

Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 15

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Abstract:

QFT for Cosmology, Achim Kempf, Lecture 15

Note Title

Solving the quantized K.G. eqn. on FRW spacetimes

Recall:

1.) We obtain the solution $\hat{\phi}(x,t)$ through the ansatz

$$\hat{\phi}(x,t) = \sum_{\mathbf{k}} u_{\mathbf{k}}(x,t) a_{\mathbf{k}} + u_{\mathbf{k}}^*(x,t) a_{\mathbf{k}}^* \quad (*)$$

* if we use operators $a_{\mathbf{k}}$ obeying $[a_{\mathbf{k}}, a_{\mathbf{k}'}^*] = \delta^3(\mathbf{k} - \mathbf{k}')$ and

* if we find classical solutions $\{u_{\mathbf{k}}(x,t)\}$ of the K.G. eqn., called mode functions, which obey:

$$\sqrt{|g|} g^{00} \sum_{\mathbf{k}} \left(u_{\mathbf{k}}(x,t) \frac{\partial}{\partial x^0} u_{\mathbf{k}}^*(x,t) - u_{\mathbf{k}}^*(x,t) \frac{\partial}{\partial x^0} u_{\mathbf{k}}(x,t) \right) = i \delta^3(\vec{x} - \vec{x}') \quad (4)$$

2.) Then, we can use the $\{a_k\}$ to build a convenient basis in the Hilbert space:

□ Namely: $|0\rangle$ is the vector obeying $a_k|0\rangle = 0$

□ The other basis vectors are:

$$a_k^+|0\rangle, \dots, \frac{1}{\sqrt{m!}}(a_k^+)^m|0\rangle, \dots, a_k^+ a_l^+|0\rangle, \dots$$

$$\frac{1}{\sqrt{m_1!}} \dots \frac{1}{\sqrt{m_m!}} (a_{k_1}^+)^{m_1} \dots (a_{k_m}^+)^{m_m} |0\rangle, \dots \text{ etc.}$$

3.) Choosing a different set of classical solutions $\{\tilde{u}_k(x,t)\}$ which obey (C) yields the same $\hat{\phi}(x,t)$, namely

$$\hat{\phi}(x,t) = \sum_k \tilde{u}_k(x,t) \tilde{a}_k + \tilde{u}_k^+(x,t) \tilde{a}_k^+$$

but the basis of vectors $|\tilde{0}\rangle, \tilde{a}_k^+|\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_k^+|\tilde{0}\rangle, \dots$ is a different basis. Recall: Stone von Neumann theorem.

Application to FRW spacetime

- For convenience (namely, to avoid a "friction"-type term) we aim to solve not for $\hat{\phi}(x,t)$ directly, but instead for:

$$\hat{\chi}(\eta, x) := a(\eta) \hat{\phi}(\eta, x)$$

- In terms of $\hat{\chi}(\eta, x)$ the quantum K.G. eqn. reads:

$$\hat{\chi}''(\eta, x) - \Delta \hat{\chi}(\eta, x) + \left(m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)} \right) \hat{\chi}(\eta, x) = 0$$

- *Note:* This is a partial differential equation because both time and space derivatives occur.

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□ **Note:** This is a partial differential equation because both time and space derivatives occur.

□ **Observation:** The derivatives $\frac{\partial}{\partial x_i}$ become multiplication operators ik_i under spatial Fourier transform.

□ **Plan:** Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more manageable set of ordinary DEs.

▣ Observation: The derivatives $\frac{\partial}{\partial x_i}$ become multiplication operators ik_i under spatial Fourier transform.

▣ Plan: Before trying to solve it, use Fourier to transform the K.G. eqn. from a partial DE into a more manageable set of ordinary DEs.

