Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 17

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Abstract:

Note Title

QFT for Cosmology, Achim Kempf, Lecture 17

From the particle picture to the wave picture

So far: Spacetime dynamics can produce particles.

When? When mode oscillators $\omega_k(t)$ changes nonadiabatically fast: $\omega_k(t)'/\omega_k(t)^2 \gg 1$.

In cosmology? No: particles mostly produced conventionally from inflation potential at the end of inflation.

Now: Spacetime dynamics can enhance quantum field fluctuations!

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When? When $\omega_k(\eta)$ becomes imaginary.

Recall: $\omega_k^2(\eta) := k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}$

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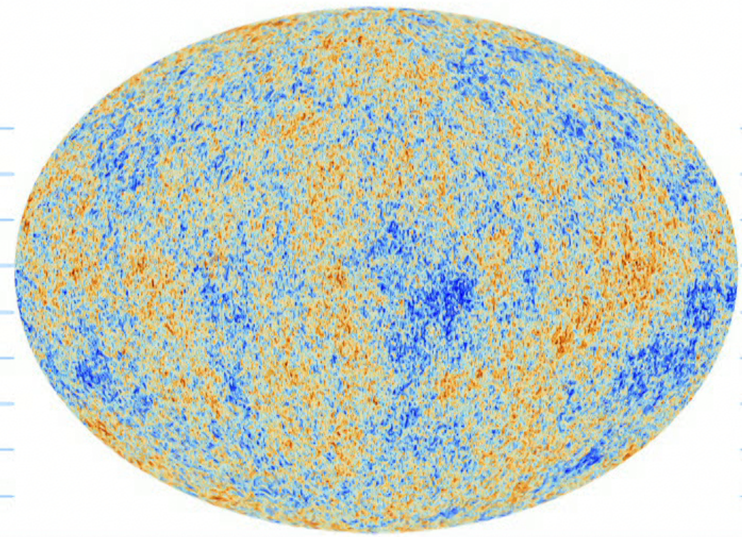
Quantum field fluctuations

Do the amplitudes $\phi(x,t)$ of a quantum field necessarily quantum fluctuate?



Yes: $\square$ Consider a real-valued function $f: \mathbb{R}^3 \to \mathbb{R}$ and a time η_0.

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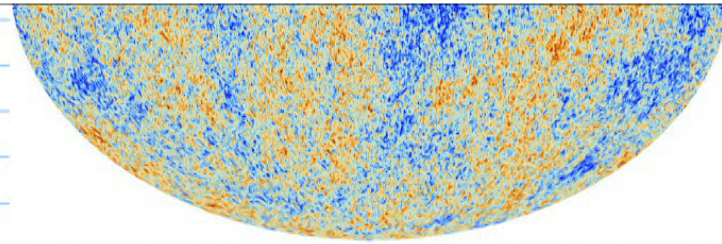

Yes: ▫ Consider a real-valued function $f: \mathbb{R}^3 \to \mathbb{R}$ and a time η_0.

▫ Then, we define the state $|f\rangle$ as the joint eigenstate of all operators $\hat{\phi}(x, \eta_0)$ with eigenvalues $f(x)$:

$$\hat{\phi}(x, \eta_0)\,|f\rangle = f(x)\,|f\rangle$$

▫ There are no quantum fluctuations of the quantum field $\hat{\phi}$ if the system is in such a state $|f\rangle$:

Expectation value: $\bar{\phi}(x,\eta_0) = \langle f | \hat{\phi}(x,\eta_0) | f \rangle$

$$= f(x) \langle f | f \rangle$$

$$= f(x)$$

Variance: $\Delta\phi^2(x,\eta_0) = \langle f | \left(\hat{\phi}(x,\eta_0) - \bar{\phi}(x,\eta_0)\right)^2 | f \rangle$

$$= \langle f | \left(f(x) - f(x)\right)^2 | f \rangle$$

$$= 0 \quad \text{i.e. no fluctuations.}$$

But, can such states $|f\rangle$ occur in practice?

No: The reason is that for these states:

$$\langle f | \hat{H}^{(x)}(\eta_0) | f \rangle = \infty$$ Exercise: Show this.

