Title: PSI 2017/2018 - Quantum Gravity - Lecture 6

Date: Mar 26, 2018 10:15 AM

URL: http://pirsa.org/18030043

Abstract:

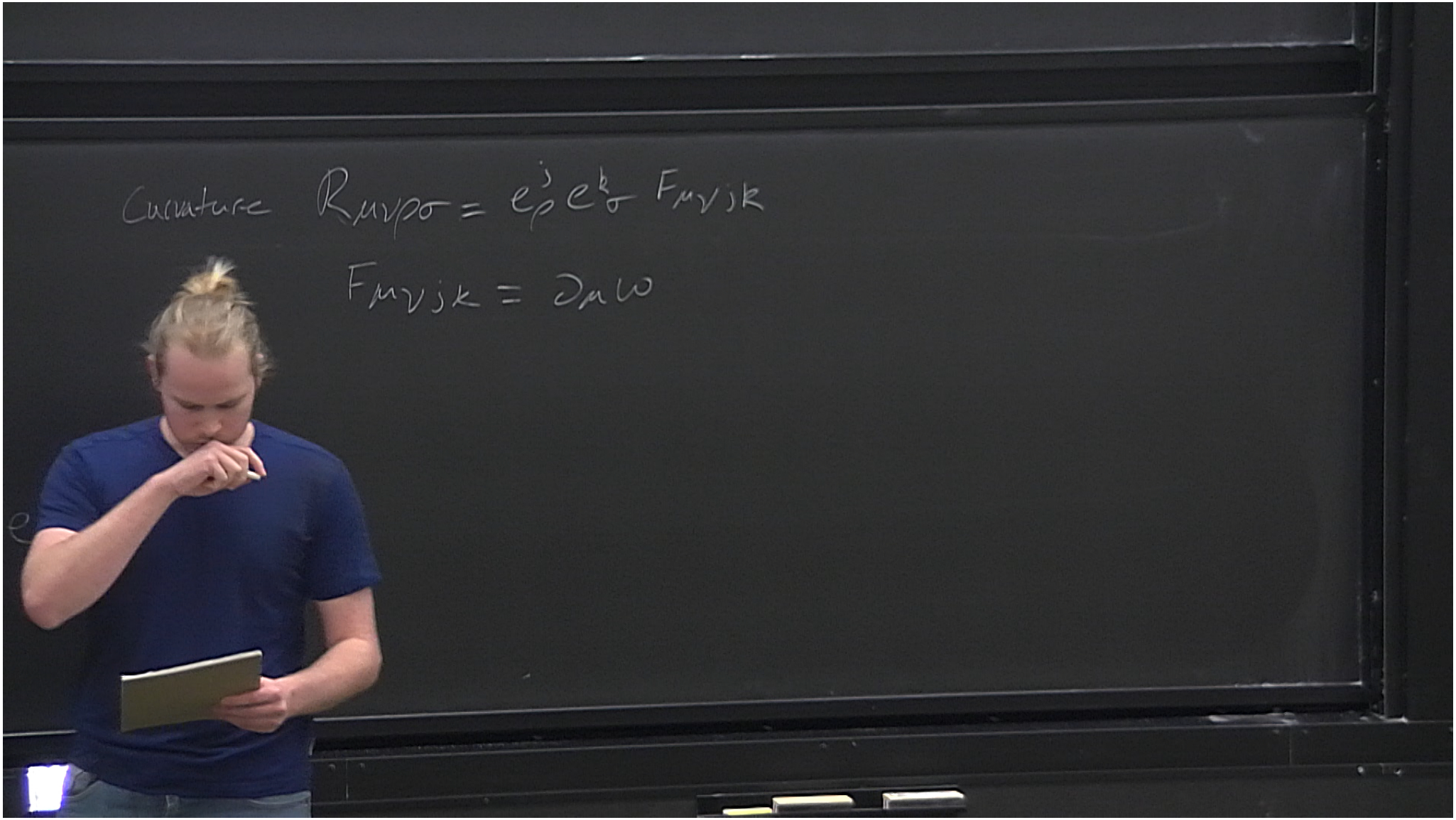
3D Gravity via Triads

$g_{\mu\nu} = e^i_\mu e^j_\nu \, s_{ij}$ $\quad SO(3)$

CoV Diff D_μ, Spin connection $\omega_\mu{}^i{}_j$,

Compatibility $D_\mu e^i_\nu = 0 \Rightarrow \omega_\mu{}^i{}_j = -e^\nu_j \nabla_\mu e^i_\nu$

Metricity $\nabla_\mu (g_{\alpha\beta}) = 0 \Rightarrow \omega_{\mu(ij)} = 0$


Curvature $R_{\mu\nu\rho\sigma} = e^j_\rho e^k_\sigma F_{\mu\nu jk}$
$F_{\mu\nu jk} = \partial_\mu \omega$