▣ Define: $\hat{\chi}_k(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}(\eta, x) e^{-ikx} d^3x$

i.e.: $\hat{\chi}(\eta, x) = \int \frac{1}{(2\pi)^{3/2}} \hat{\chi}_k(\eta) e^{ikx} d^3k$

▣ Analogously:

$$\hat{\pi}_k^{(x)}(\eta) := \int \frac{1}{(2\pi)^{3/2}} \hat{\pi}^{(x)}(\eta, x) e^{-ikx} d^3x$$

□ Thus, in terms of $\hat{x}_k(\eta)$, the K.G. eqn. reads:

$$\hat{x}_k''(\eta) + \left(k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)} \right) \hat{x}_k(\eta) = 0 \quad (\text{EOM})$$

⇒ for each *comoving* Fourier mode k the K.G. eqn. is the eqn. of a harmonic oscillator with time-dependent frequency

$$\hat{x}_k''(\eta) + \omega_k^2(\eta) \hat{x}_k(\eta) = 0$$

$$\text{with: } \omega_k(\eta) := \sqrt{k^2 + m^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)}}$$

Remark: It is not surprising that the frequency of each comoving mode is changing because its physical wavelength is changing too.

In extreme cases:

The frequency $\omega_k(\eta)$ may become imaginary, namely if $a''(\eta)$ is large enough, i.e., if the expansion is rapid enough. Note that the discriminant also depends on k , i.e., some modes may have imaginary frequencies while others don't.

□ Exercise:

* Show that $\hat{\mathcal{X}}_k^+(\eta) = \hat{\mathcal{X}}_{-k}(\eta)$, $\hat{\Pi}_k^{(cc)+}(\eta) = \hat{\Pi}_{-k}^{(cc)}(\eta)$ (HC)

* Show that

$$[\hat{\mathcal{X}}_k(\eta), \hat{\Pi}_{k'}(\eta)] = i \delta^3(k+k') \quad (\text{CCR})$$

$$\text{i.e. } [\hat{\mathcal{X}}_k(\eta), \hat{\Pi}_{k'}^+(\eta)] = i \delta^3(k-k')$$

□ In order to solve EoM, HC, CCR for $\hat{x}_k(\eta)$, we make this ansatz:

$$\hat{x}_k(\eta) := \frac{1}{\sqrt{2}} \left(v_k^*(\eta) a_k + v_k(\eta) a_k^\dagger \right) \quad (A)$$

convenient later

□ Exercise: Express the mode functions $u_k(\eta, x)$ of (*) in terms of the functions $v_k(\eta)$.

□ Proposition: The ansatz (A)

1.) solves the hermiticity condition (HC) by construction.

2.) solves the (EoM), if the $v_k(\eta)$ each solve (EoM) as (complex!) number-valued functions:

(Note: The equation depends only on $|k|$, not on the direction of k . Thus if $v_k(z)$ is a solution for one k then it is solution for all k' with $|k'| = |k|$. \Rightarrow We can and will choose $v_{k'}(z) = v_k(z)$)

$$v_k''(z) + \left(k^2 + m^2 a^2(z) - \frac{a''(z)}{a(z)} \right) v_k(z) = 0 \quad (M)$$

3.) the commutation relations (CCR) if the v_k are chosen such that they also obey:

$$v_k'(z) v_k^*(z) - v_k(z) v_k'^*(z) = 2i \quad (W)$$

□ Exercise:

a) Prove the proposition.

b.) Assume that $v_k(z)$ is any solution of (EOM). Show that if (W) holds at one time then it holds at all time.

Conclusion:

In order to obtain the solution $\hat{\phi}(\gamma, x)$, we do:

- A) Find for each mode $k \in \mathbb{R}^3$ a solution $v_k(\gamma)$ to (M), i.e., a solution to the classical harmonic oscillator with time-dependent frequency.
- B) Make sure $v_k(\gamma)$ obeys (W), if need be by multiplying with a constant. (Recall exercise b))
- C) Build a basis in the Hilbert space:
 $a_k |0\rangle = 0$, $a_k^\dagger |0\rangle$, $a_k^\dagger a_k^\dagger |0\rangle$, etc...

