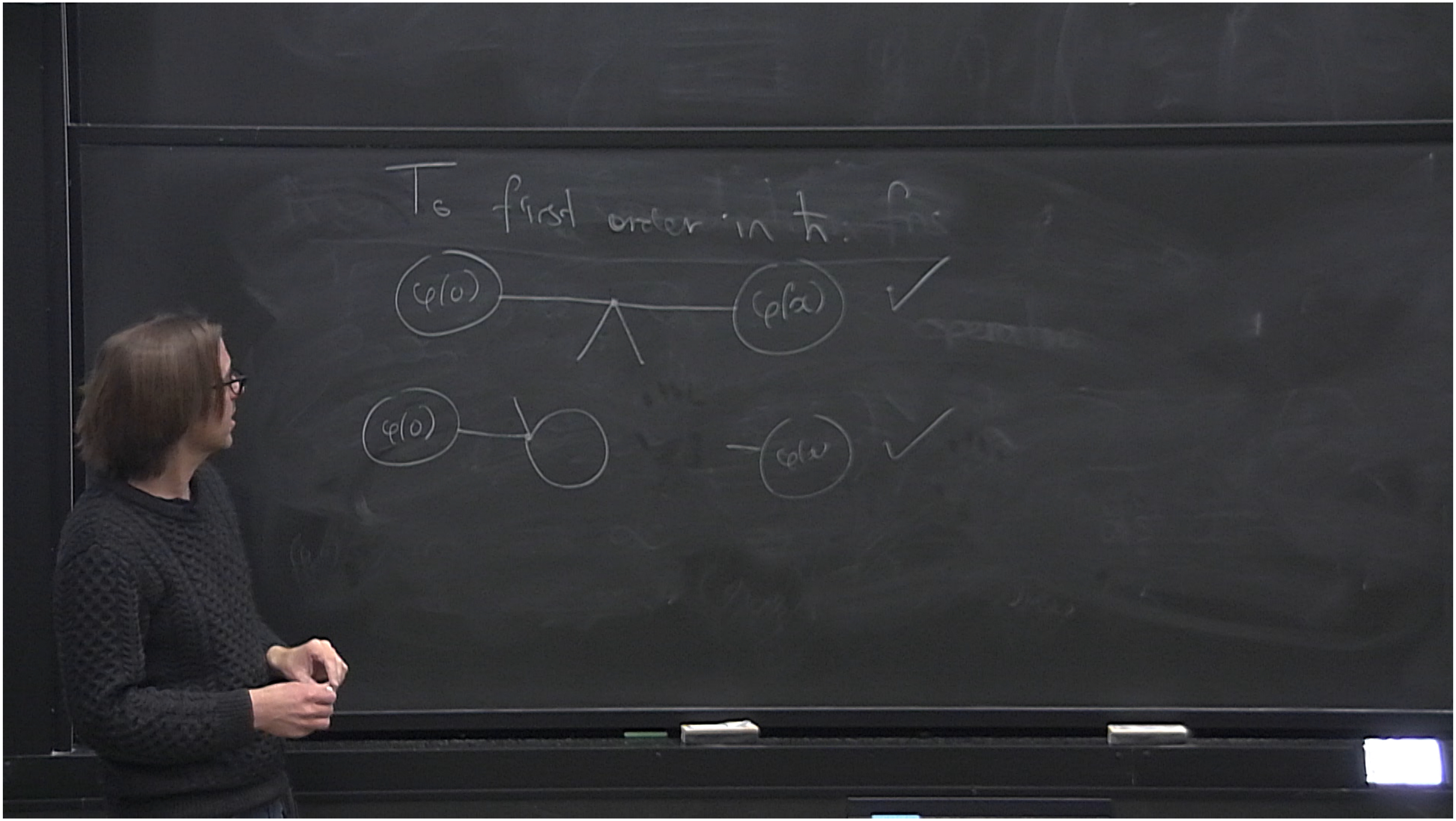
Result: a function of φ, which i

