Title: PSI 2017/2018 - Quantum Integrable Models - Lecture 1

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Abstract:

Vertex models of Statistical mechanics

A <u>vertex model</u> is a classical statistical model which lives on a planar square lattice

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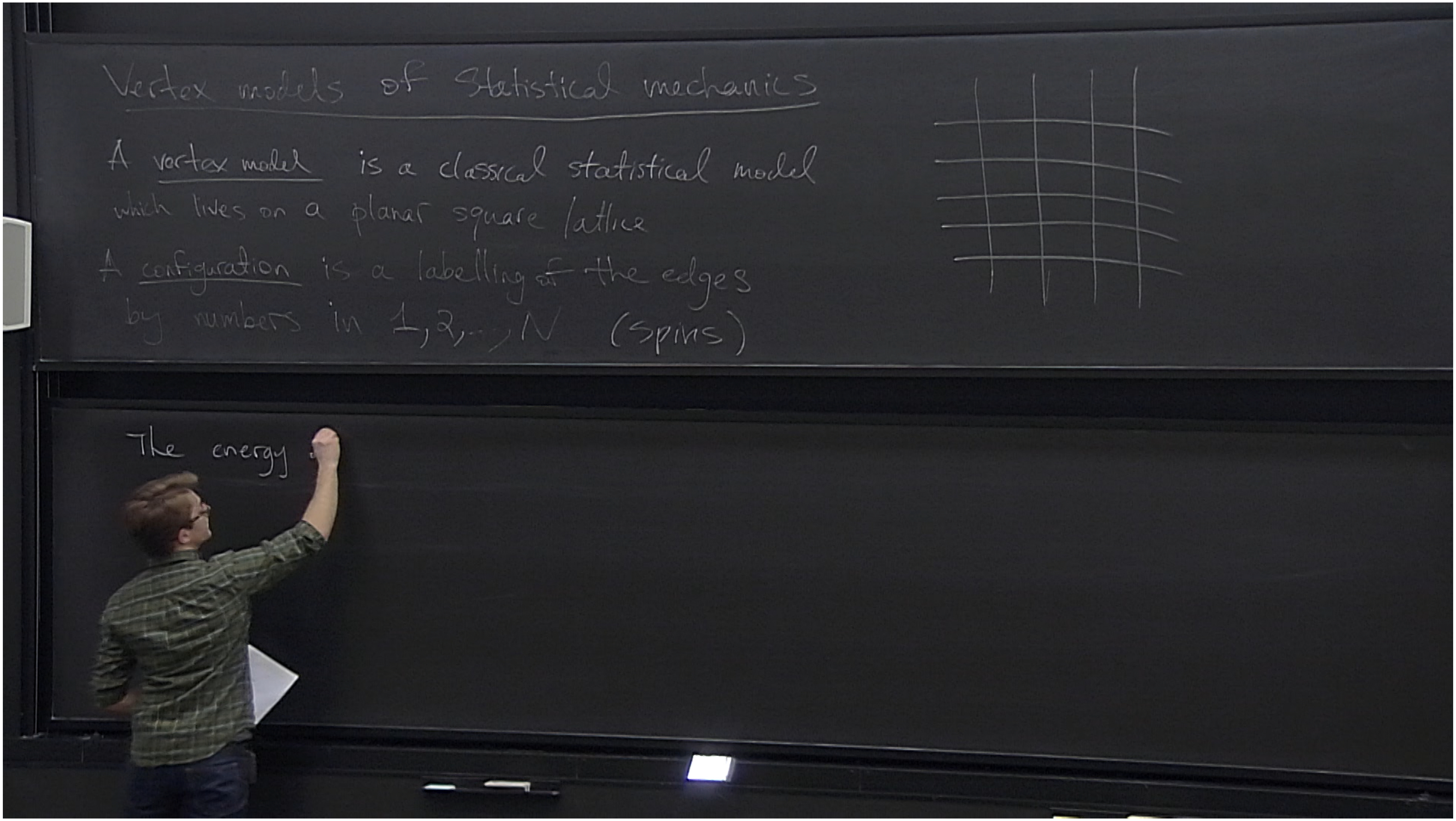
A configuration is a labelling of the edges by numbers in $1, 2, \ldots, N$ (spins)


Vertex models of Statistical mechanics
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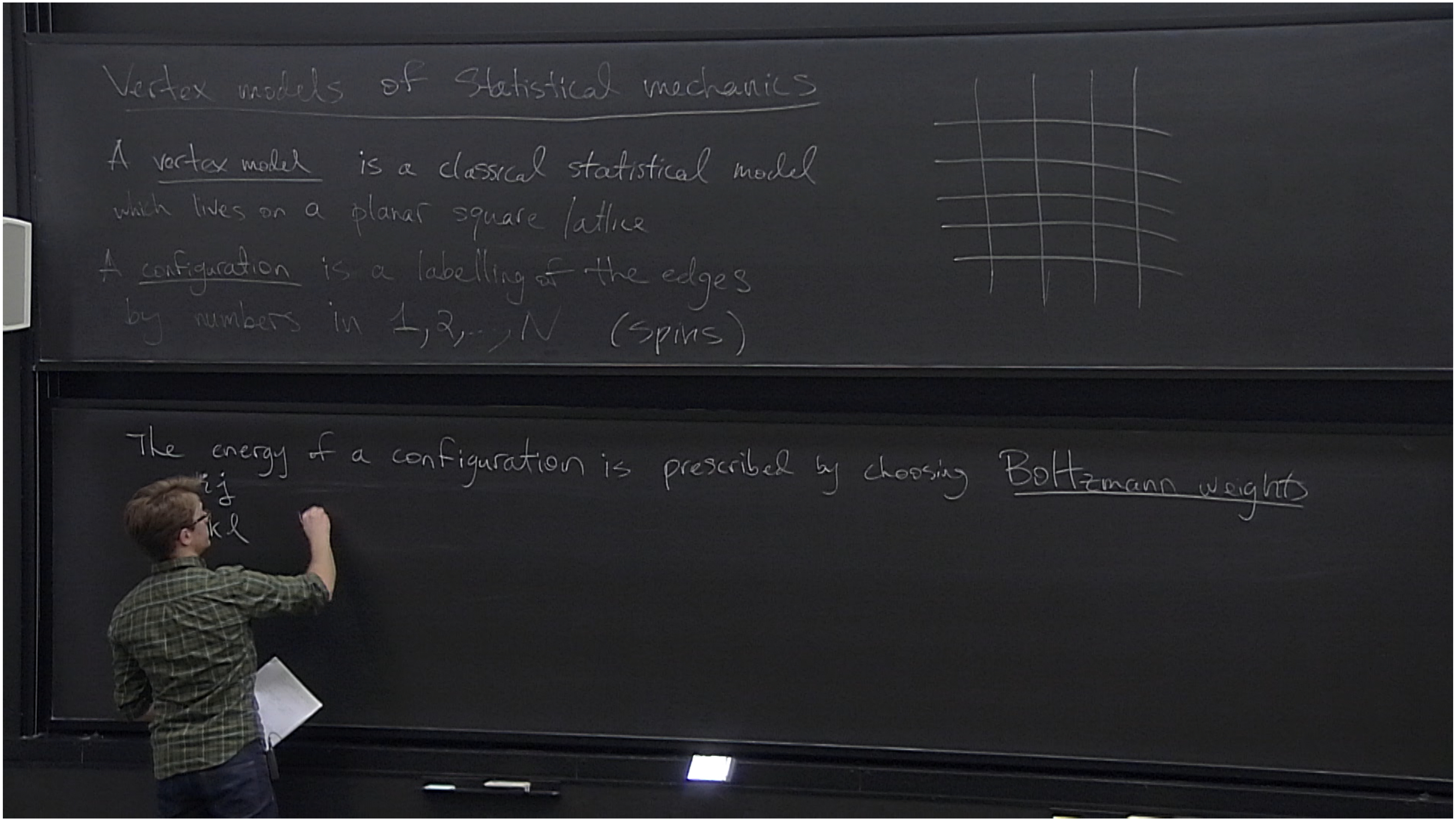
The energy