Choice of mode solutions $\{V_k(z)\}$

- For each choice, say $\{V_k(z)\}_{k \in \mathbb{R}^3}$ or $\{\tilde{V}_k(z)\}_{k \in \mathbb{R}^3}$ we obtain the same $\hat{\phi}(x,t)$ but the bases

$$|0\rangle, a_k^+ |0\rangle, a_k^+ a_{k'}^+ |0\rangle, \dots$$

and

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_{k'}^+ |\tilde{0}\rangle, \dots$$

will of course be different.

- We will often find it convenient to use the basis $|0\rangle, a_k^+ |0\rangle, a_k^+ a_{k'}^+ |0\rangle, \dots$ that comes with one set of mode functions $\{V_k(z)\}$ at one time (say initially) and then the basis $|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_{k'}^+ |\tilde{0}\rangle, \dots$ of some $\{\tilde{V}_k(z)\}$ later.

Why?

□ In the Heisenberg picture, the system's state vector is always the same Hilbert space vector.

□ But the observables evolve in time!

⇒ The meanings of all Hilbert space vectors change over time

⇒ We may, e.g., choose a set $\{v_n\}$ whose vector $|0\rangle$ happens to be the vacuum state at one time and we may choose another set $\{\tilde{v}_n\}$ whose vector $|\tilde{0}\rangle$ happens to be the vacuum state at another time.

□ How many possible choices of

$$\{v_k(\gamma)\}_{k \in \mathbb{R}^3}, \{\tilde{v}_k(\gamma)\}_{k \in \mathbb{R}^3}, \{\hat{\tilde{v}}_k(\gamma)\}_{k \in \mathbb{R}^3}, \dots$$

do exist?

□ Let us consider each mode, k , separately:

□ The solution space of (M) , for fixed k ,

$$v_k''(\gamma) + \left(k^2 + m^2 a^2(\gamma) - \frac{a''(\gamma)}{a(\gamma)} \right) v_k(\gamma) = 0$$

has of course 2 complex dimensions.

— Note: Every solution obeying (W) must be complex-valued. Why?

□ If v_k is a complex-valued solution, then

$$v_k \text{ and } v_k^*$$

form a basis in the solution space.



Every solution, \tilde{v}_k , is a linear combination of v_k, v_k^* , i.e., there must exist $\alpha, \beta \in \mathbb{C}$, so that:

$$\tilde{v}_k(\gamma) = \alpha_k v_k(\gamma) + \beta_k v_k^*(\gamma)$$

□ The actual dimensionality is 3!

The solution space thus has 4 real dimensions, but one real dimension is lost because the solutions v_k, \tilde{v}_k , etc must also obey (W), i.e.:

$$v_k'(\gamma) v_k^*(\gamma) - v_k(\gamma) v_k'^*(\gamma) = 2i \quad (W)$$

(i.e. $\text{Im}(v'v^*) = 1$, which is only one real equation)

□ Proposition:

Assume v_k obeys (W). Then, \tilde{v}_k defined through

$$\tilde{v}_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta) \quad (B)$$

also obeys (W), iff the coefficients α_k, β_k obey:

$$|\alpha_k|^2 - |\beta_k|^2 = 1$$

□ Proof: Exercise

⇒ we easily obtain:
$$\begin{aligned} \hat{\mathcal{C}}_k(\eta) &= \frac{1}{\sqrt{2}} (v_k^*(\eta) a_k + v_k(\eta) a_{-k}^+) \\ &= \frac{1}{\sqrt{2}} (\tilde{v}_k^*(\eta) \tilde{a}_k + \tilde{v}_k(\eta) \tilde{a}_{-k}^+) \\ &= \dots \end{aligned} \quad \left. \vphantom{\hat{\mathcal{C}}_k(\eta)} \right\} (P)$$

□ Terminology: Such a transformation from one choice $\{v_k\}, a_k$ and corresponding basis

$$|0\rangle, a_k^+ |0\rangle, a_k^+ a_k^+ |0\rangle, \dots$$

to some $\{\tilde{v}_k\}, |\tilde{0}\rangle, \tilde{a}_k$ and their basis

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots$$

is called a "Bogolubov transformation".