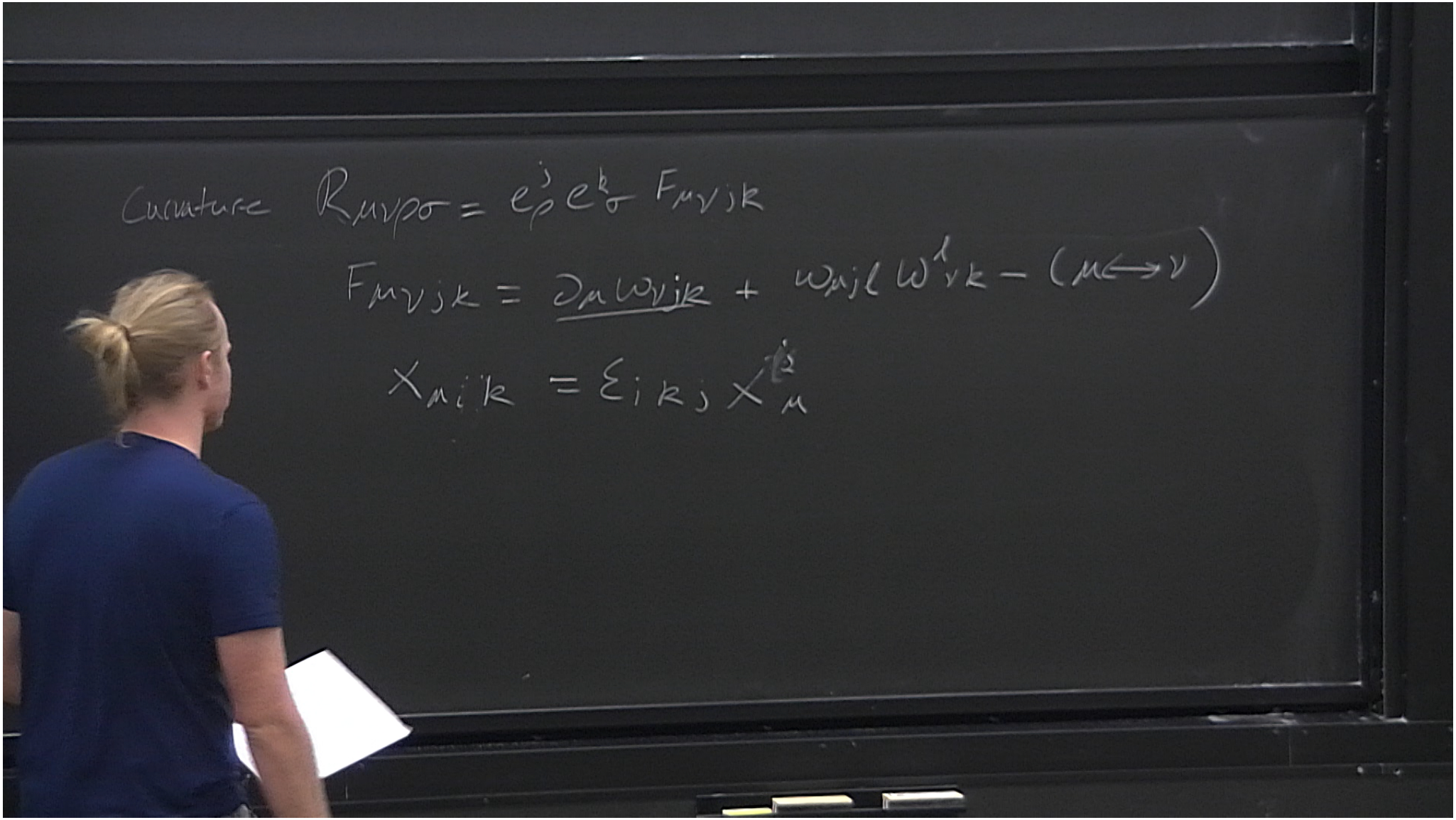
3D Gravity via Triads

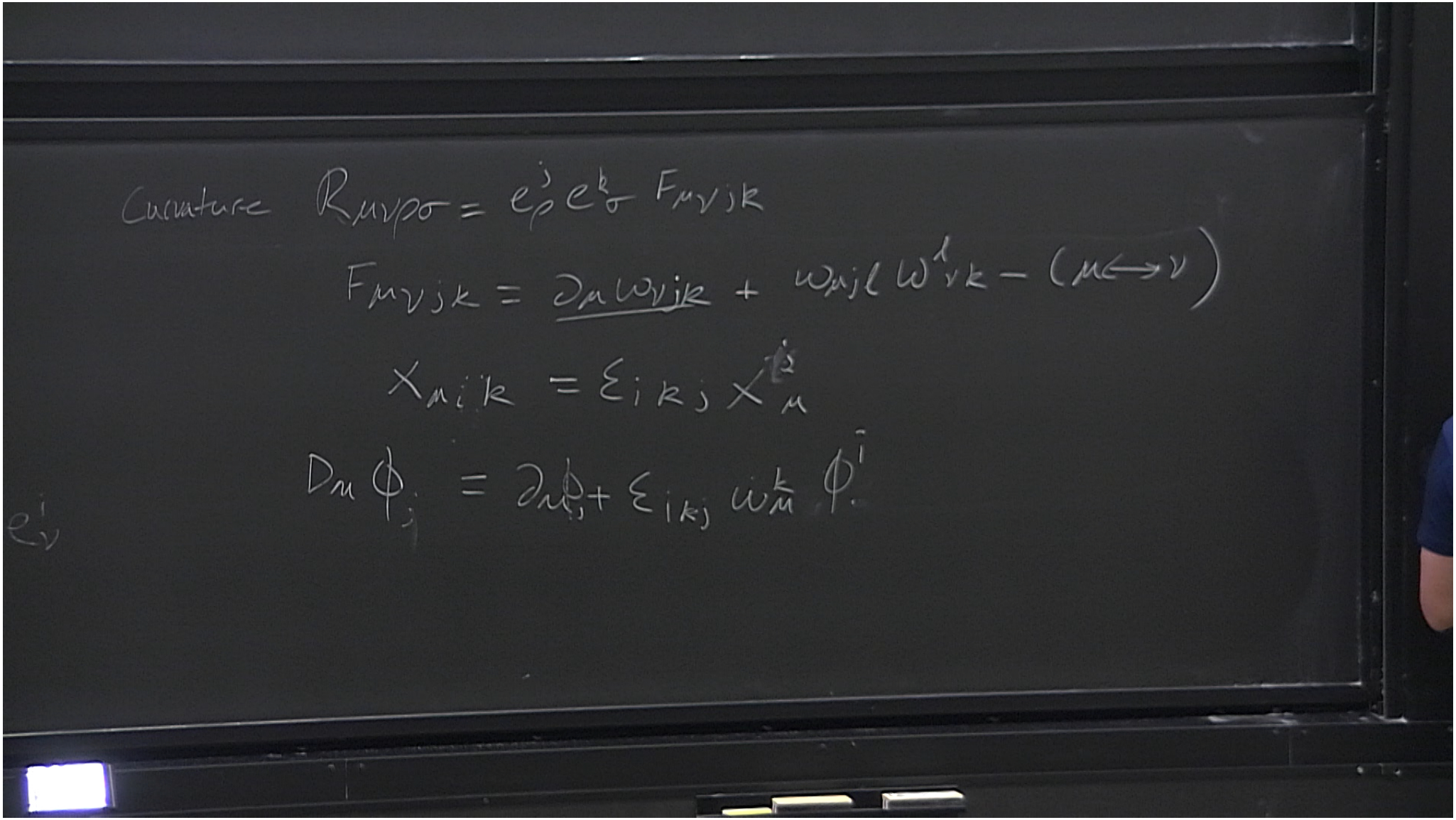
$g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}$ $SO(3)$