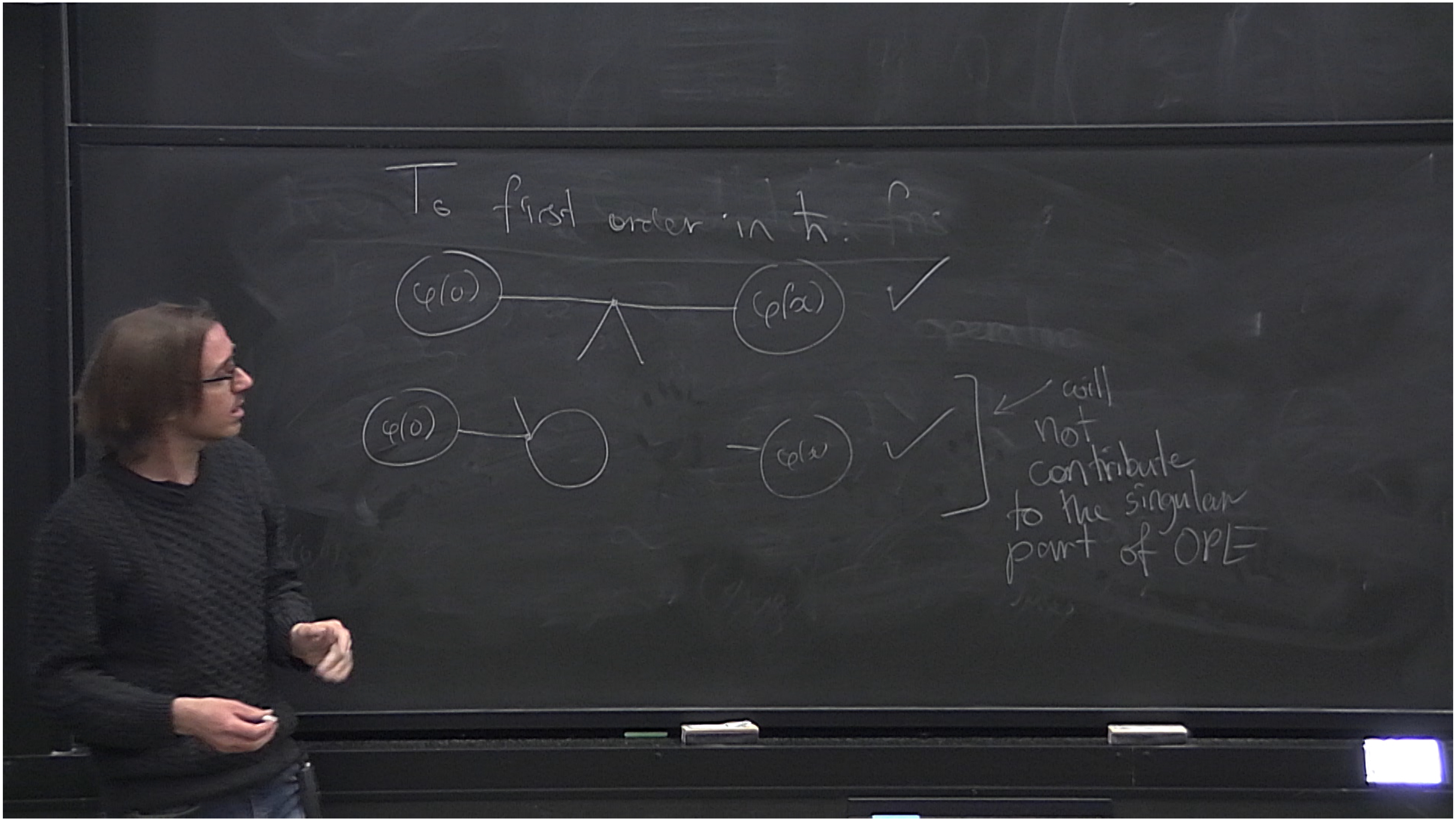
Internal edges: $\hbar$

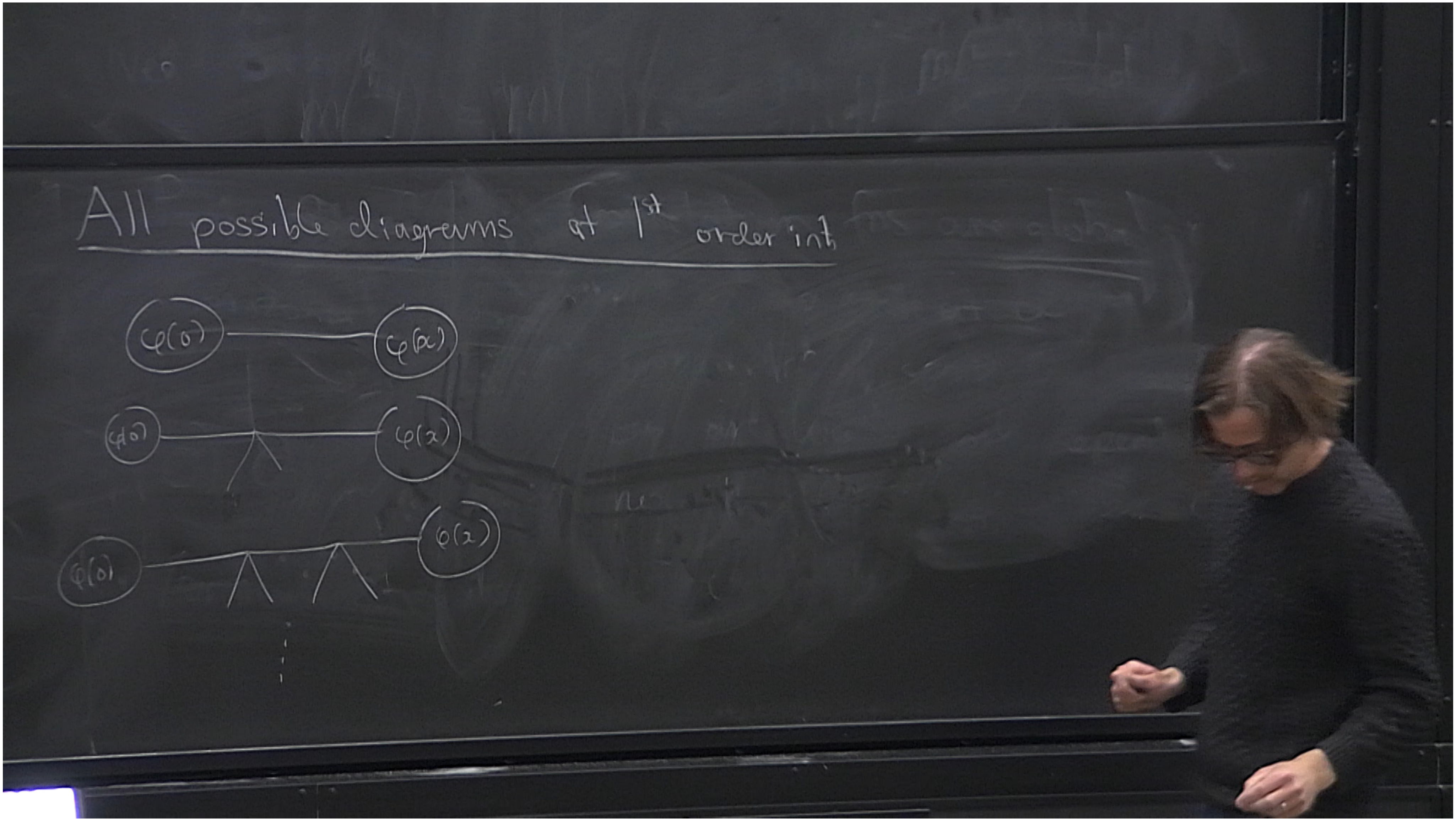