Vertex models of Statistical mechanics
A vertex model is a classical statistical model
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A configuration is a labelling of the edges
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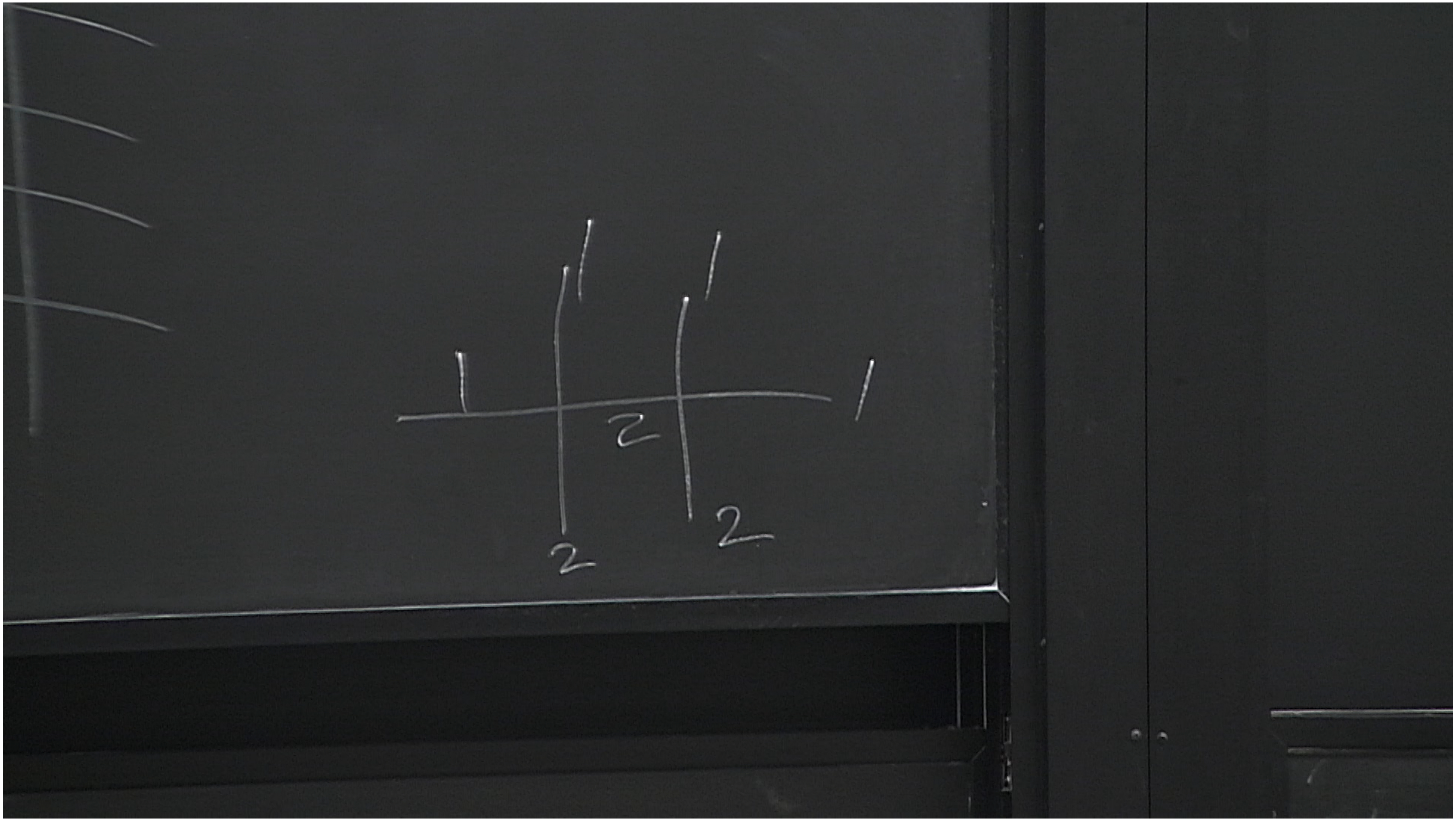
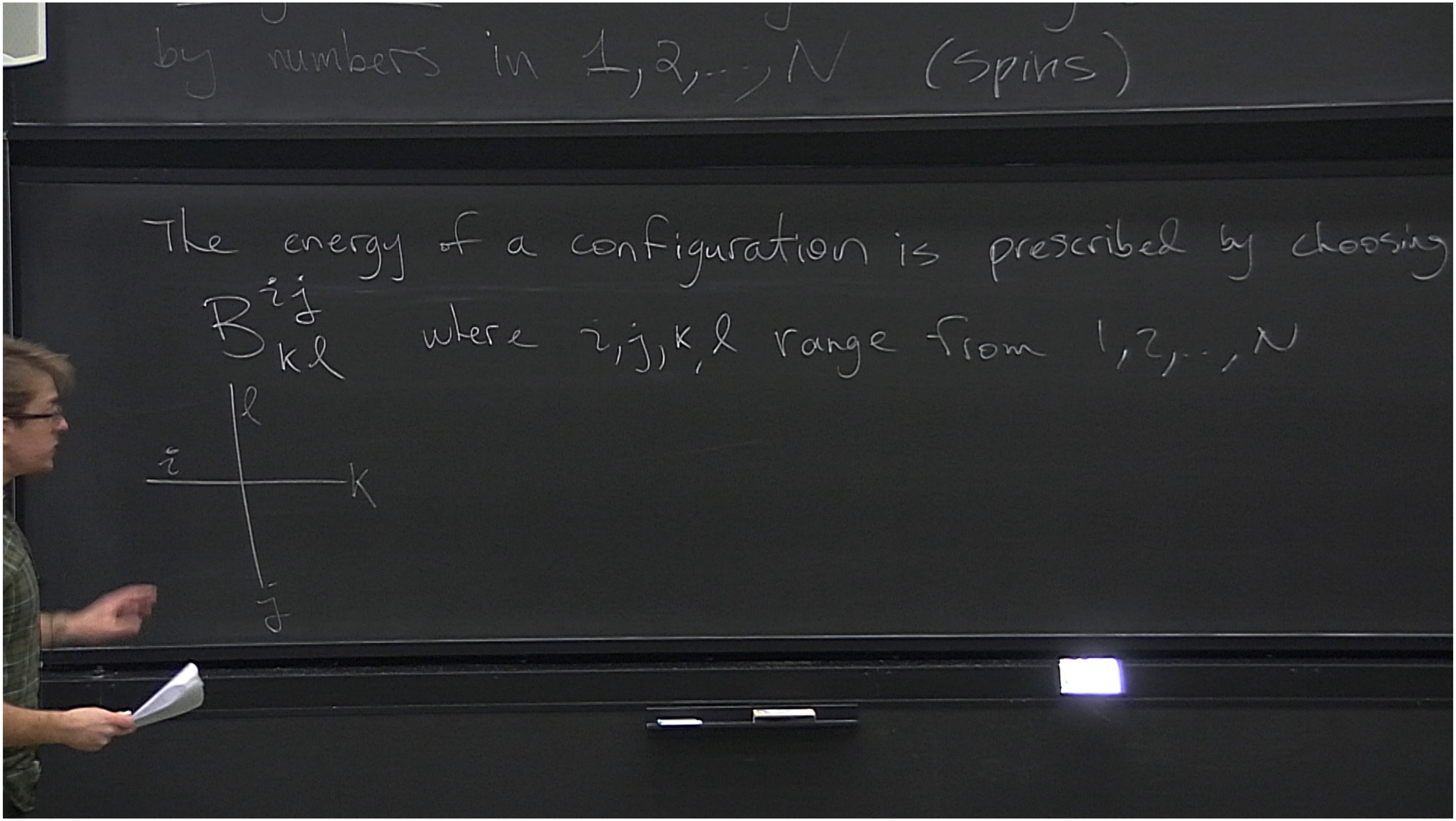
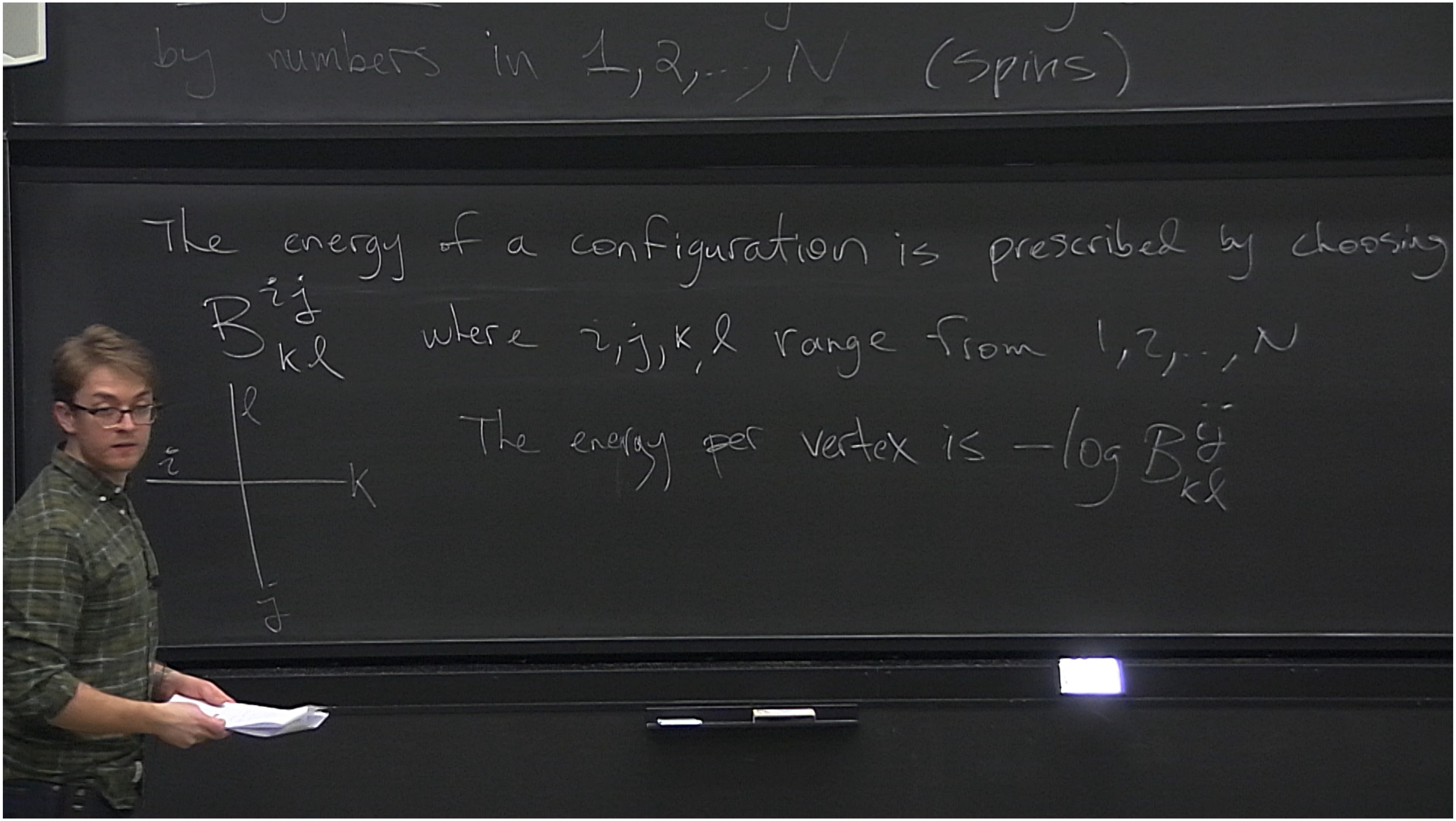
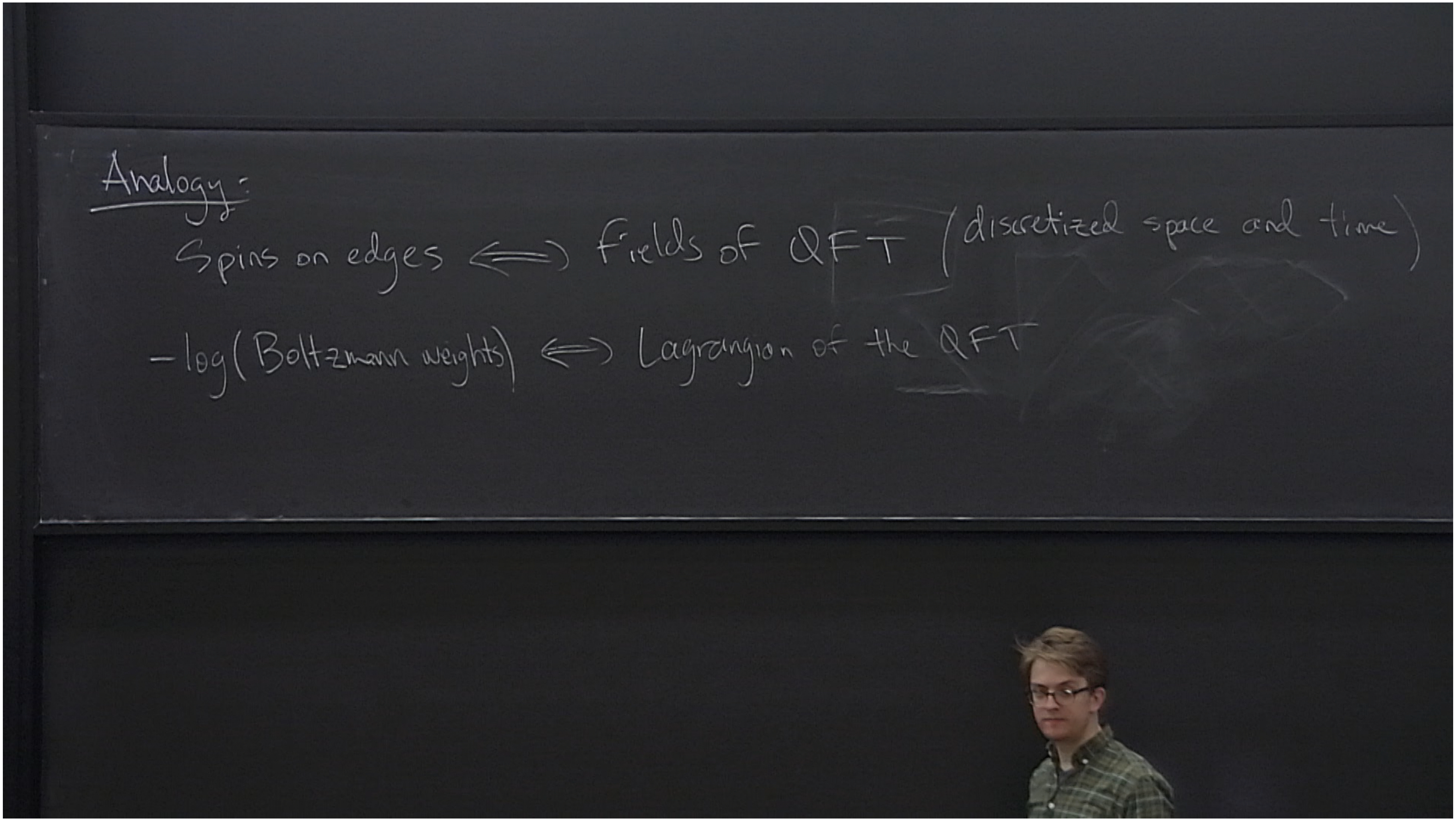
The energy of a configuration is prescribed by choosing Boltzmann weights
i j
k l