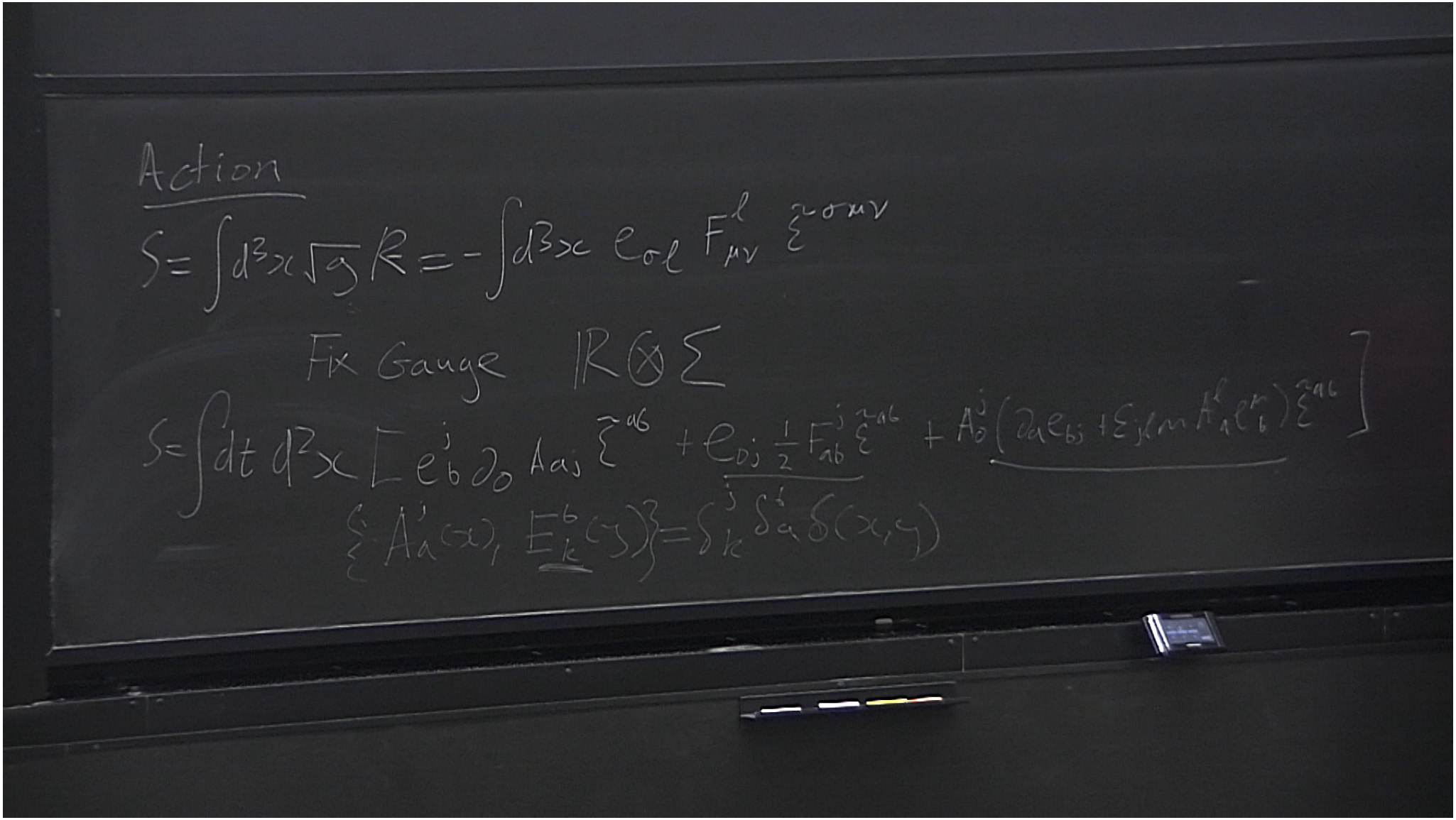
Cov. Diff D_μ, Spin connection $\omega_\mu{}^i{}_j$

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Curvature $R_{\mu\nu\rho\sigma} = e^j_\rho e^k_\sigma F_{\mu\nu jk}$
$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} + \omega_{\mu jl}\omega^l_{\nu k} - (\mu \leftrightarrow \nu)$
$\chi_{\mu ik} = \varepsilon_{ikj}\chi^j_\mu$


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$\chi_{\mu ik} = \varepsilon_{ikj}\chi^j_\mu$
$D_\mu \phi_j = \partial_\mu \phi_j + \varepsilon_{ikj}\omega^k_\mu \phi^i$
e^i_ν


Action
$S=\int d^3x\sqrt{g}R=-\int d^3x\, e_{0l}F^{l}_{\mu\nu}\tilde{\varepsilon}^{0\mu\nu}$
Fix Gauge $R\otimes\Sigma$
$S=\int dt\, d^2x\left[E^{j}_{b}\partial_0 A_{aj}\tilde{\varepsilon}^{ab}+e_{0j}\frac{1}{2}F^{j}_{ab}\tilde{\varepsilon}^{ab}+A^{j}_{0}\left(\partial_a e_{bj}+\varepsilon_{jlm}A^{l}_{a}e^{m}_{b}\right)\tilde{\varepsilon}^{ab}\right]$
$\{A^{i}_{a}(x),E^{b}_{k}(y)\}=\delta^{i}_{k}\delta^{b}_{a}\delta(x,y)$

$$S = \frac{1}{2}\int e e^I F^I_{\mu\nu}\, \tilde{\varepsilon}^{\mu\nu}\, d^3x .$$

Symmetries ? → internal rotations : $\delta^R_{\lambda^i} e^j_\nu = ?$, $\delta^R_{\lambda^i} A^j_\nu = ?$

→ diffeo. $\delta^D_{\xi^\mu} e^j_\nu$; $\delta^D_{\xi^\mu} A^j_\nu$

→ translation symmetry

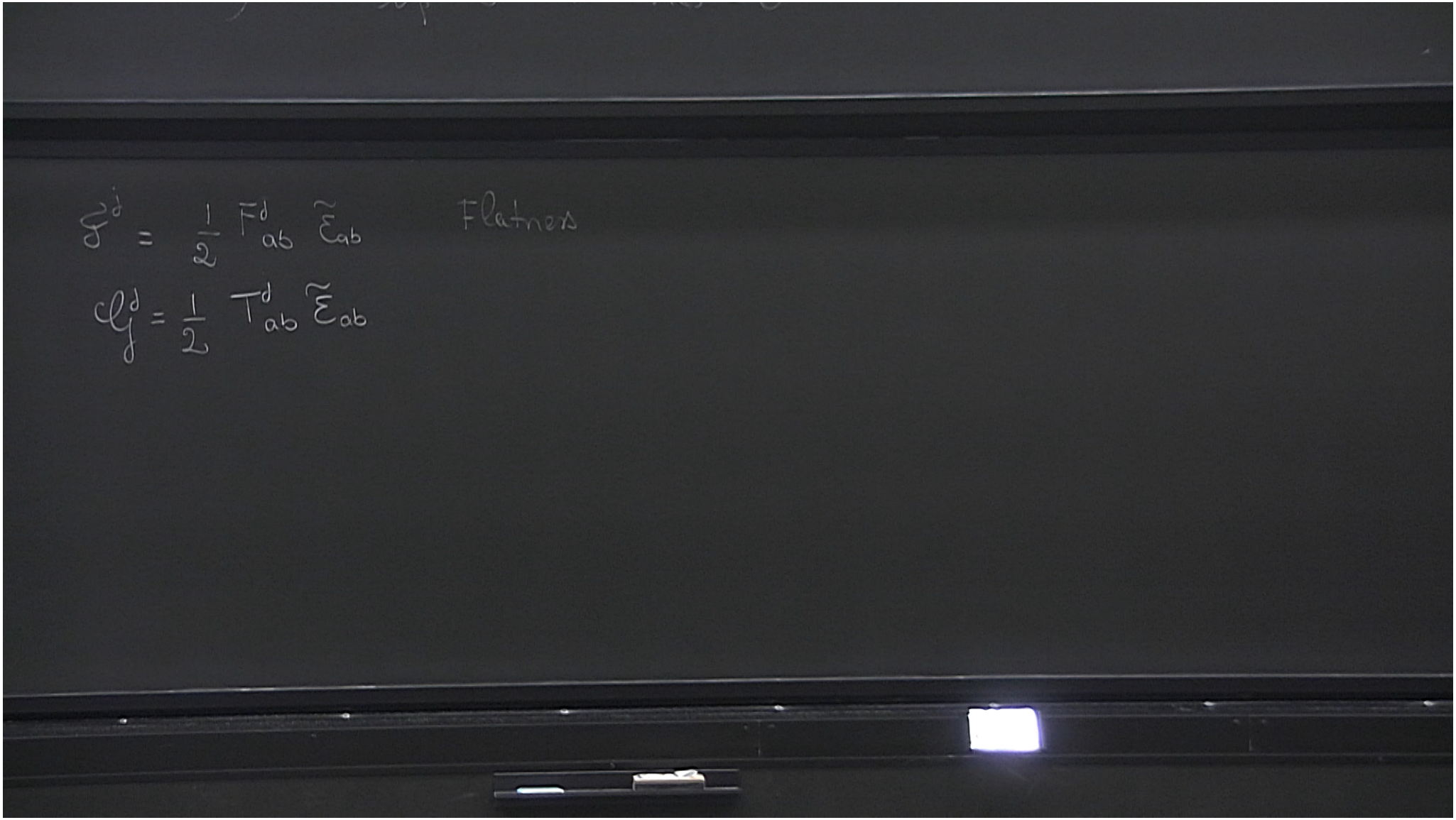
translation → $\delta^T_N e^j_\mu = D_\mu N^j$ (parametrized by scalar N with an internal index)

