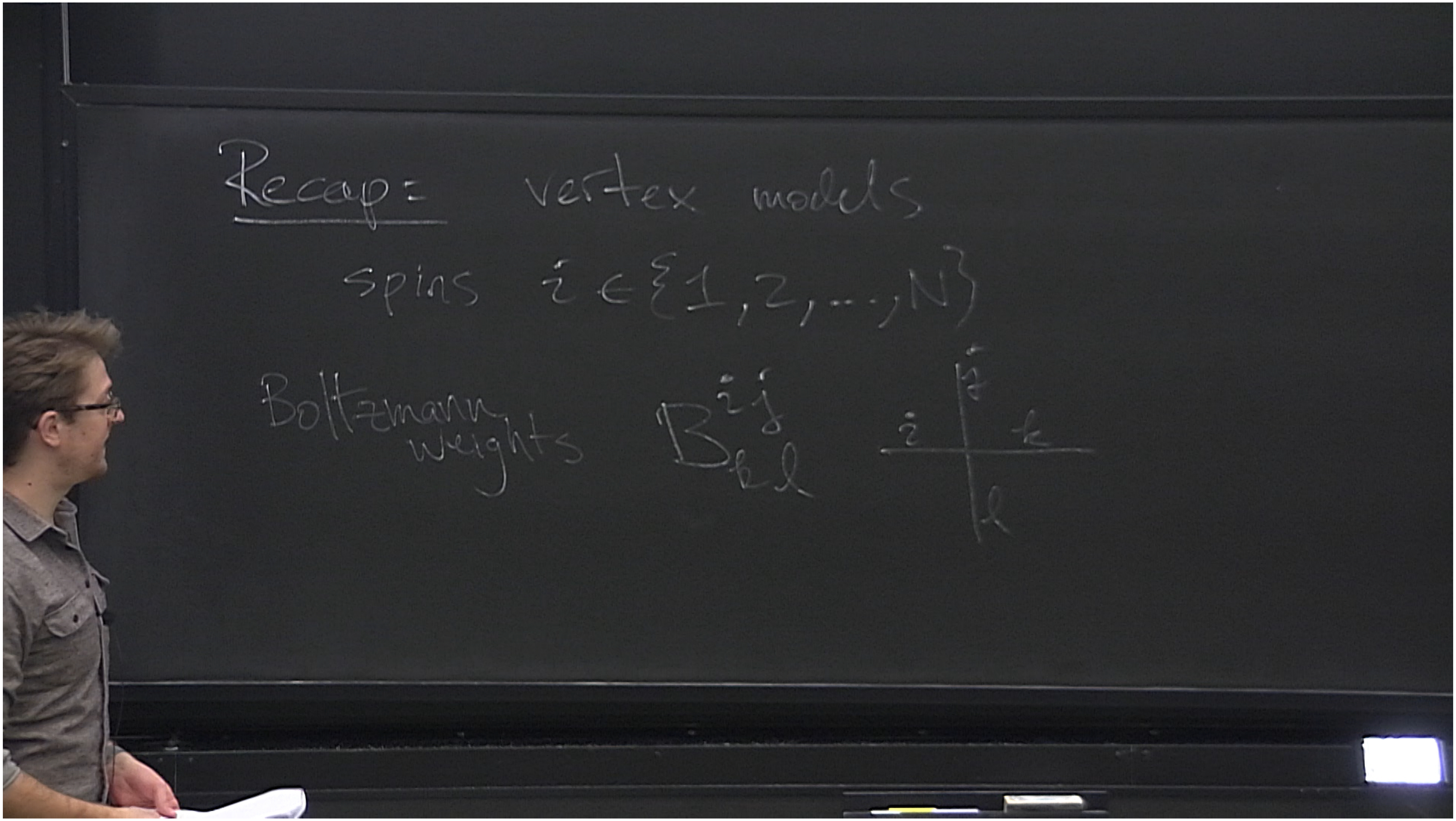
Title: PSI 2017/2018 - Quantum Integrable Models - Lecture 2

Date: Mar 20, 2018 11:30 AM

URL: http://pirsa.org/18030060

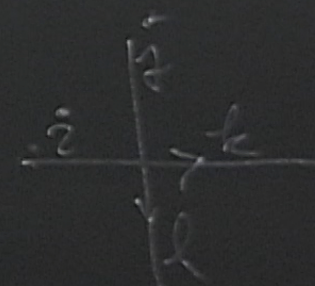
Abstract:


Recap: vertex models
spins $i \in \{1, 2, \dots, N\}$
Boltzmann weights B^{ij}_{kl}
j
i
k
l

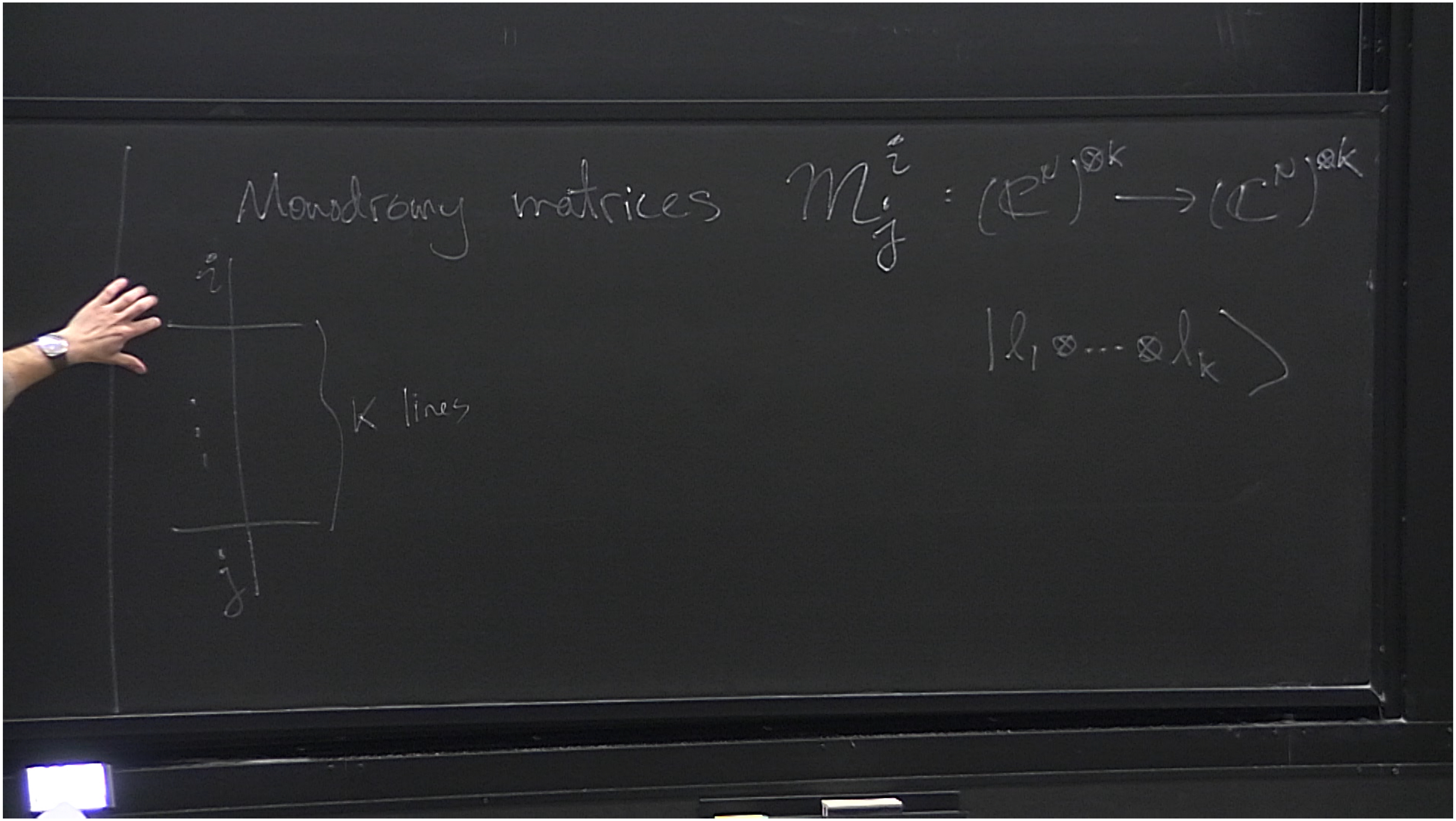
Recap: vertex models

spins $i \in \{1, 2, \ldots, N\}$

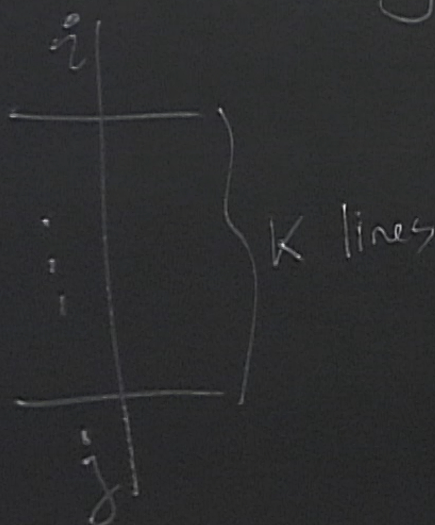
Boltzmann weights B^{ij}_{kl}



think of as $\underset{\text{left}}{\mathbb{C}^N} \otimes \underset{\text{up}}{\mathbb{C}^N} \longrightarrow \underset{\text{right}}{\mathbb{C}^N} \otimes \underset{\text{down}}{\mathbb{C}^N}$


Monodromy matrices $M^i_j : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$
i
j
k lines
$|l_1 \otimes \cdots \otimes l_k\rangle$

Monodromy matrices $M^i_j : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$

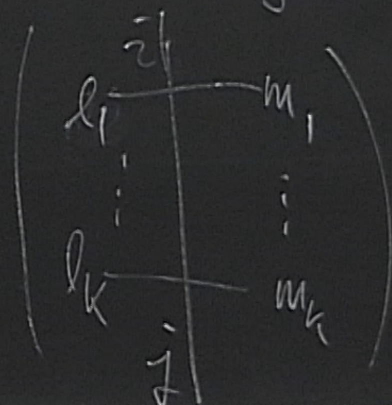


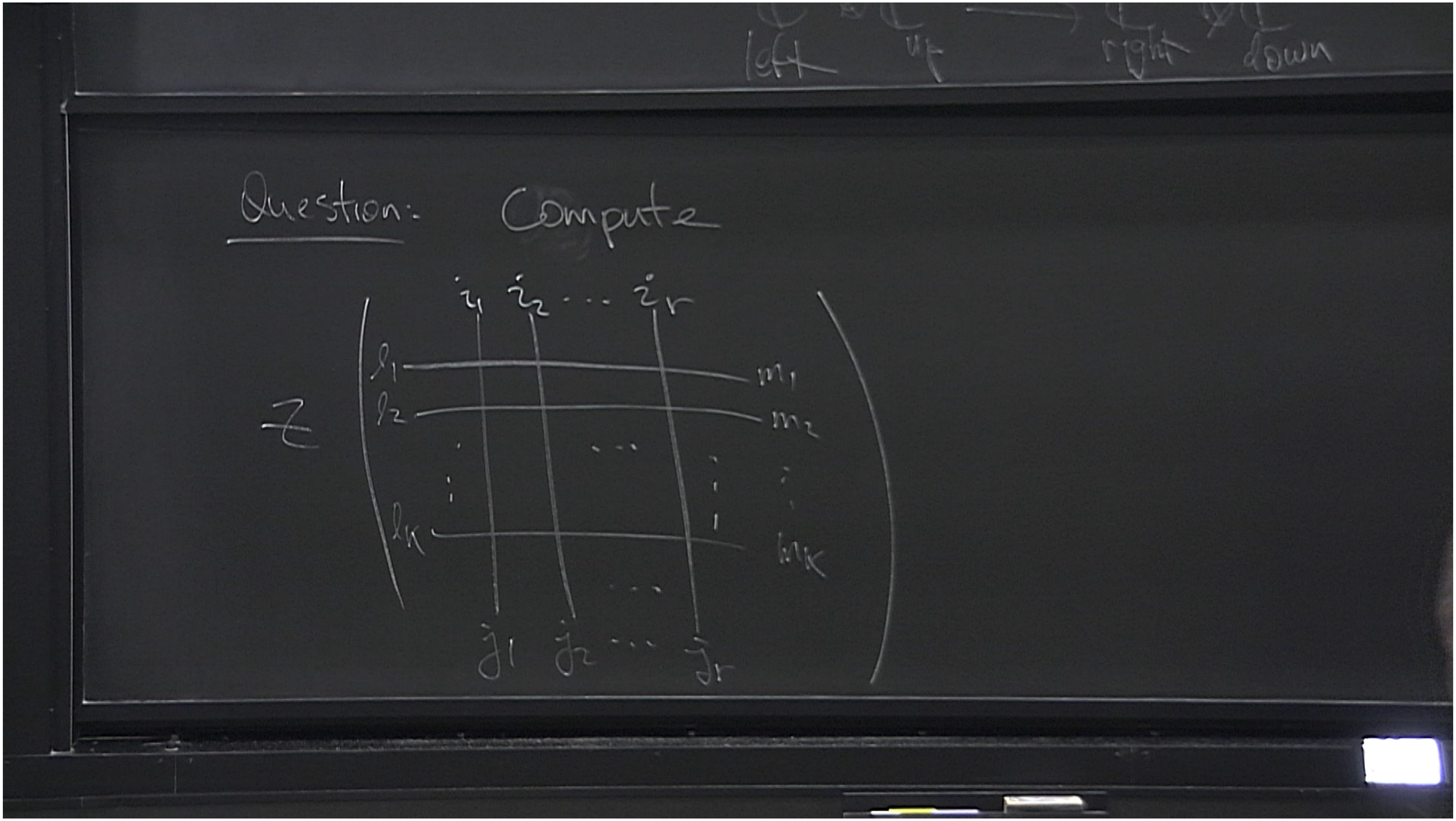
$$\langle m_1 \otimes \cdots \otimes m_k | M^i_j | l_1 \otimes \cdots \otimes l_k \rangle$$

Monodromy matrices $\mathcal{M}^i_j : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$

k lines

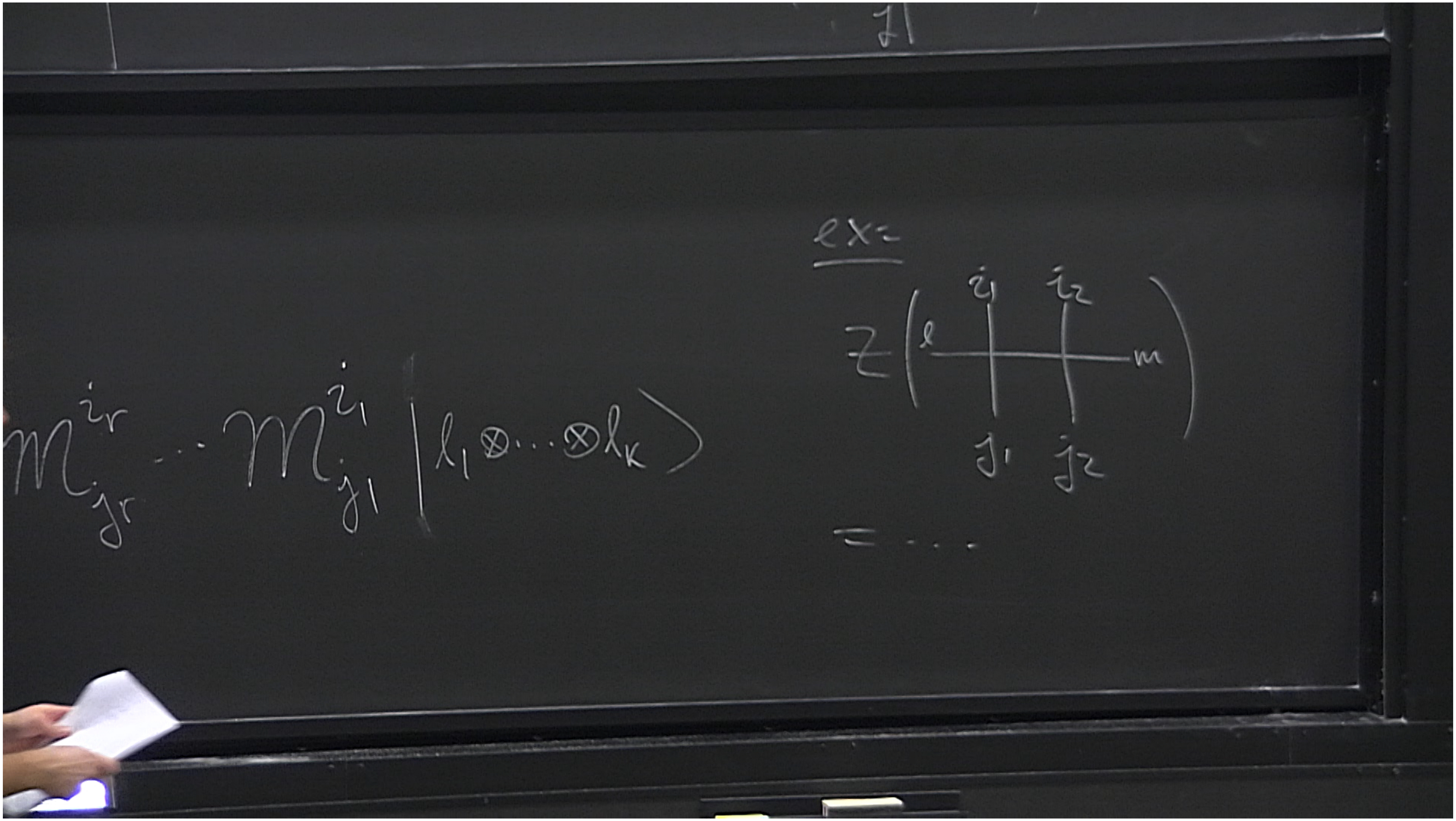
$$\langle m_1 \otimes \dots \otimes m_k | \mathcal{M}^i_j | l_1 \otimes \dots \otimes l_k \rangle = \sum$$

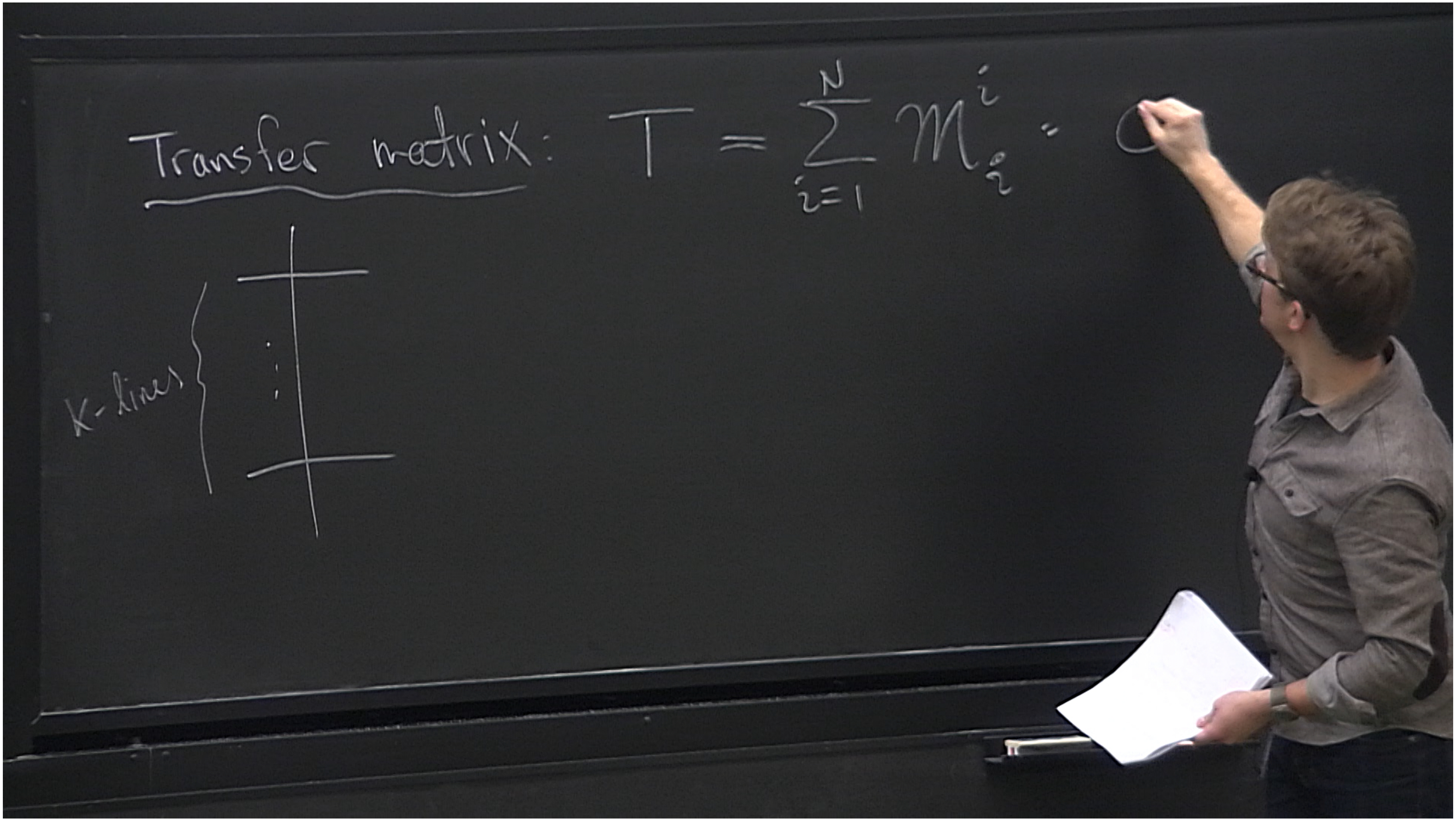


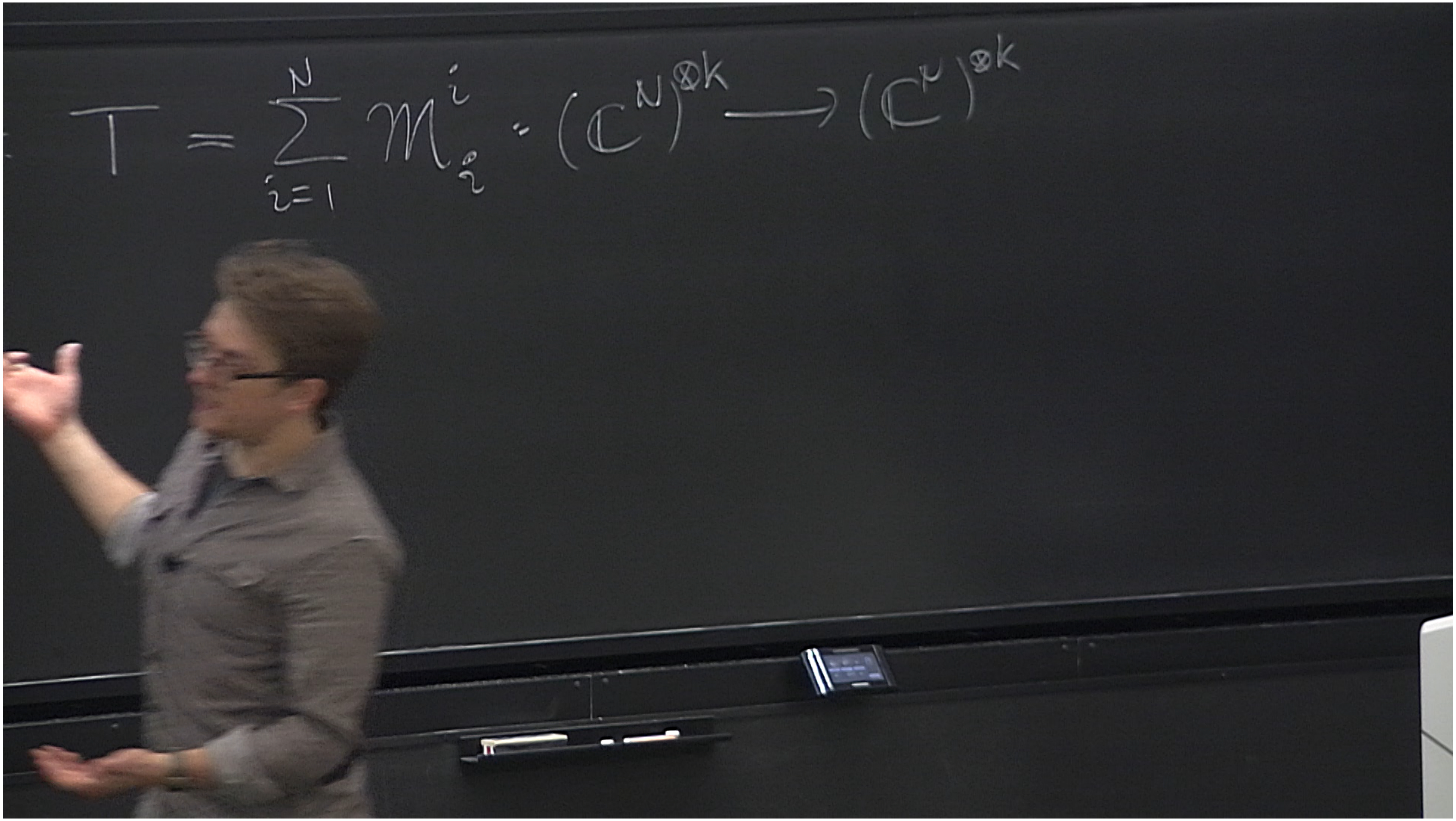

left
up
right
down
Question: Compute
Z
i1 i2 ... ir
l1
l2
lk
m1
m2
mk
j1 j2 ... jr