1
1
1
1
2
2
2


by numbers in 1, 2, ..., N (spins)
The energy of a configuration is prescribed by choosing
$B^{ij}_{k\ell}$ where i, j, k, ℓ range from $1, 2, \ldots, N$
ℓ
i
k
j

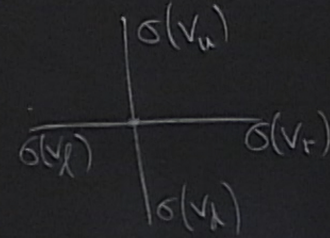

by numbers in 1, 2, ..., N (spins)
The energy of a configuration is prescribed by choosing
B^{ij}_{kl} where i, j, k, l range from 1, 2, ..., N
l
i
k
j
The energy per vertex is $-\log B^{ij}_{kl}$


Analogy:
Spins on edges $\Longleftrightarrow$ fields of QFT (discretized space and time)
$-\log$(Boltzmann weights) $\Longleftrightarrow$ Lagrangian of the QFT

$-\log(\text{Boltzmann weights}) \Longleftrightarrow$ Lagrangian of the QFT

Given a configuration σ, we define energy $E(\sigma)$ by

$$e^{-E(\sigma)} = \prod_{\substack{\text{vertices}\\ v}} B^{\sigma(v_\ell)\,\sigma(v_d)}_{\sigma(v_r)\,\sigma(v_u)} \longleftarrow \text{vertex } v$$



$-\log(\text{Boltzmann weights}) \iff$ Lagrangian of the QFT

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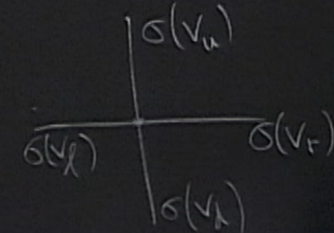
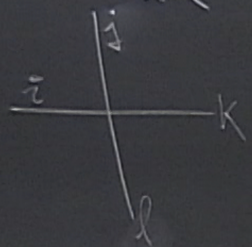


B_{kl} where i,j,k,l range from $1,2,\ldots,N$



The energy per vertex is $-\log B^{ij}_{kl}$

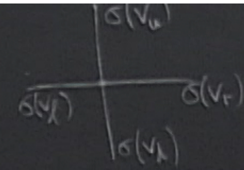
Partition function: Define

$$Z = \sum_{\substack{\text{configurations} \\ \sigma}} e^{-E(\sigma)} = \sum_{\sigma} \prod_{v} B(v,\sigma) = \sum_{\sigma} \exp\left(-\sum_{v} \left(-\log(B(v,\sigma))\right)\right)$$

$$\boxed{Z = \sum_{\sigma} \exp\left(-\sum_{v} E(v,\sigma)\right)}$$

$E(v,\sigma) = -\log B(v,\sigma)$

$$e^{-E(\sigma)} = \prod_{\substack{\text{vertices} \\ v}} B^{\sigma(v_\ell)\,\sigma(v_u)}_{\sigma(v_r)\,\sigma(v_d)} \longleftarrow \text{vertex } v$$



<u>Analogy</u>: If have a QFT on a space-time manifold $\mathcal{M}$,
with fields φ, Lagrangian $\mathcal{L}$,

$$Z = \int_\varphi \exp\Big(-\int_{\mathcal{M}} \mathcal{L}(\varphi)\Big)$$

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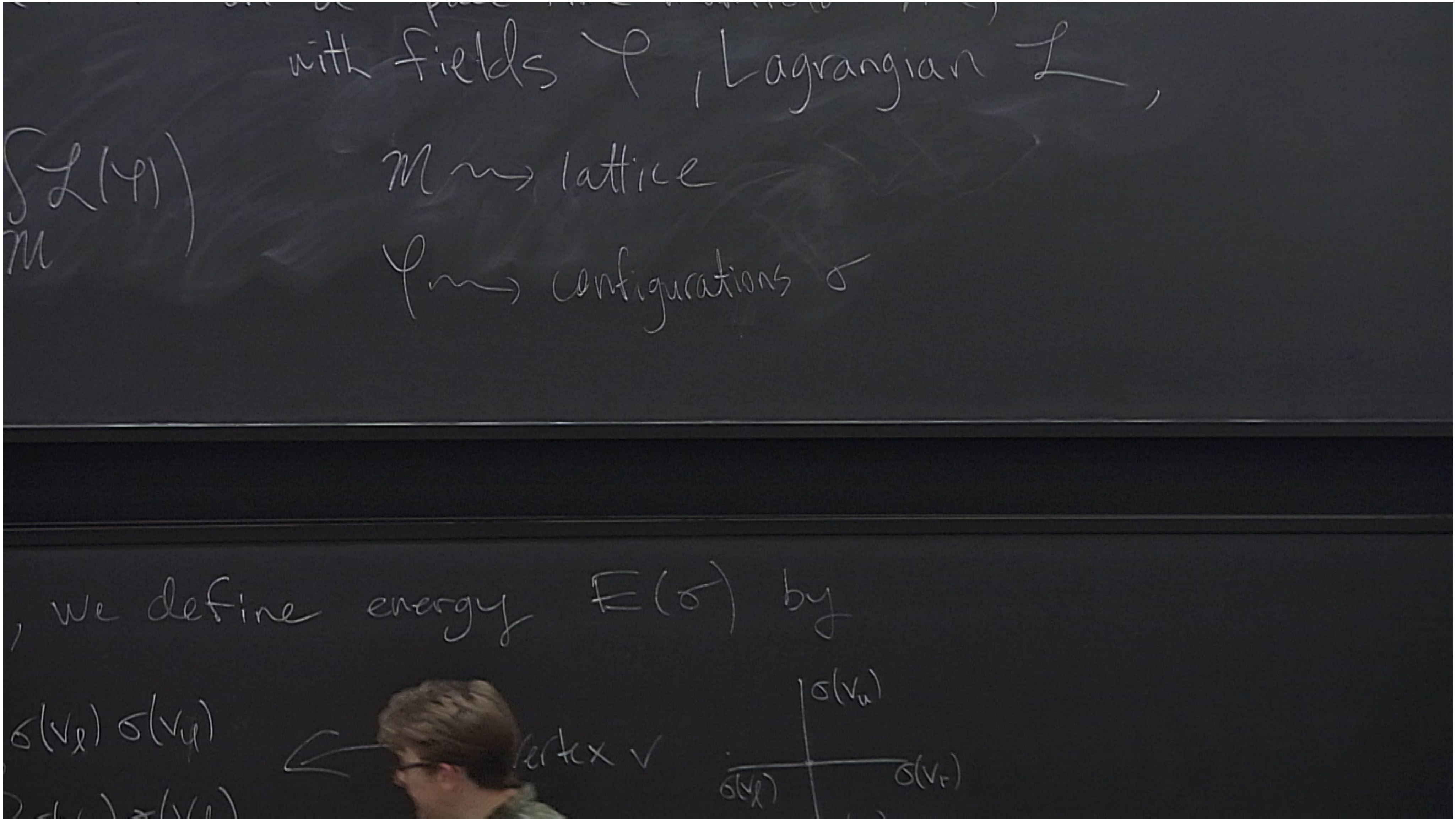
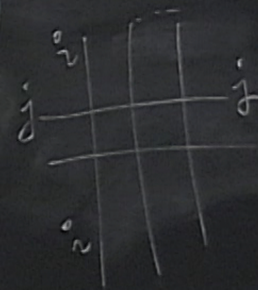

with fields φ, Lagrangian L,
M ⇝ lattice
φ ⇝ configurations σ
we define energy E(σ) by
σ(vℓ) σ(vu)
ertex v
σ(vu)
σ(vℓ)
σ(vr)

$\varphi \rightsquigarrow$ configurations

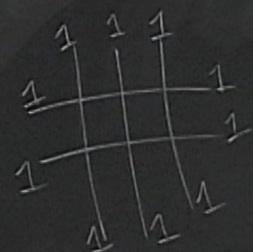
<u>Boundary conditions</u>: might restrict which configurations are allowed

1) doubly-periodic boundary conditions



i, j, etc. arbitrary

2) Fixing values at boundary

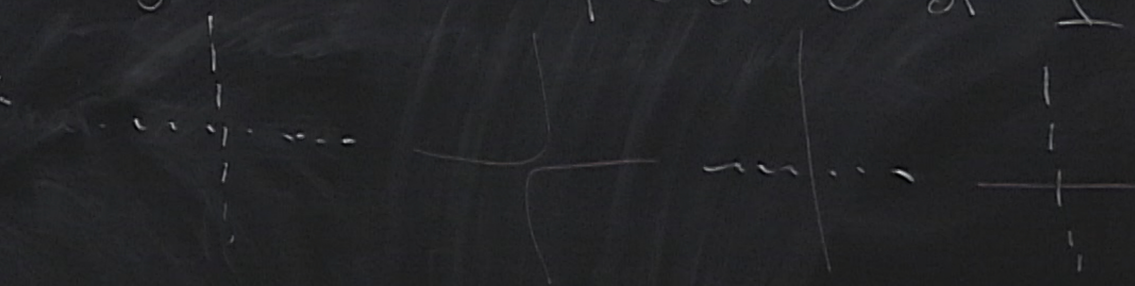


$$Z = \sum_{\sigma} \exp\Big(- \sum_{v}$$

<u>example:</u> The 6-vertex model

$N=2$, so each edge carries a spin of 0 or 1

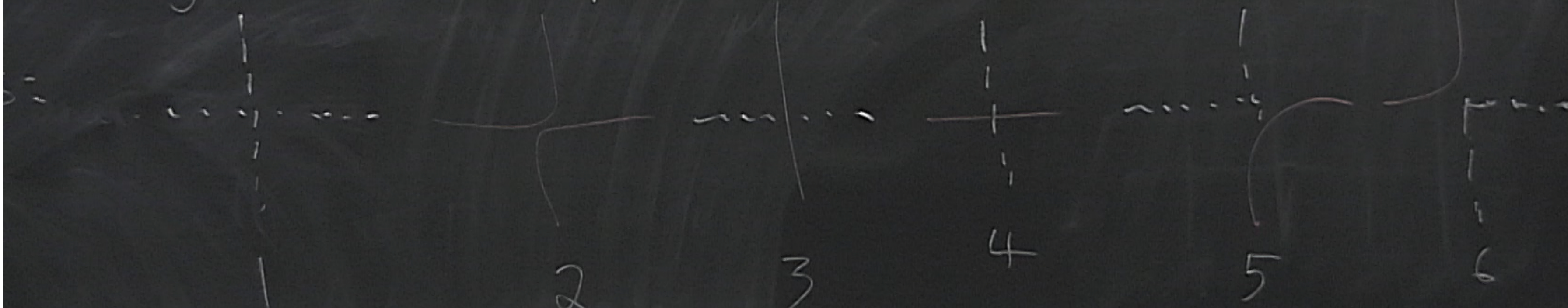
Allowed configurations:



$$Z = \sum_{\sigma} \exp\left(-\sum_{v} E(v,\sigma)\right)$$

6-vertex model

[E]ach edge carries a spin of 0 or 1



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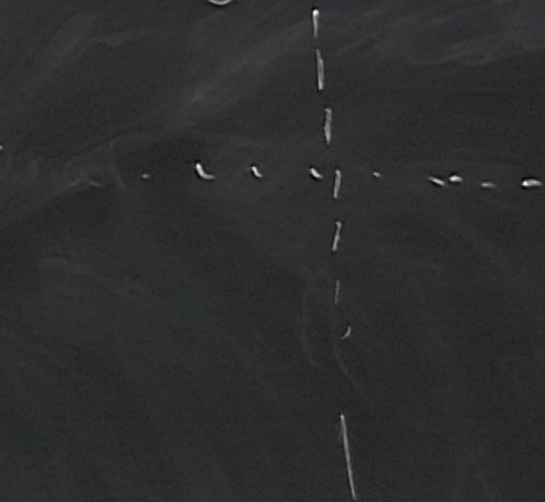
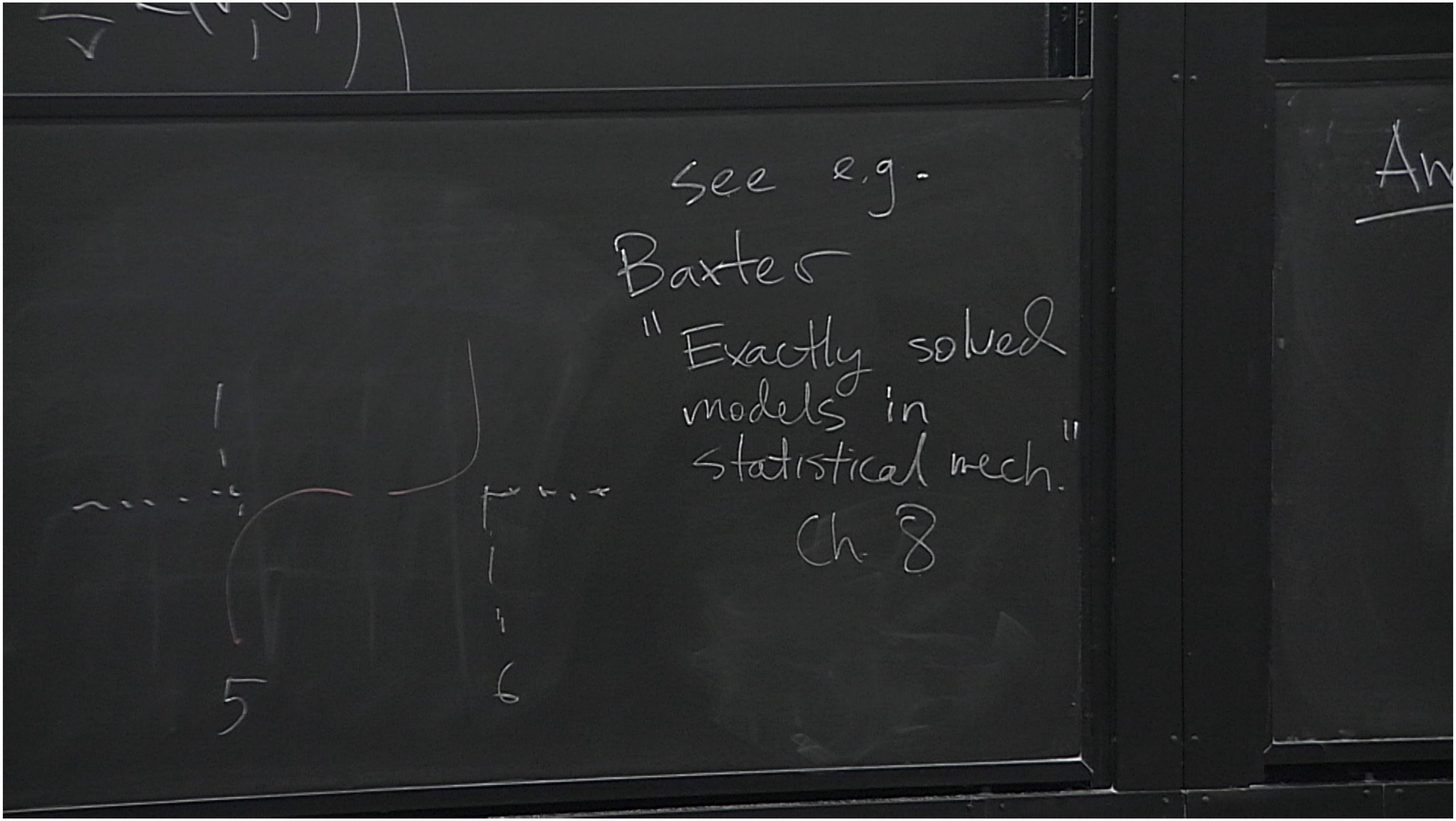
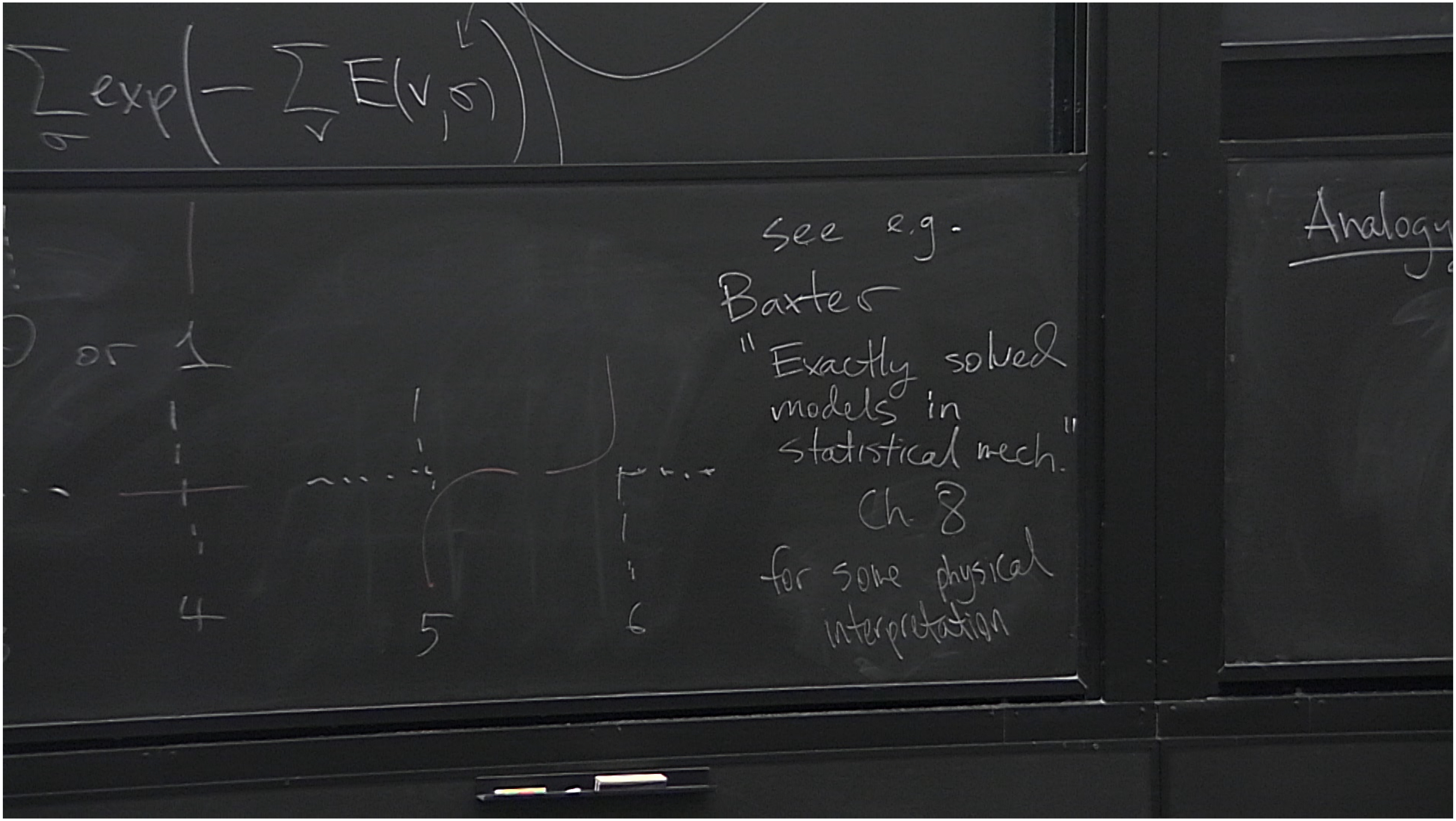
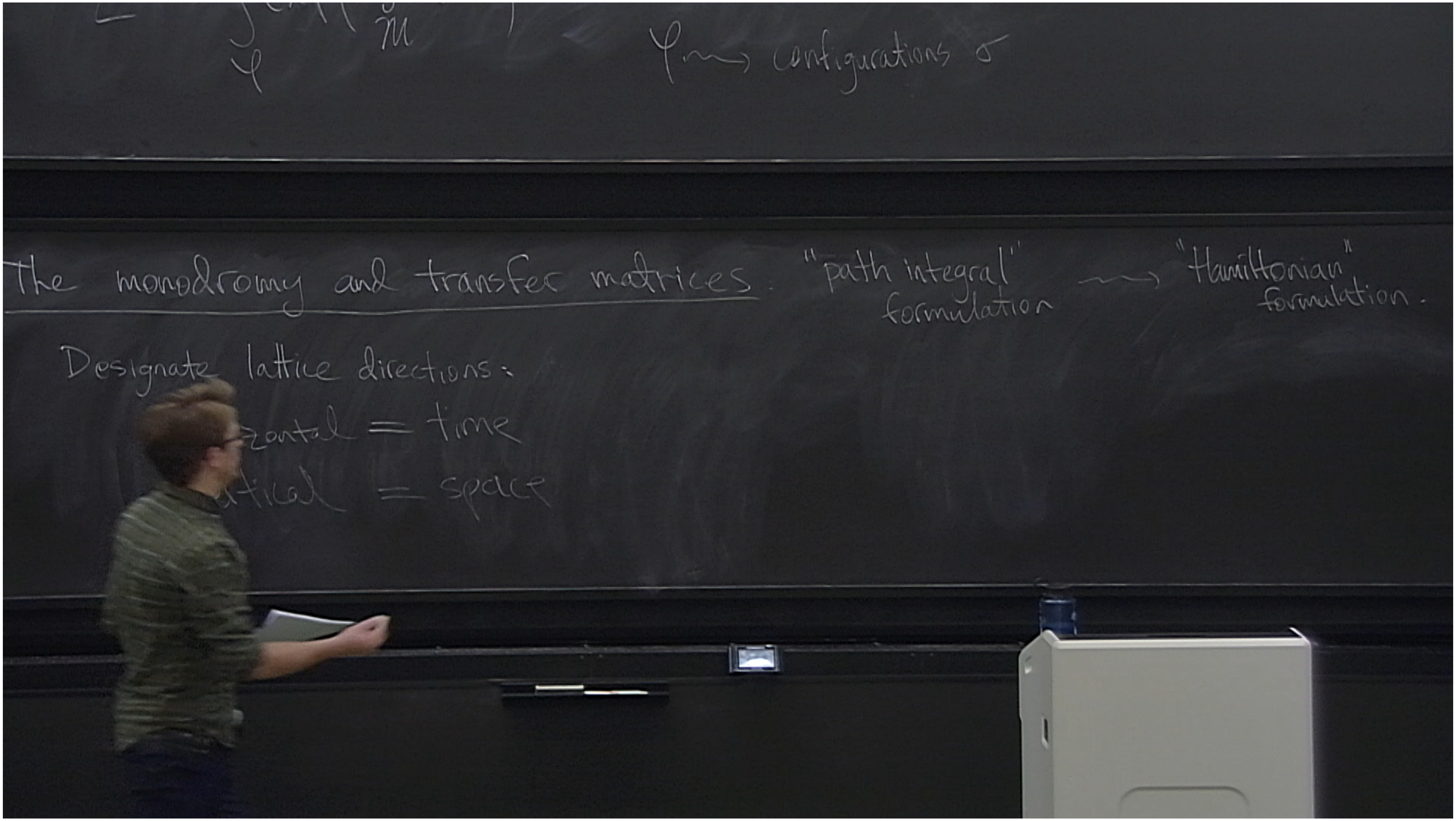