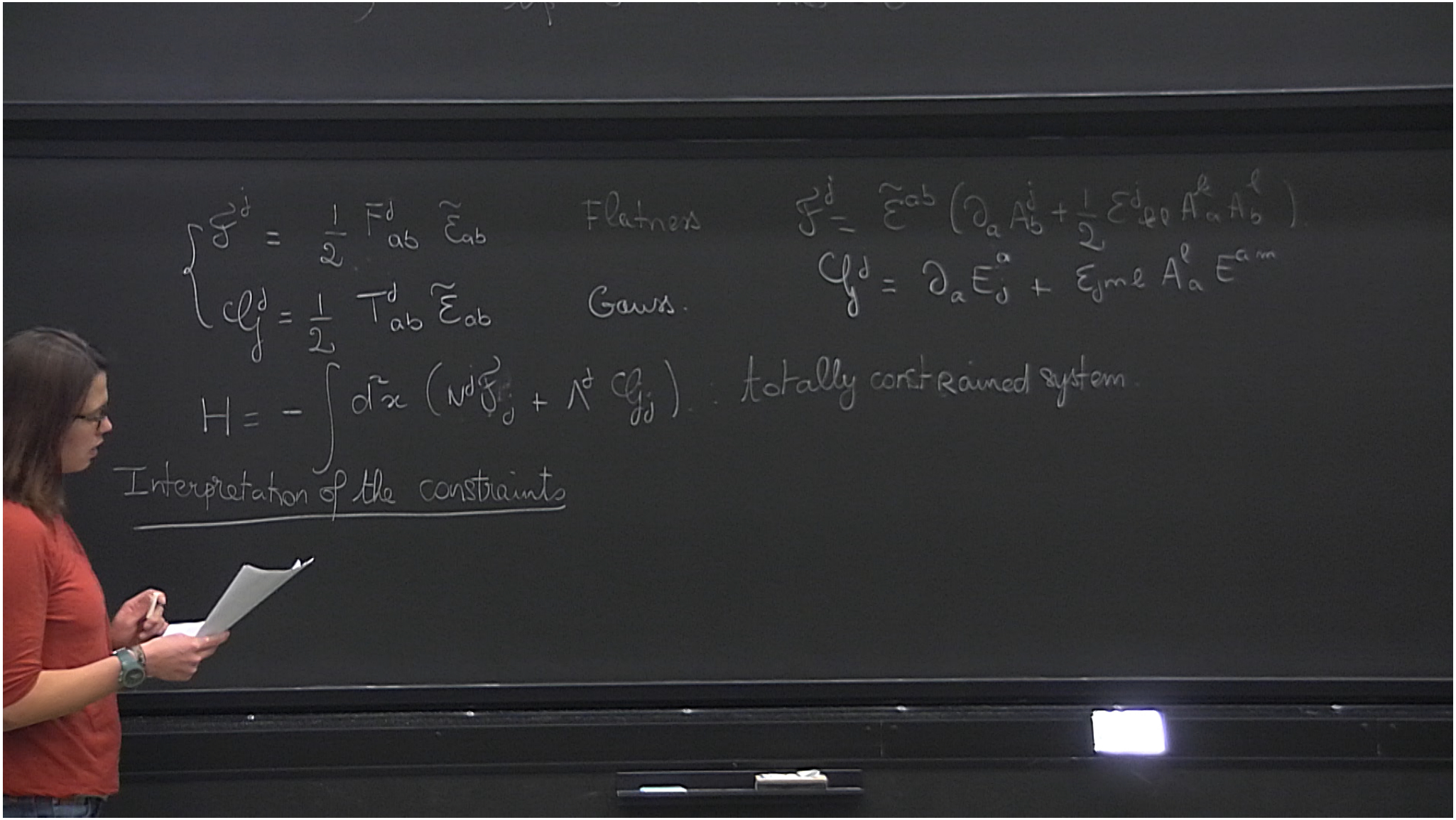
$\delta^T_N A^j_\mu = 0$

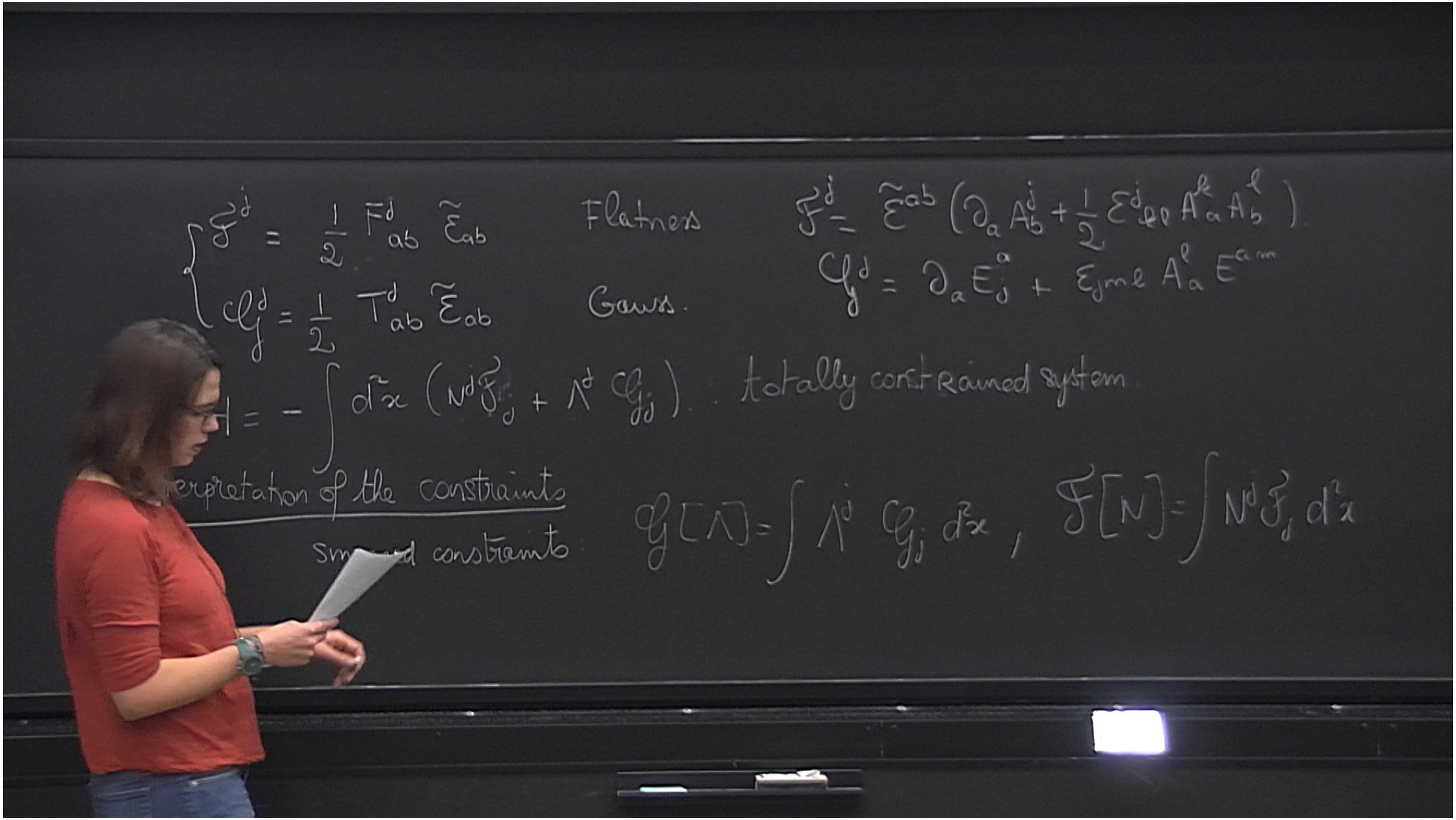
→ replacing e^j_μ by $\delta^T_\mu e^j_\mu$ in the action

+ integration by part : $D_\mu F = 0$ (Bianchi identity)

⇒ action invariant under translation


$\mathcal{S}^d = \frac{1}{2} F^d_{ab} \tilde{\Sigma}_{ab}$
Flatness
$\mathcal{Q}^d = \frac{1}{2} T^d_{ab} \tilde{\Sigma}_{ab}$


$\mathcal{F}^i = \frac{1}{2} F^i_{ab} \tilde{\Sigma}_{ab}$ Flatness
$\mathcal{G}^i = \frac{1}{2} T^i_{ab} \tilde{\Sigma}_{ab}$ Gauss.
$\mathcal{F}^i = \tilde{\Sigma}^{ab} (\partial_a A^i_b + \frac{1}{2} \varepsilon^i{}_{kl} A^k_a A^l_b)$
$\mathcal{G}^i = \partial_a E^a_i + \varepsilon_{ijk} A^j_a E^{a\,m}$
$H = -\int d^3x \, (N^i \mathcal{F}_i + \Lambda^i \mathcal{G}_i)$ totally constrained system
Interpretation of the constraints


$\mathcal{F}^i = \frac{1}{2} F^i_{ab} \tilde{E}_{ab}$ Flatness
$\mathcal{G}^i = \frac{1}{2} T^i_{ab} \tilde{E}_{ab}$ Gauss.
$\mathcal{F}^i = \tilde{E}^{ab} (\partial_a A^i_b + \frac{1}{2} \varepsilon^i{}_{kl} A^k_a A^l_b)$
$\mathcal{G}^i = \partial_a E^a_i + \varepsilon_{ijk} A^j_a E^{ak}$
$H = -\int d^3x (N^i \mathcal{F}_i + \Lambda^i \mathcal{G}_i)$ totally constrained system
Interpretation of the constraints
Smeared constraints
$\mathcal{G}[\Lambda] = \int \Lambda^i \mathcal{G}_i \, d^3x$, $\mathcal{F}[N] = \int N^i \mathcal{F}_i \, d^3x$