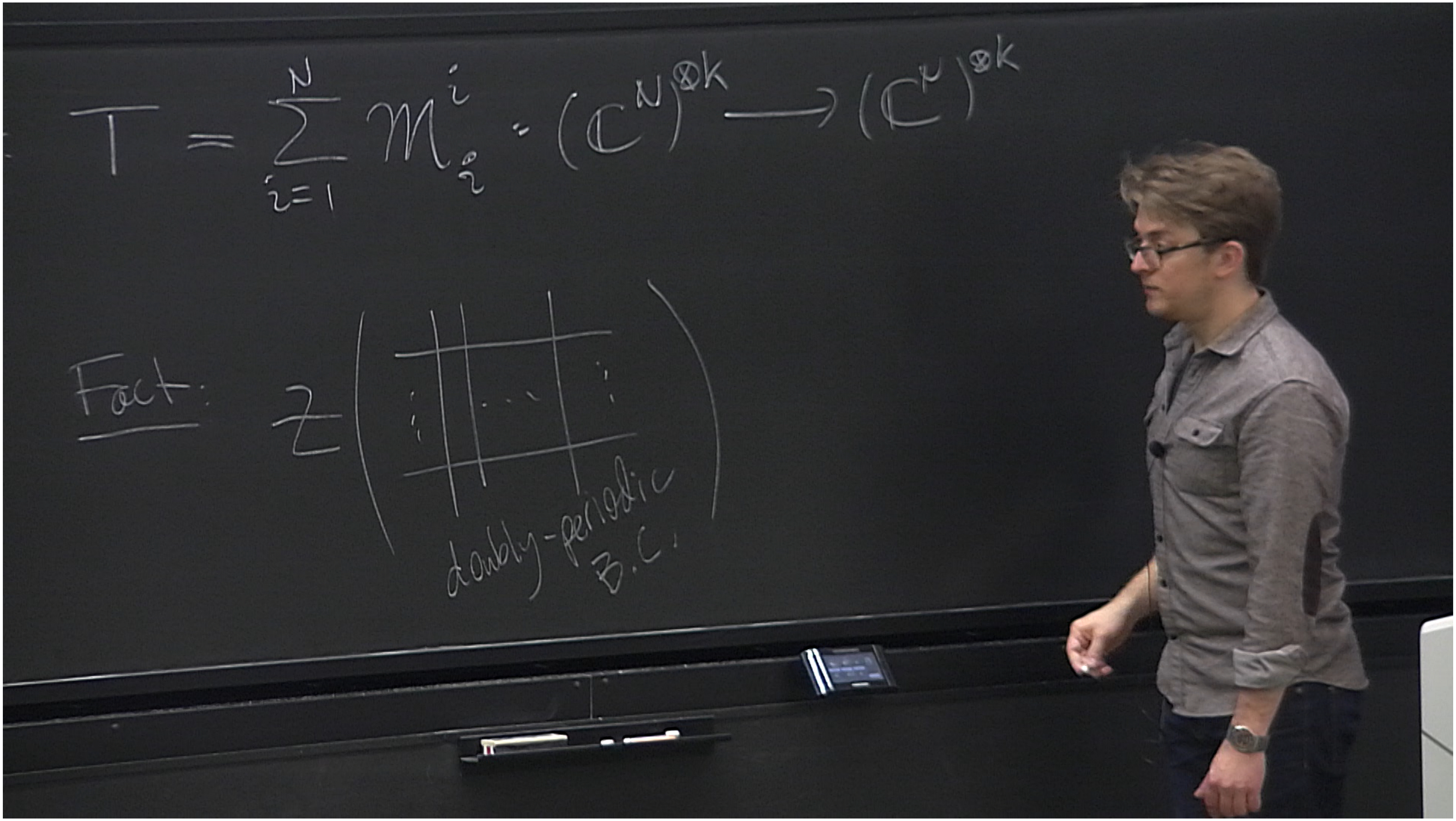
up right down

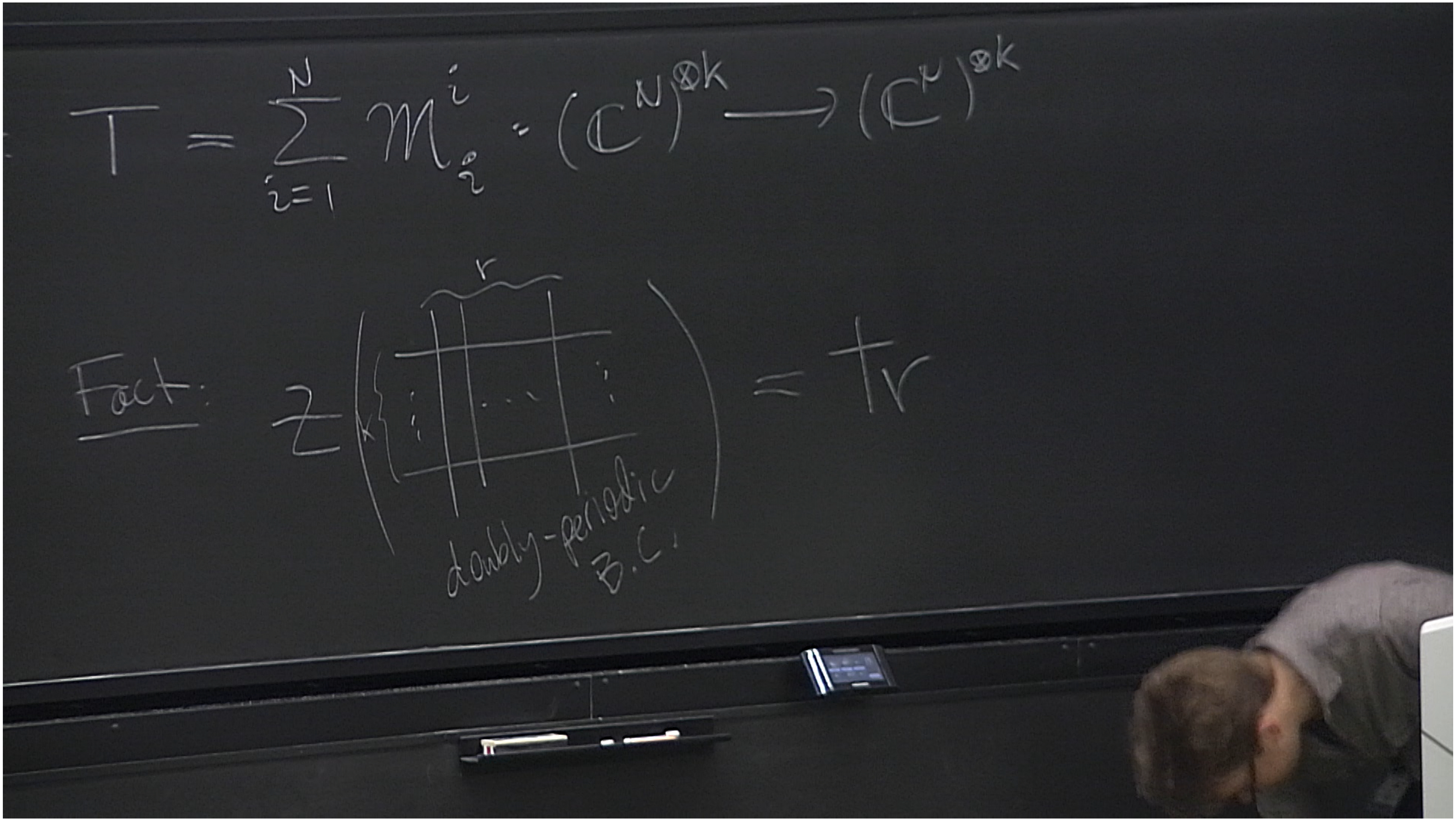
$$= \langle m_1 \otimes \cdots \otimes m_k | M^{i_r}_{j_r} \cdots M^{i_1}_{j_1} | l_1 \otimes \cdots \otimes l_k \rangle$$


$M^{i_r}_{j_r} \cdots M^{i_1}_{j_1} | l_1 \otimes \cdots \otimes l_k \rangle$
ex:
$Z($
i_1 i_2
l m
j_1 j_2
$)$
$= \cdots$

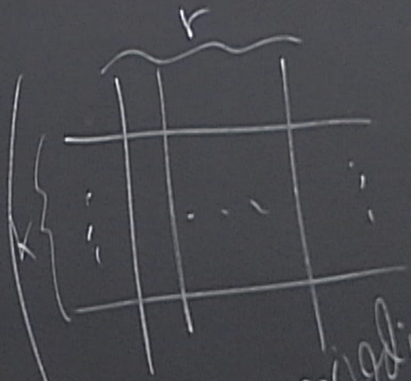

Transfer matrix: $T = \sum_{i=1}^{N} M_i^i =$
k-lines


$T = \sum_{i=1}^{N} \mathcal{M}^{i}_{i} : (\mathbb{C}^{N})^{\otimes k} \longrightarrow (\mathbb{C}^{N})^{\otimes k}$


$T = \sum_{i=1}^{N} m^{i}_{i} : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$
Fact:
Z
doubly-periodic
B.C.


$T = \sum_{i=1}^{N} M_i^i : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$
r
Fact:
Z
k
$= \mathrm{tr}$
doubly-periodic B.C.

Ex: $T = \sum_{i=1}^{N} \mathcal{M}^{i}_{\mathring{i}} : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$

Fact: $Z\Bigg($ 

$\Bigg) = \mathrm{tr}_{(\mathbb{C}^N)^{\otimes k}}\left(T^r\right)$

doubly-periodic B.C.

Yang-Baxter equation and commuting transfer matrices

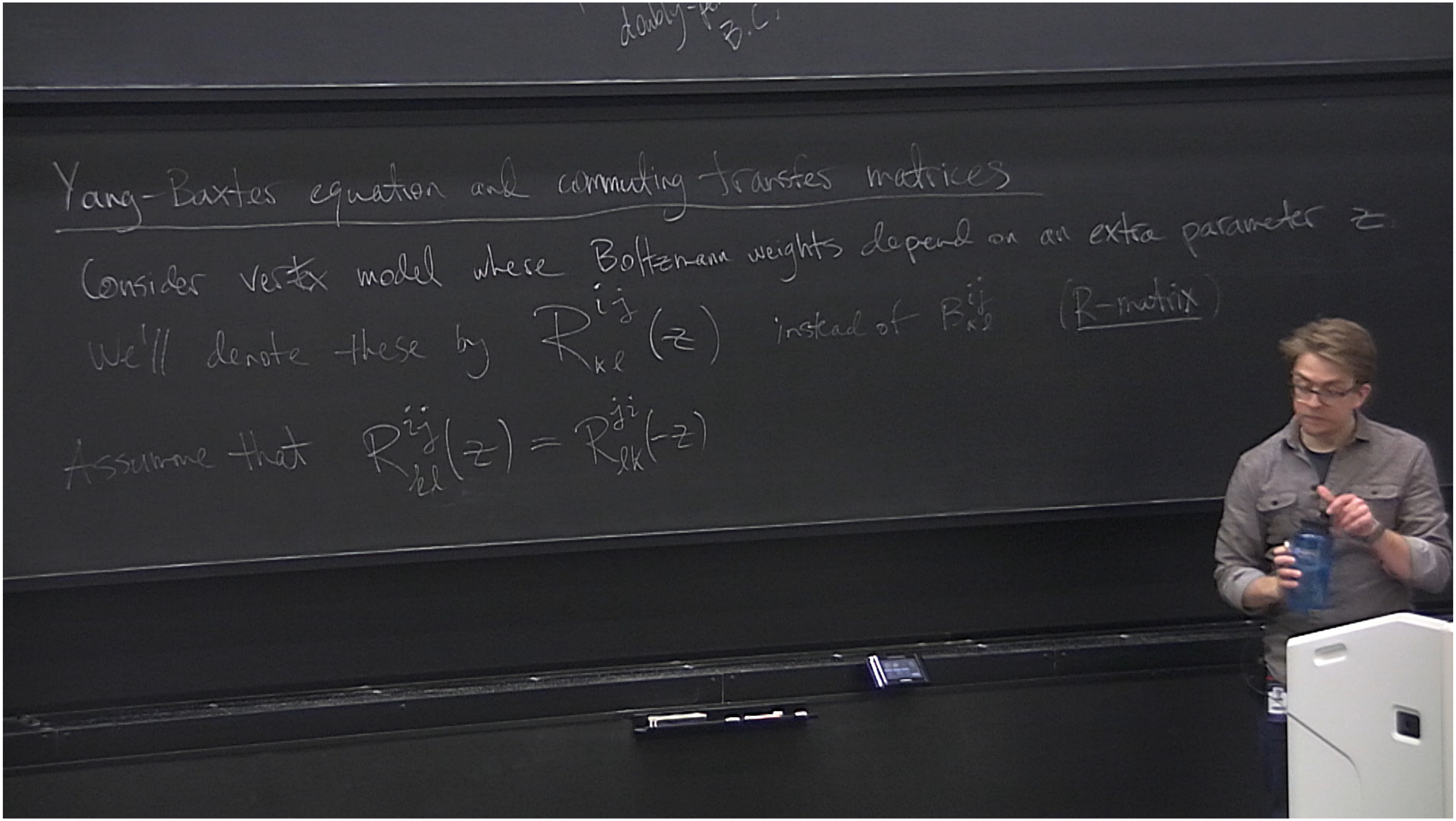
Consider vertex model where Boltzmann weights depend on an extra parameter z

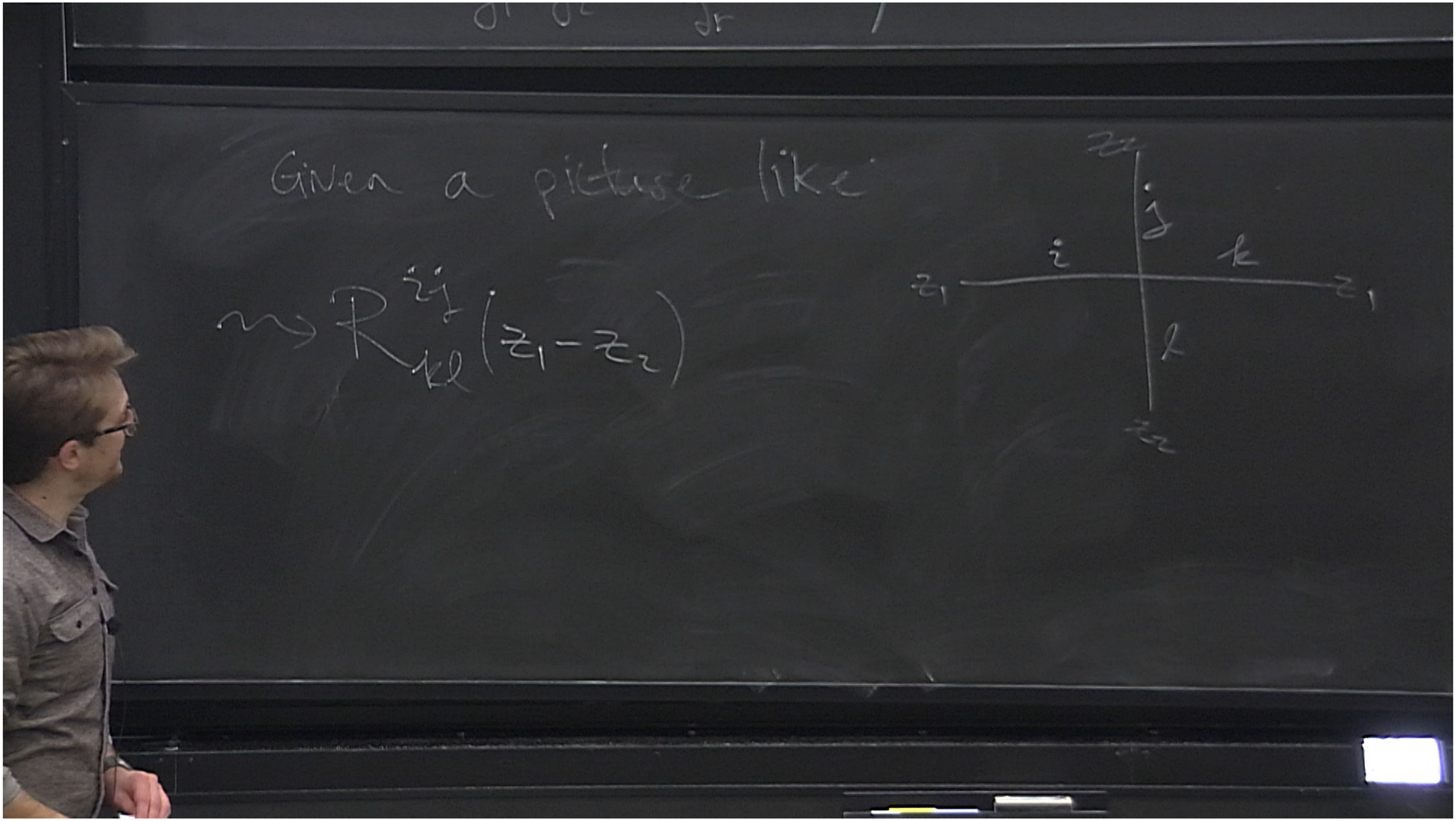
doubly-pe B.C.

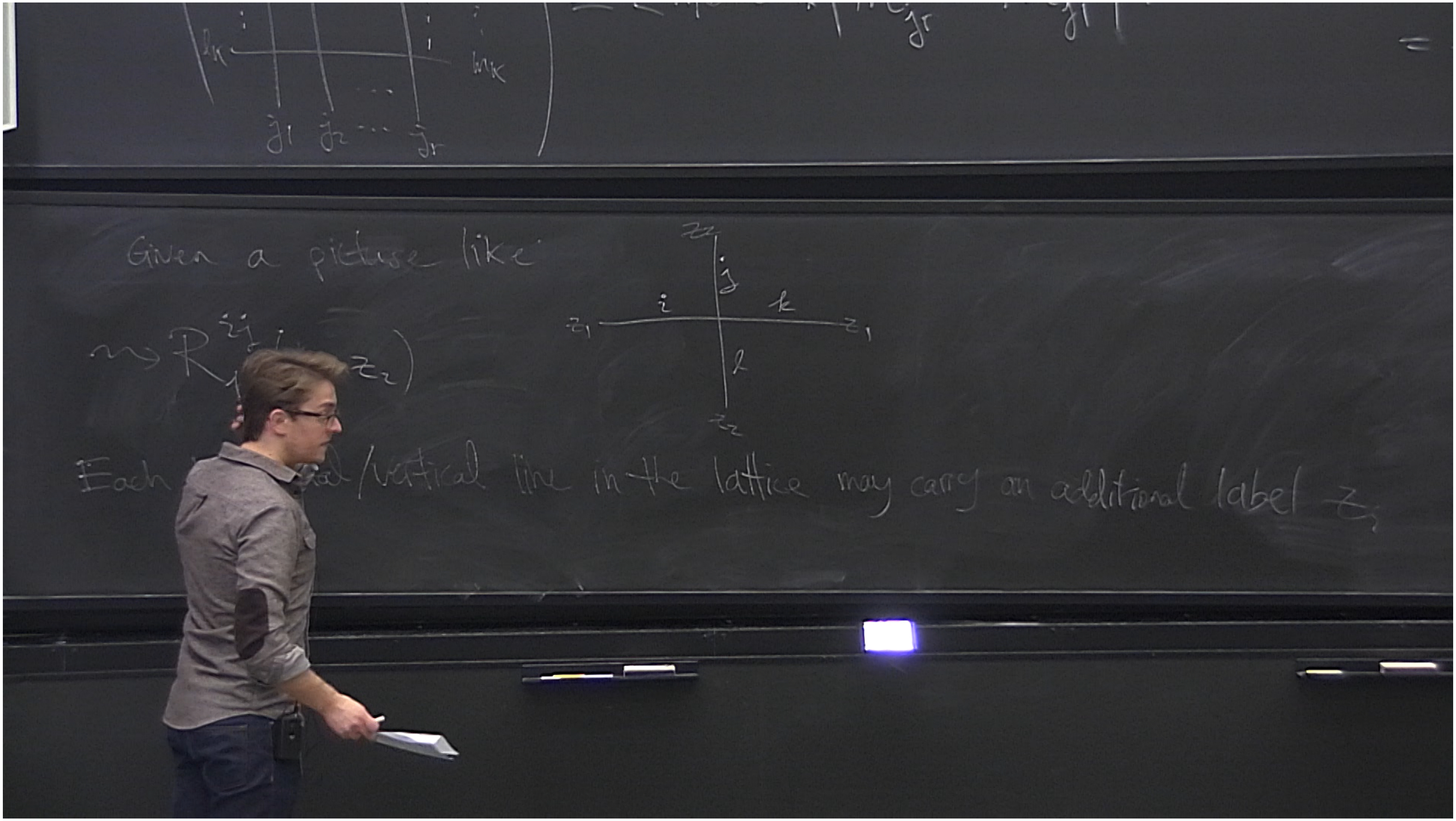
Yang-Baxter equation and commuting transfer matrices

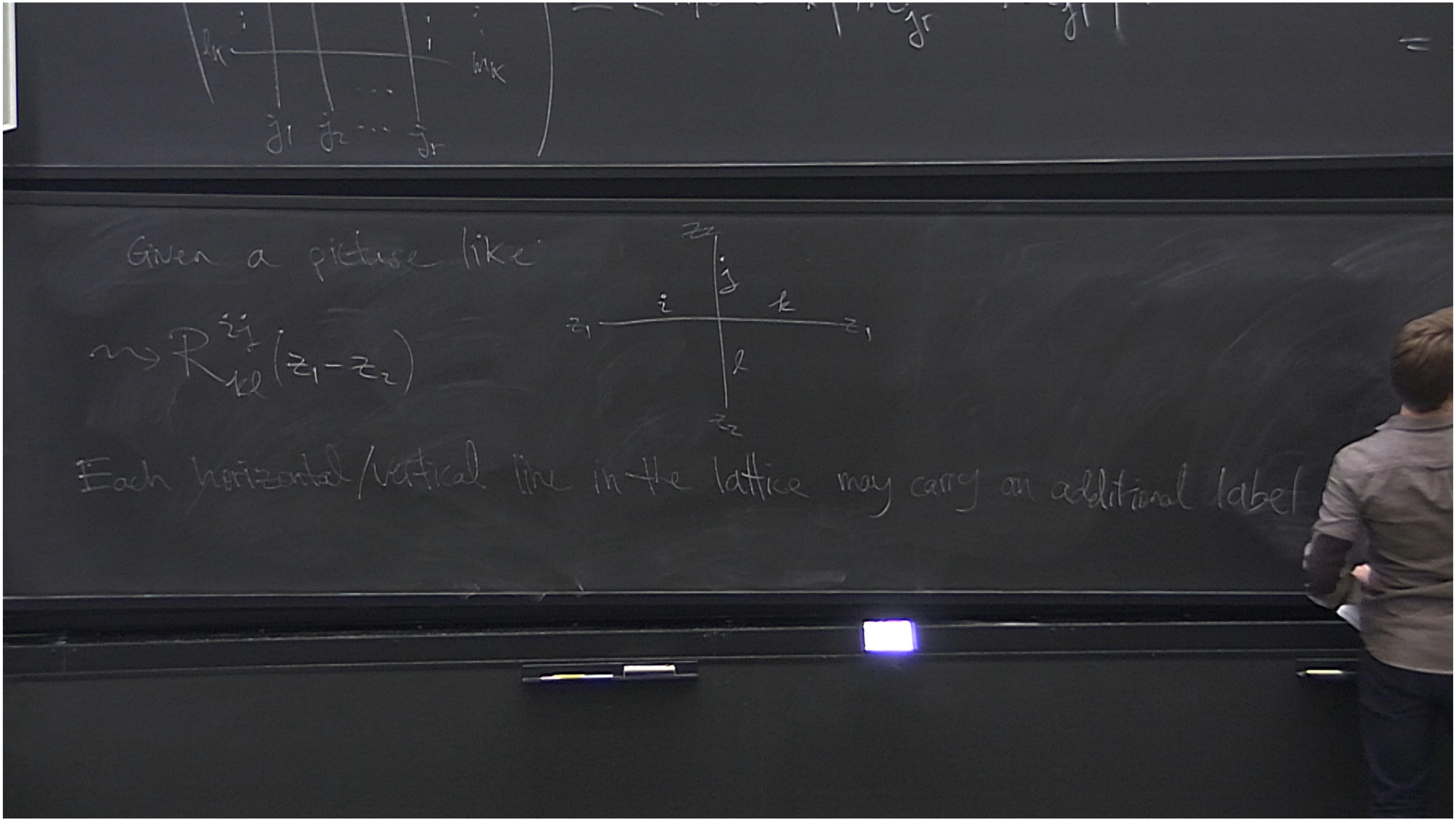
Consider vertex model where Boltzmann weights depend on an extra parameter z.