Internal Vertices: $\hbar^{-1}$

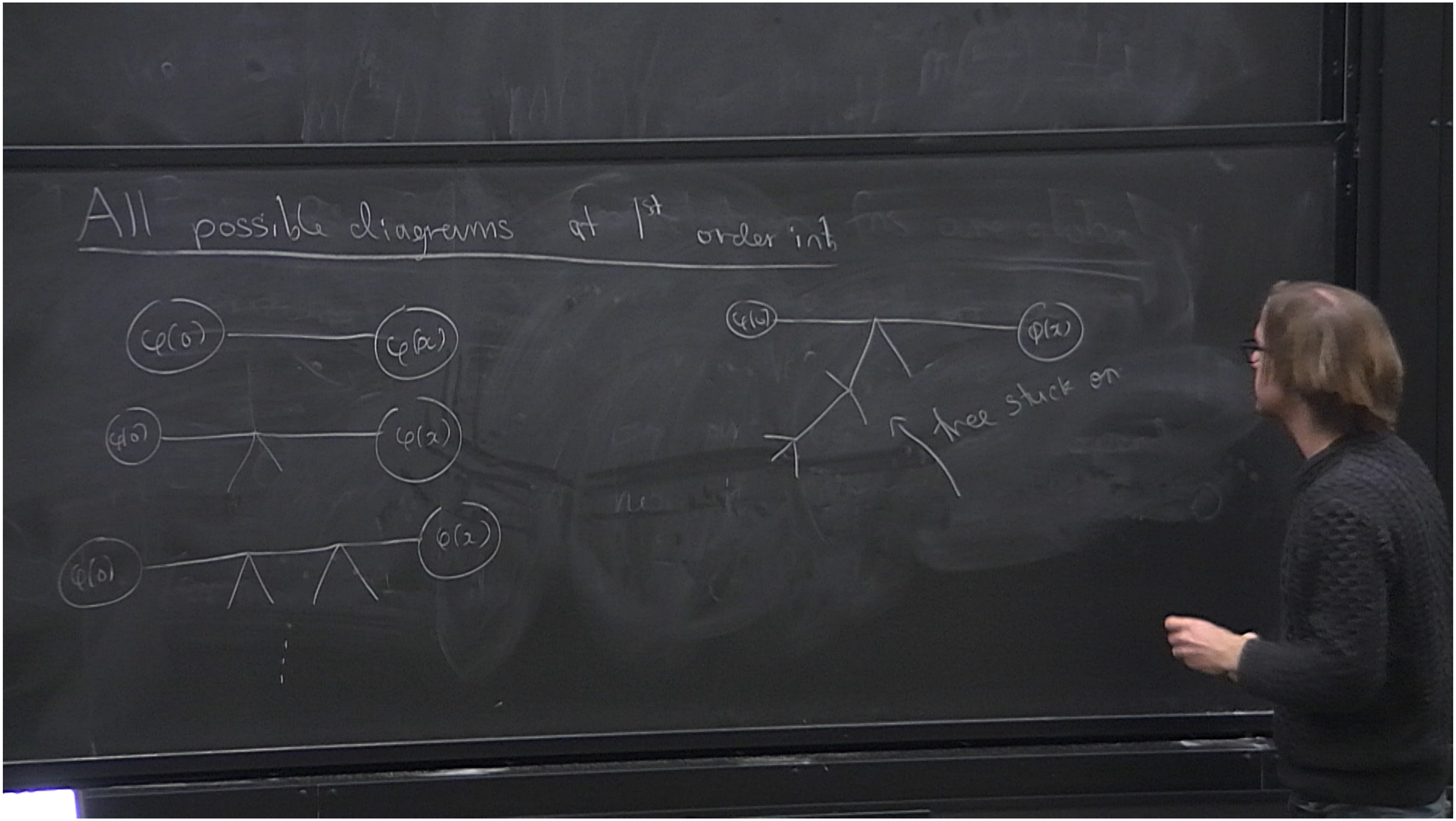