Boltzmann weights

$\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_6$


see e.g.
Baxter
"Exactly solved models in statistical mech."
Ch. 8
5
6


$\sum_\sigma \exp\left(-\sum_v E(v,\sigma)\right)$
or 1
4
5
6
see e.g.
Baxter
"Exactly solved models in statistical mech."
Ch. 8
for some physical interpretation
Analogy

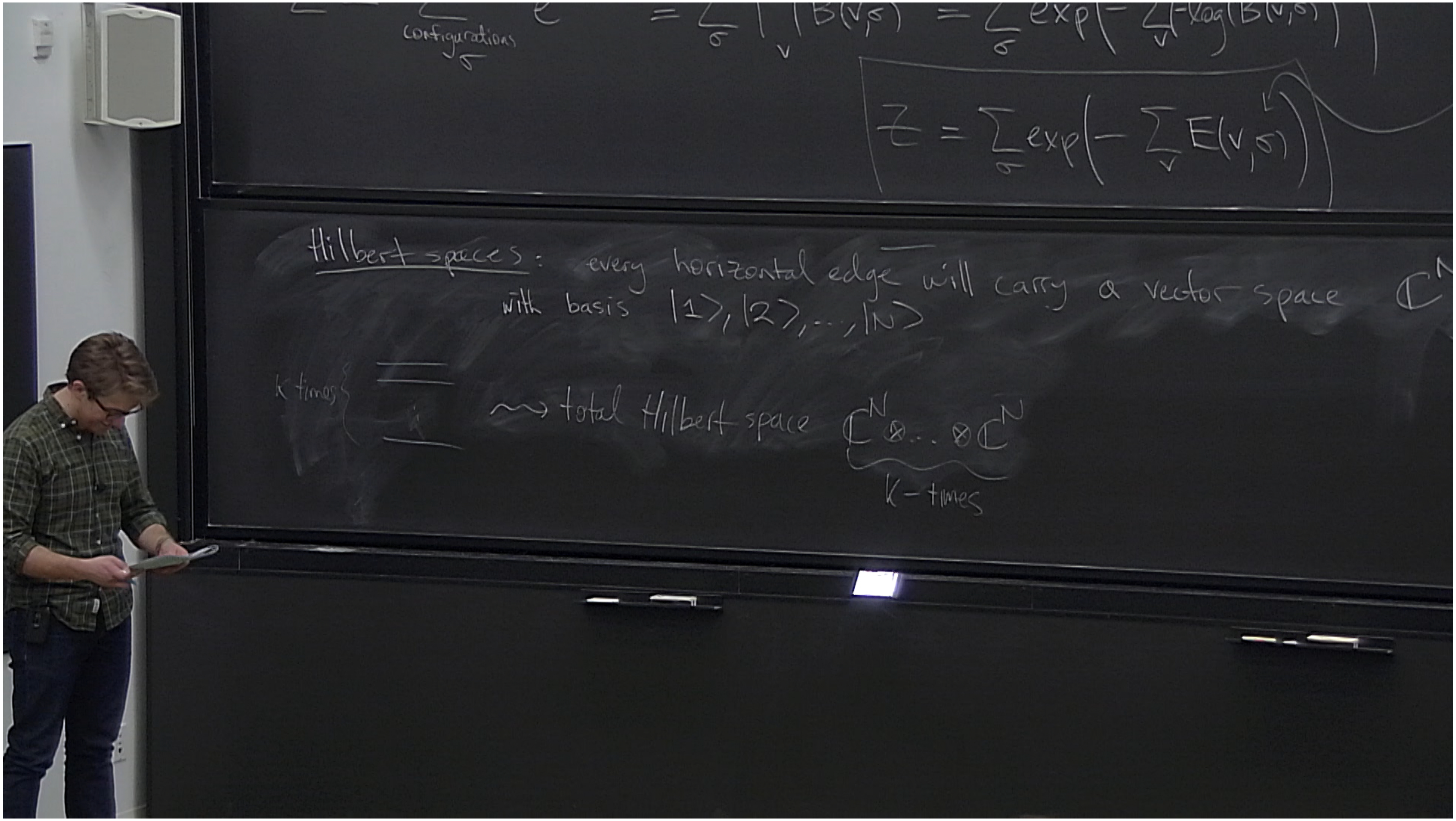

φ ⇝ configurations σ
The monodromy and transfer matrices: "path integral" formulation ⇝ "Hamiltonian" formulation.
Designate lattice directions:
= time
= space

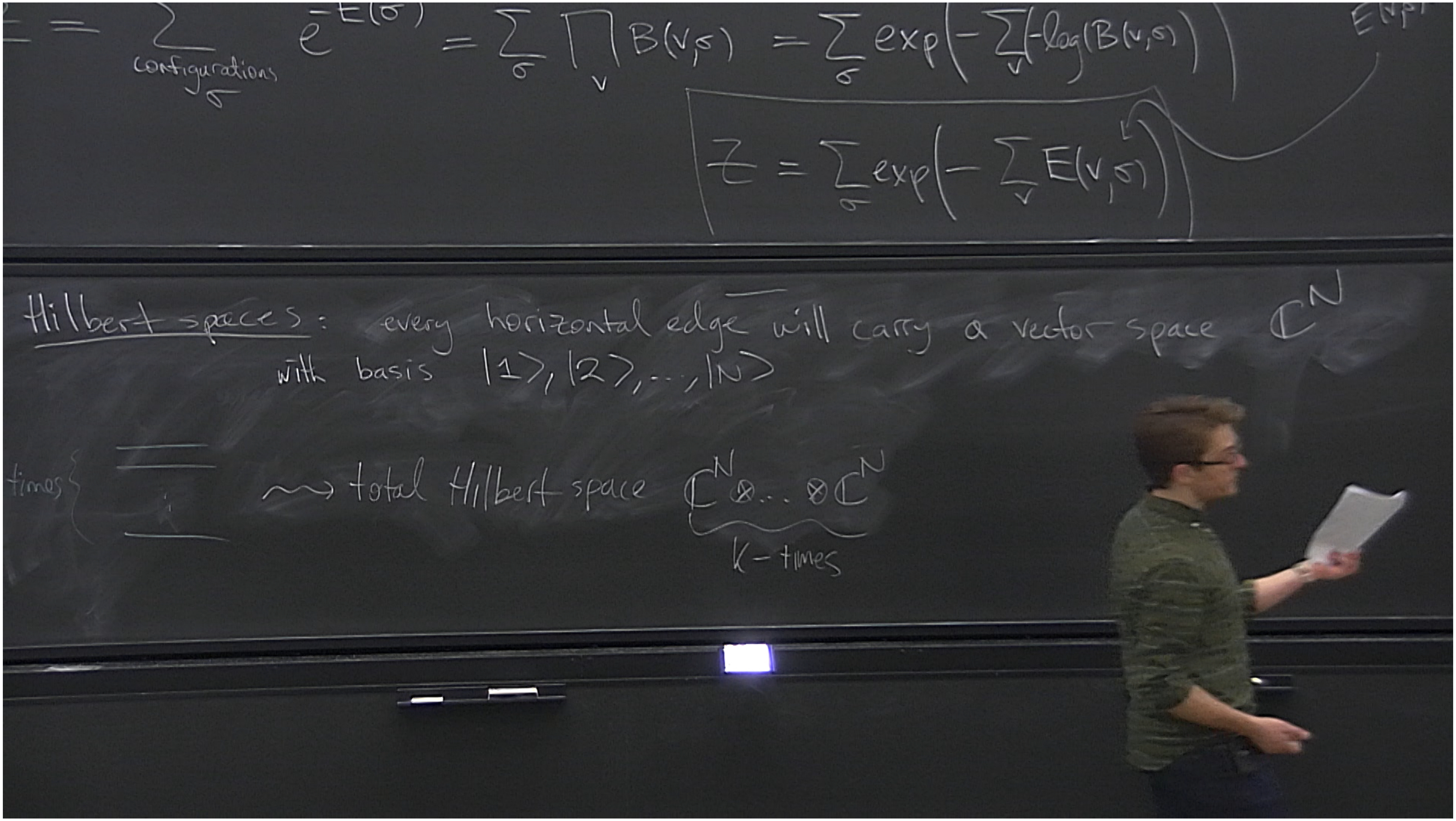
The monodromy and transfer matric

Designate lattice directions:

Horizontal = time

Vertical = space

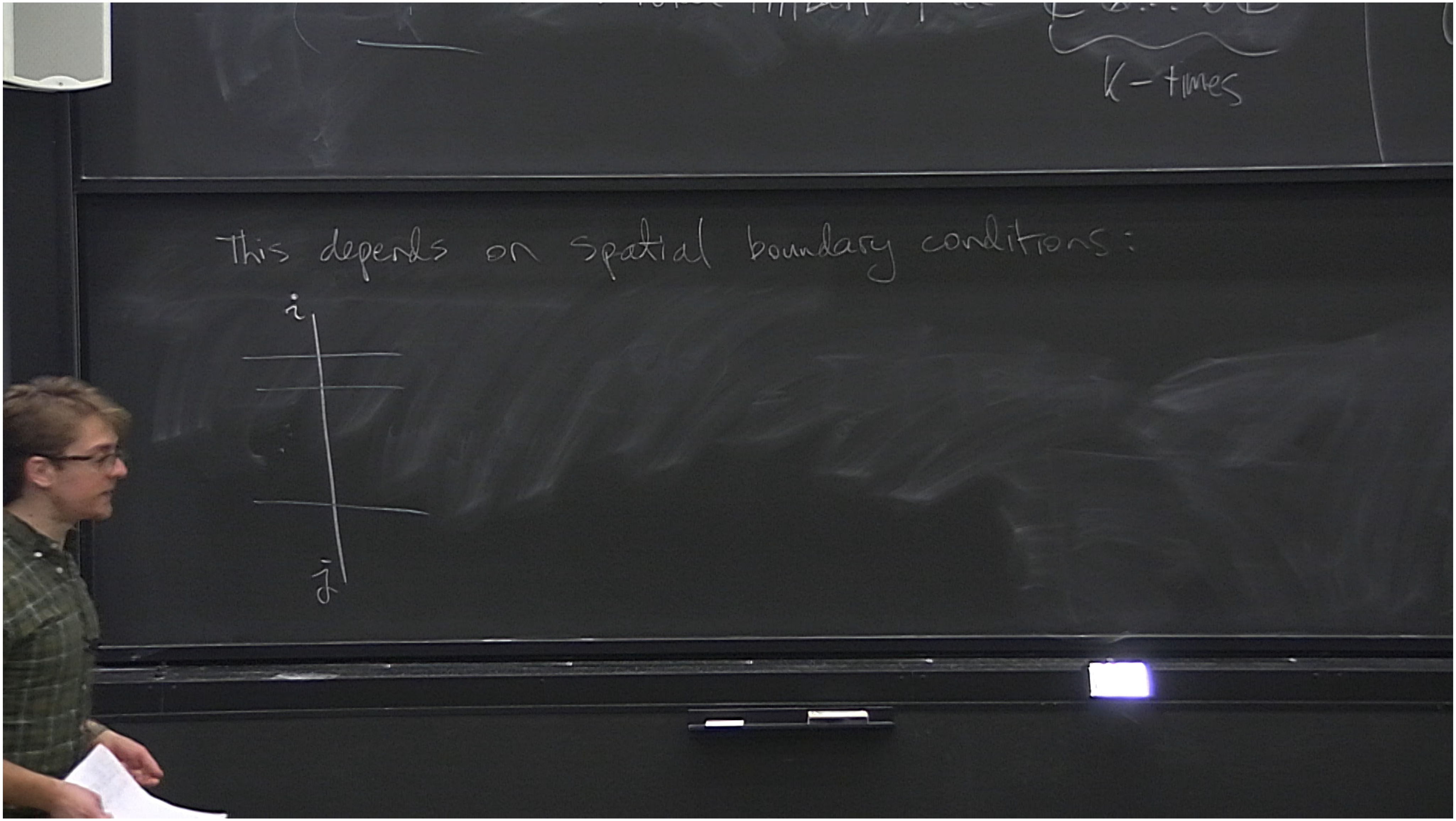

configurations σ
Z = Σ_σ exp(− Σ_v E(v,σ))
Hilbert spaces: every horizontal edge will carry a vector space ℂ^N
with basis |1>, |2>, ..., |N>
k times
total Hilbert space ℂ^N ⊗ ... ⊗ ℂ^N
k-times