We'll denote these by $R^{ij}_{k\ell}(z)$ instead of $B^{ij}_{k\ell}$ (R-matrix)

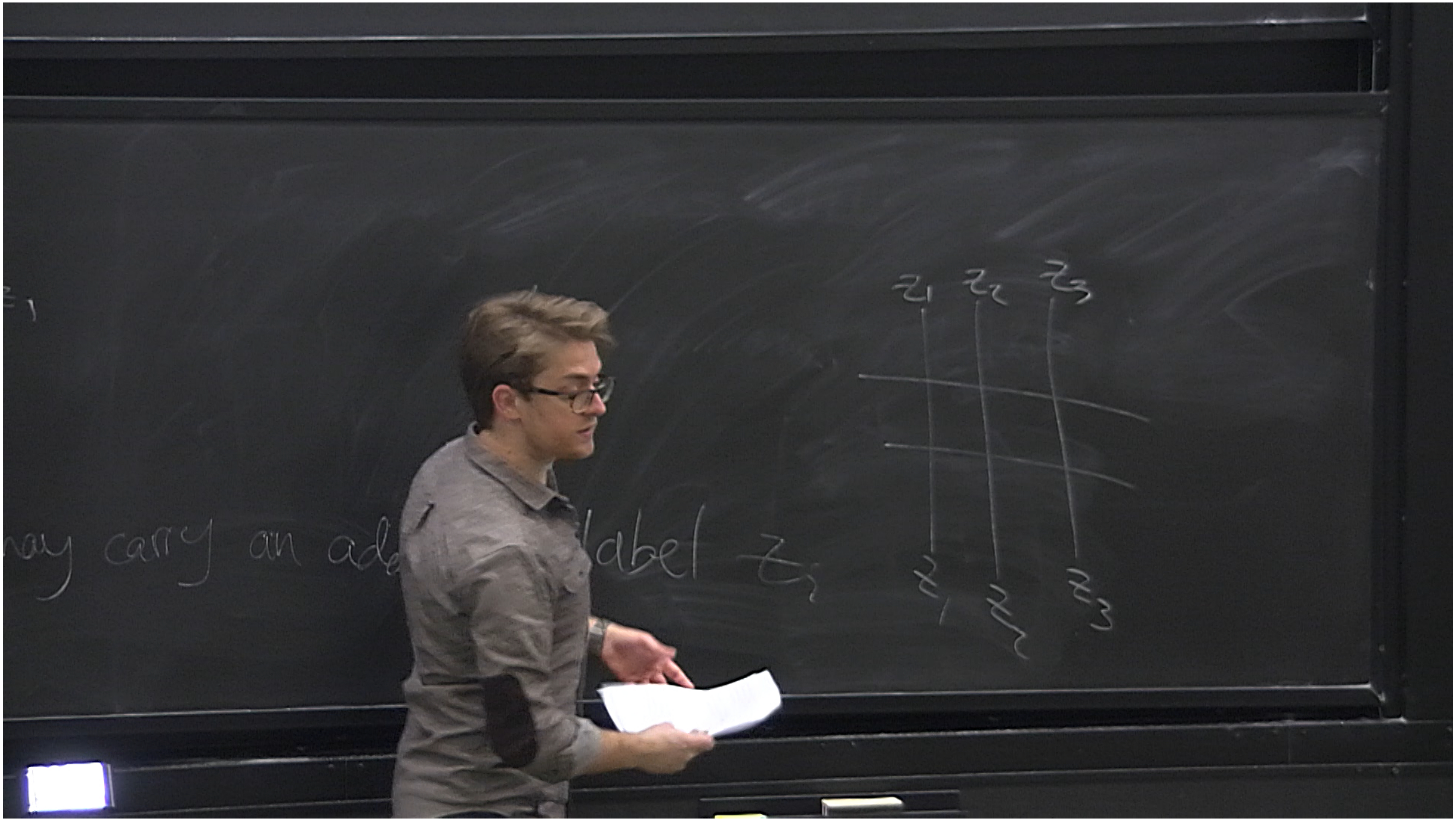

Yang-Baxter equation and commuting transfer matrices
Consider vertex model where Boltzmann weights depend on an extra parameter z.
We'll denote these by $R^{ij}_{kl}(z)$ instead of B^{ij}_{kl} (R-matrix)
Assume that $R^{ij}_{kl}(z) = R^{ji}_{lk}(-z)$

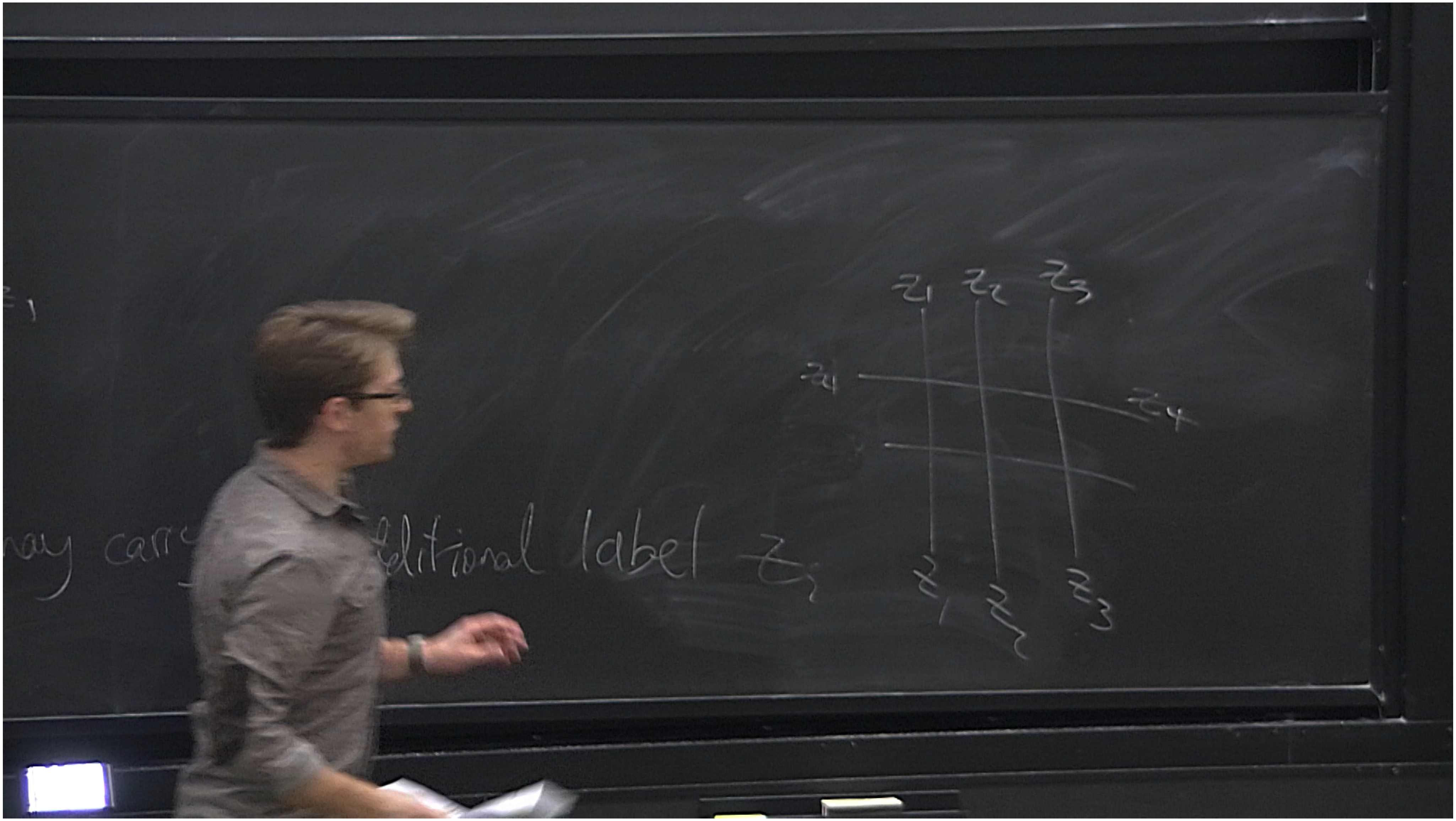

Given a picture like
$\rightsquigarrow R^{ij}_{kl}(z_1 - z_2)$
z_2
j
i
k
z_1
z_1
l
z_2


Given a picture like
Each
/vertical line in the lattice may carry an additional label



Given a picture like
$\rightsquigarrow R^{ij}_{kl}(z_1 - z_2)$
z_2
j
i
k
z_1
z_1
l
z_2
Each horizontal/vertical line in the lattice may carry an additional label
$j_1 \; j_2 \cdots j_r$

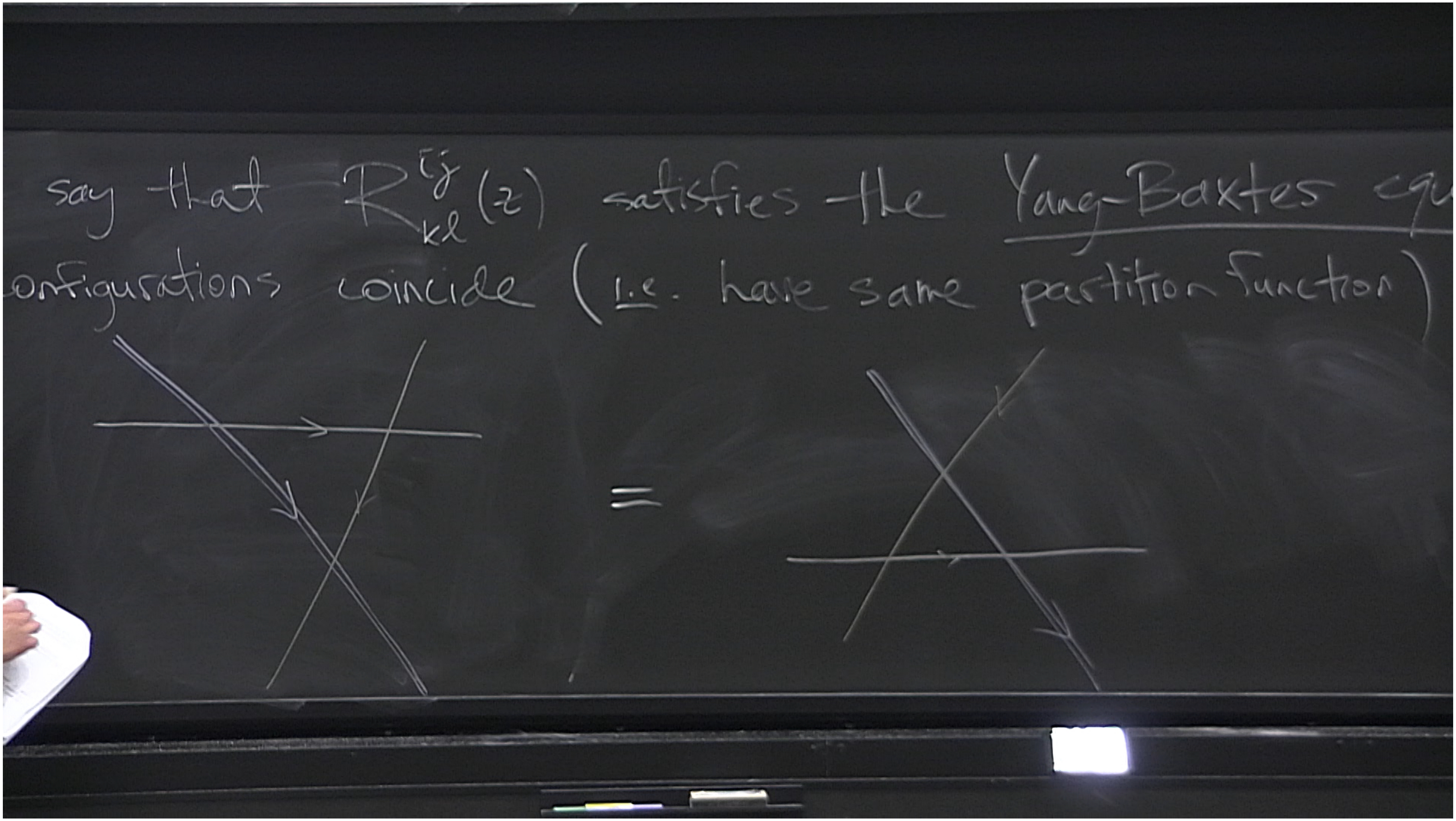

carry an
label z_i
z_1 z_2 z_3
z_1 z_2 z_3

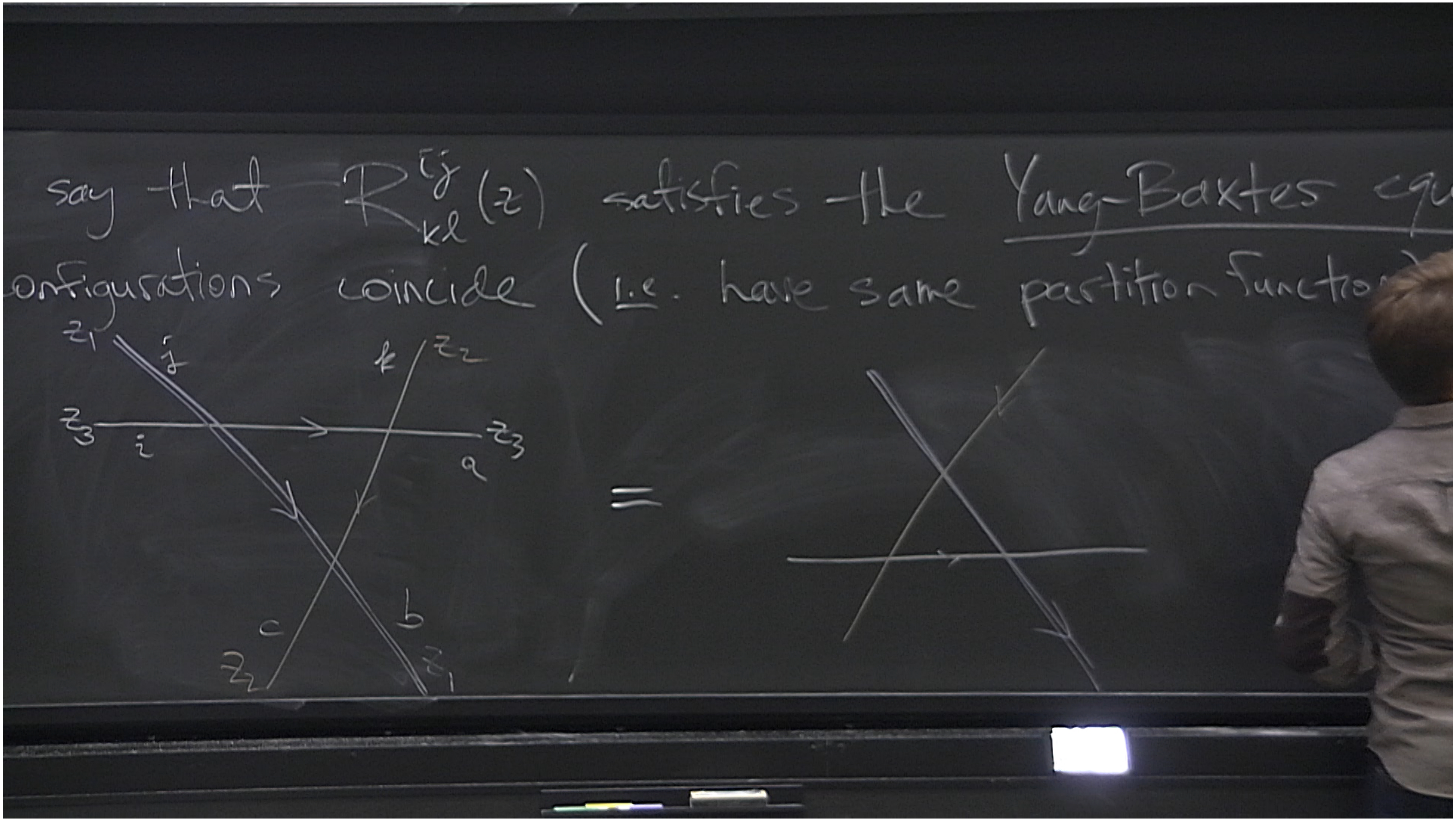

label z_i
z_1 z_2 z_3
z_4 z_4
z_1 z_2 z_3

Each horizontal/vertical line in the lattice may carry an additional label z_i

We say that $R^{ij}_{kl}(z)$ satisfies the Yang-Baxter equation if the following configurations coincide (i.e. have same partition function) say that $R^{ij}_{kl}(z)$ satisfies the <u>Yang-Baxter equ</u>

onfigurations coincide (i.e. have same partition function)


say that $R^{ij}_{kl}(z)$ satisfies the Yang-Baxter equ
onfigurations coincide (i.e. have same partition function)
=

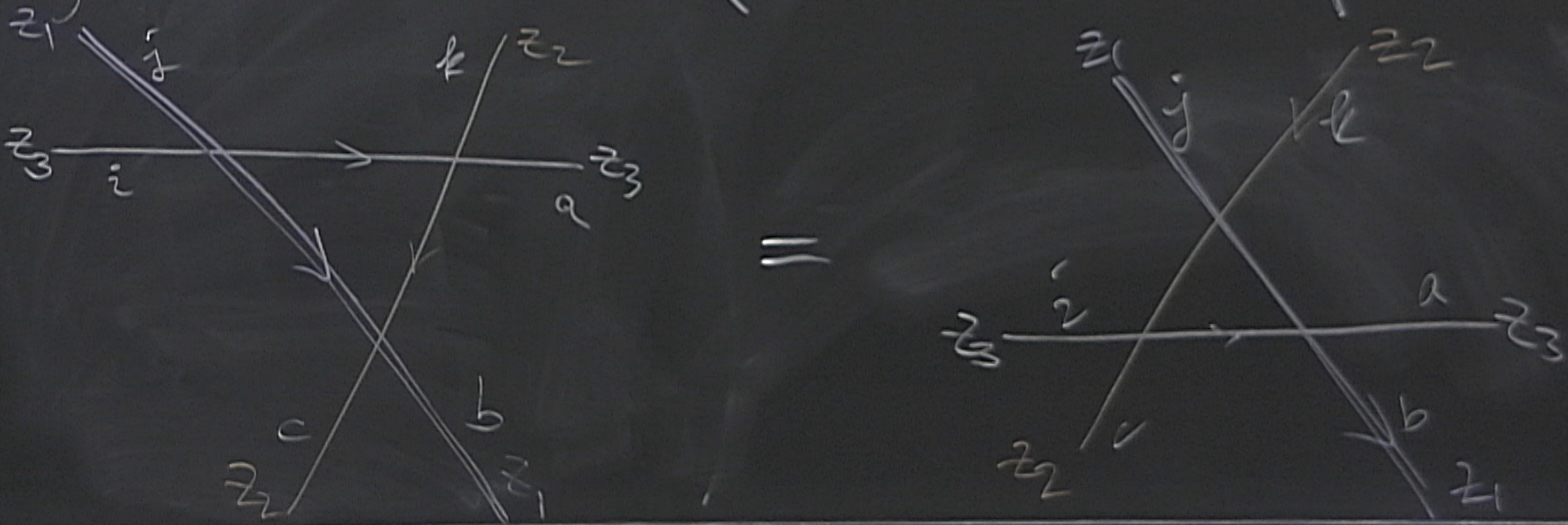

say that $R^{ij}_{kl}(z)$ satisfies the Yang-Baxter equ
onfigurations coincide (i.e. have same partition functio
z_1
j
k
z_2
z_3
i
z_3
a
=
c
b
z_2
z_1

ang-Baxter equation if the following

partition function), for any fixed i, j, k, a, b, c

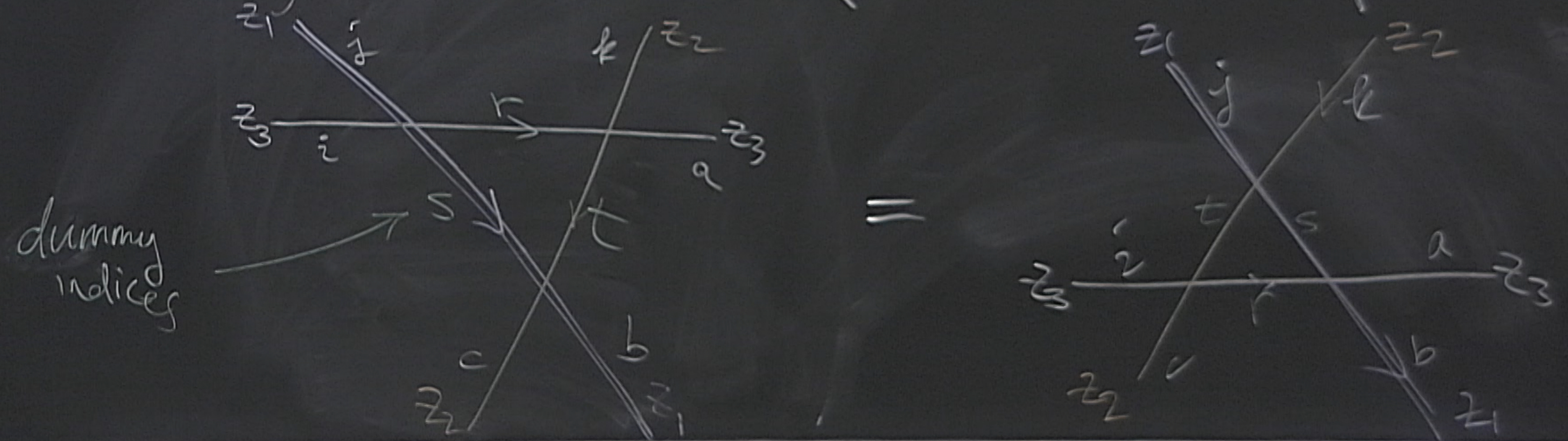
say that $R^{ij}_{kl}(z)$ satisfies the Yang-Baxter

configurations coincide (i.e. have same partition functio



We say that $R^{ij}_{kl}(z)$ satisfies the Yang-Baxte

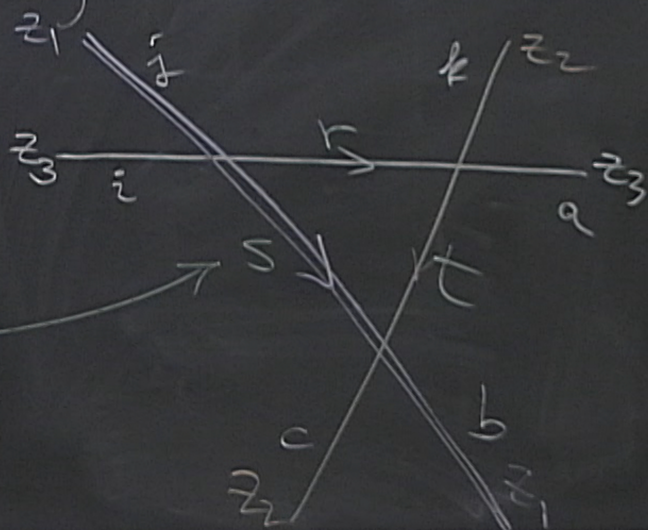
configurations coincide (i.e. have same partition fun