$\hbar^{k}$

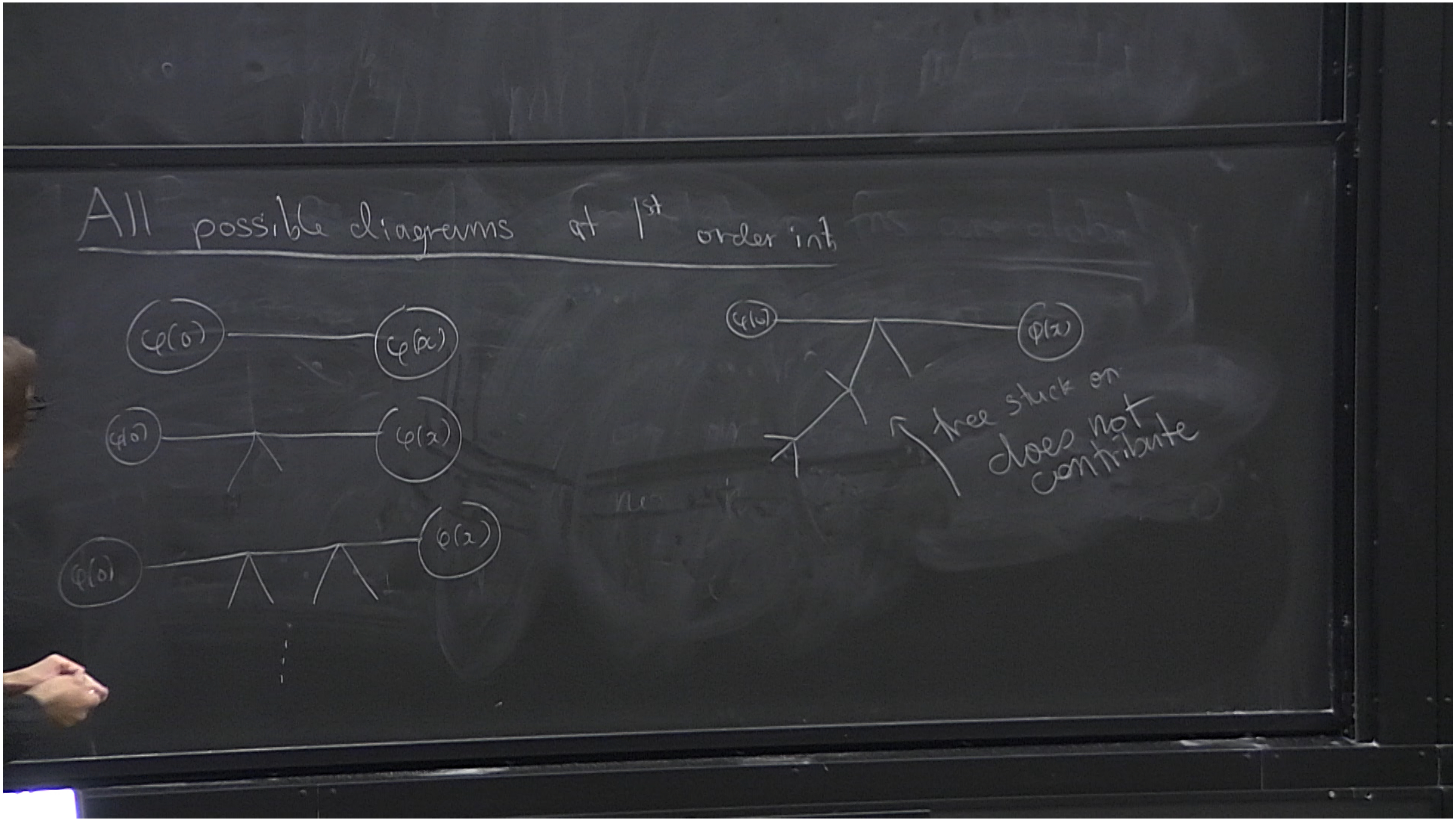

To first order in ħ:
φ(0)
φ(x)