$Z = \sum_{\text{configurations } \sigma} e^{-E(\sigma)} = \sum_\sigma \prod_v B(v,\sigma) = \sum_\sigma \exp\left(-\sum_v -\log(B(v,\sigma))\right)$
$E(v,\sigma)$
$Z = \sum_\sigma \exp\left(-\sum_v E(v,\sigma)\right)$
Hilbert spaces: every horizontal edge will carry a vector space $\mathbb{C}^N$
with basis $|1\rangle, |2\rangle, \ldots, |N\rangle$
times
$\rightsquigarrow$ total Hilbert space $\underbrace{\mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N}_{k\text{-times}}$

y a vector space $\mathbb{C}^N$

Discrete time evolution gives an operator on these Hilbert spaces

(analog of e^H, H Hamiltonian)


k - times
This depends on spatial boundary conditions:
i
j

k - times

...nds on spatial boundary conditions:

monodromy matrix: $M^{i}_{j} : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$

Total Hilbert space $\underbrace{\mathbb{C} \otimes \dots \otimes \mathbb{C}}_{k\text{-times}}$

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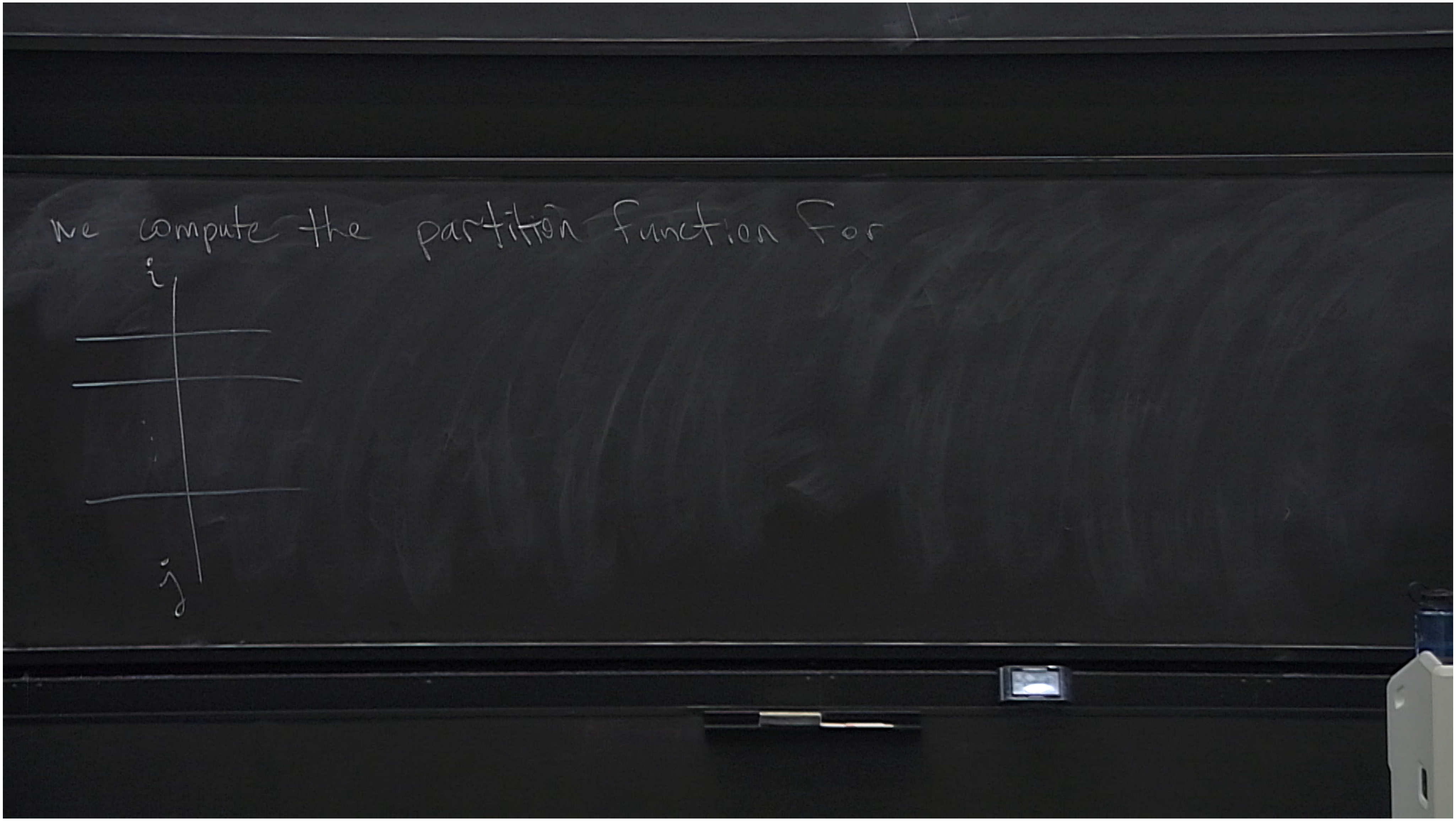
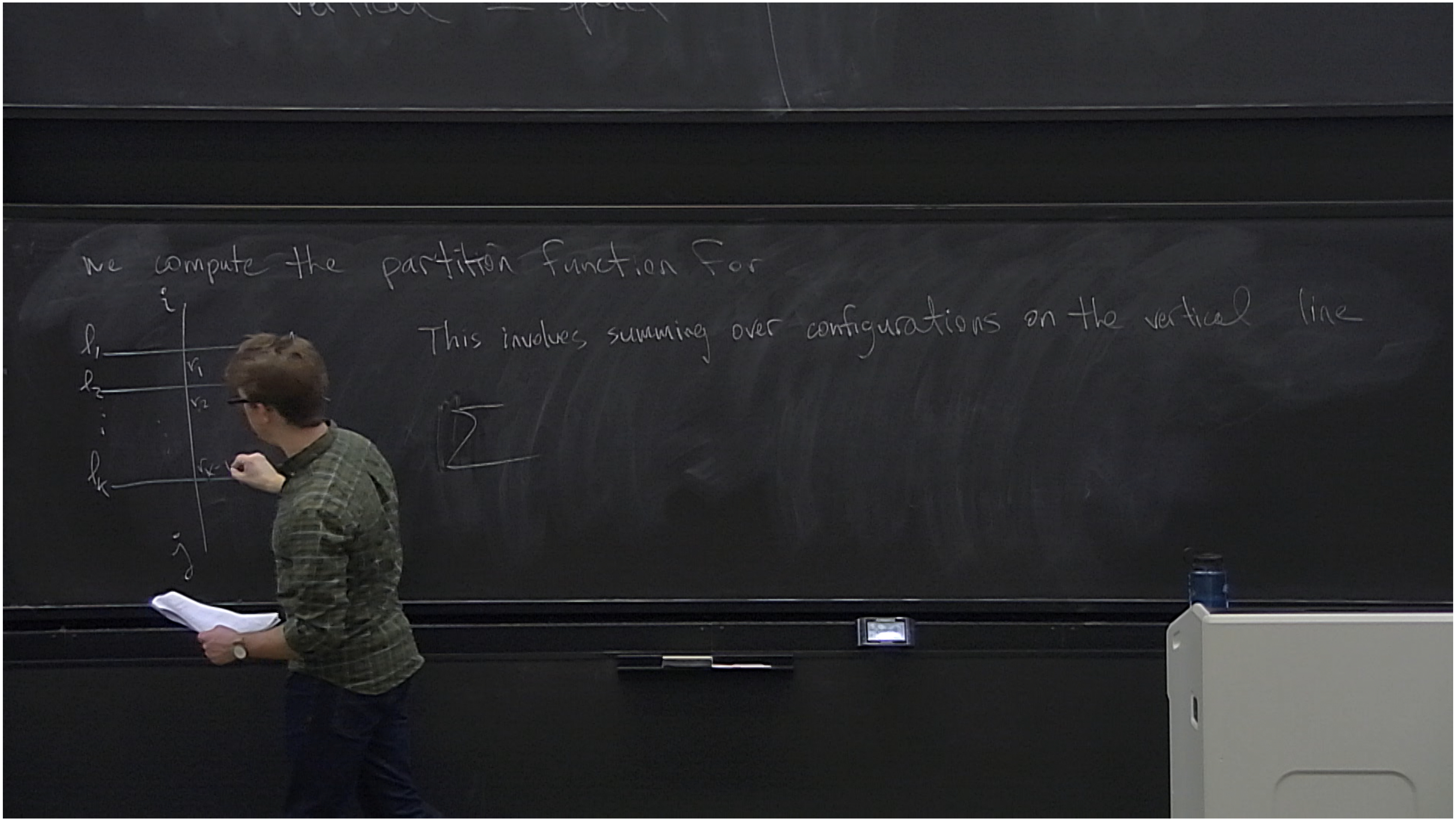
on spatial boundary conditions:

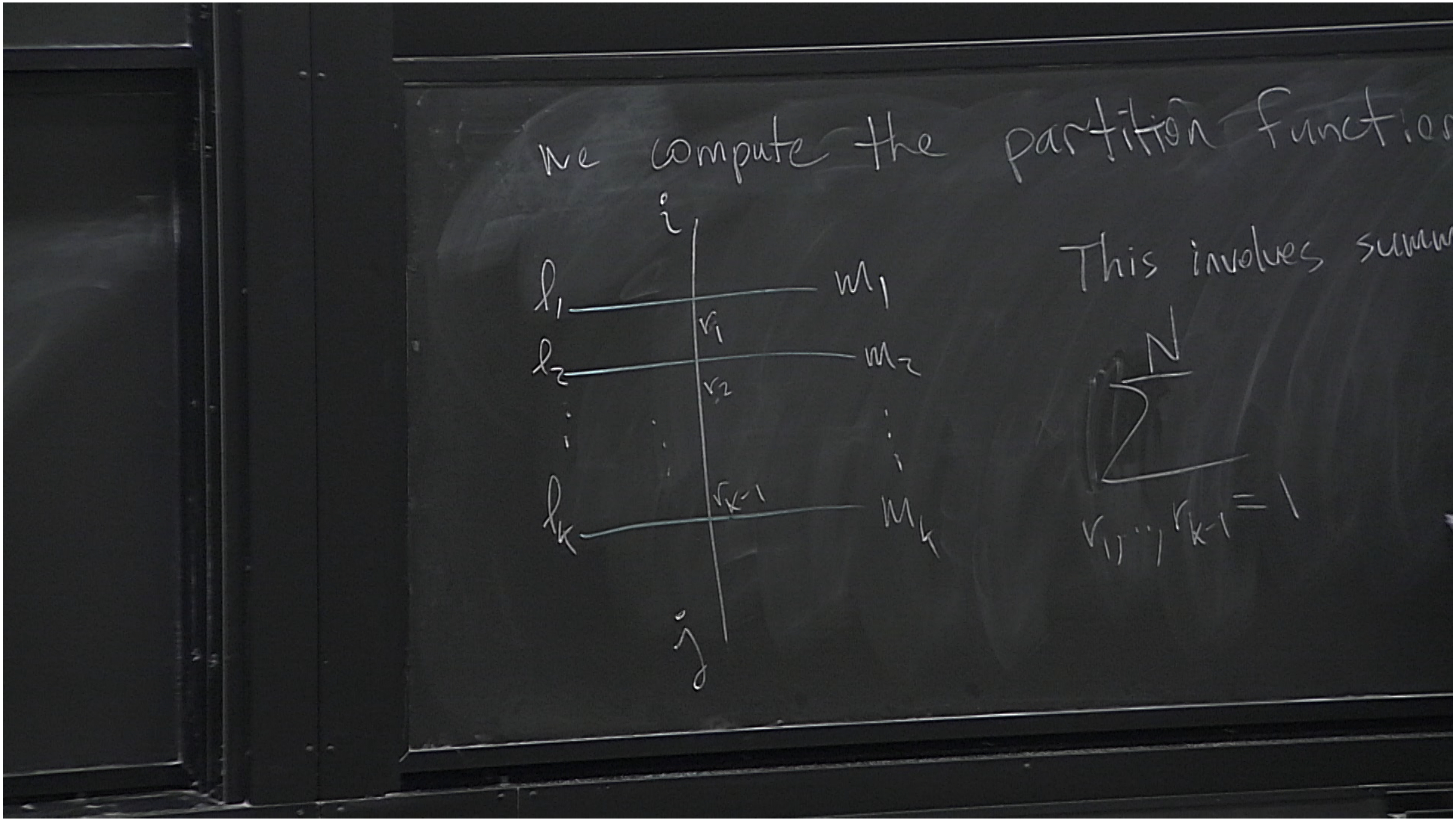
<u>monodromy matrix</u>: $M^{i}_{j} : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$

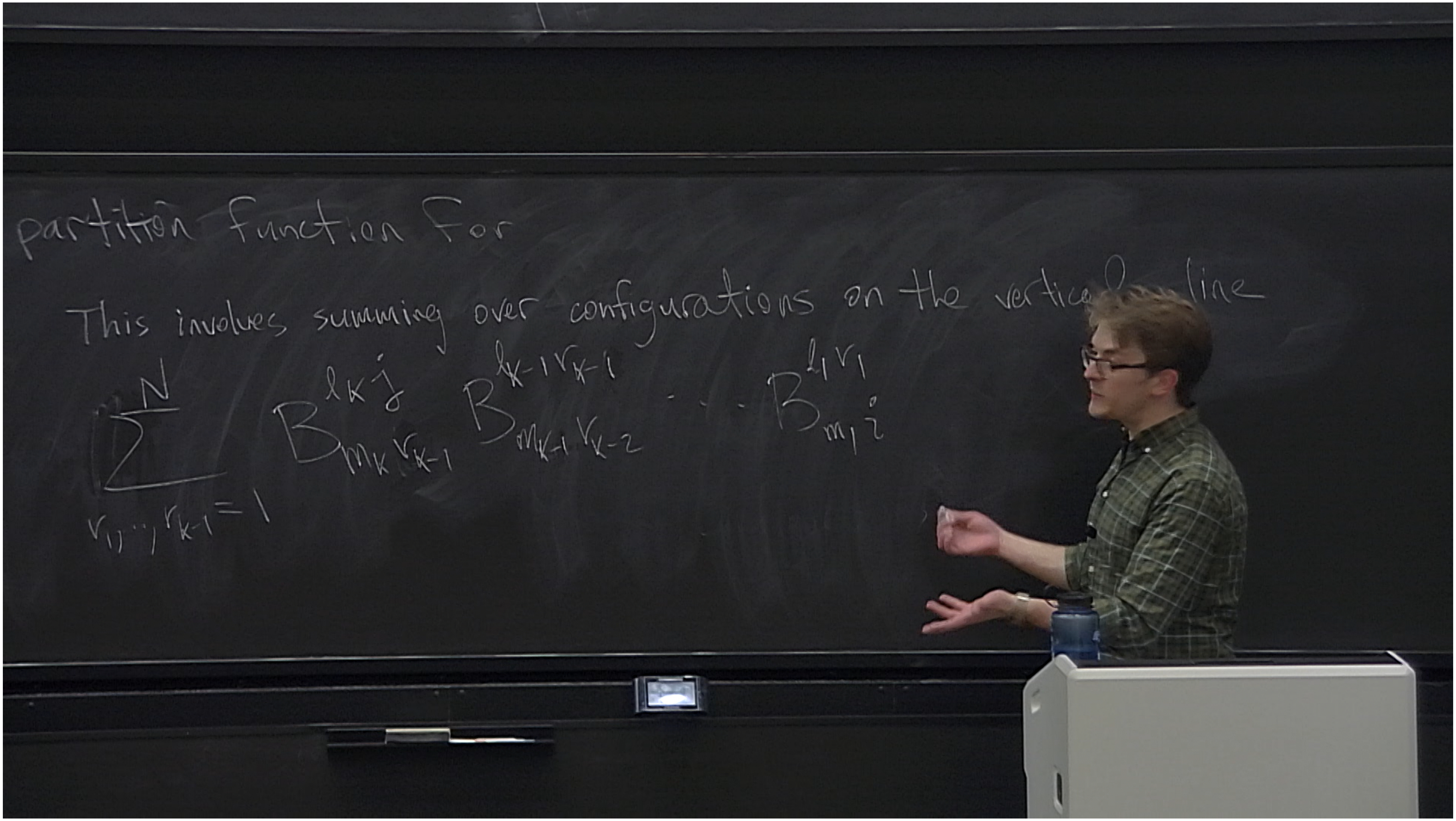
To compute the matrix entries, between incoming state $|l_1\rangle \otimes \dots \otimes |l_k\rangle$

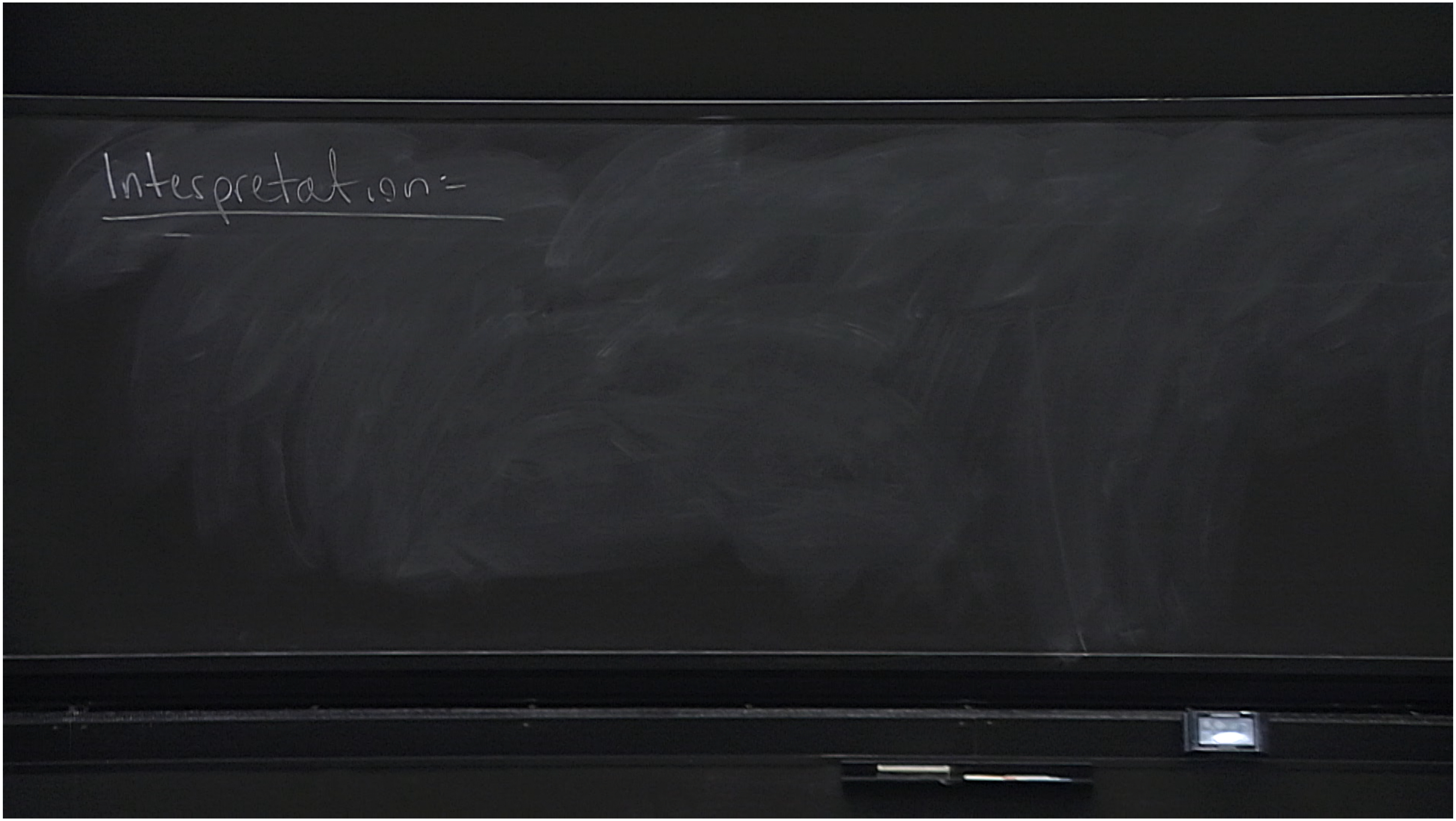
and outgoing state $|m_1\rangle \otimes \dots \otimes |m_k\rangle$


we compute the partition function for
i
j


we compute the partition function for
This involves summing over configurations on the vertical line
i
j
l_1
l_2
l_k
r_1
r_2
$\sum$


we compute the partition
This involves
i
j
l1 l2 lk
m1 m2 mk
r1 r2 rk-1
N
r1, .., rk-1 = 1


partition function for
This involves summing over configurations on the vertical line
$\sum_{r_1,\dots,r_{k-1}=1}^{N} B^{l_k j}_{m_k r_{k-1}} B^{l_{k-1} r_{k-1}}_{m_{k-1} r_{k-2}} \cdots B^{l_1 r_1}_{m_1 i}$


Interpretation:

Interpretation: View B^{ij}_{kl} as a linear operator

$$\mathbb{C}^N \otimes \mathbb{C}^N \longrightarrow \mathbb{C}^N \otimes \mathbb{C}^N$$