Hint: For these states, $\Delta\phi = 0$ and so $\Delta\pi^{(x)} = \infty$

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Hint: For these states, $\Delta\phi = 0$, and so $\Delta\pi^{(x)} = \infty$

But $\hat{H}^{(x)}$ contains a term $\hat{\pi}^2$...

Exercise:

What is the analogue of this observation in the case of the harmonic oscillator

$\Longrightarrow$ * Even the state $|f\rangle$ with $f(x)=0$ for all x has ∞ energy and is, therefore, not accessible.

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Exercise:

What is the analogue of this observation in the case of the harmonic oscillator in quantum mechanics?

$\Longrightarrow$ * Even the state $|f\rangle$ with $f(x)=0$ for all x has ∞ energy and is, therefore, not accessible.

* Thus, all finite energy states do possess quantum fluctuations

How to calculate the amount of quantum field fluctuations?

- It is not realistic to measure all operators $\hat{\phi}(x)$ individually.
- Realistically, we could at most hope to measure an average of the values of $\hat{\phi}$ over regions $B \subset \mathbb{R}^3$ of not too small volume $V = L \times L \times L$:

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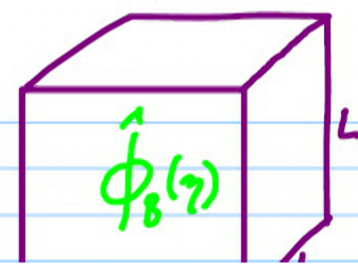
$$\hat{\phi}_B(\eta) := \int_{\mathbb{R}^3} \hat{\phi}(x,\eta)\, W(x)\, d^3x$$

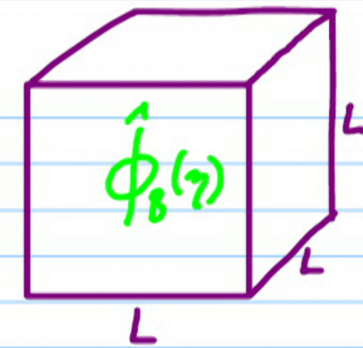
with "window function" W



$$W(x) = \begin{cases} \approx 0 & \text{for all } x \notin B \end{cases}$$

with "window function" W

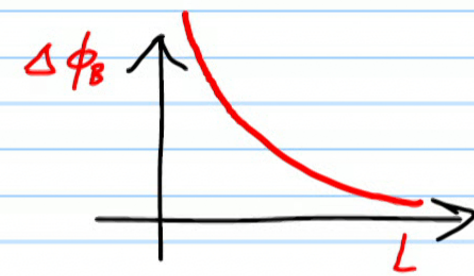
$$W(x) = \begin{cases} \approx 0 & \text{for all } x \notin B \\ \approx V^{-1} & \text{for all } x \in B \end{cases}$$



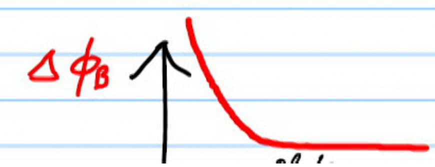
(we'll also allow B to be spherical)

Plan:

- Calculate the typical amplitude of quantum fluctuations as a function of their spatial size.



- Calculate how this relationship is affected by



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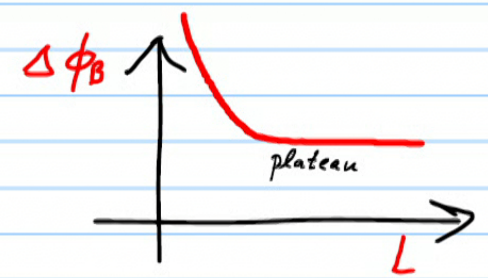
Plan:

- Calculate the typical amplitude of quantum fluctuations as a function of their spatial size.

- Calculate how this relationship is affected by cosmic expansion.



Quantum field fluctuations in FRW spacetime

- Choose conformal time η and comoving coordinates x.
- Choose a region B of size $L \times L \times L$.