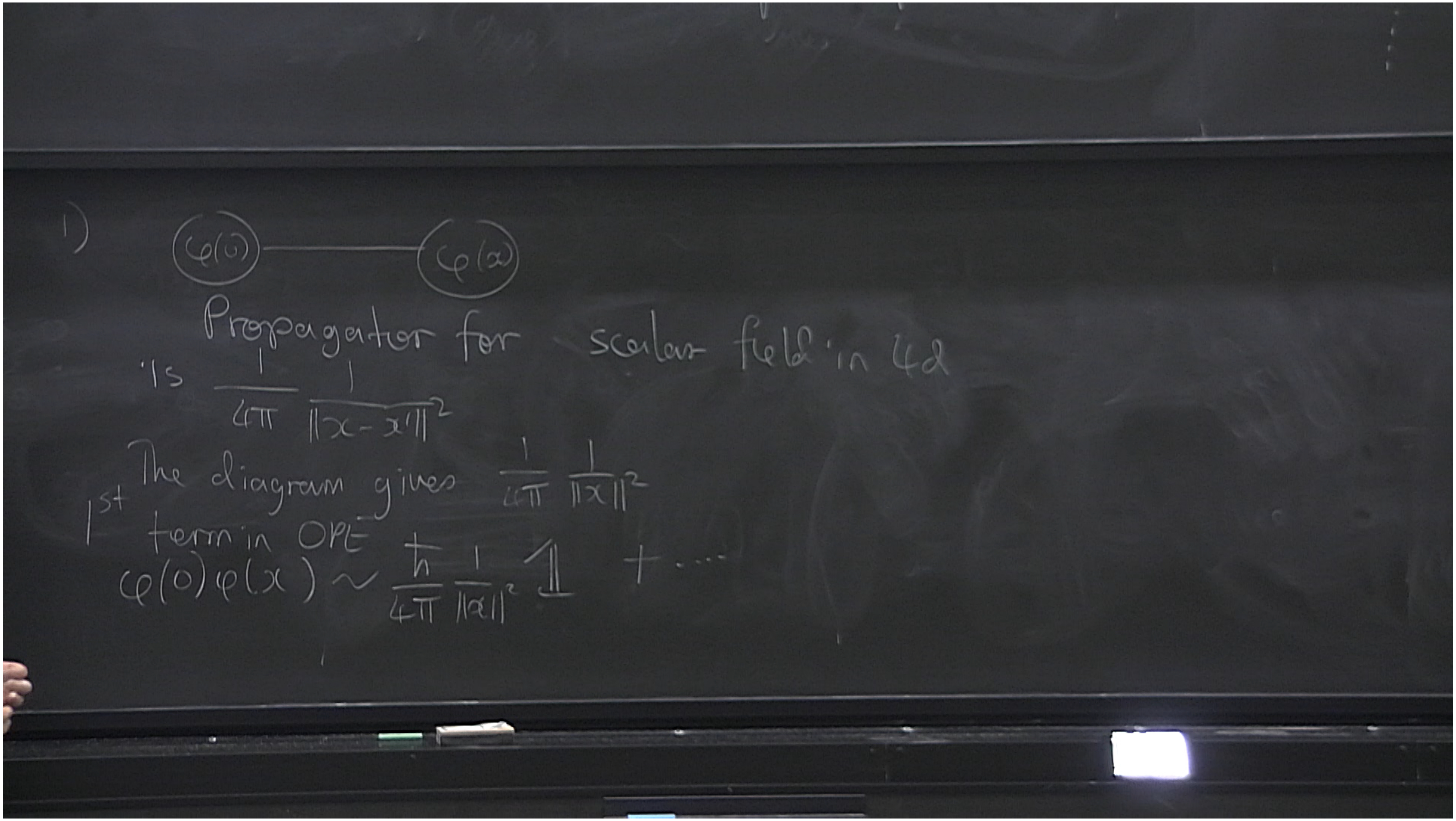

To first order in ħ:
φ(0)
φ(x)
φ(0)


To first order in ħ:
φ(0)
φ(x)
φ(0)
φ(x)
will not contribute to the singular part of OPE

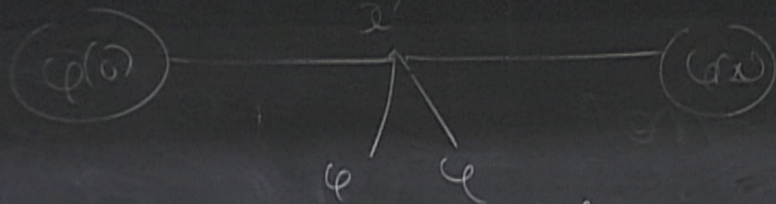

All possible diagrams at 1st order int
φ(0)
φ(x)
φ(0)
φ(x)
φ(0)
φ(x)


All possible diagrams at 1st order int
φ(0)
φ(x)
tree stuck on


All possible diagrams at 1st order int
φ(0)
φ(x)
φ(0)
φ(x)
φ(0)
φ(x)
φ(0)
φ(x)
tree stuck on
does not contribute

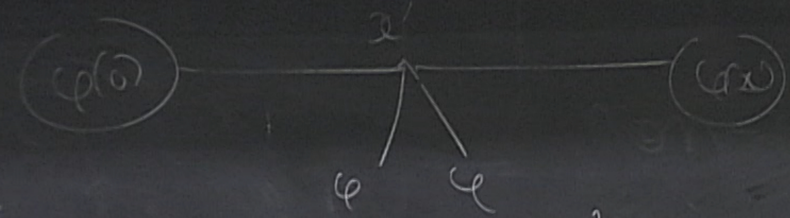

1)
$\varphi(0)$ $\varphi(x)$
Propagator for scalar field in 4d
is $\frac{1}{4\pi}\frac{1}{\|x-x'\|^2}$
The diagram gives $\frac{1}{4\pi}\frac{1}{\|x\|^2}$
1st term in OPE
$\varphi(0)\varphi(x) \sim \frac{\hbar}{4\pi}\frac{1}{\|x\|^2}\mathbb{1} + \dots$