$$\begin{cases} \mathcal{F}^i = \frac{1}{2} F^i_{ab} \tilde{\Sigma}_{ab} & \text{Flatness} \\ \mathcal{G}^i = \frac{1}{2} T^i_{ab} \tilde{\Sigma}_{ab} & \text{Gauss.} \end{cases}$$

$$\mathcal{F}^i = \tilde{\Sigma}^{ab} \left(\partial_a A^i_b + \frac{1}{2} \varepsilon^i{}_{kl} A^k_a A^l_b \right)$$

$$\mathcal{G}^i = \partial_a E^a_i + \varepsilon_{ijk} A^j_a E^{ak}$$

$$H = -\int d^3x \left(N^i \mathcal{F}_i + \Lambda^i \mathcal{G}_i \right)$$ totally constrained system.

Interpretation of the constraints

smeared constraints

$$\mathcal{G}[\Lambda] = \int \Lambda^i \mathcal{G}_i \, d^3x, \quad \mathcal{F}[N] = \int N^i \mathcal{F}_i \, d^3x$$

$$\sqrt{g}R = -\int d^3x\, e_{0\ell} F^{\ell}_{\mu\nu} \tilde{\varepsilon}^{0\mu\nu}$$

Fix Gauge $\quad \mathbb{R} \otimes \Sigma$

Lagrange multipliers

$$\int_{\Sigma} \left[e^{j}_{b} \partial_0 A_{aj} \tilde{\varepsilon}^{ab} + e_{0j} \tfrac{1}{2} F^{j}_{ab} \tilde{\varepsilon}^{ab} + A^{j}_{0} \left(\partial_a e_{bj} + \varepsilon_{jkm} A^{k}_{a} e^{m}_{b} \right) \tilde{\varepsilon}^{ab} \right]$$

$$\{ A^{j}_{a}(x), E^{b}_{k}(y) \} = \delta^{j}_{k} \delta^{b}_{a} \delta(x,y)$$

δ_{diff} e$^{a}_{\mu}$; δ_{diff} F$^{i}_{\mu\nu}$

diffeo. $\rightarrow$ translation symmetry

translation $\rightarrow$

$\delta^{T}_{N} e^{a}_{\mu} = D_{\mu} N^{a}$ $\longrightarrow$ replacing e^{a}_{μ} by $\delta^{T}_{p} e^{a}_{\mu}$ in the action

+ integration by part: $D_{\mu} F = 0$ (Bianchi id

$\Rightarrow$ action invariant under translation.