  Note: In proper coordinates, this is a box of increasing size:

$$a(\eta)L \times a(\eta)L \times a(\eta)L$$

- Assume that at η_0 the system's state, $|\Omega\rangle$, is the vacuum state:

$$|\Omega\rangle = |vac_{\eta_0}\rangle$$

- We choose the mode functions $v_k(\eta)$ so that $|vac_{\eta_0}\rangle = |0\rangle$ with:

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$$\hat{x}_k(\eta) = \frac{1}{\sqrt{2}}\left(v_k^*(\eta)\, a_k + v_k\, a_{-k}^+\right) \text{ and } a_k|0\rangle = 0$$

⇝ The expectation value of the region-averaged field at a time $\eta \geq \eta_0$:

$$\bar{\phi}_B(\eta) = \langle \Omega | \hat{\phi}_B(\eta) | \Omega \rangle$$

$$= \langle vac_{\eta_0} | \hat{\phi}_B(\eta) | vac_{\eta_0} \rangle$$

$$= \langle 0 | \int_{\mathbb{R}^3} \hat{\phi}(x,\eta)\, W(x)\, d^3x \,| 0 \rangle$$

$$= \langle 0 | \frac{1}{a(\eta)} \int_{\mathbb{R}^3} \hat{\chi}(x,\eta)\, W(x)\, d^3x \,| 0 \rangle$$

$$= \frac{1}{a(\eta)} \int_{\mathbb{R}^3} \langle 0 | \frac{1}{\sqrt{2}} \left(v_k^*(\eta)\, \cancel{a_k} + v_k(\eta)\, \cancel{a_{-k}^+} \right) W(x) | 0 \rangle$$

$$= 0$$

⇒ The average amplitude of $\hat{\phi}_B$ vanishes in the vacuum state.

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$\Rightarrow$ The average amplitude of $\hat{\phi}_B$ vanishes in the vacuum state.

The quantum fluctuations

While $\bar{\phi}_B(\eta)$ vanishes, measurement outcomes for $\hat{\phi}_B(\eta)$ are not fully predictable because subject to fluctuations around zero with this standard deviation:

$$\Delta \phi_B^2(\eta) = \langle \Omega | \left(\hat{\phi}_B(\eta) - \cancel{\bar{\phi}_B(\eta)}\right)^2 | \Omega \rangle$$

$$= \langle 0 | \hat{\phi}_B(\eta)^2 | 0 \rangle$$

$$= \frac{1}{a(\eta)^2} \langle 0 | \left(\int_{\mathbb{R}^3} \hat{x}(x,\eta)\, W(x)\, d^3x \right)^2 | 0 \rangle$$

$$= \quad \ldots$$ Exercise: fill in the steps

$$= \langle 0 | \hat{\phi}_0(\eta)^2 | 0 \rangle$$

$$= \frac{1}{a(\eta)^2} \langle 0 | \left(\int_{\mathbb{R}^3} \hat{\chi}(x,\eta)\, W(x)\, d^3x \right)^2 | 0 \rangle$$

$=$ ... Exercise: fill in the steps

$$= \frac{1}{2a(\eta)^2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |v_k(\eta)|^2 |\widetilde{W}(k)|^2 d^3k$$

Fourier transform of the

$$= \frac{1}{2a(\eta)^2} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |v_k(\eta)|^2 |\widetilde{W}(k)|^2 d^3k$$

Fourier transform of the window function $W(x)$.

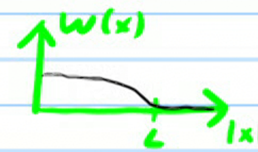
Assume now for simplicity that B is spherical with radius L. Then use spherical coordinates:

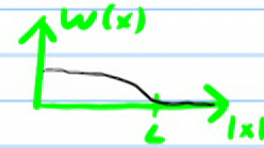
$$= \frac{1}{2a(\eta)^2} \frac{1}{(2\pi)^3} \int_0^\infty k^2 \, 4\pi \, |v_k(\eta)|^2 \, \widetilde{W}(k) \, dk$$

$k = \sqrt{k_1^2 + k_2^2 + k_3^2}$

Notice the dimension dependence of the integral's measure!