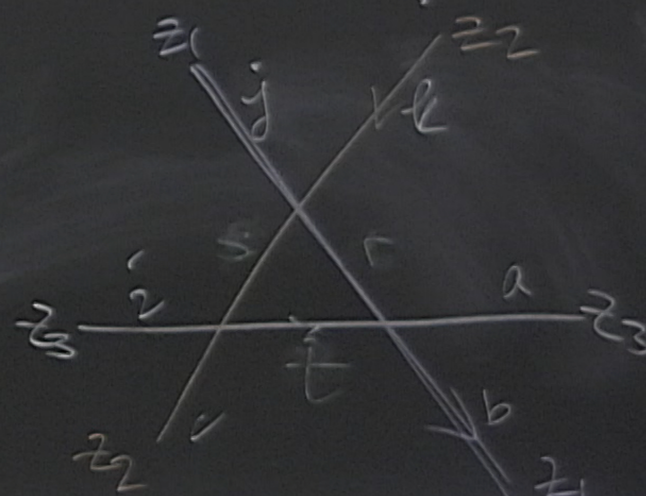
We say that $R^{ij}_{kl}(z)$ satisfies the Yang-Baxte

configurations coincide (i.e. have same partition fun

dummy indices



$=$

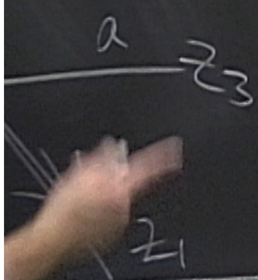


ang-Baxter equation if the following

partition function), for any fixed i, j, k, a, b, c

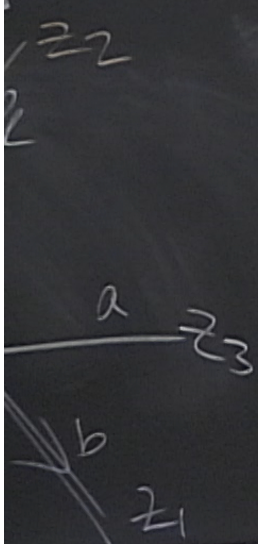
z_2

$$R^{ij}_{rs}(z_3-z_1)\,R^{rk}_{at}(z_3-z_2)\,R^{st}_{bc}(z_1-z_2)$$



...ang-Baxter equation if the following

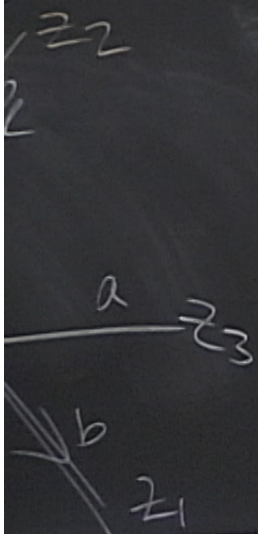
...partition function), for any fixed i, j, k, a, b, c



$$\sum_{r,s,t} R^{ij}_{rs}(z_3-z_1)\, R^{rk}_{at}(z_3-z_2)\, R^{st}_{bc}(z_1-z_2)$$

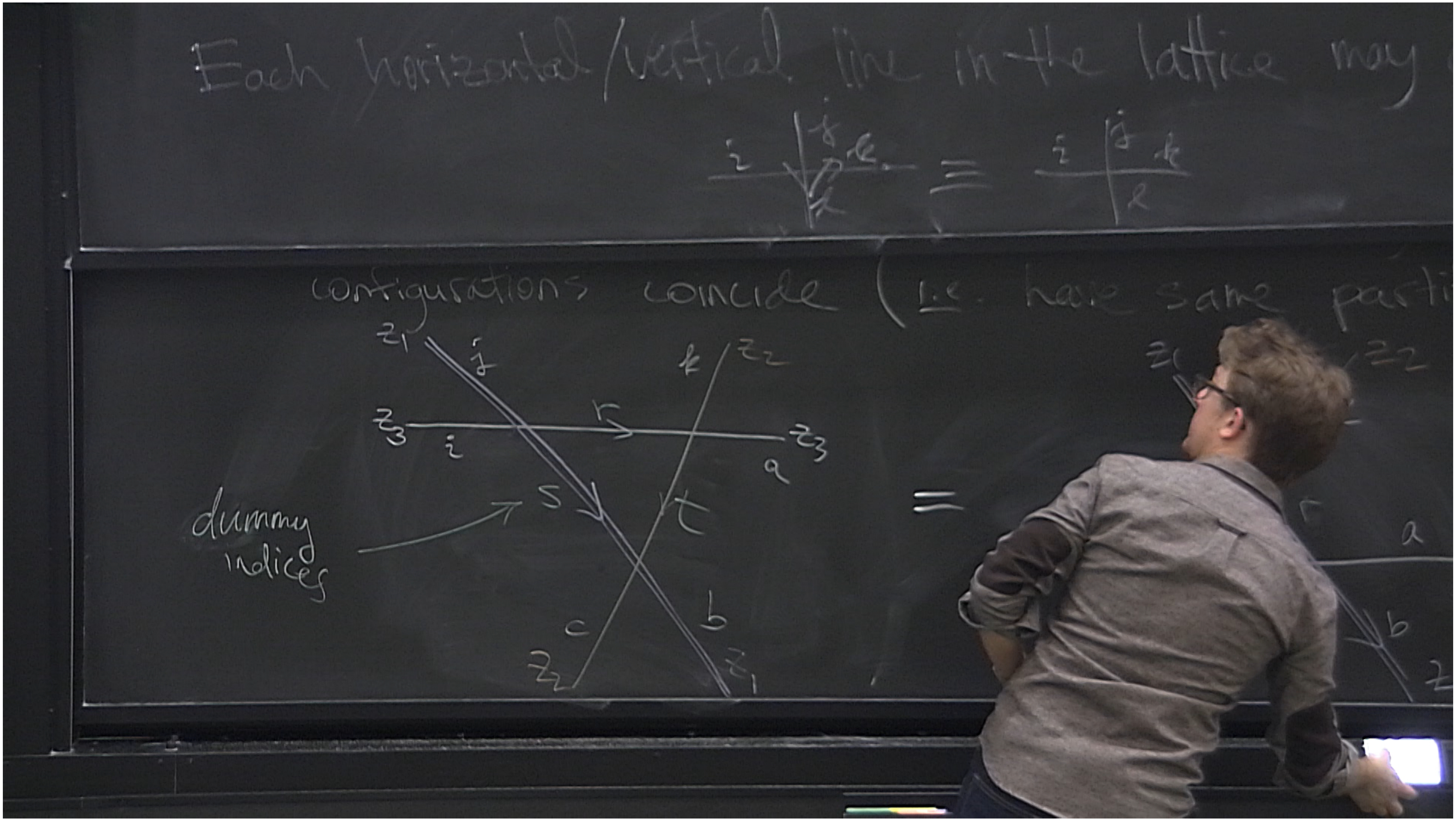
ang-Baxter equation if the following

partition function), for any fixed i, j, k, a, b, c



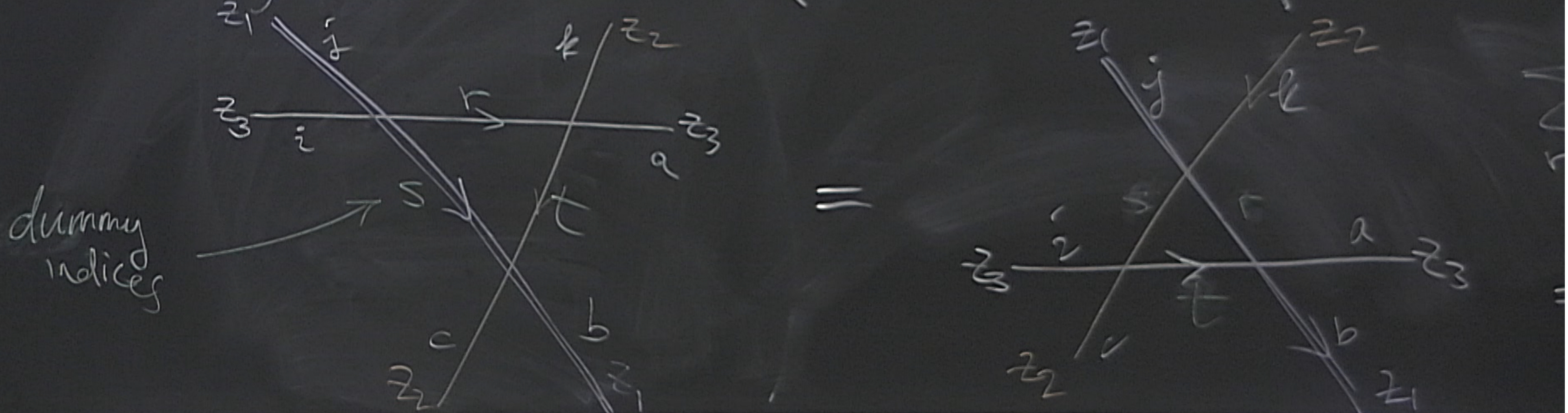
$$\sum_{r,s,t} R^{ij}_{rs}(z_3-z_1)\, R^{rk}_{at}(z_3-z_2)\, R^{st}_{bc}(z_1-z_2)$$

$$= \sum_{r,s,t} R^{jk}_{rs}(z_1-z_2)\, R^{is}_{tc}(z_3-z_2)\, R^{tr}_{ab}(z_3-z_1)$$


Each horizontal/vertical line in the lattice may
configurations coincide (i.e. have same parti
dummy indices
z1 j k z2
z3 i r z3 a
s t
c b
z2 z1

We say that $R^{ij}_{kl}(z)$ satisfies the Yang-Baxt

configurations coincide (i.e. have same partition fu

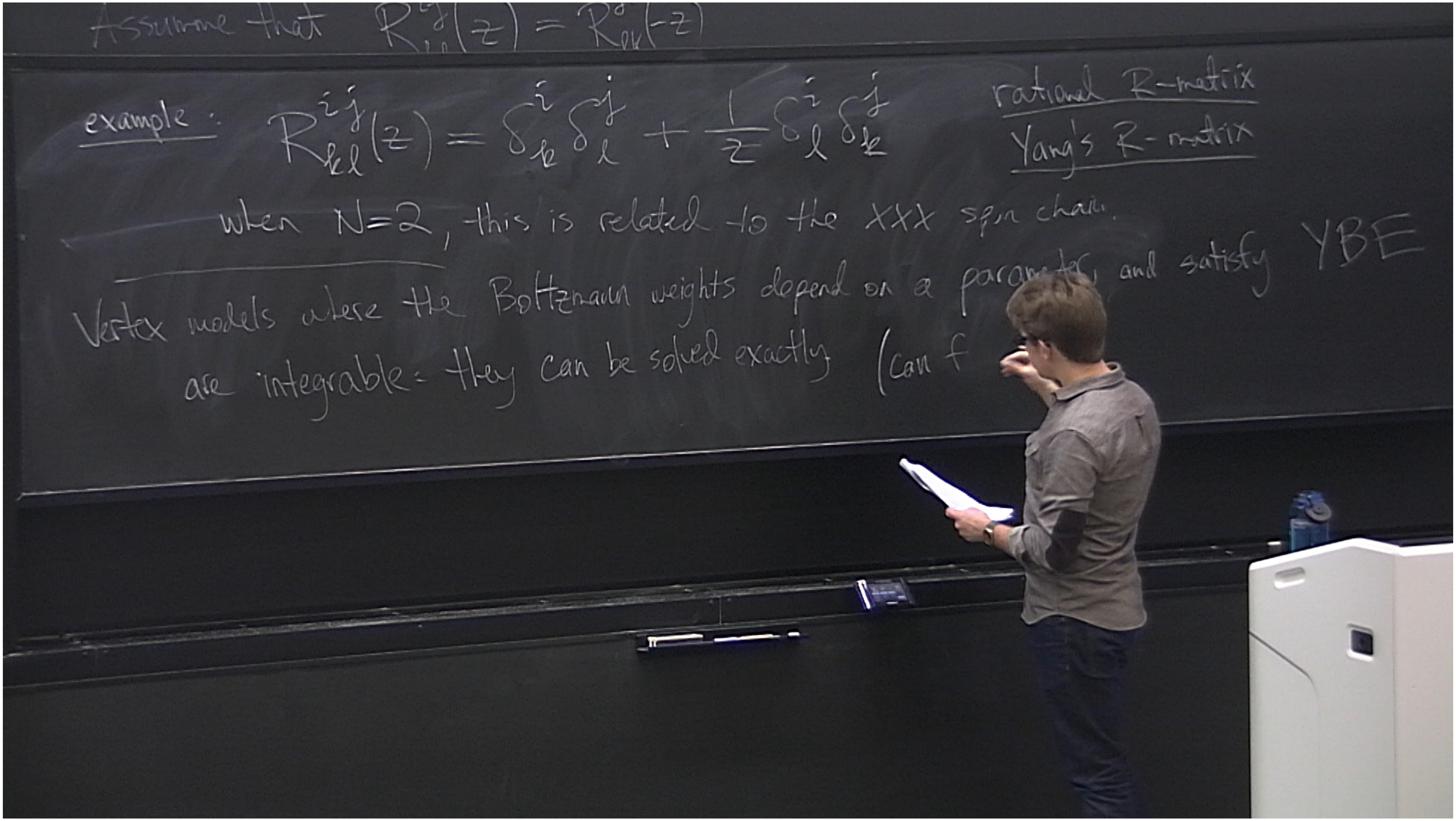


example:

$$R^{ij}_{kl}(z) = \delta^i_k \delta^j_l + \frac{1}{z} \delta^i_l \delta^j_k$$

rational R-matrix

Yang's R-matrix

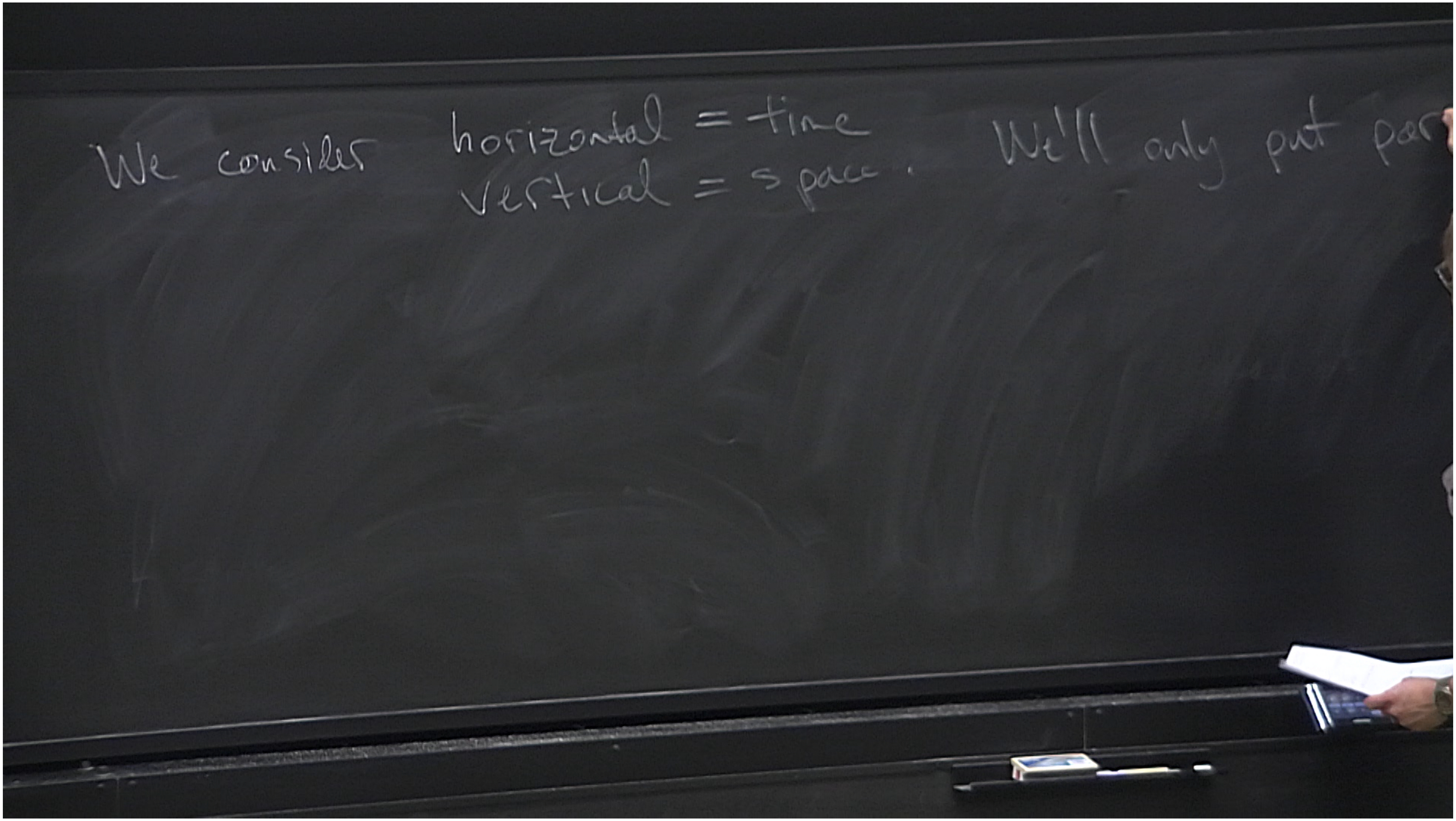

Assume that $R^{ij}_{kl}(z) = R^{ji}_{lk}(-z)$
example: $R^{ij}_{kl}(z) = \delta^i_k \delta^j_l + \frac{1}{z} \delta^i_l \delta^j_k$
rational R-matrix
Yang's R-matrix
when N=2, this is related to the XXX spin chain.
Vertex models where the Boltzmann weights depend on a parameter, and satisfy YBE
are integrable: they can be solved exactly (can f

Assume that $R^{ij}_{kl}(z) = R^{ji}_{lk}(-z)$

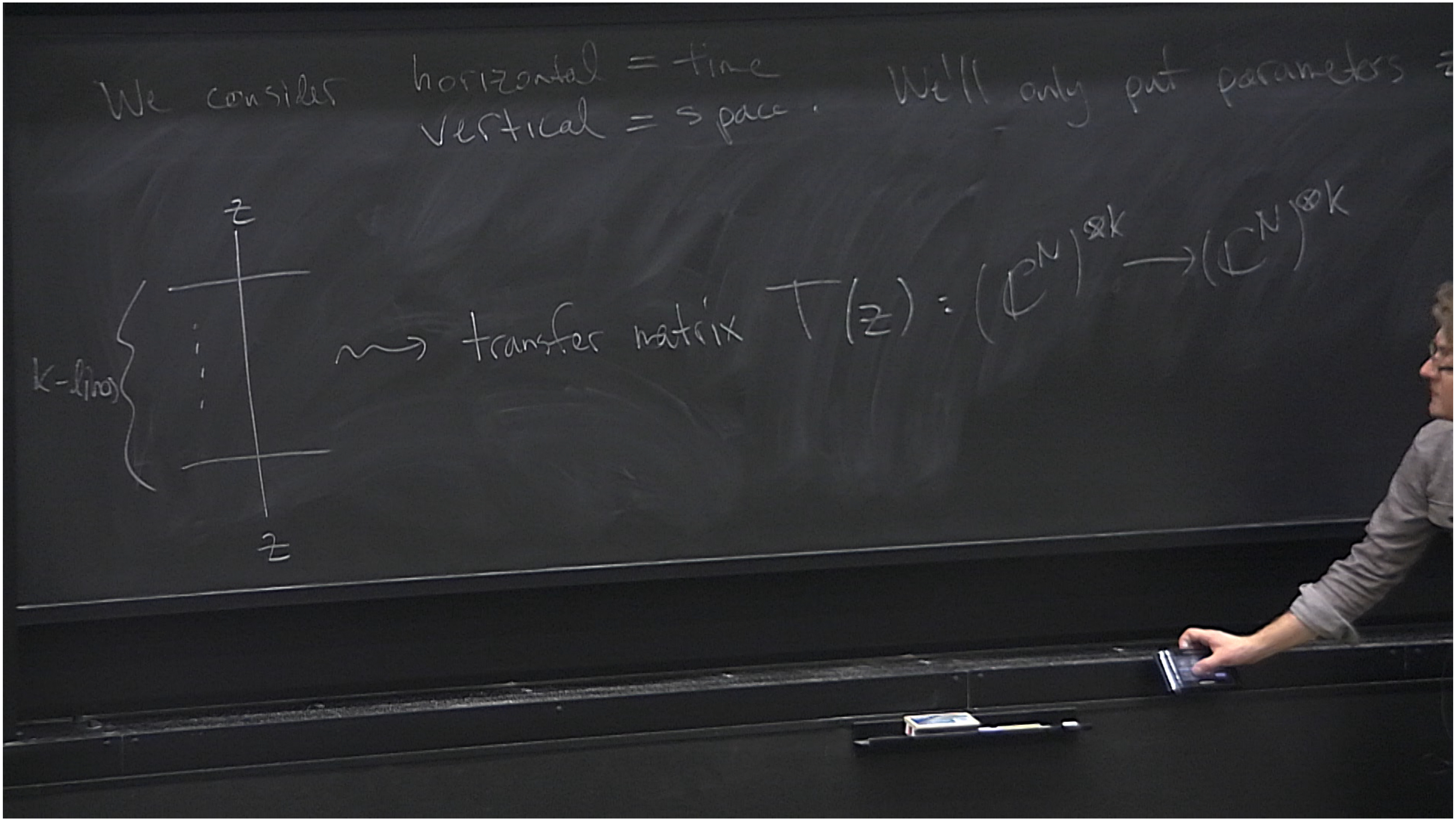
example : $R^{ij}_{kl}(z) = \delta^i_k \delta^j_l + \frac{1}{z} \delta^i_l \delta^j_k$ rational R-matrix / Yang's R-matrix