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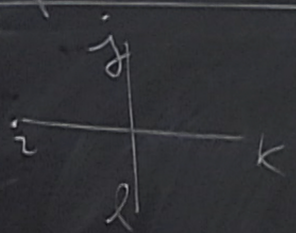
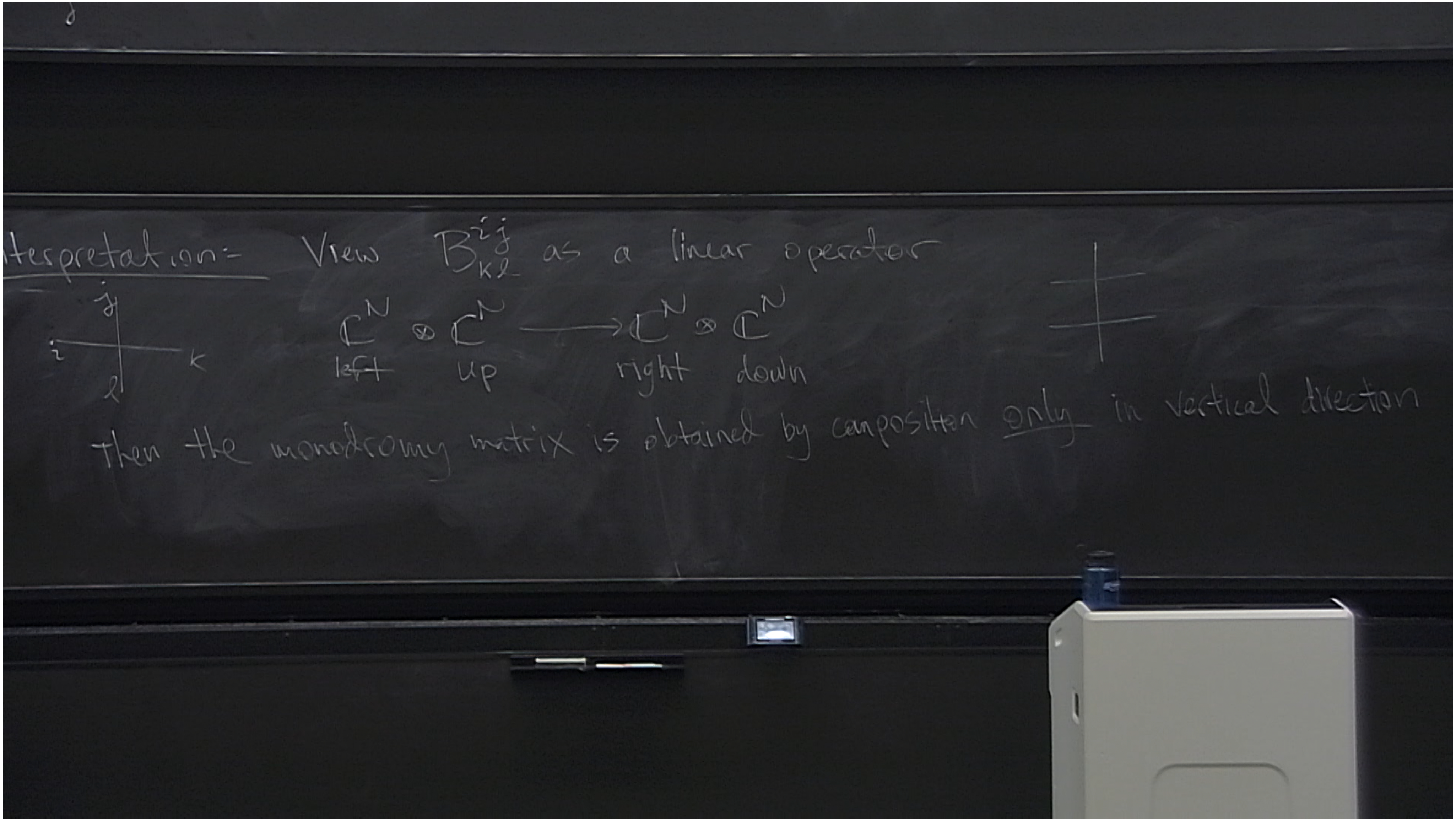
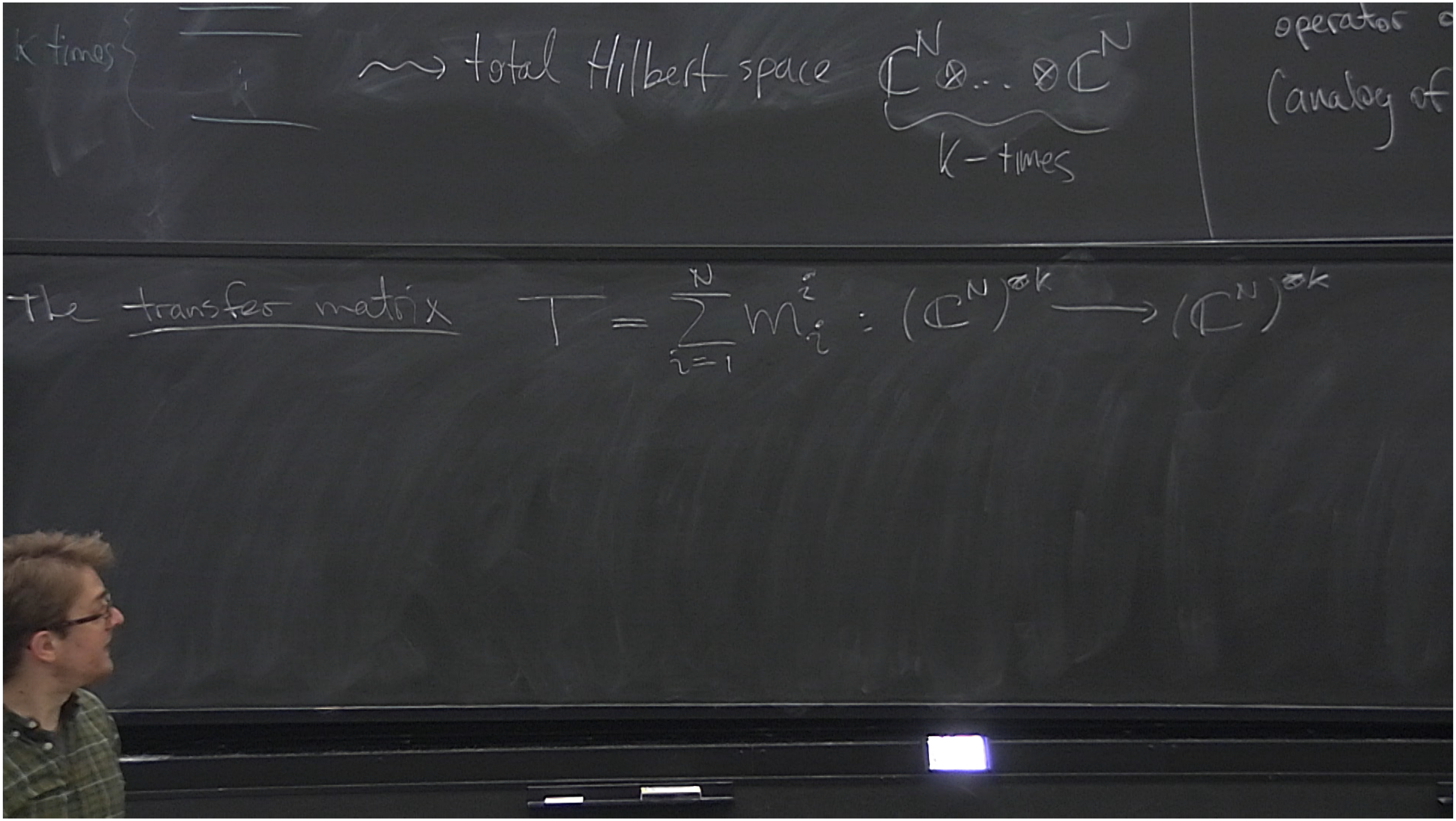
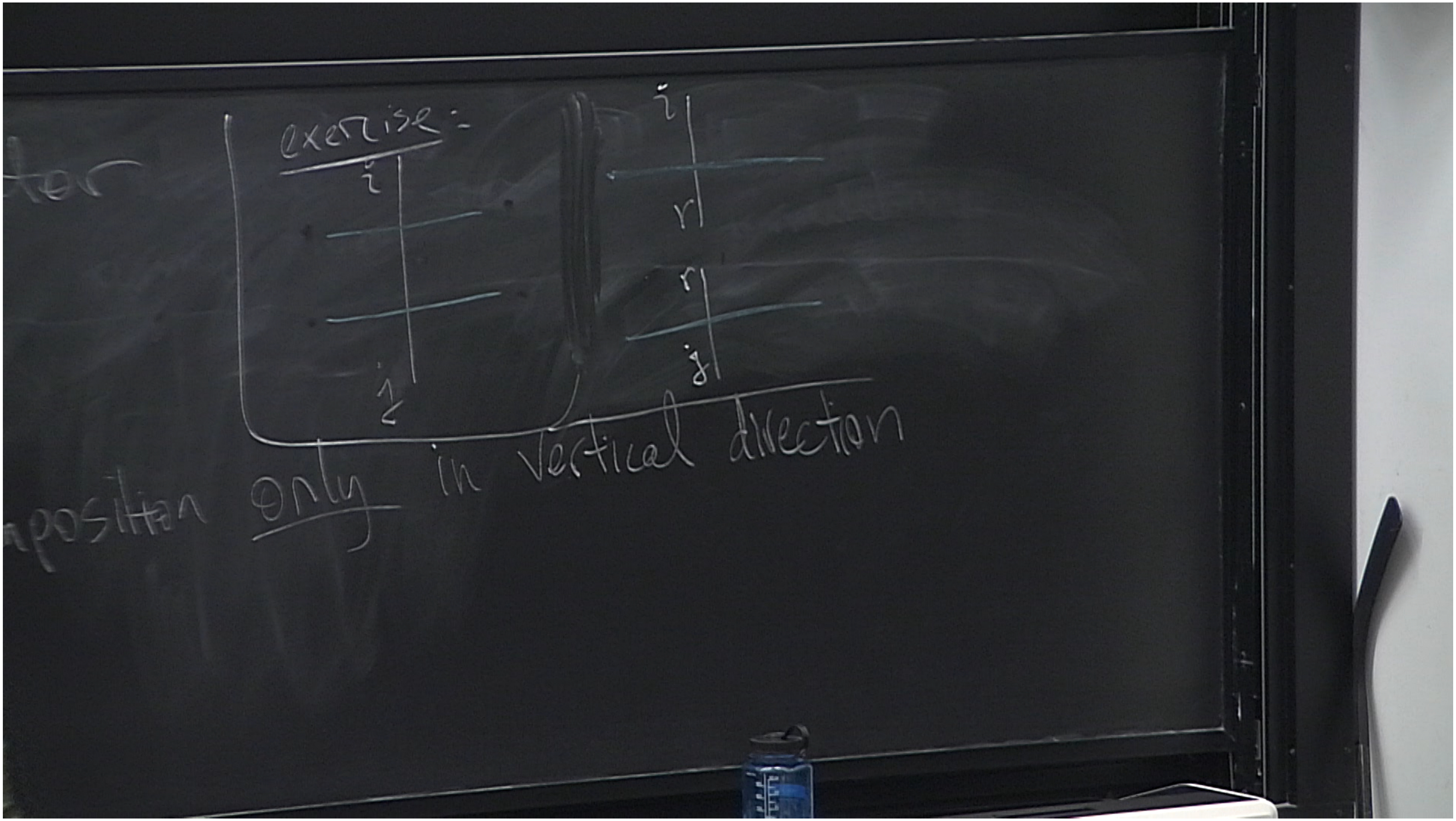


$$\underset{\text{left}}{\mathbb{C}^N} \otimes \underset{\text{up}}{\mathbb{C}^N} \longrightarrow \underset{\text{right}}{\mathbb{C}^N} \otimes \underset{\text{down}}{\mathbb{C}^N}$$


terpretation:- View B^{ij}_{kl} as a linear operator
j
i
k
l
$\mathbb{C}^N \otimes \mathbb{C}^N \longrightarrow \mathbb{C}^N \otimes \mathbb{C}^N$
left up right down
Then the monodromy matrix is obtained by composition only in vertical direction


k times
$\rightsquigarrow$ total Hilbert space $\underbrace{\mathbb{C}^N \otimes \dots \otimes \mathbb{C}^N}_{k\text{-times}}$
operator
(analog of
The transfer matrix $T = \sum_{i=1}^{N} m_i^i : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes k}$


exercise:
only in vertical direction