Approximation: Consider that:



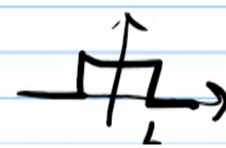
Approximation: Consider that: 

↑ typical scale is L

(using Fourier) $\Rightarrow$ We can assume that, roughly:

$$\tilde{W}(k) \approx 0 \text{ for } |k| > \frac{2\pi}{L}$$

$\frac{\sin(kL)}{kL}$

Example: If $W(x) =$ 

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and we approximate that $\tilde{W}(k) \approx$ 

$2\pi/L$

$$\Longrightarrow \Delta \phi_B^2(\eta) \approx \frac{1}{4\pi^2 a(\eta)^2} \int_0^{2\pi/L} k^2 \, |v_k(\eta)|^2 \, dk$$

In the integral, the values of $|v_k(\eta)|^2$ for small k are suppressed by k^2.

$\Rightarrow$ Can approximately replace $|v_k(\eta)|$ by its value at $k = 2\pi/L$:

$$\Delta \phi_B^2(\eta) \approx \frac{1}{4\pi^2 a(\eta)^2} \int_0^{2\pi/L} k^2 \, |v_{\frac{2\pi}{L}}(\eta)|^2 \, dk$$

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$$\Delta\phi_B^2(\eta) \approx \frac{1}{4\pi^2} \frac{(2\pi)^3}{3L^3} \left| \frac{v_{\frac{2\pi}{L}}(\eta)}{a(\eta)} \right|^2$$

$$= \frac{2\pi}{3L^3} \left| w_{\frac{2\pi}{L}}(\eta) \right|^2$$

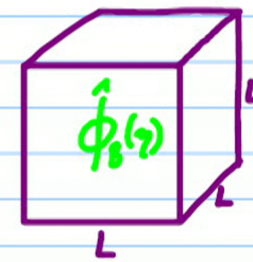
Here, w is the mode function of $\hat{\phi}$, because $\frac{\hat{\chi}(\eta)}{a(\eta)} = \hat{\phi}(\eta)$.

Conclusion:

Assume that at a time η_0 the vacuum state was the state $|0\rangle$ corresponding to $\{v_k\}$.

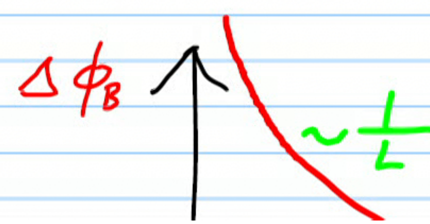
Then, the typical amplitude of fluctuations of size L at an arbitrary time η is:

$$\Delta\phi_B^2(\eta) \approx \frac{2\pi}{3L^3}\left|\frac{V_{\frac{2\pi}{L}}(\eta)}{a(\eta)}\right|^2 = \frac{2\pi}{3L^3}\left|W_{\frac{2\pi}{L}}(\eta)\right|^2$$



Special case: Minkowski space (massless field)

- $v_k(\eta) = w_k(\eta)$
- $\eta = t$
- $|v_k(t)|^2 = \frac{1}{\omega_k(t)} = \frac{1}{|k|} = \frac{L}{2\pi}$



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$$\Rightarrow \Delta\phi_B^2 \approx \frac{1}{3L^2} \quad \Rightarrow \Delta\phi_B \approx \frac{1}{\sqrt{3}}\frac{1}{L}$$

How to describe field quantum fluctuations using correlators

A primer on classical fluctuations:

- Assume $n(t)$ is a Ω-bandlimited gaussian white noise signal, (all frequencies occur to same amount)

How to describe field quantum fluctuations using correlators

A primer on classical fluctuations:

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- Assume $n(t)$ is a Ω-bandlimited gaussian white noise signal, i.e., a random signal with gaussian distributed amplitudes, filtered to leave only frequencies in the interval $[-\Omega, \Omega]$.