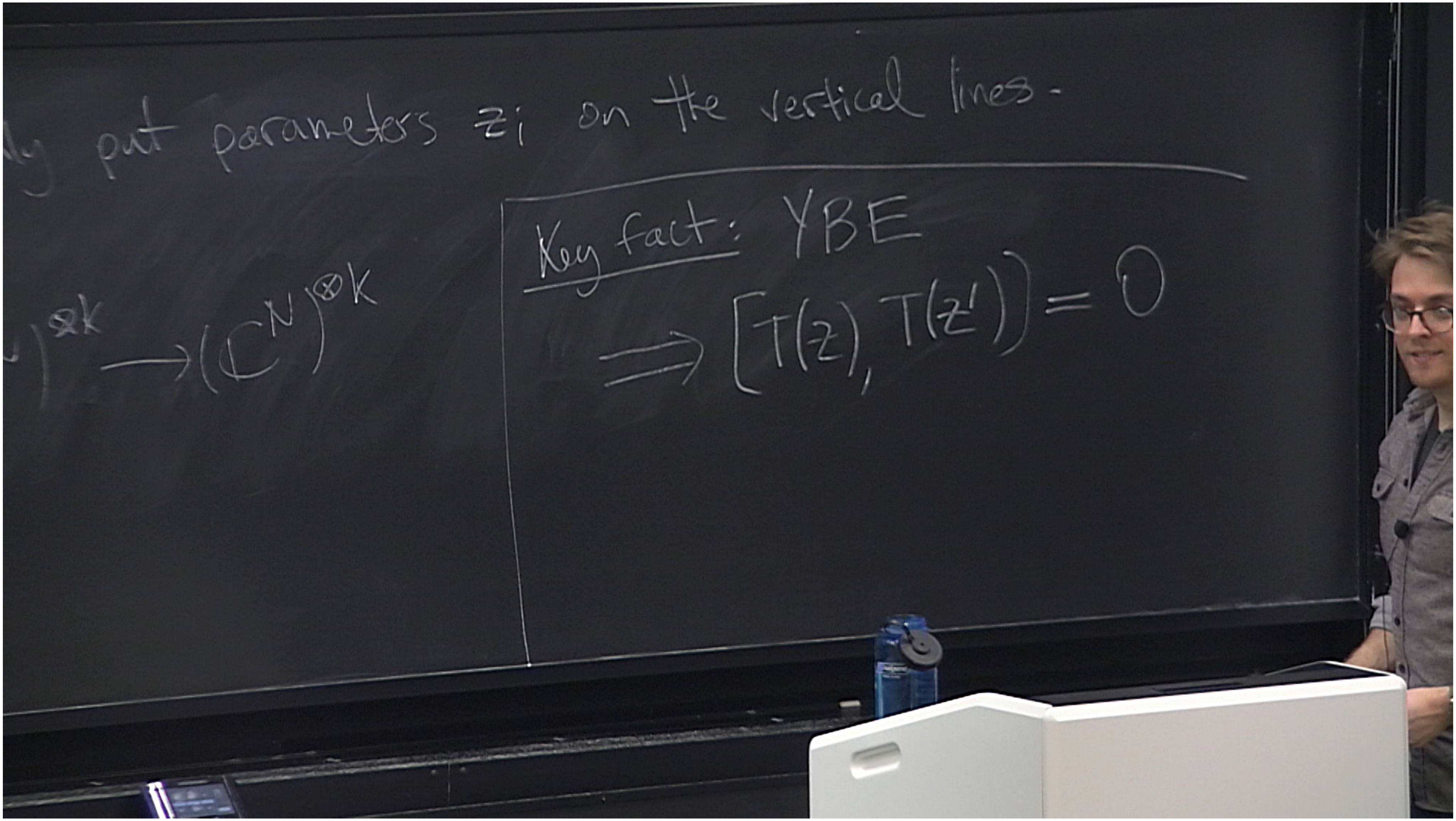
when $N=2$, this is related to the XXX spin chain.

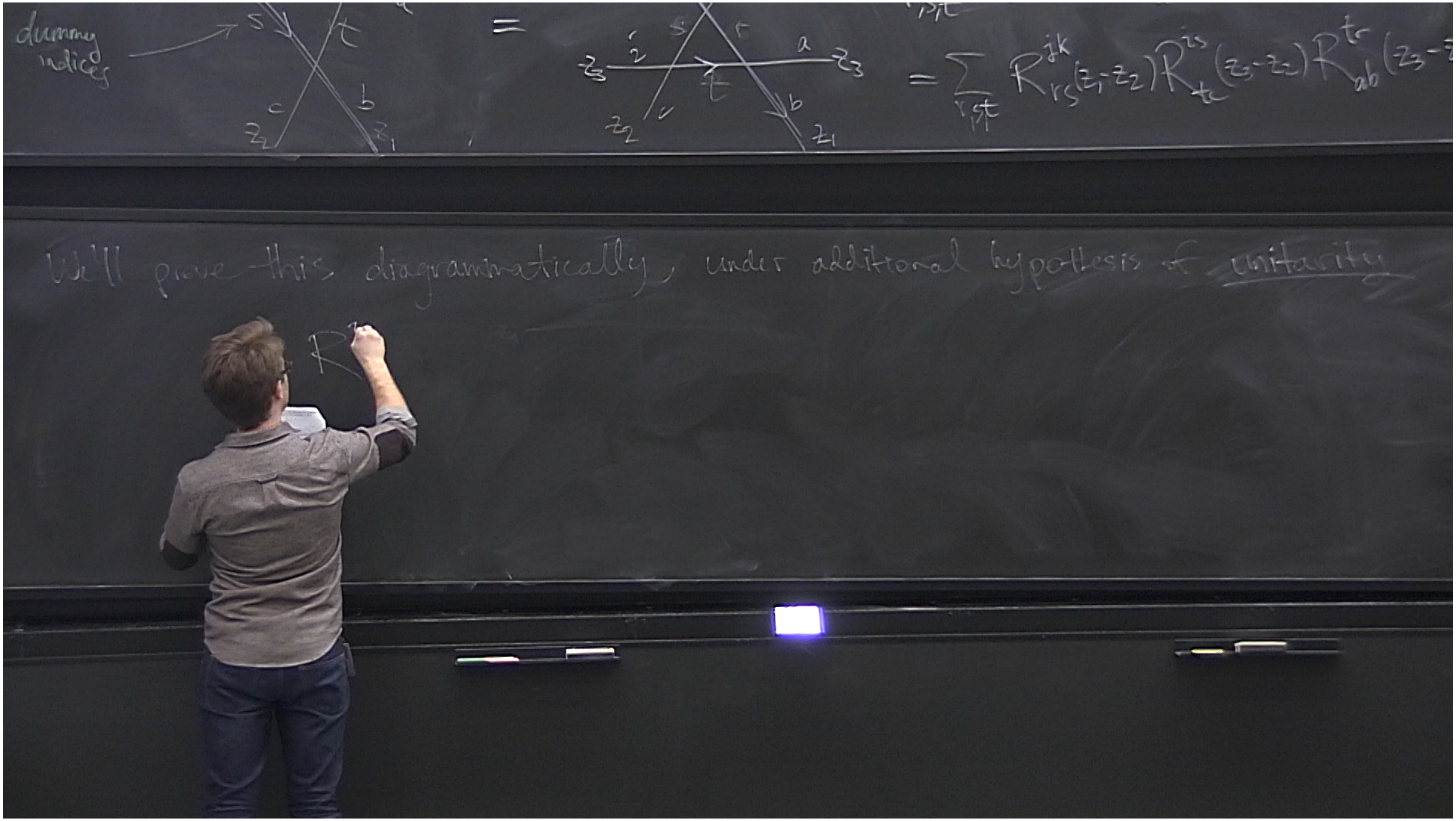
Vertex models where the Boltzmann weights depend on a parameter, and satisfy YBE are integrable: they can be solved exactly (can find infinitely many conserved currents)


We consider horizontal = time
vertical = space.
We'll only put

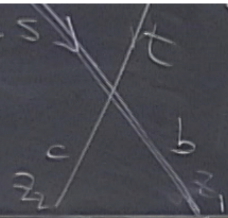
...ly put parameters z_i on the vertical lines.


We consider horizontal = time
vertical = space.
We'll only put parameters
z
k-lines
transfer matrix T(z) : (ℂ^N)^⊗k → (ℂ^N)^⊗k
z

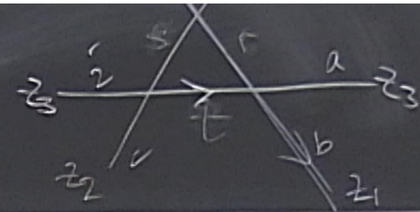

put parameters z_i on the vertical lines.
Key fact: YBE
$\Longrightarrow [T(z), T(z')] = 0$
$)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$


dummy indices
We'll prove this diagrammatically, under additional hypothesis of unitarity

dummy indices



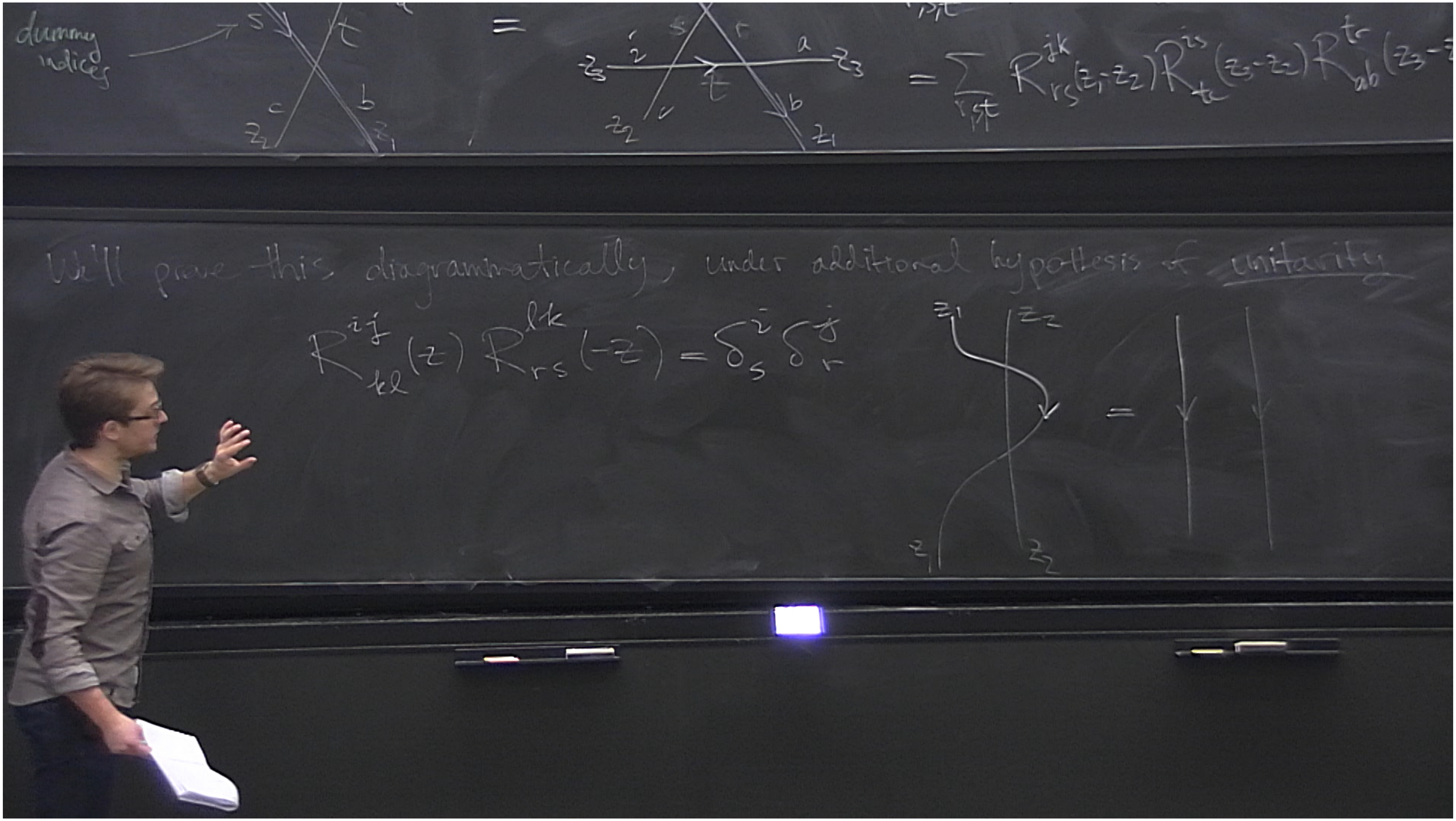
=

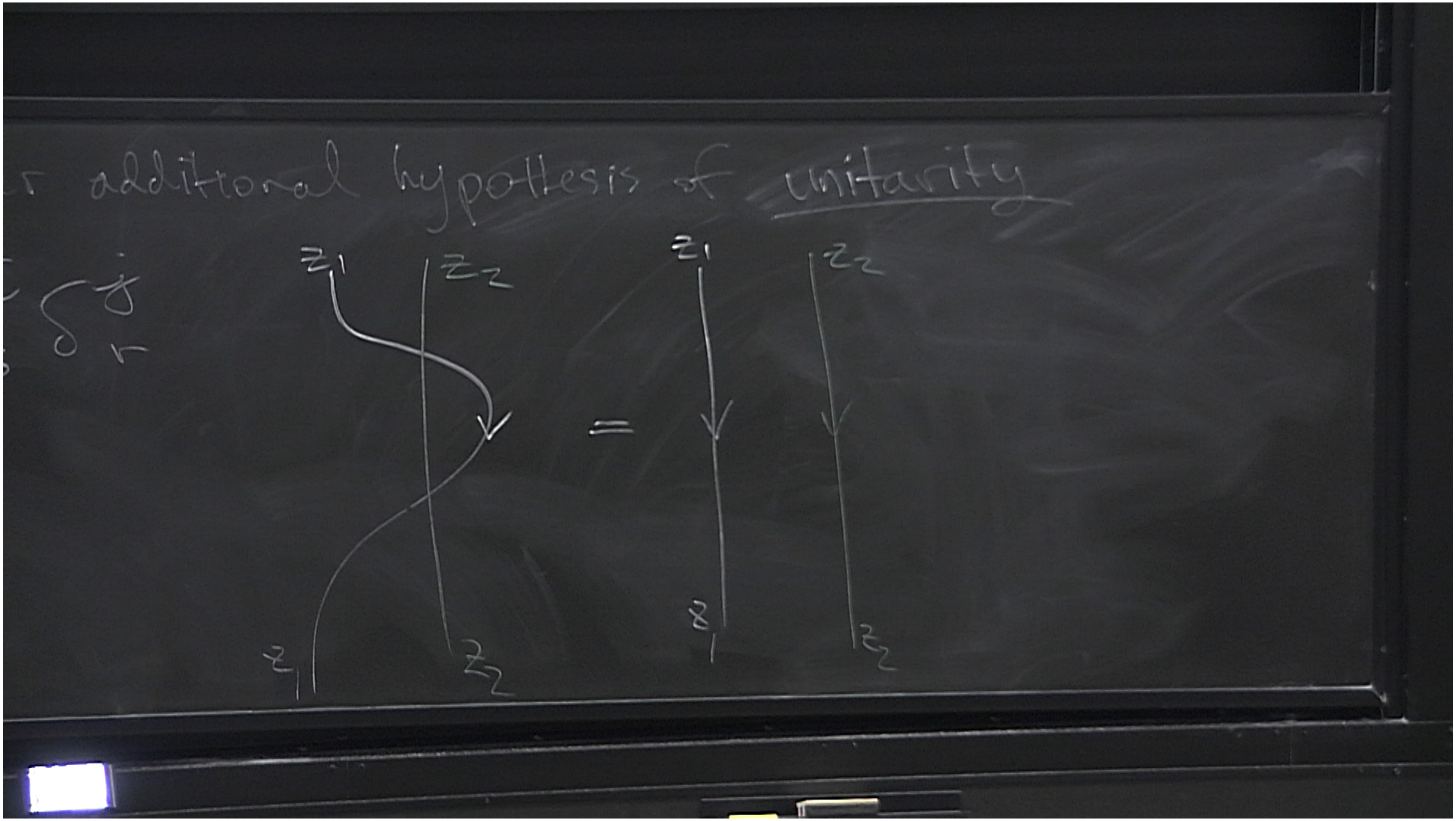


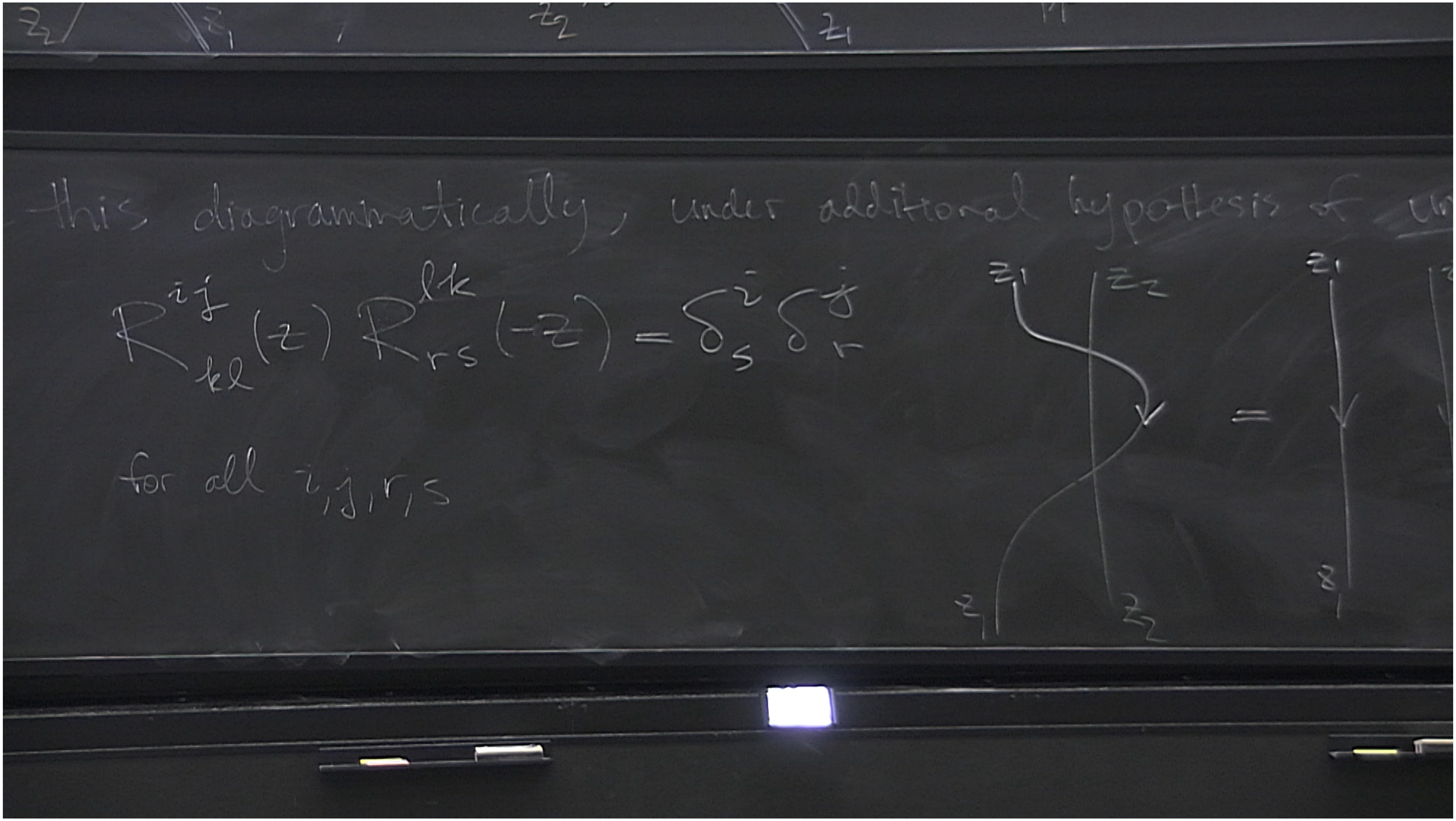
$= \sum_{r,s,t} R^{jk}_{rs}(z_1-z_2) R^{is}_{tc}(z_3-z_2) R^{tr}_{ab}(z_3-$

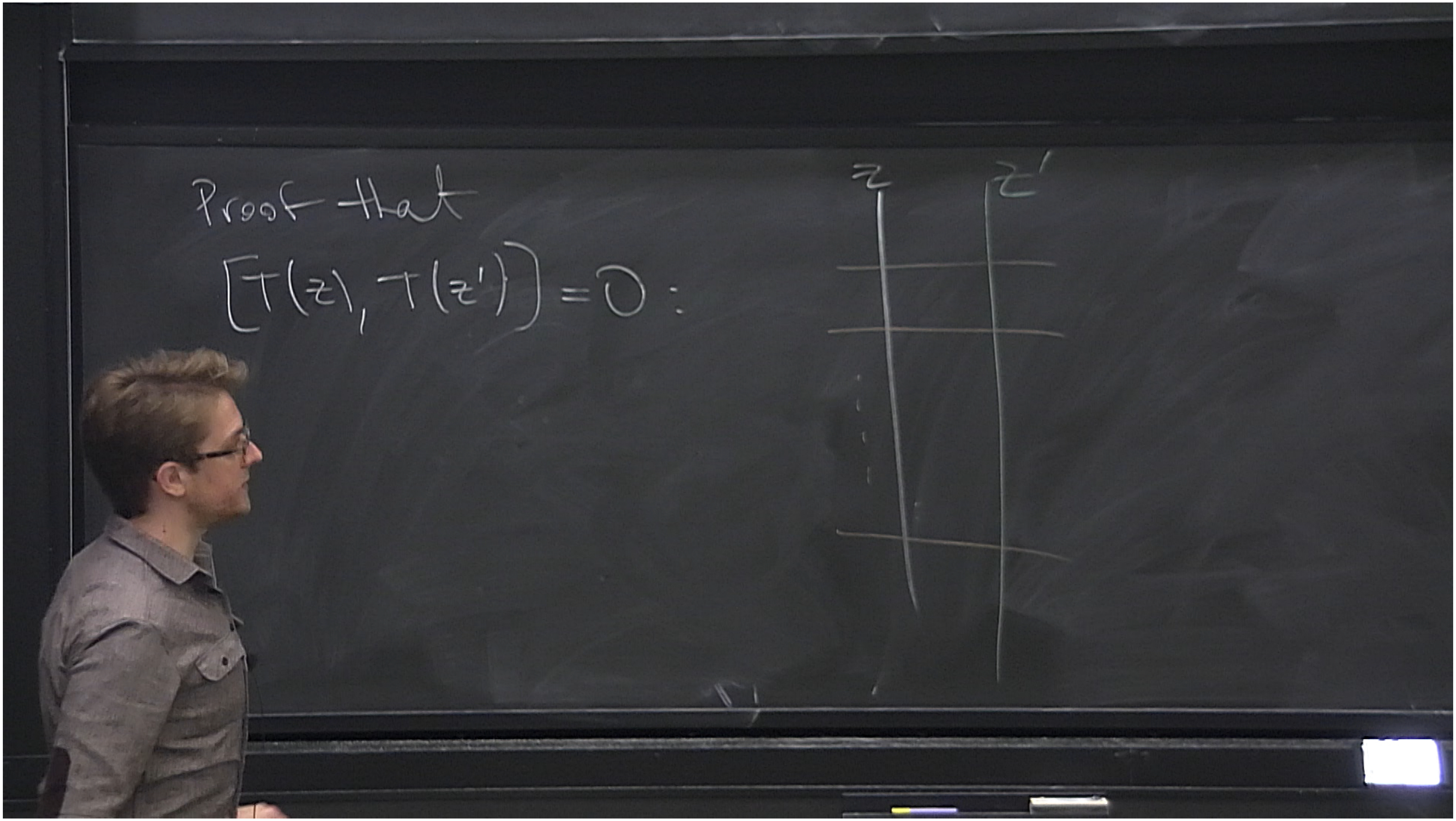
We'll prove this diagrammatically, under additional hypothesis of unitarity