2)



$$\frac{1}{(4\pi)^2}\int_{x'} \frac{1}{\|x'\|^2}\,\frac{1}{\|x-x'\|^2}\,\varphi(x')^2$$

2)

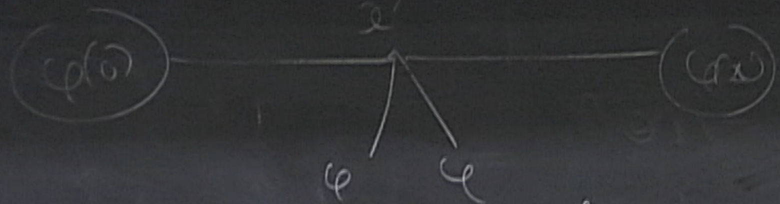


$$\frac{1}{(4\pi)^2}\int_{x'} \frac{1}{\|x'\|^2}\,\frac{1}{\|x-x'\|^2}\,\varphi(x')^2$$

Expand $\varphi(x') = \varphi(0) + x'_i \partial_i \varphi(0) + \cdots$

Insert the expansion into the integral:

2)



$$\frac{1}{(4\pi)^2}\int_{x'} \frac{1}{\|x'\|^2}\,\frac{1}{\|x-x'\|^2}\,\varphi(x')^2 \, d^4x'$$

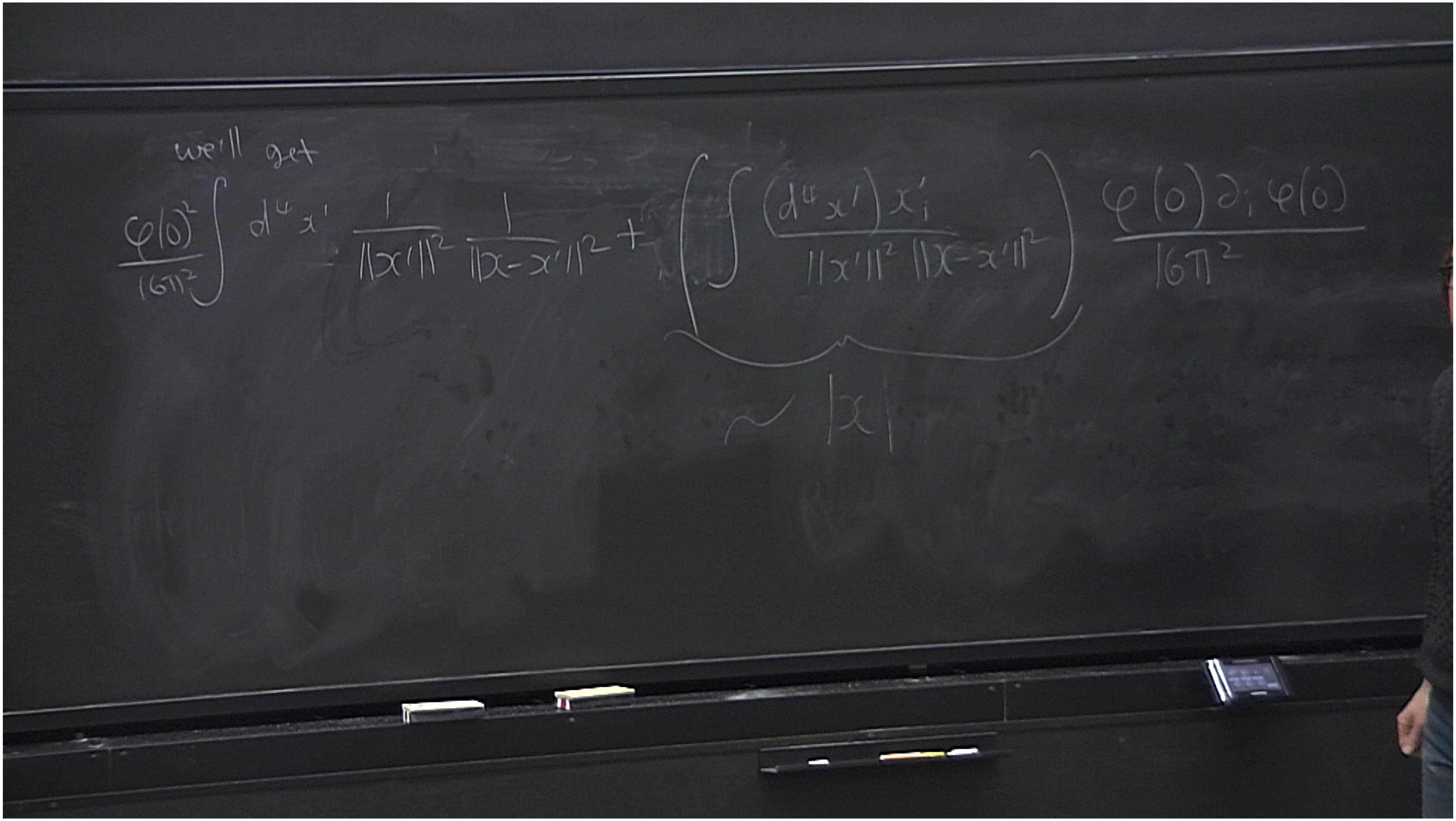
Expand $\varphi(x') = \varphi(0) + x'_i \partial_i \varphi(0) + \cdots$