- Then, for an ensemble of such noise signals, one can show:

$$\overline{n(t)} = 0 \qquad \forall t$$

"2-point correlator": $$\overline{n(t)\,n(t+L)} = c\,\frac{\sin(\Omega L)}{\Omega L} \qquad \forall t$$

This noise is ergodic, i.e. we could instead average over all t:

$$\underbrace{\overline{n(t)\,n(t+L)}}_{\text{"Auto-correlator"}} = \int f(t)\, f(t+L)\, dt$$

(suitably regularize if non-normalizable, e.g. $\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} f(t)\, f(t+L)\, dt$)

$$= \iiint \tilde{f}(\omega)\,\tilde{f}(\omega')\, e^{i\omega t}\, e^{i\omega'(t+L)}\, dt\, d\omega\, d\omega'$$

$$= \iiint \underbrace{e^{it(\omega+\omega')}\, dt}_{=(2\pi)^{-1}\delta(\omega+\omega')}\ \tilde{f}(\omega)\,\tilde{f}(\omega')\, e^{i\omega' L}\, d\omega\, d\omega'$$

$$= \frac{1}{2\pi}\int \tilde{f}(\omega)\,\tilde{f}(-\omega)\, e^{i\omega L}\, d\omega$$

$$= \frac{1}{2\pi}\int \underbrace{|\tilde{f}(\omega)|^2}_{\text{"Spectral power function"}}\, e^{i\omega L}\, d\omega$$

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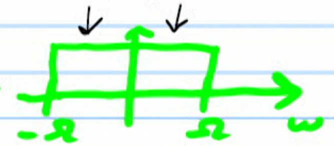
$$= \frac{1}{2\pi}\int \overbrace{|\tilde{f}(\omega)|^2}^{\text{"Spectral power function"}}\,e^{i\omega L}\,d\omega$$

$\Rightarrow$ ☐ Autocorrelation and power spectrum are a Fourier pair!

Recall: flatness of spectrum means noise is "white"

☐ For white bandlimited noise: $|\tilde{f}(\omega)|^2 =$ 

Exercise: Show that its Fourier transform is indeed $\sin(\Omega L)/\Omega L$.

The 2-point correlator in QFT:

$$\langle 0|\hat{\phi}(\vec{x},\eta)\,\hat{\phi}(\vec{x}+\vec{L},\eta)|0\rangle = \frac{1}{a(\eta)^2}\langle 0|\hat{\chi}(\vec{x},\eta)\hat{\chi}(\vec{x}+\vec{L},\eta)|0\rangle$$

Exercise: use mode expansion of $\hat{\chi}$ and spherical coordinates to derive:

$$= \frac{1}{a(\eta)^2}\int_0^\infty \frac{k^2\,dk}{4\pi^2}\,\frac{\sin(kL)}{kL}\,|v_k(\eta)|^2 \qquad \text{with } k=|\vec{k}|,\ L=|\vec{L}|.$$

Notice dimension dependence of the integral's measure!

Observe: $\frac{\sin(kL)}{kL} \approx \begin{cases} 1 & \text{for } k<1/L \\ 0 & \text{for } k>1/L \end{cases}$

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Observe: $k^2\,\frac{\sin(kL)}{kL}$ has largest amplitude around $k \approx \frac{2\pi}{L}$

$\Rightarrow$ Estimate:

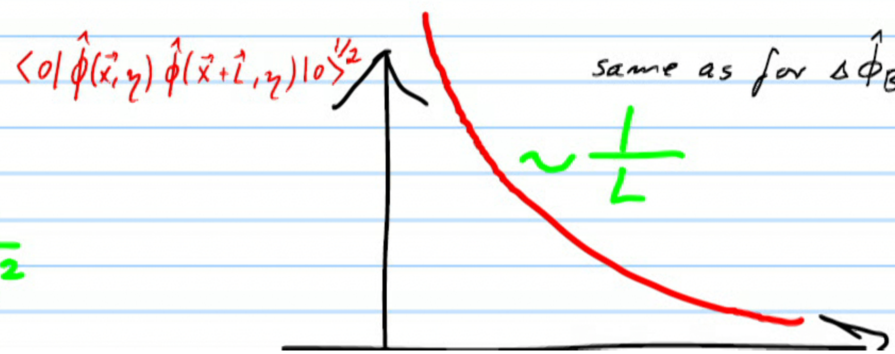
$$\langle 0|\hat{\phi}(\vec{x},\eta)\,\hat{\phi}(\vec{x}+\vec{L},\eta)|0\rangle \approx a(\eta)^{-2}\int_0^{2\pi/L}\frac{k^2}{4\pi^2}\left|V_{\frac{2\pi}{L}}(\eta)\right|^2$$