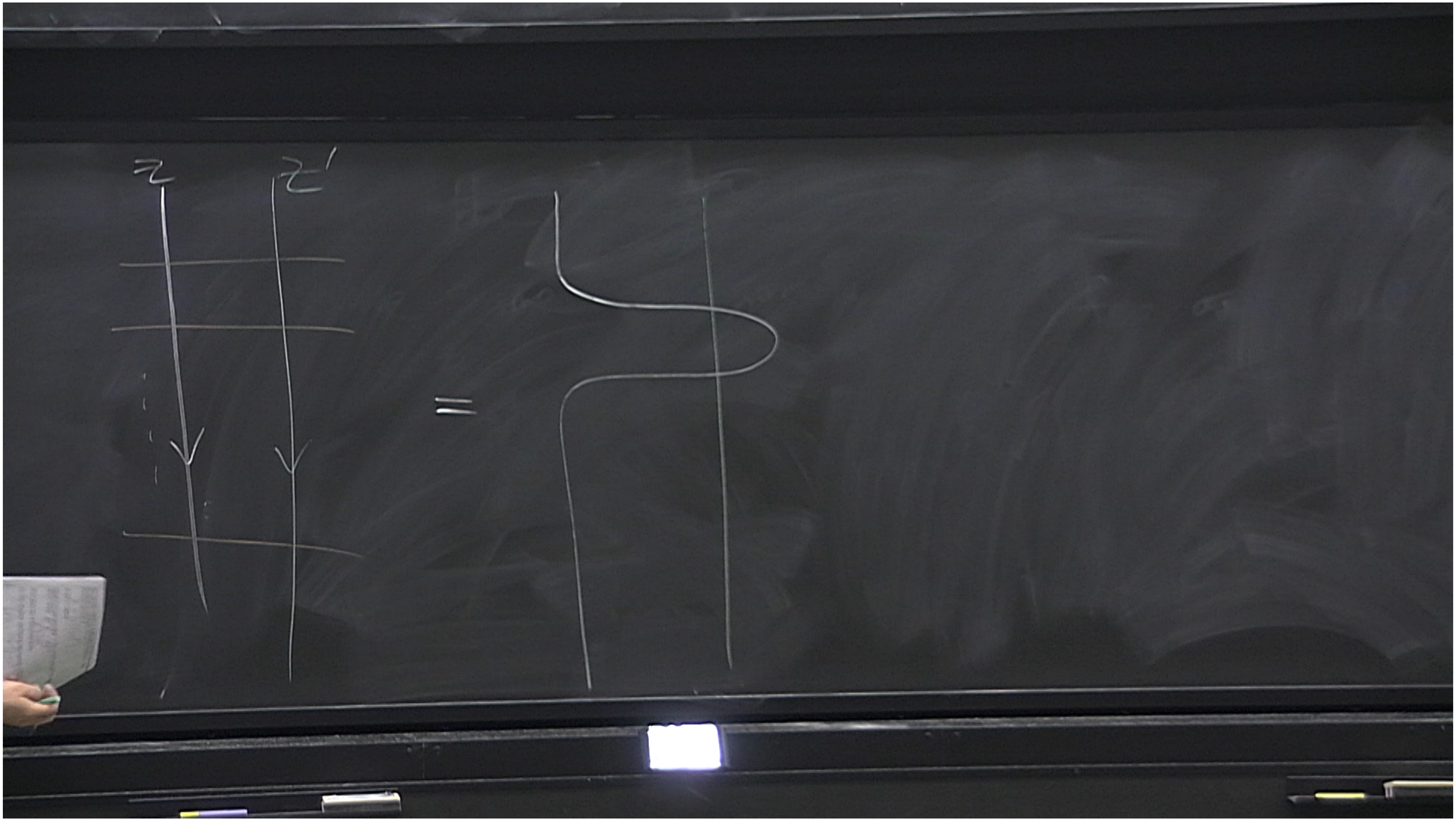
$$R^{ij}_{kl}(z)\, R^{lk}_{rs}(-z) = \delta^i_s \delta^j_r$$

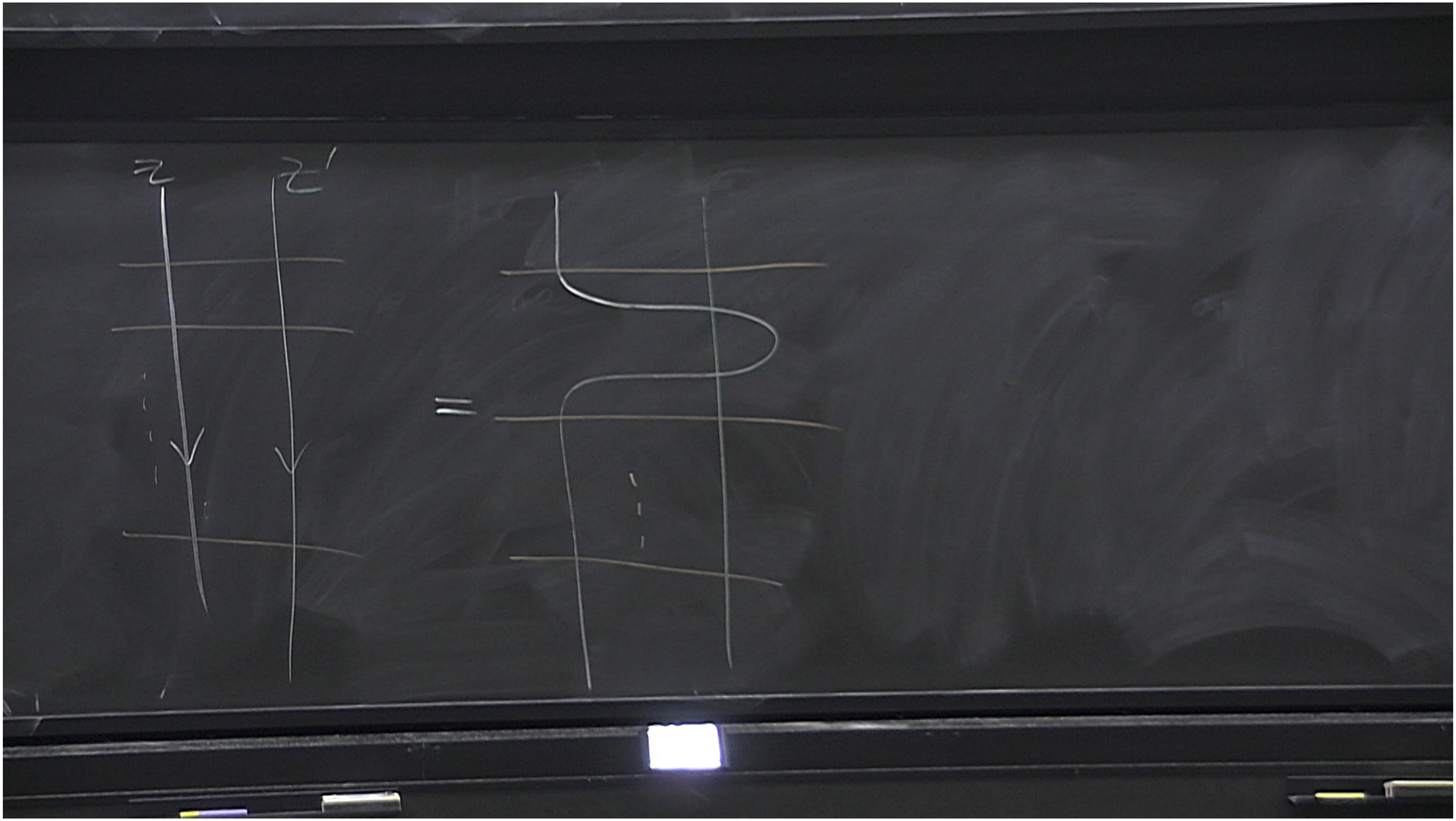

dummy indices
$= \sum_{r,s,t} R^{jk}_{rs}(z_1 - z_2) R^{is}_{tc}(z_3 - z_2) R^{tr}_{ab}(z_3 -$
We'll prove this diagrammatically, under additional hypothesis of unitarity
$R^{ij}_{kl}(z) R^{lk}_{rs}(-z) = \delta^i_s \delta^j_r$

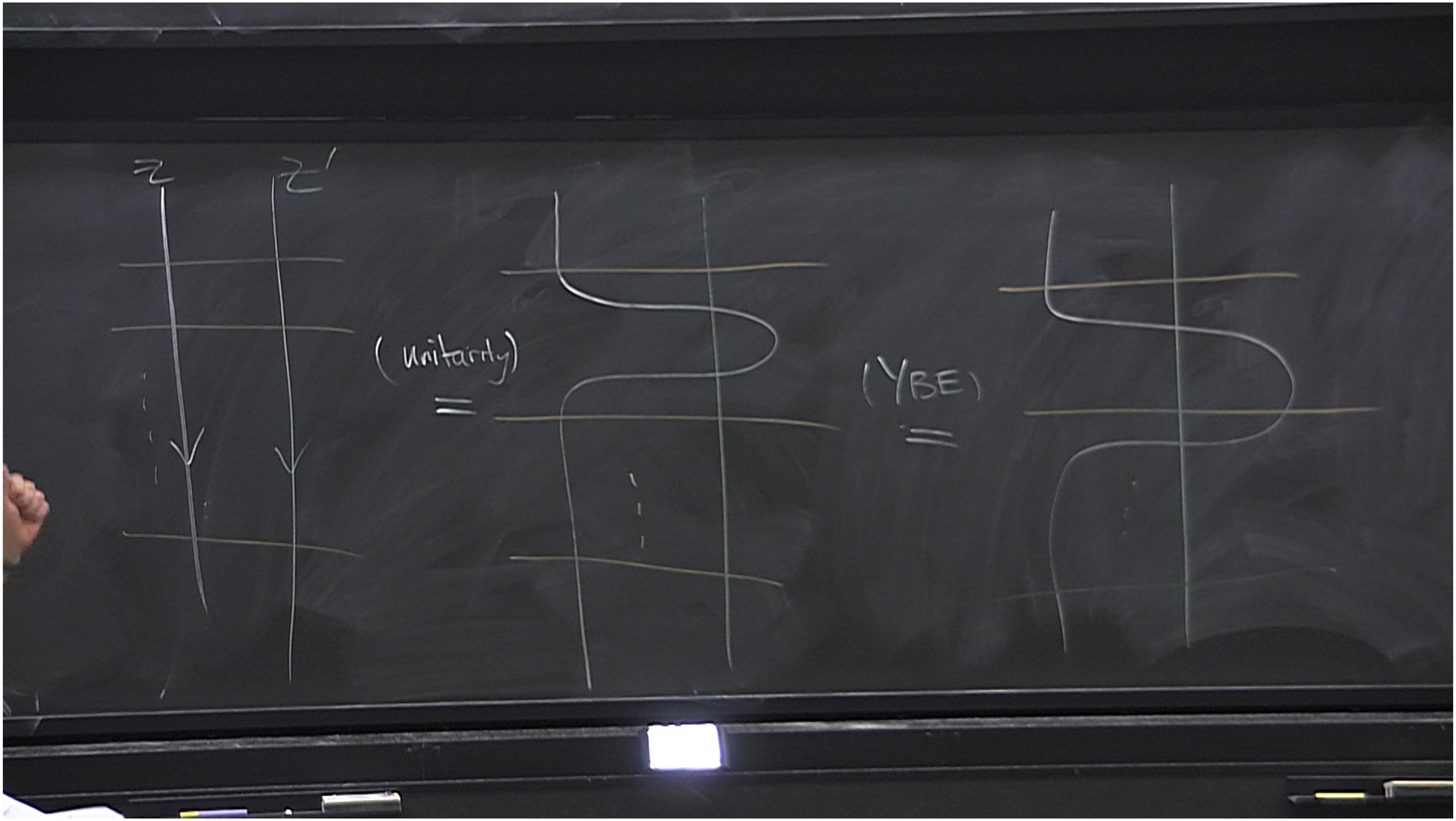

additional hypothesis of unitarity
z1
z2
=
z1
z2
z1
z2
z1
z2

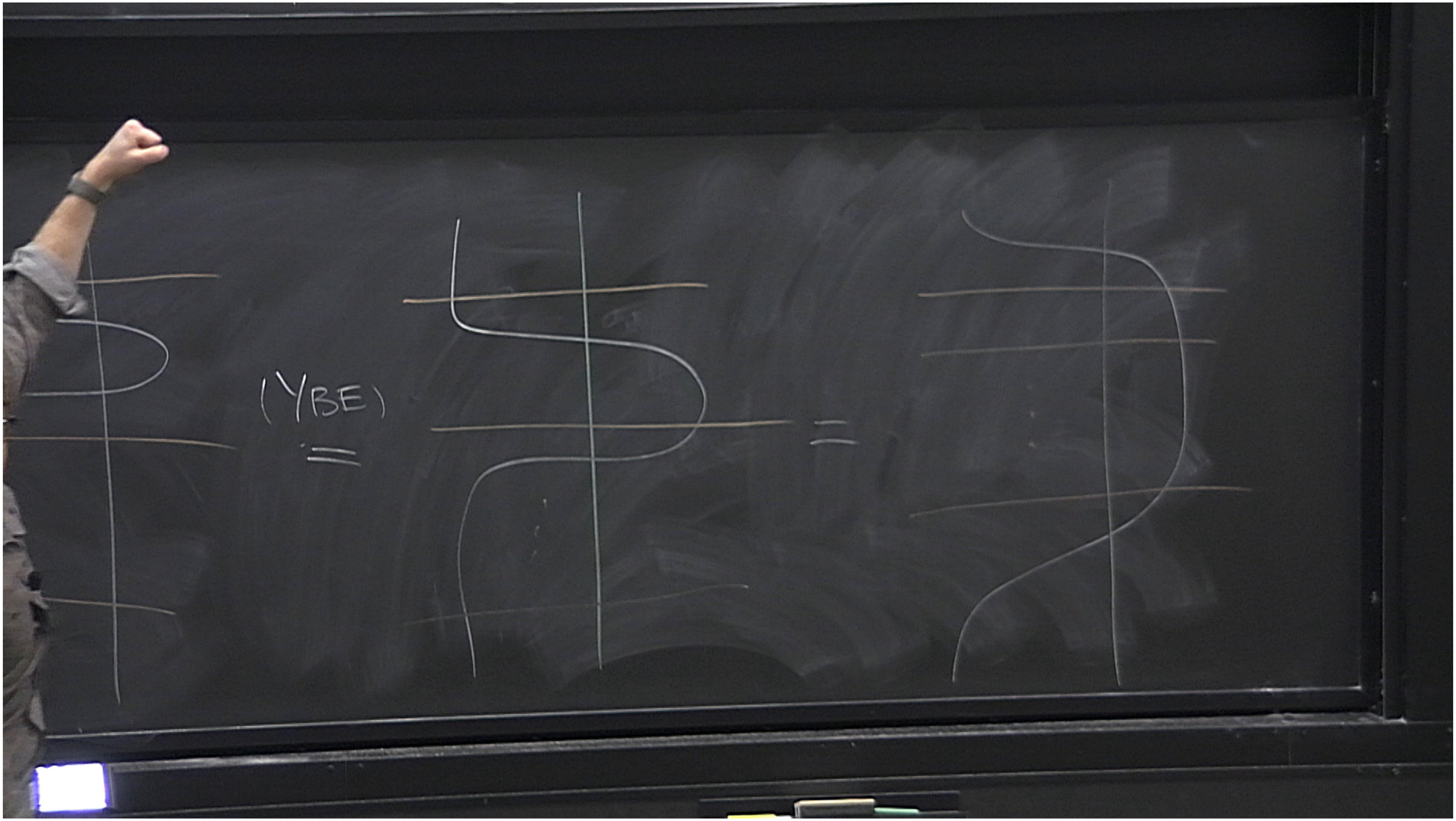

this diagrammatically, under additional hypothesis of un
$R^{ij}_{kl}(z)\, R^{lk}_{rs}(-z) = \delta^{i}_{s}\, \delta^{j}_{r}$
for all i, j, r, s
z_1
z_2
=

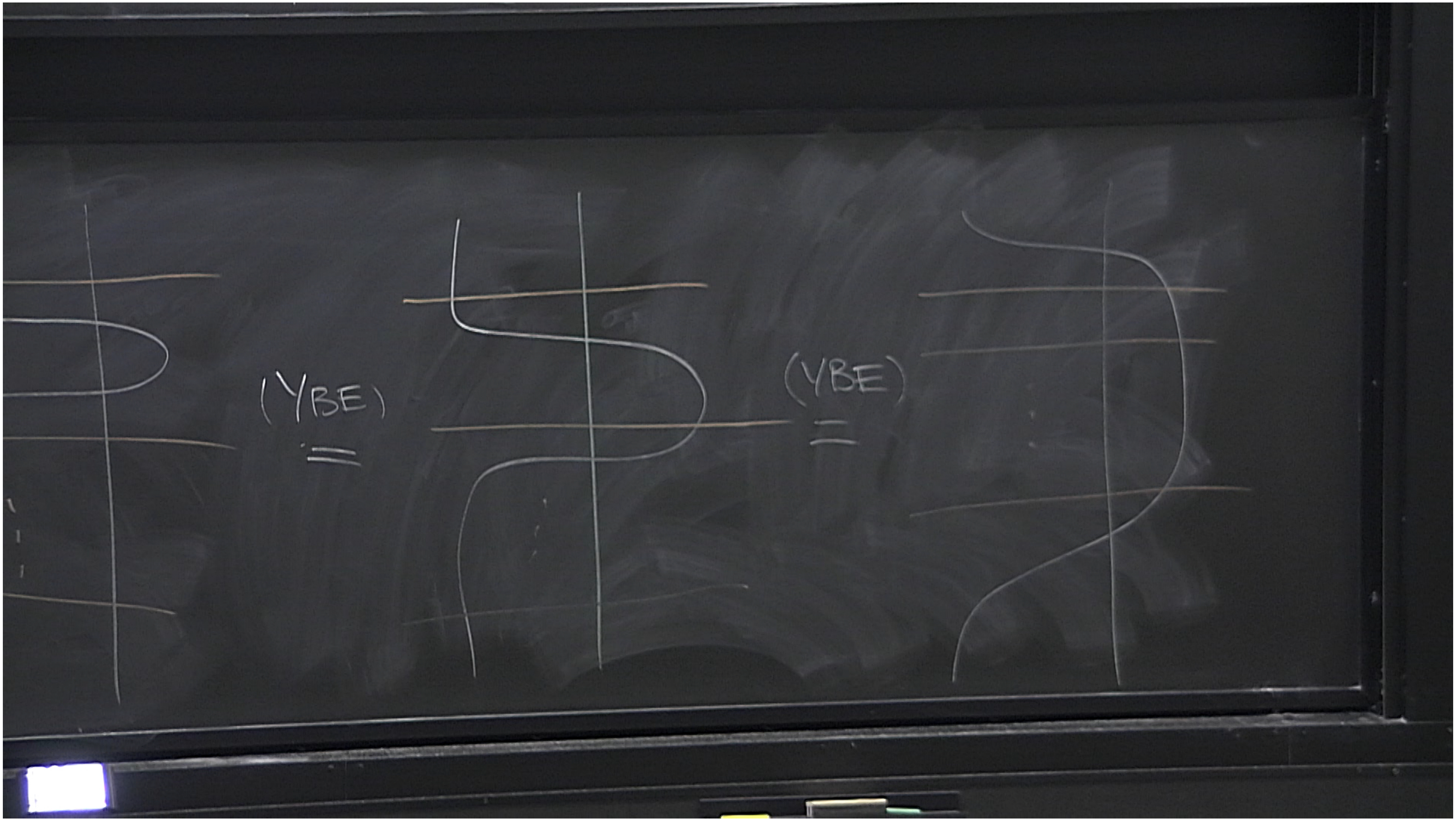

Proof that
[T(z), T(z')] = 0 :
z
z'