Insert the expansion into the integral:

we'll get

we'll get

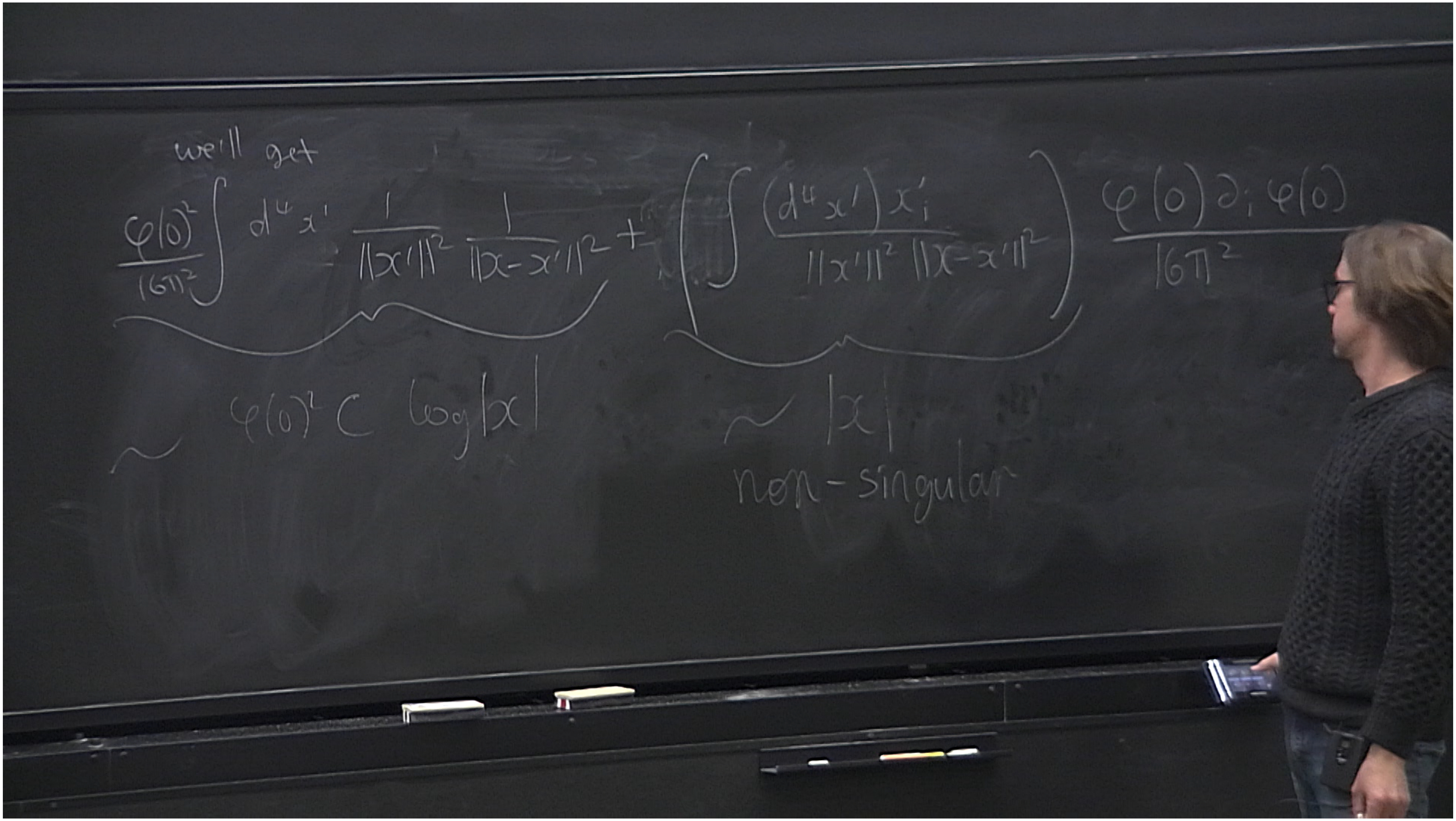
$$\frac{\varphi(0)^2}{16\pi^2}\int d^4x' \, \frac{1}{\|x'\|^2}\,\frac{1}{\|x-x'\|^2} + \frac{\partial_i\varphi(0)\,\varphi(0)}{16\pi^2}\int \frac{(d^4x')\,x'_i}{\|x'\|^2\,\|x-x'\|^2}$$