$$\approx a(\eta)^{-2}\frac{k^3}{12\pi^2}|V_k(\eta)|^2\Big|_{k=\frac{2\pi}{L}}$$

Special case: Minkowski space

Mode function: $|V_k|^2 = \frac{1}{|k|}$

$$\Rightarrow \langle 0|\hat{\phi}(\vec{x},\eta)\,\hat{\phi}(\vec{x}+\vec{L},\eta)|0\rangle \approx \frac{1}{3}\frac{1}{L^2}$$



same as for $\Delta\hat{\phi}_B$!

Special case: Minkowski space

Mode function: $|v_k|^2 = \frac{1}{|k|}$

$\Rightarrow \langle 0|\hat{\phi}(\vec{x},\eta)\,\hat{\phi}(\vec{x}+\vec{L},\eta)|0\rangle \approx \frac{1}{3}\frac{1}{L^2}$

We notice: The variance in a box scales like the correlator!
Both are good measures of the fluctuations.

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Definition: We define the so-called **Fluctuation Spectrum** at time η as a function of k:

$$\boxed{\delta\phi_k(\eta) := a(\eta)^{-1} k^{3/2} |v_k(\eta)|} \qquad k = \frac{2\pi}{L}$$

Special case Minkowski space with massive field:

- Scale factor: $a(\eta) = 1$ for all η
- Mode functions:

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$$v_k(\eta) = \frac{1}{\sqrt{\omega_k}} e^{i\eta\omega_k} \quad \text{with} \quad \omega_k = \sqrt{k^2 + m^2}$$

$\Rightarrow$ The fluctuation spectrum is: (recall: $k = \frac{2\pi}{L}$)

$$\delta\phi_k = \frac{k^{3/2}}{(m^2 + k^2)^{1/4}} \approx \begin{cases} k & \text{for } k \to \infty \\ \frac{k^{3/2}}{\sqrt{m}} & \text{for } k \to 0 \end{cases}$$

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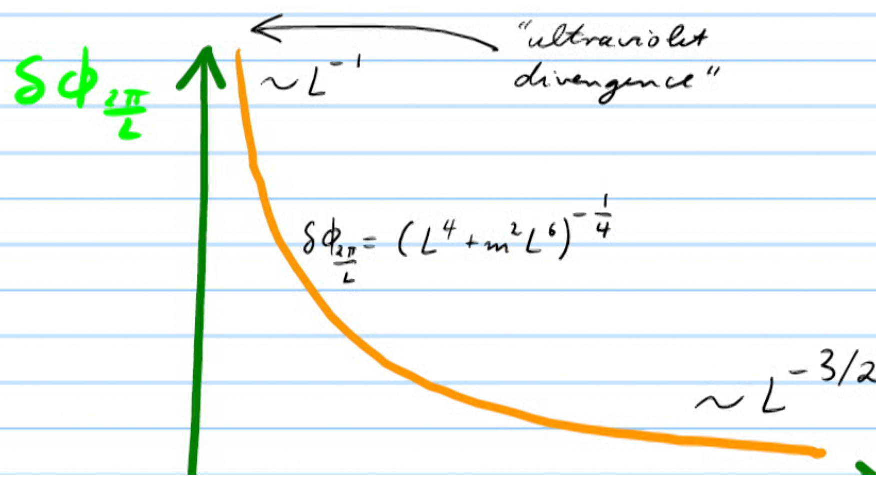
and as a function of L it is: $\delta\phi_{\frac{2\pi}{L}} = \frac{(2\pi)^{3/2} L^{-3/2}}{\left(\frac{4\pi}{L^2} + m^2\right)^{1/4}}$