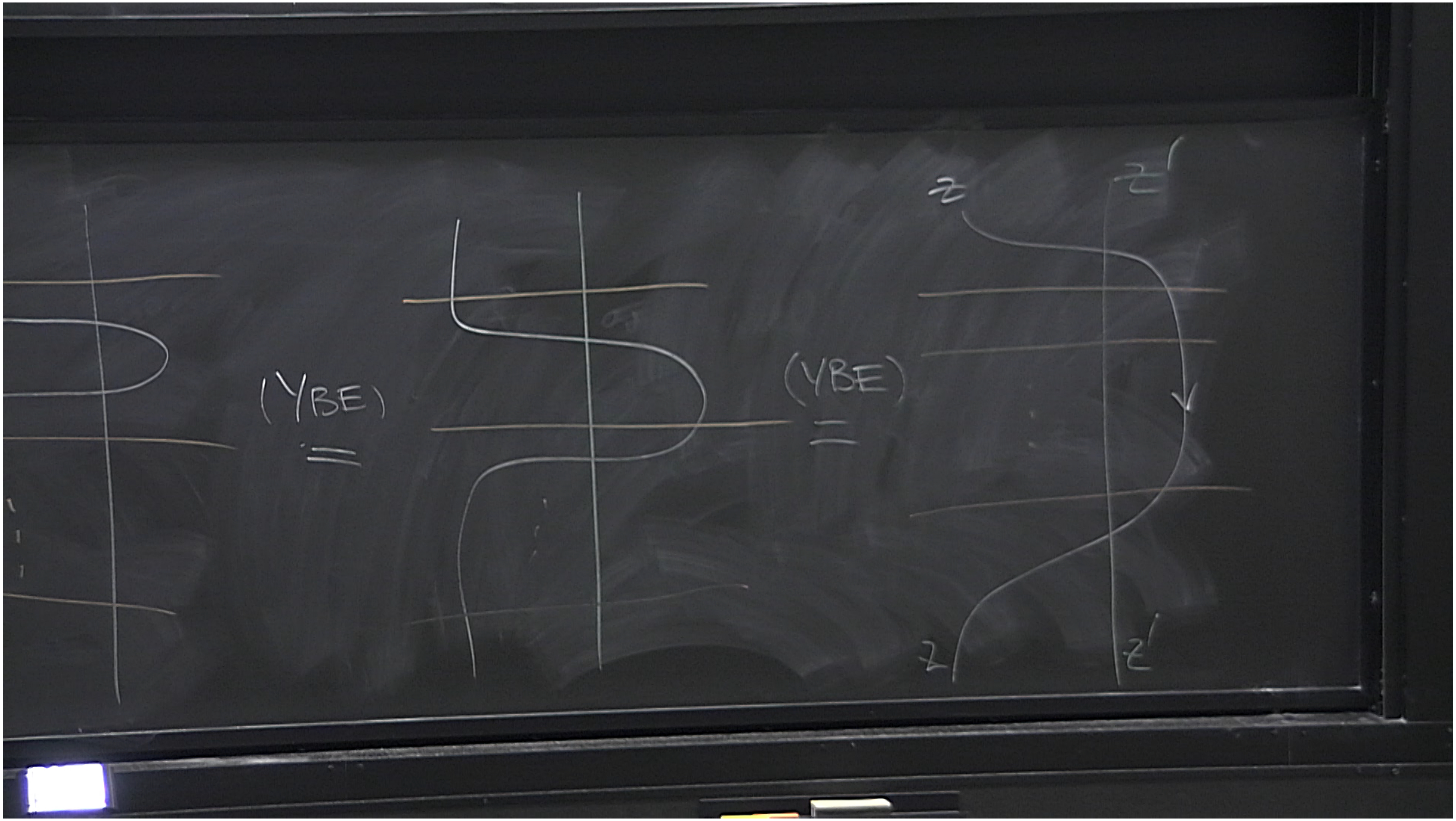

z
z'
=


z
z'
=

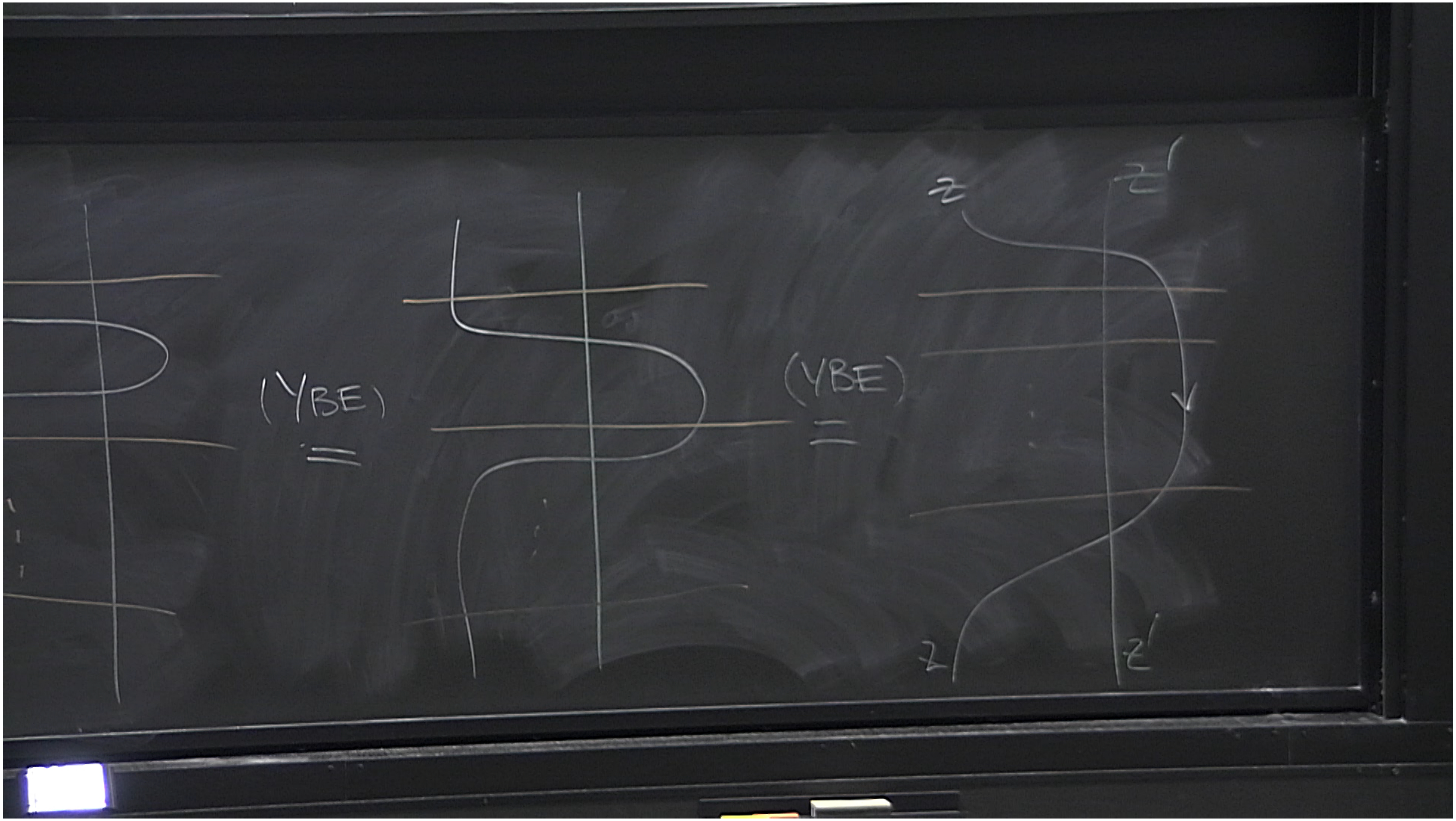

z
z'
(unitarity)
=
(YBE)
=

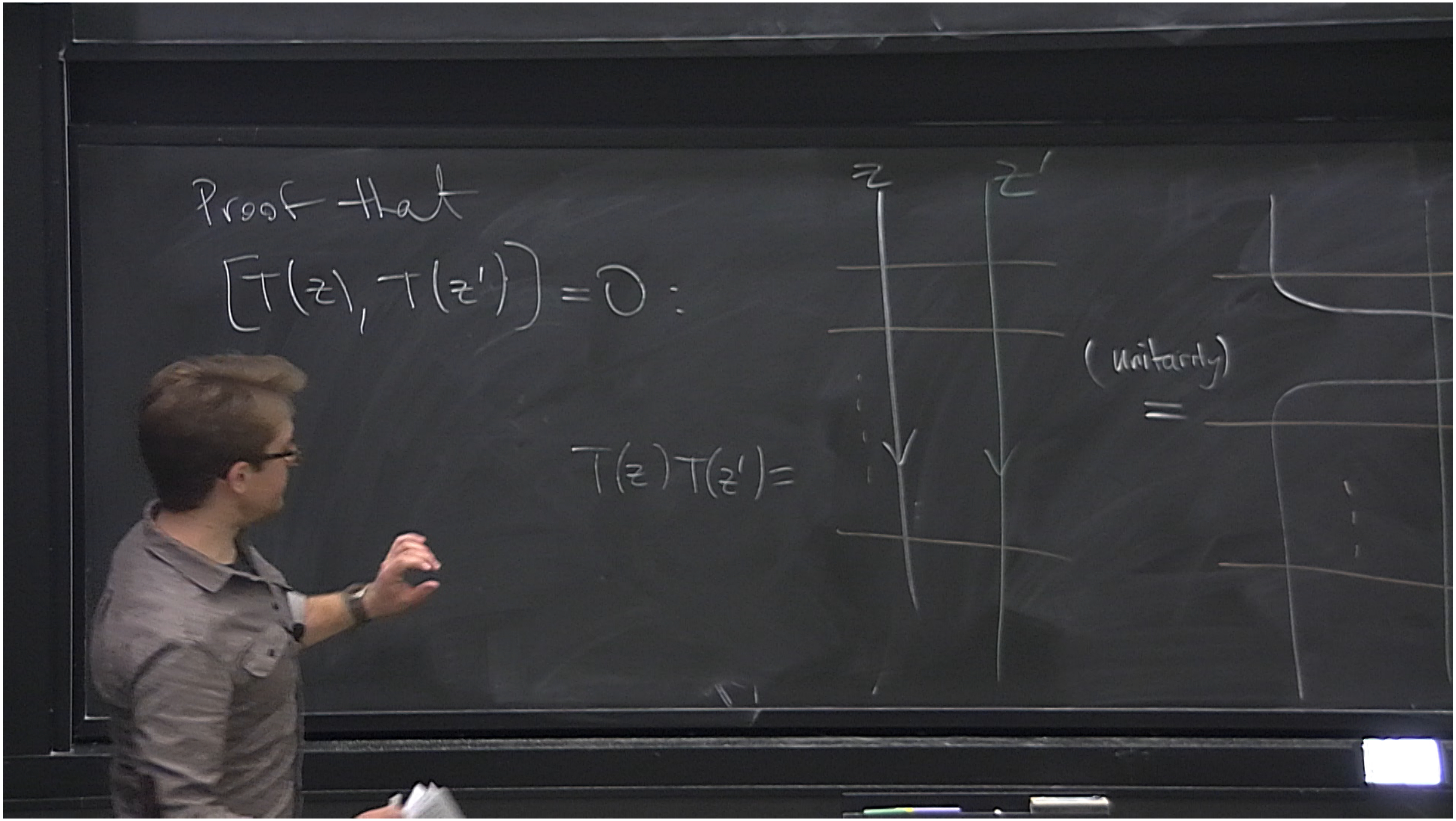

(YBE)
=
=

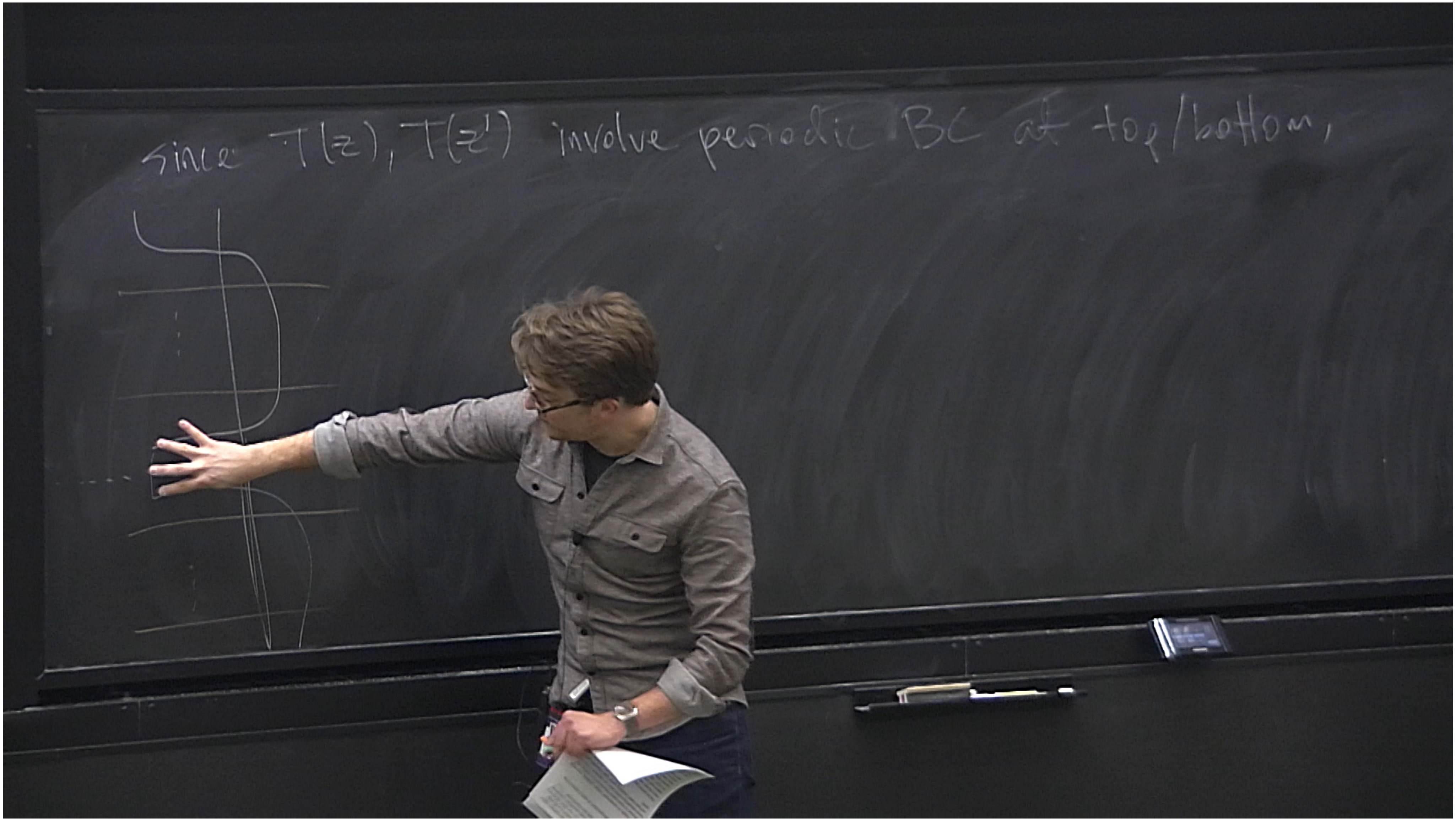

(YBE)
=
(YBE)
=

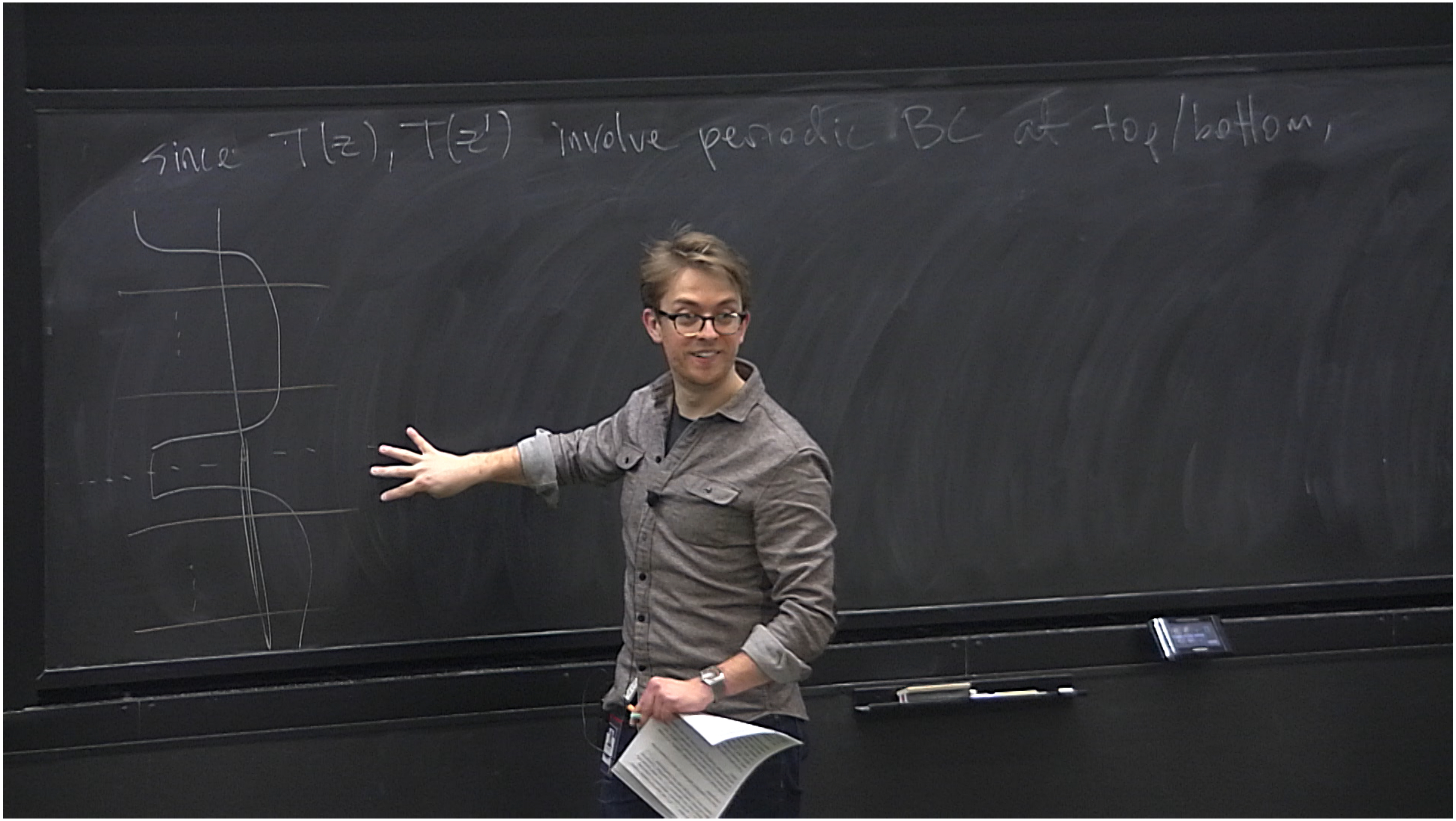


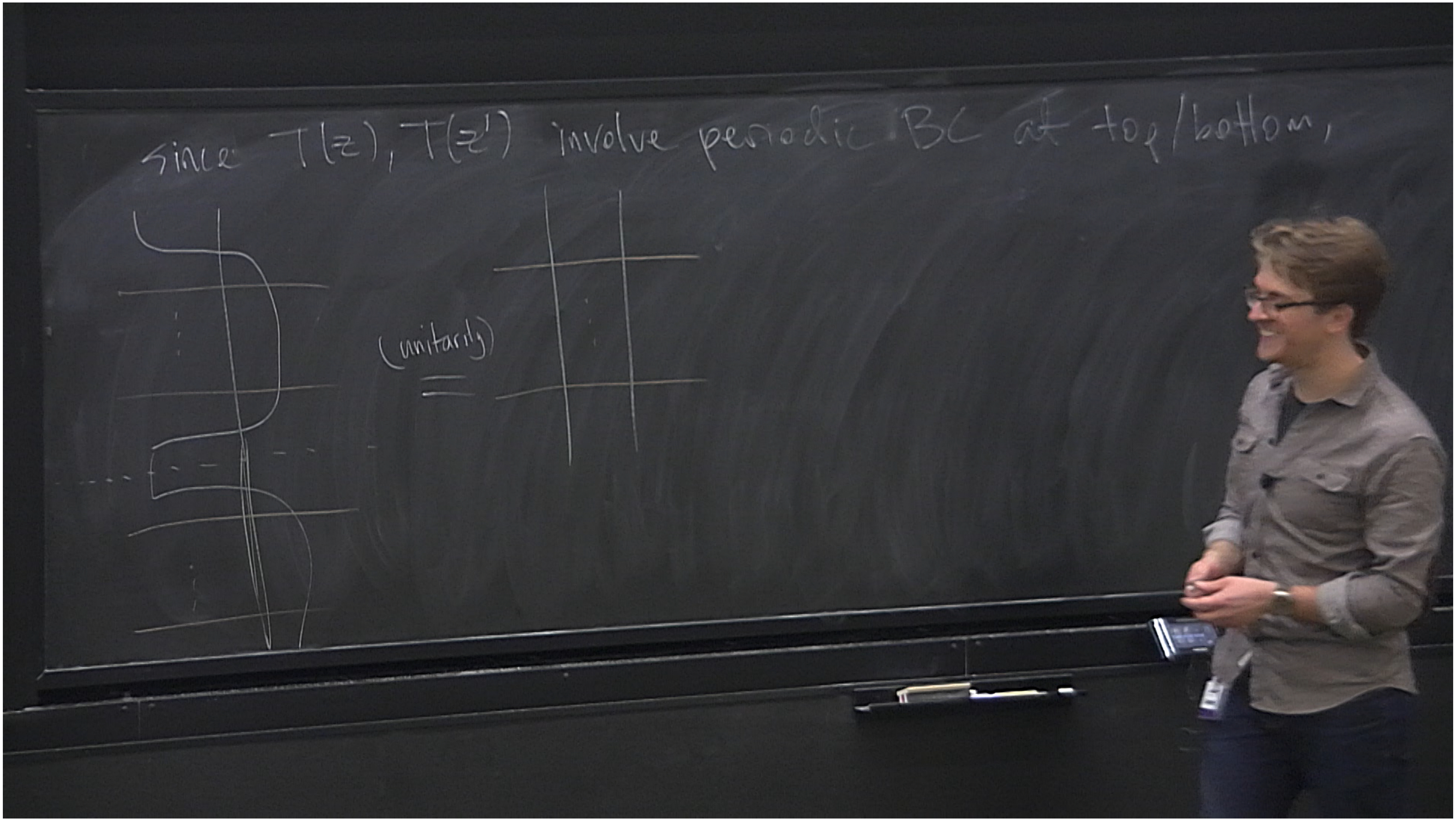
(YBE)
=
(YBE)
=
z
z'
z
z'

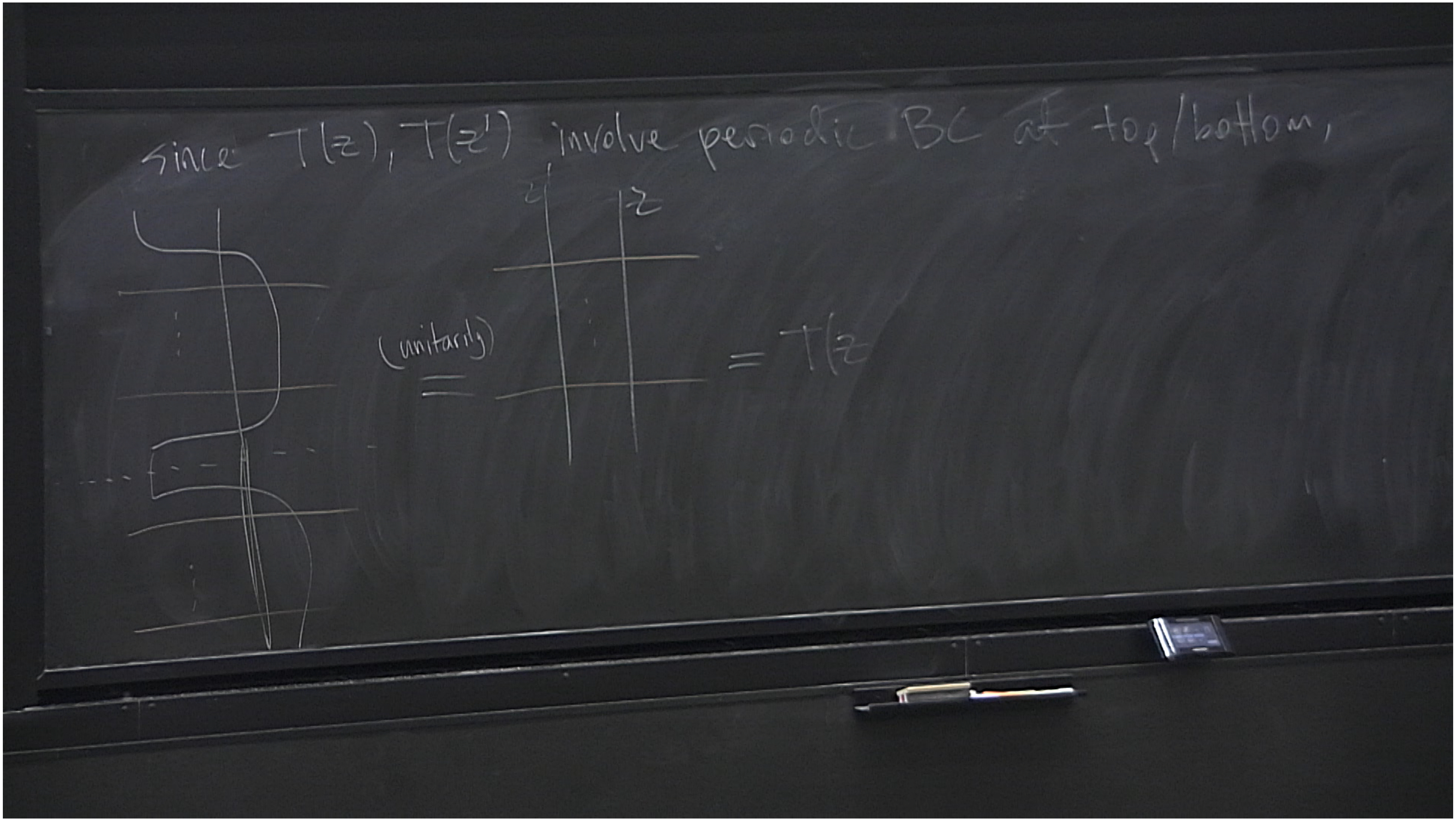

(YBE)
=
(YBE)
=
z
z'
z
z'