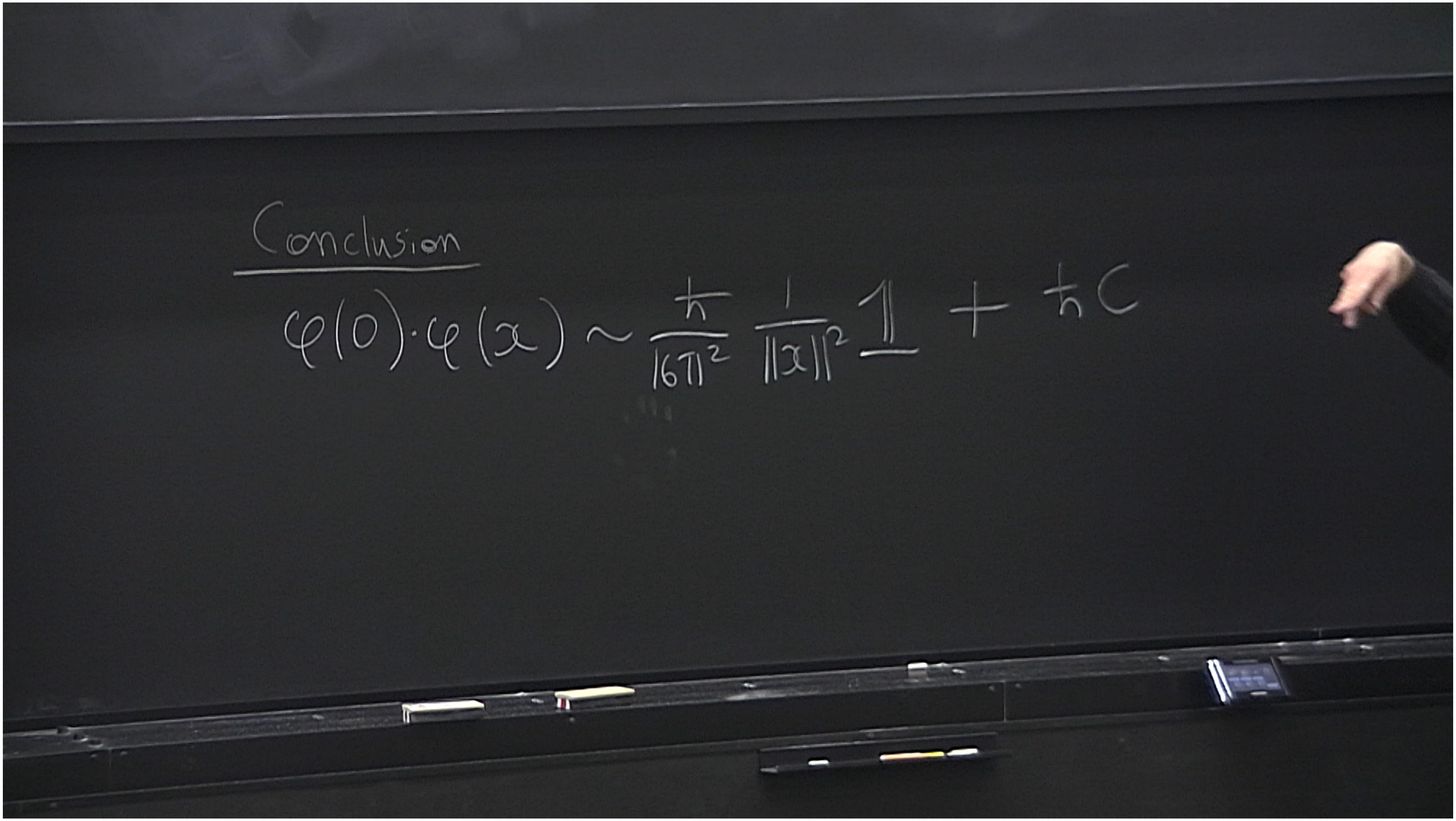

we'll get
$\frac{\varphi(0)^2}{16\pi^2}\int d^4x' \frac{1}{\|x'\|^2}\frac{1}{\|x-x'\|^2} + \underbrace{\left(\int \frac{(d^4x')\,x'_i}{\|x'\|^2\,\|x-x'\|^2}\right)}_{\sim |x|} \frac{\varphi(0)\,\partial_i\varphi(0)}{16\pi^2}$

we'll get

$$\frac{\varphi(0)^2}{16\pi^2}\int d^4x' \, \frac{1}{\|x'\|^2}\,\frac{1}{\|x-x'\|^2} + \underbrace{\left(\int \frac{(d^4x')\,x'_i}{\|x'\|^2\,\|x-x'\|^2}\right)}_{\sim\, |x|} \frac{\varphi(0)\,\partial_i\varphi(0)}{16\pi^2}$$

non-singular


we'll get
$\frac{\varphi(0)^2}{16\pi^2}\int d^4x' \frac{1}{\|x'\|^2}\frac{1}{\|x-x'\|^2} + \left(\int \frac{(d^4x')\,x'_i}{\|x'\|^2\,\|x-x'\|^2}\right)\frac{\varphi(0)\,\partial_i\varphi(0)}{16\pi^2}$
$\sim \varphi(0)^2 C \log|x|$
$\sim |x|$
non-singular


Conclusion
$\varphi(0)\cdot\varphi(x) \sim \frac{\hbar}{16\pi^2}\frac{1}{\|x\|^2}\underline{\mathbb{1}} + \hbar C$

Conclusion

$$\varphi(0)\cdot\varphi(x) \sim \frac{\hbar}{16\pi^2}\frac{1}{\|x\|^2}\underline{\mathbb{1}} + \hbar\, C\left(\log\|x\|\right)\varphi(0)^2 + O(\hbar^2)$$