Fix Gauge $\mathbb{R} \otimes \Sigma$

$$\int dt\, d^2x \left[E^j_b \partial_0 A_{aj} \tilde{\xi}^{ab} + e_{0j} \underbrace{\tfrac{1}{2} F^j_{ab} \tilde{\xi}^{ab}}_{\xi^j} + A^j_0 \underbrace{\left(\partial_a e_{bj} + \varepsilon_{jkm} A^k_a e^m_b \right) \tilde{\xi}^{ab}}_{\xi^j} \right]$$

Lagrange multipliers

$$\{ A^j_a(x), E^b_k(y) \} = \delta^j_k \delta^b_a \delta(x,y)$$

translation → diffeo.

translation symmetry → $\delta_{AN} e^a_\mu$; $\delta_{AN} \dots$

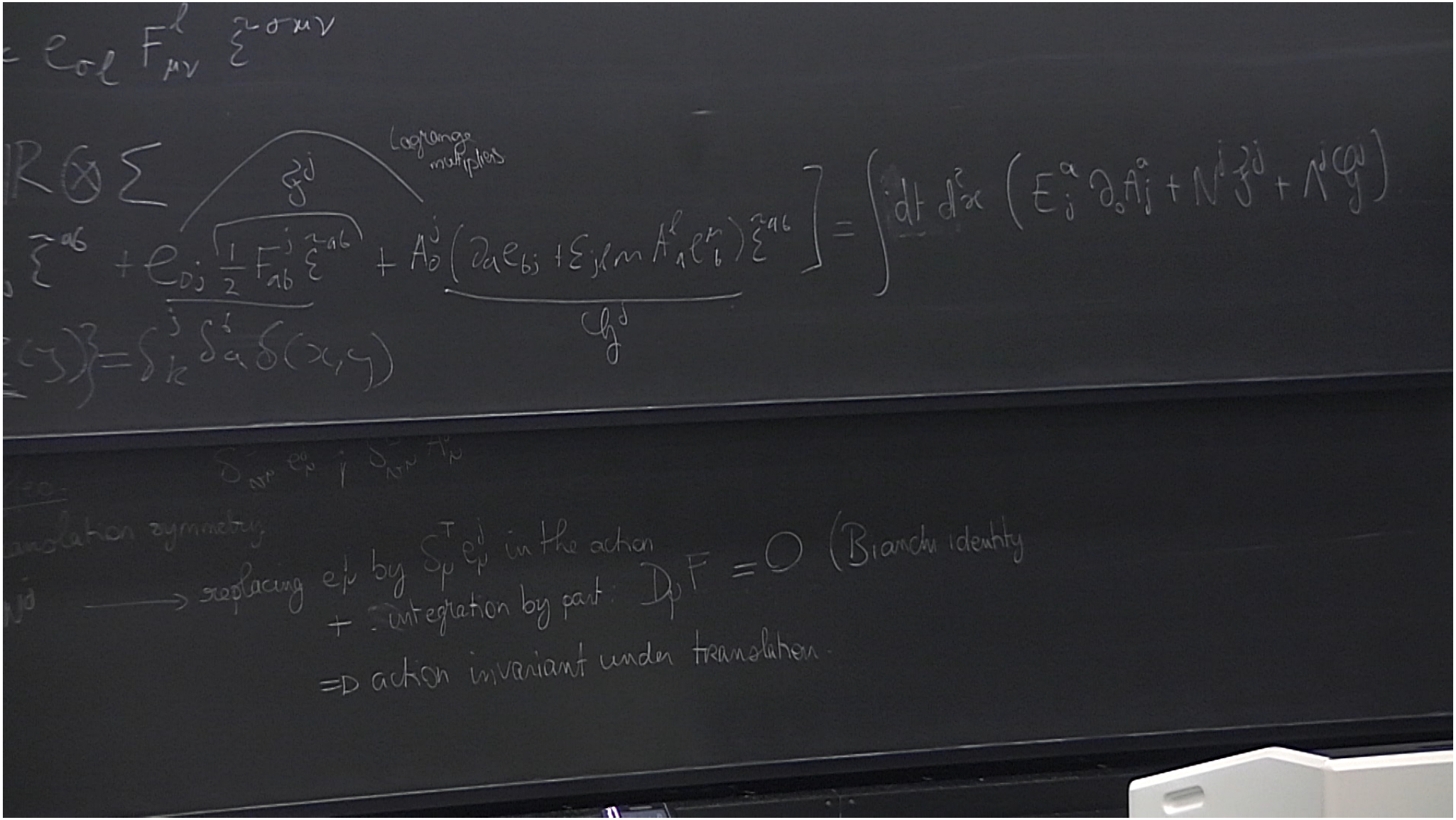
$\delta^T_N e^j_\mu = D_\mu N^j$ ⟶ replacing e^j_μ by $\delta^T_\mu e^j_\mu$ in the action