Proof that
[T(z), T(z')] = 0 :
T(z) T(z') =
z
z'
(unitarity)
=


since T(z), T(z') involve periodic BC at top/bottom,


since T(z), T(z') involve periodic BC at top/bottom,

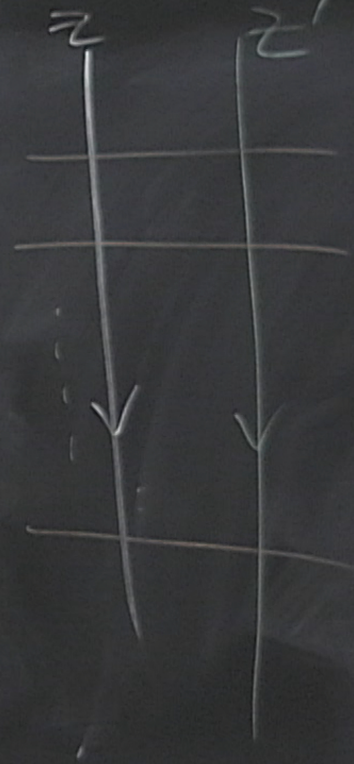

since T(z), T(z') involve periodic BC at top/bottom,
(unitarity)
=


since T(z), T(z') involve periodic BC at top/bottom,
z'
z
(unitarily)
=
= T(z

Proof that

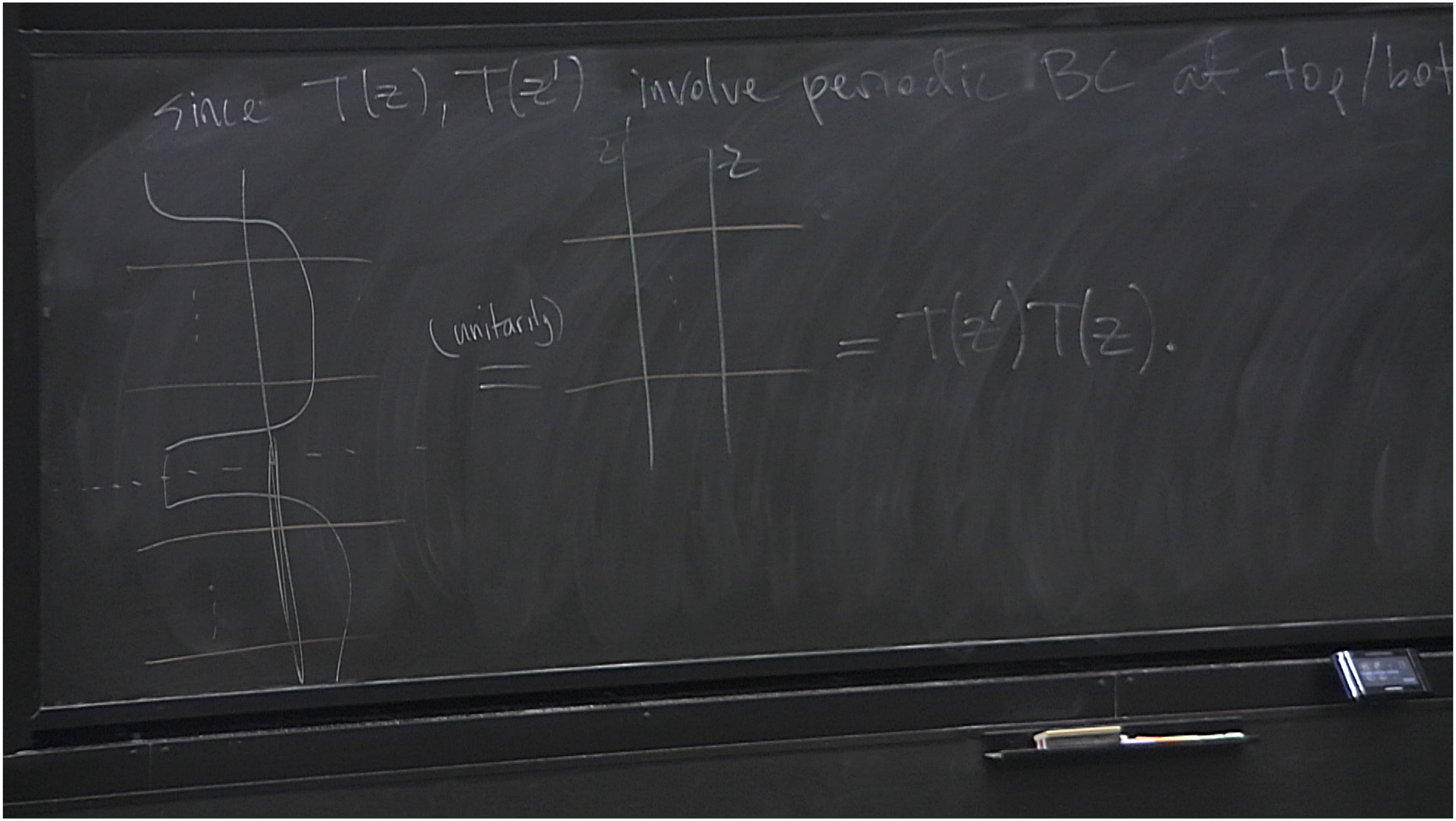
$$[T(z), T(z')] = 0:$$

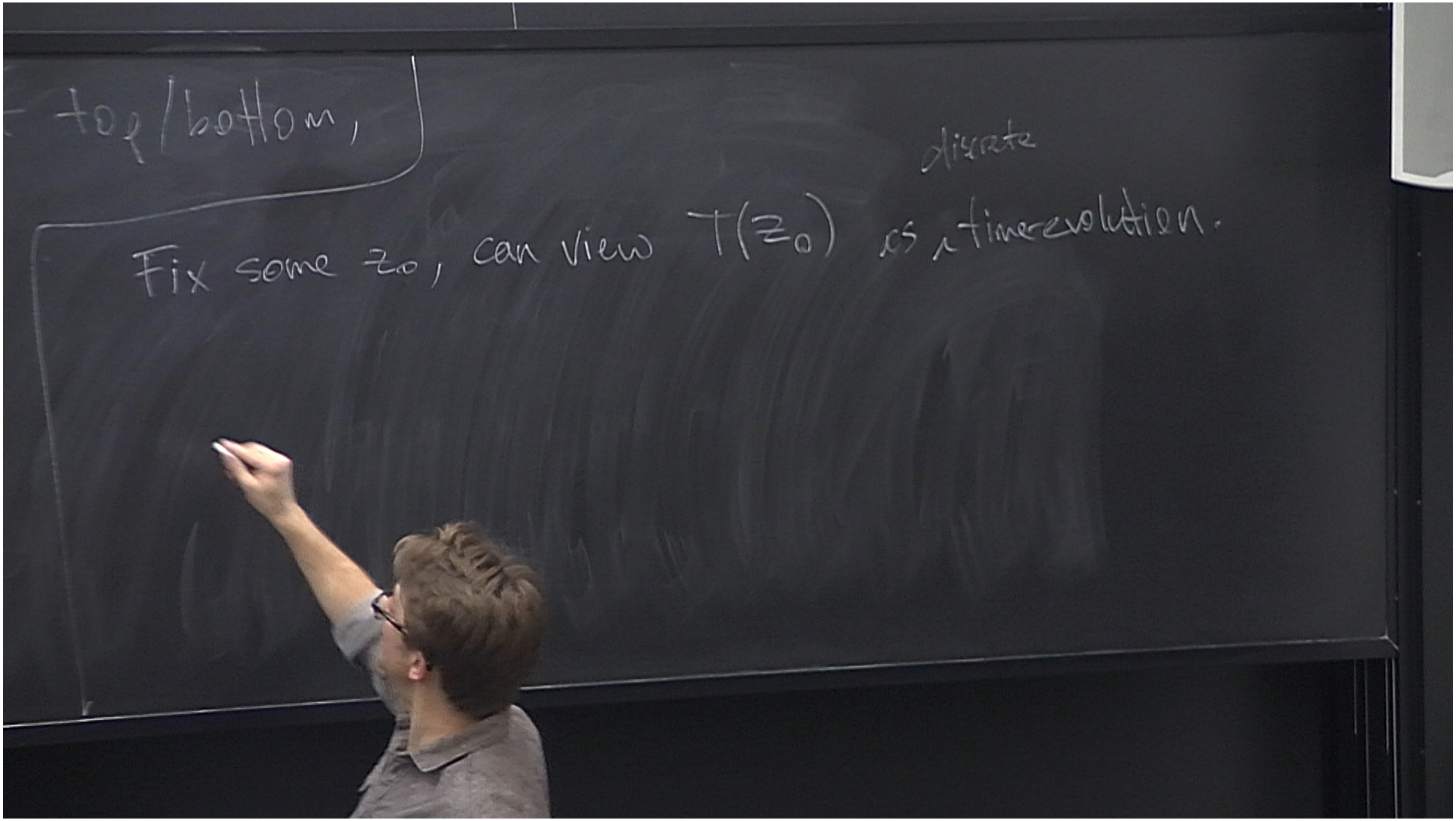
$$T(z')\,T(z) =$$

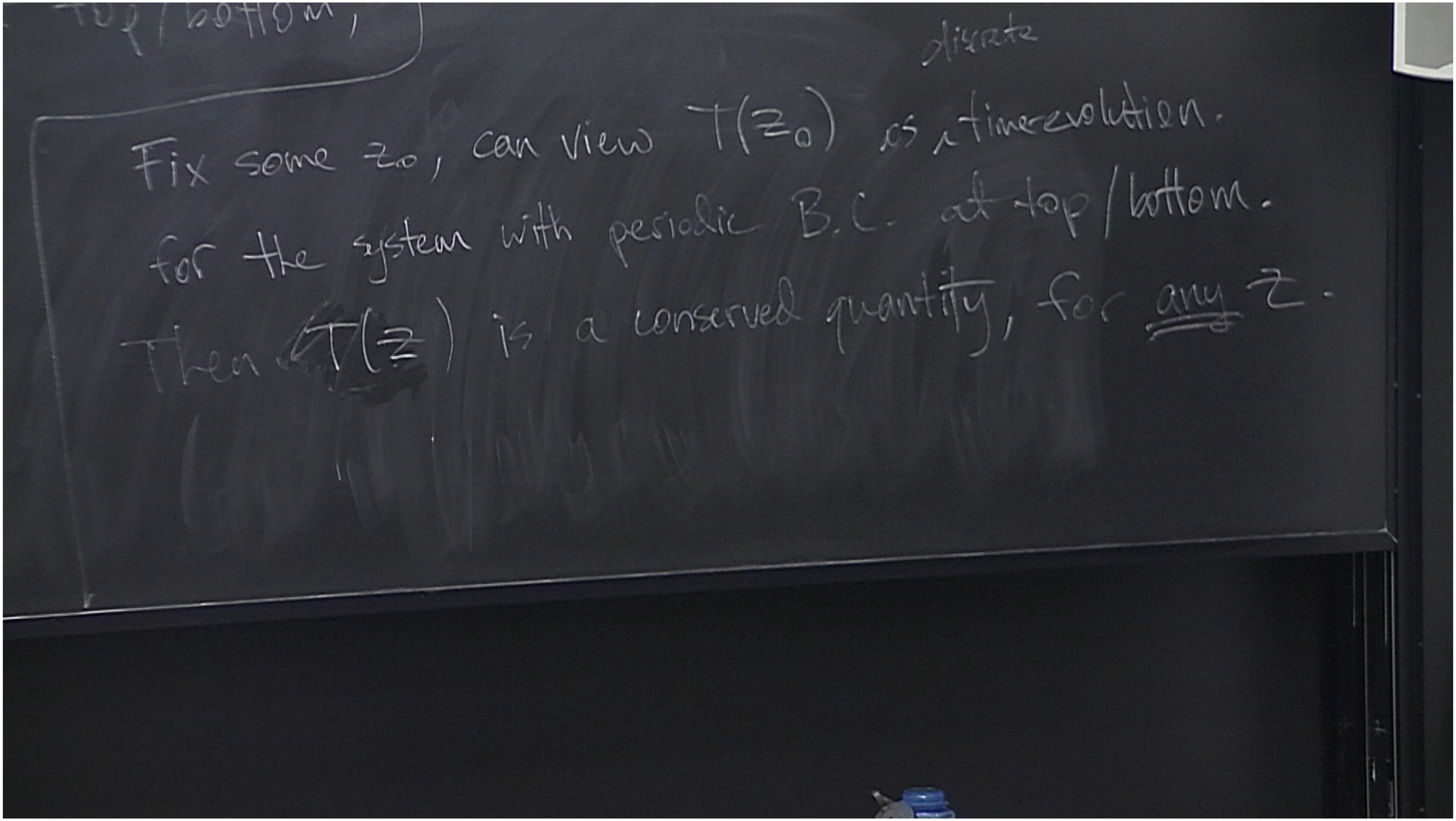


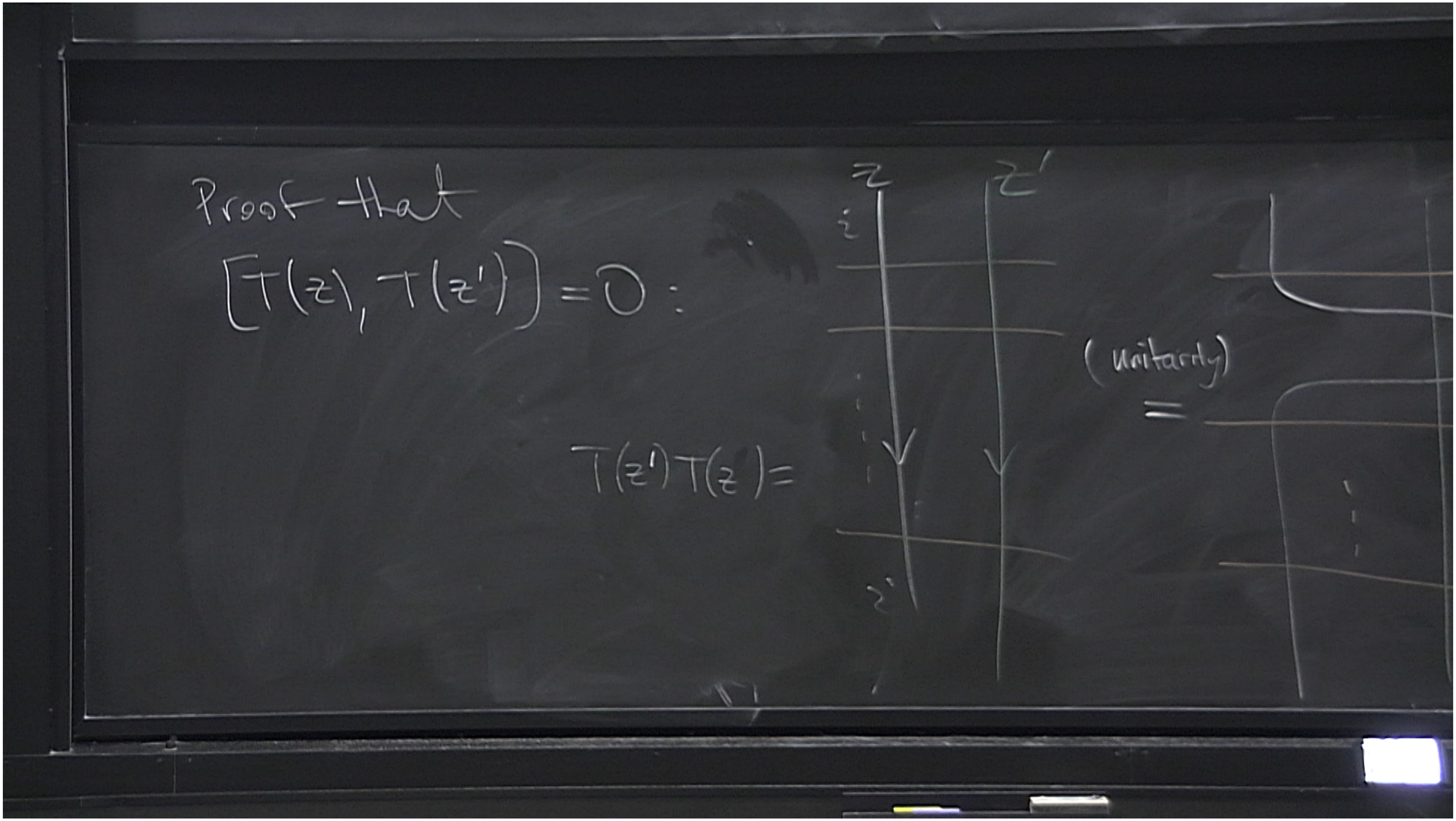
(unitarity)

$$=$$


since T(z), T(z') involve periodic BC at top/bot
z'
z
(unitarily)
=
= T(z')T(z).


top/bottom,
discrete
Fix some z_0, can view $T(z_0)$ as time evolution.

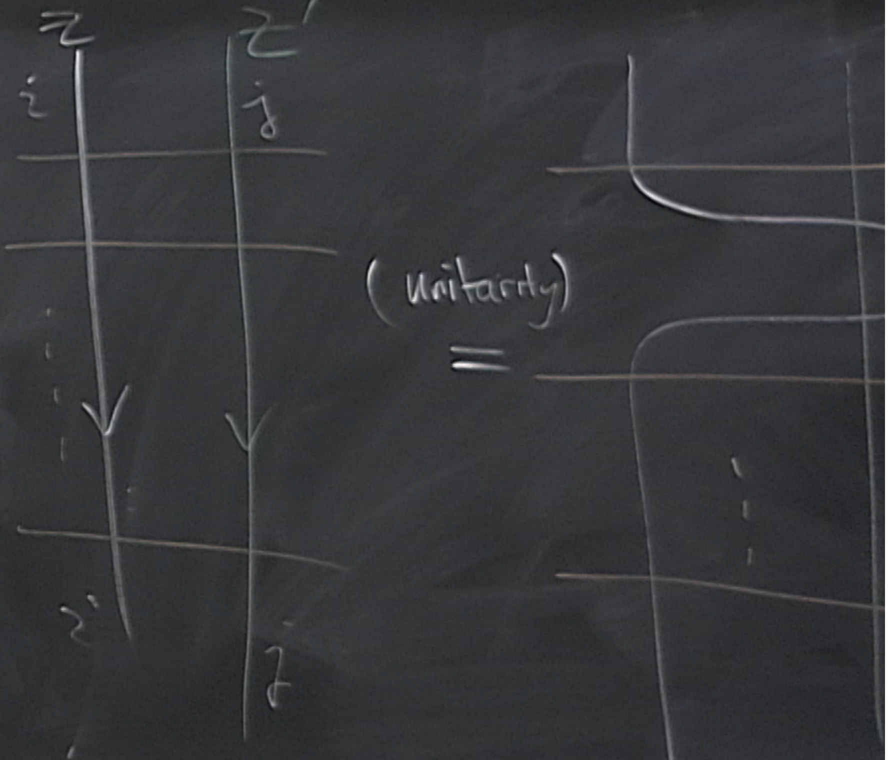

top/bottom,
discrete
Fix some z_0, can view $T(z_0)$ as a time evolution.
for the system with periodic B.C. at top/bottom.
Then $T(z)$ is a conserved quantity, for any z.

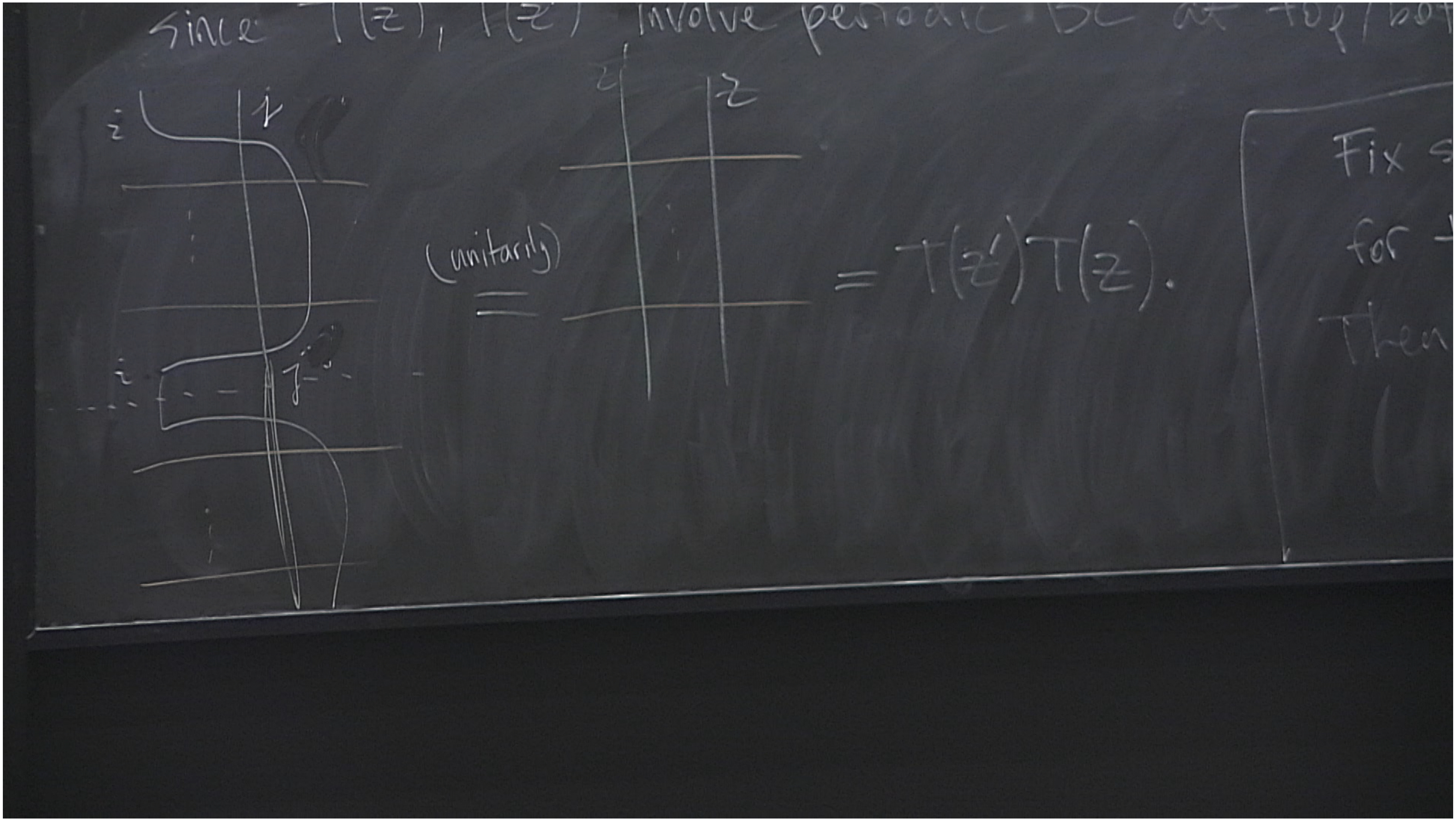

Proof that
$[T(z), T(z')] = 0$:
(unitarity)
$T(z')T(z) =$
z
z'

Proof that

$$[T(z), T(z')] = 0 :$$

$$T(z')\,T(z) =$$




Since T(z), T(z') involve periodic BC at
(unitarity)
= T(z') T(z).
z'
z
i
j
Fix
for
Then