Strategy: We have two tasks now:

* Make Bogolubov Hilbert basis transforms explicit.

(E.g. so that $|0\rangle$ is, at least at one time, the vacuum.)

* Find out when which choice of $\{v_k\}$ is convenient.

Bogolubov transformations of Hilbert bases

□ How can we express the basis vectors

$$|\tilde{0}\rangle, \tilde{a}_k^+ |\tilde{0}\rangle, \frac{1}{\sqrt{2!}} \tilde{a}_k^2 |\tilde{0}\rangle, \dots, \tilde{a}_k^+ \tilde{a}_k^+ |\tilde{0}\rangle, \dots$$

as linear combinations of the basis

$$|0\rangle, a_k^+ |0\rangle, \frac{1}{\sqrt{2!}} a_k^2 |0\rangle, \dots, a_k^+ a_k^+ |0\rangle, \dots ?$$

□ Proposition: Equations (B) & (P) yield: $a_k = d_k^+ \tilde{a}_k + \beta_k \tilde{a}_{-k}^+$

Proof: Exercise.

□ Now we observe that $a_k |0\rangle = 0$ becomes:

$$(d_k^+ \tilde{a}_k + \beta_k \tilde{a}_{-k}^+) |0\rangle = 0$$

Proof: Exercise.

□ Now we observe that $a_k |0\rangle = 0$ becomes:

$$(d_k^* \tilde{a}_k + \beta_k \tilde{a}_{-k}^*) |0\rangle = 0$$

□ Try to solve for $|0\rangle$ using ansatz: $|0\rangle := \left(\prod_k f_k(\tilde{a}_k^+, \tilde{a}_{-k}^+) \right) |\tilde{0}\rangle$

□ Proposition:

$$|0\rangle = \left[\prod_k \frac{1}{|d_k|^{1/2}} e^{-\frac{\beta_k}{2d_k^*} \tilde{a}_k^+ \tilde{a}_{-k}^+} \right] |\tilde{0}\rangle \quad (\text{T})$$

← needed for normalization

□ Proof: Exercise.

Hint: Idea $\rho \lambda a^+ - \rho \lambda a^+ + 2 \rho \lambda a^+$

□ Proposition:

$$|0\rangle = \left[\prod_k \frac{1}{|\alpha_k|^{1/2}} e^{-\frac{\beta_k}{2\alpha_k} \hat{a}_k^+ \hat{a}_k^+} \right] |\tilde{0}\rangle \quad (T)$$

← needed for normalization

□ Proof: Exercise.

Hint: Use $a e^{\lambda a^+} = e^{\lambda a^+} a + \lambda e^{\lambda a^+} \dots$

Interpretation of (T):

□ Assume, e.g., that $|0\rangle$ and $|\tilde{0}\rangle$ are those Hilbert space vectors which happen to be the vacuum state vectors at the times η_1 and η_2 respectively.

(We will soon explore how to identify the vacuum state at any given time)

- ▢ Assume at time η_1 , the system's state $|\Omega\rangle$ is the vacuum state (in the sense of no particle state). Then it is convenient to choose the mode functions $\{v_n\}$ so that:

$$|\Omega\rangle = |0\rangle$$

At a later time, η_2 , the system is still in this state but the vacuum is then some other vector $|\tilde{0}\rangle$, which obeys $\tilde{a}_n|\tilde{0}\rangle = 0$ with mode functions $\tilde{v}_n(t)$.

- ▢ Thus, from (T) we see that the system's state, $|\Omega\rangle$, is at η_2 a state with many particles.
- ▢ **Note:** the particles have been created in $k, -k$ pairs.
- ▢ Intuitively: The expansion rips virtual particle + antiparticle pairs apart.