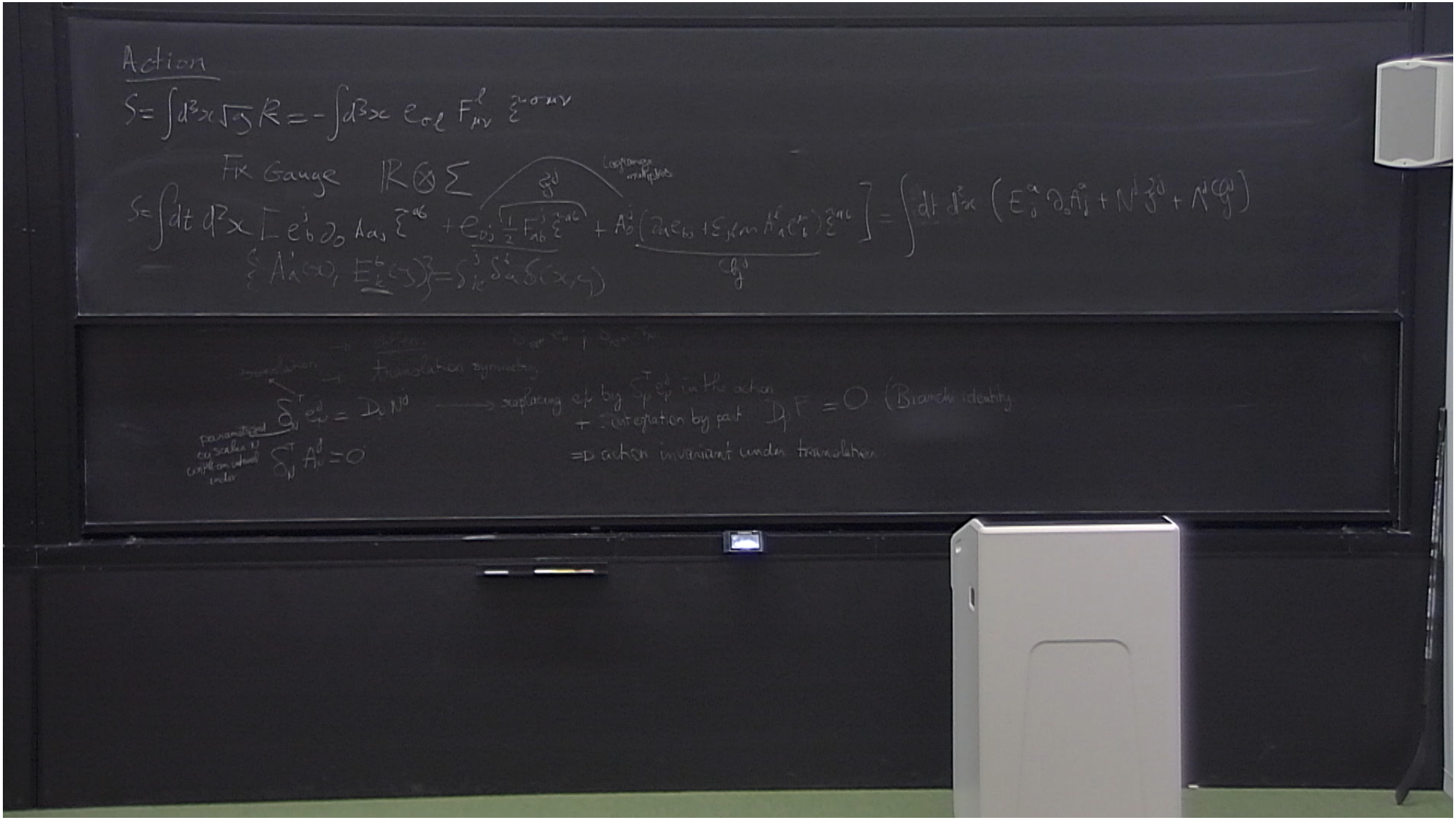
+ integration by part: $D_\mu F = 0$ (

parametrized by scalar N with an interval

$\delta^T_N A^j_\mu = 0$

⇒ action invariant under translation.


Lagrange multipliers
translation symmetry
replacing e^j_μ by $S^T e^j_\mu$ in the action
+ integration by part $D_\mu F = 0$ (Bianchi identity
=> action invariant under translation.


Action
Fix Gauge
translation
translation symmetry
integration by part
action invariant under translation

$\frac{1}{2}\varepsilon^{ijk}\varepsilon_{lmn}A^l_a A^m_b)$

$\varepsilon_{jmk}A^k_a E^{am}$

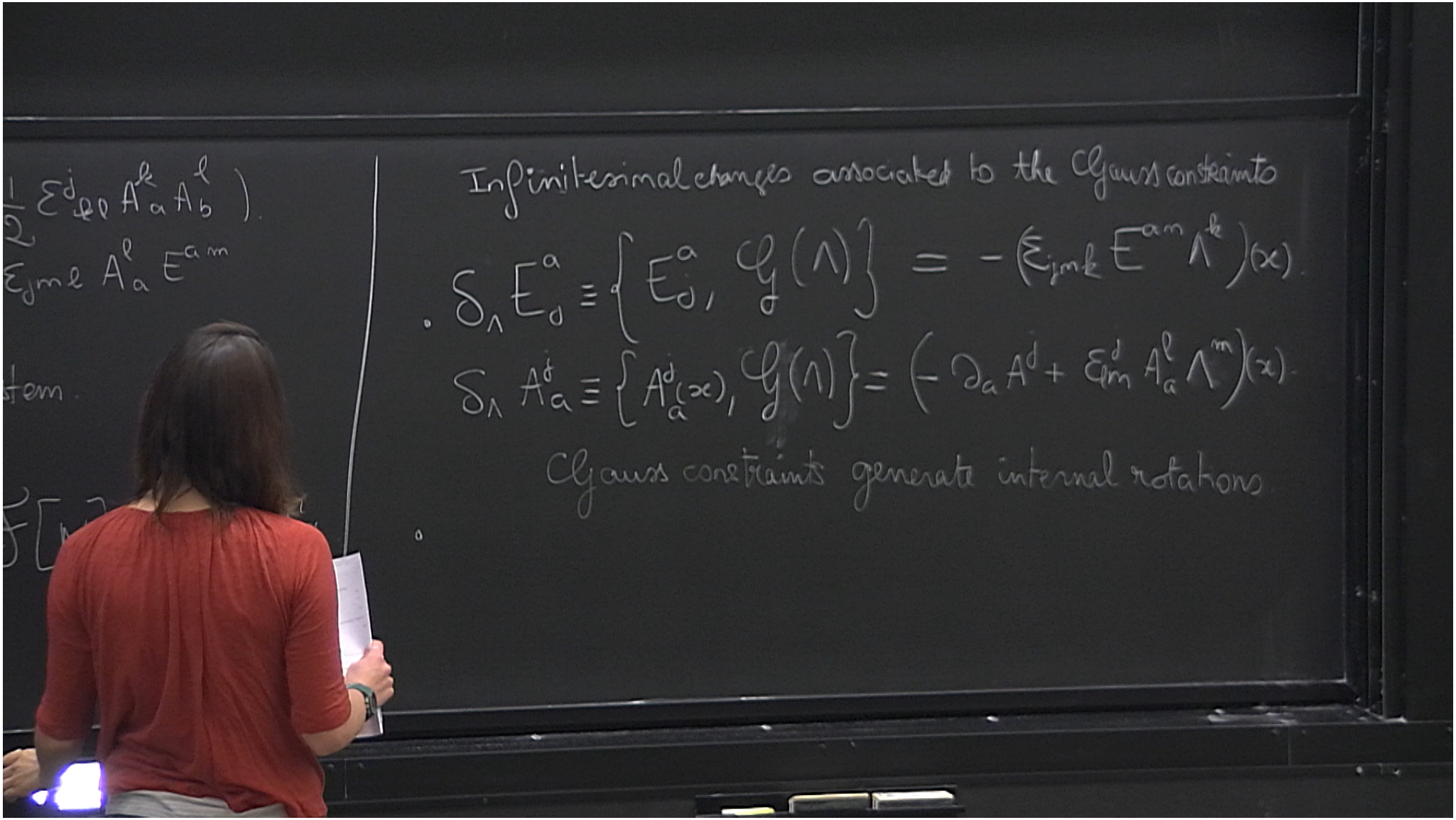
stem

$\mathcal{G}[N]=\int N^j \mathcal{G}_j\, d^3x$

Infinitesimal changes associated to the Gauss constraints

$$\delta_\Lambda E^a_j \equiv \{E^a_j, \mathcal{G}(\Lambda)\} = -(\varepsilon_{jmk}E^{am}\Lambda^k)(x)$$

$$\delta_\Lambda A^d_a \equiv \{A^d_a(x), \mathcal{G}(\Lambda)\} = (-\partial_a A^d + \varepsilon^d_{lm}A^l_a\Lambda^m)(x)$$


Infinitesimal changes associated to the Gauss constraints
$\delta_\Lambda E^a_j \equiv \{E^a_j, \mathcal{G}(\Lambda)\} = -(\varepsilon_{jmk} E^{am} \Lambda^k)(x)$
$\delta_\Lambda A^d_a \equiv \{A^d_a(x), \mathcal{G}(\Lambda)\} = (-\partial_a A^d + \varepsilon^d_{m} A^l_a \Lambda^m)(x)$
Gauss constraints generate internal rotations

$\varepsilon_{jmk} A^l_a E^{a\,m}$

tem.