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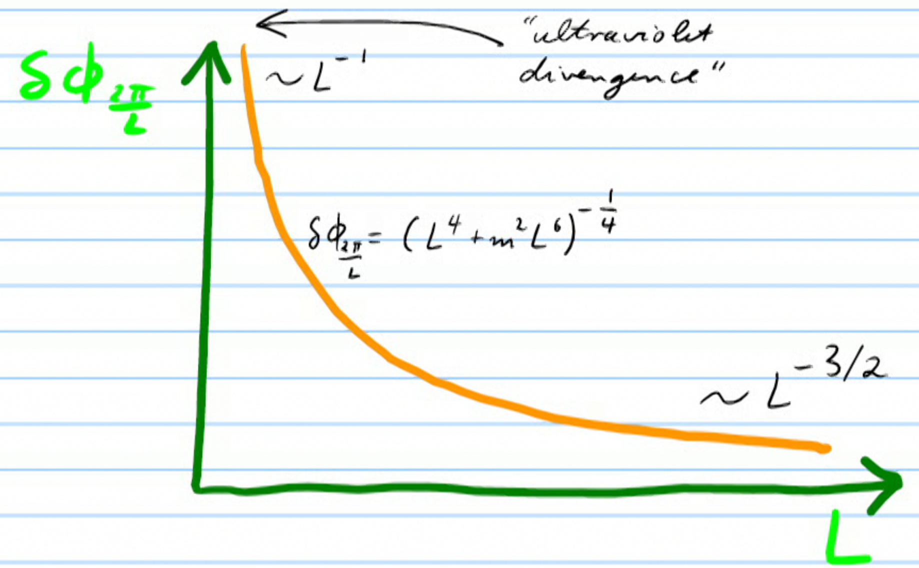


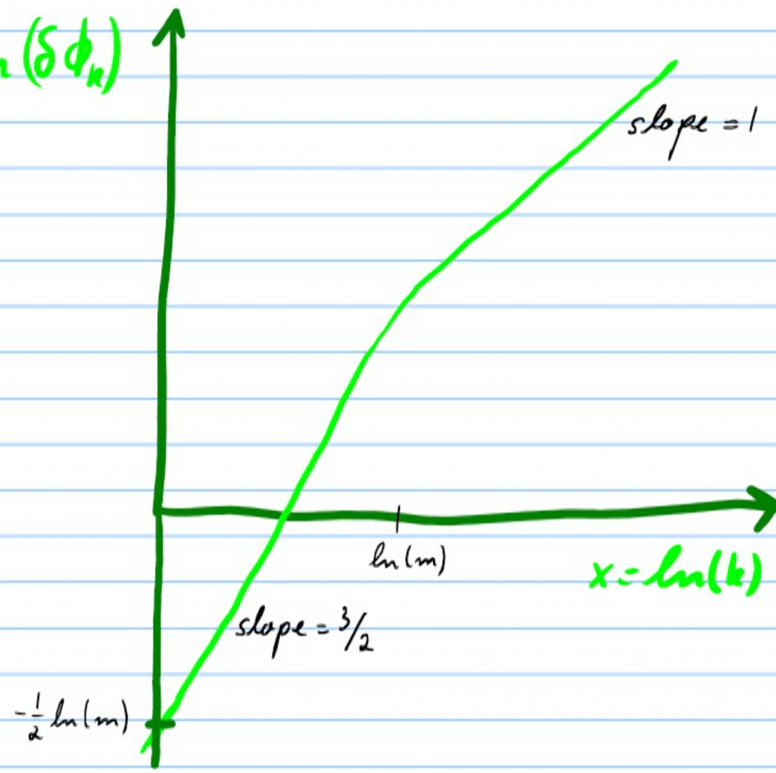
Recall "Log-Log plots":

$$\boxed{x := \ln(k), \quad y = \ln(\delta\phi_k)}$$

Here: $\ln \delta\phi_k = \ln\left(\frac{k^{3/2}}{(m^2+k^2)^{1/4}}\right) \approx \begin{cases} \ln k & \text{for } k \to \infty \\ \underbrace{\ln\left(\frac{k^{3/2}}{\sqrt{m}}\right)}_{= -\frac{1}{2}\ln(m) + \frac{3}{2}\ln k} & \text{for } k \to 0 \end{cases}$

Thus: $y \approx \begin{cases} x & \text{for } x \to \infty \\ -\frac{1}{2}\ln(m) + \frac{3}{2}x & \text{for } x \to -\infty \end{cases}$

Plot:



We notice that, in Minkowski space, large scale