$\mathcal{F}[N] = \int N^j \mathcal{F}_j \, d^3x$

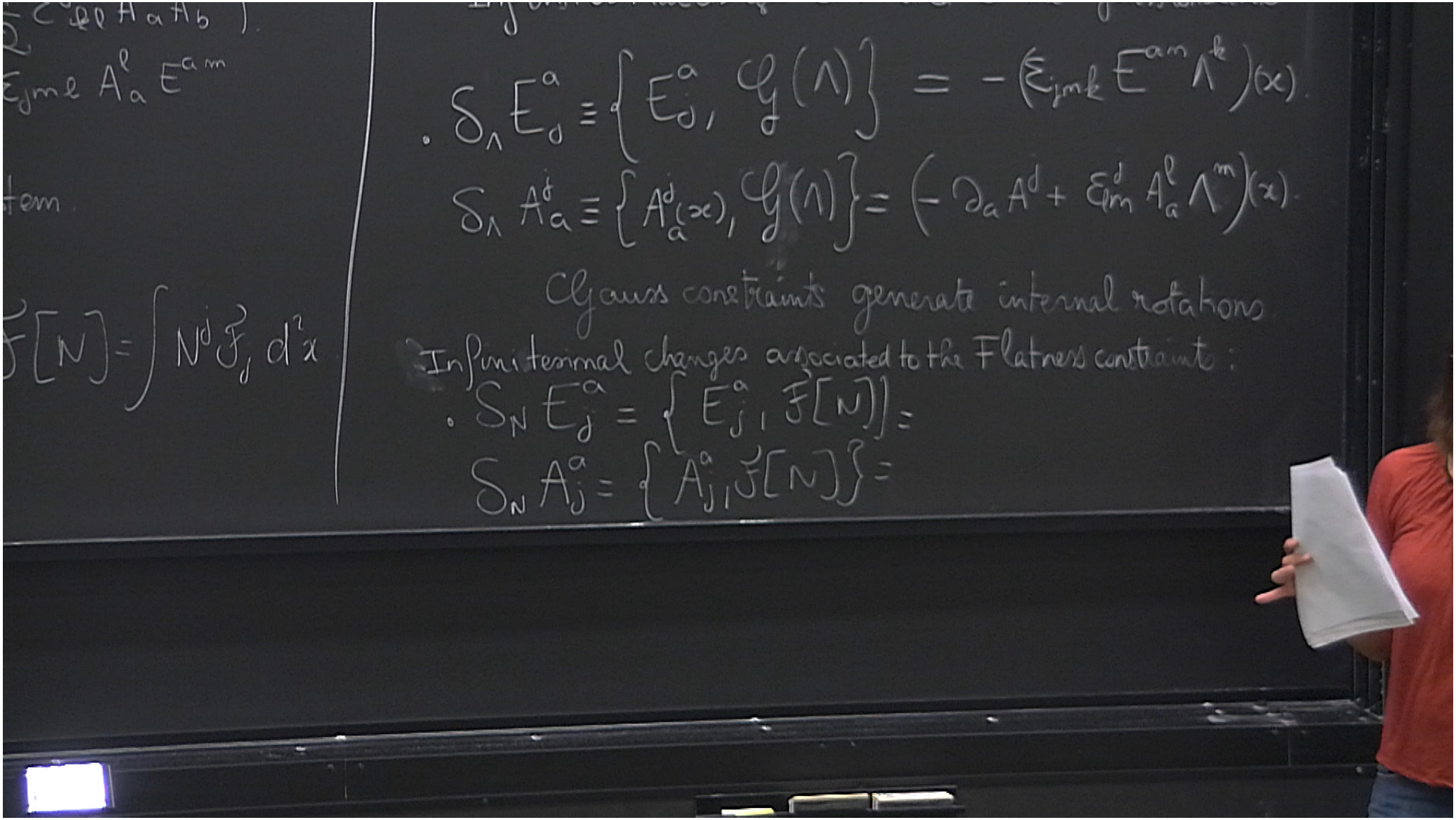
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Gauss constraints generate internal rotations

Infinitesimal changes associated to the Flatness constraint:

$\delta_N E^a_j = \{ E^a_j, \mathcal{F}[N] \} =$


$\mathcal{F}[N] = \int N^j \mathcal{F}_j \, d^3x$
$\delta_\Lambda E^a_j \equiv \{ E^a_j, \mathcal{G}(\Lambda) \} = -(\varepsilon_{jmk} E^{am} \Lambda^k)(x)$
$\delta_\Lambda A^j_a \equiv \{ A^j_a(x), \mathcal{G}(\Lambda) \} = (-\partial_a \Lambda^j + \varepsilon^j_{lm} A^l_a \Lambda^m)(x)$
Gauss constraints generate internal rotations
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$\delta_N E^a_j = \{ E^a_j, \mathcal{F}[N] \} =$
$\delta_N A^a_j = \{ A^a_j, \mathcal{F}[N] \} =$

$= \frac{1}{2} F^d_{ab} \tilde{\varepsilon}_{ab}$ Flatness $\quad \mathcal{F}^i = \tilde{\varepsilon}^{ab} \left(\partial_a A^i_b + \frac{1}{2} \varepsilon^i{}_{kl} A^k_a A^l_b \right)$

$\mathcal{G}^d = \frac{1}{2} T^d_{ab} \tilde{\varepsilon}_{ab}$ Gauss. $\quad \mathcal{G}^d = \partial_a E^a_d + \varepsilon_{dml} A^l_a E^{am}$

$= -\int d^3x \left(N^d \mathcal{F}_d + \Lambda^d \mathcal{G}_d \right)$ totally constrained system.

tation of the constraints

smeared constraints $\quad \mathcal{G}[\Lambda] = \int \Lambda^d \mathcal{G}_d \, d^3x, \quad \mathcal{F}[N] = \int N^d \mathcal{F}_d \, d^3x$

Infin

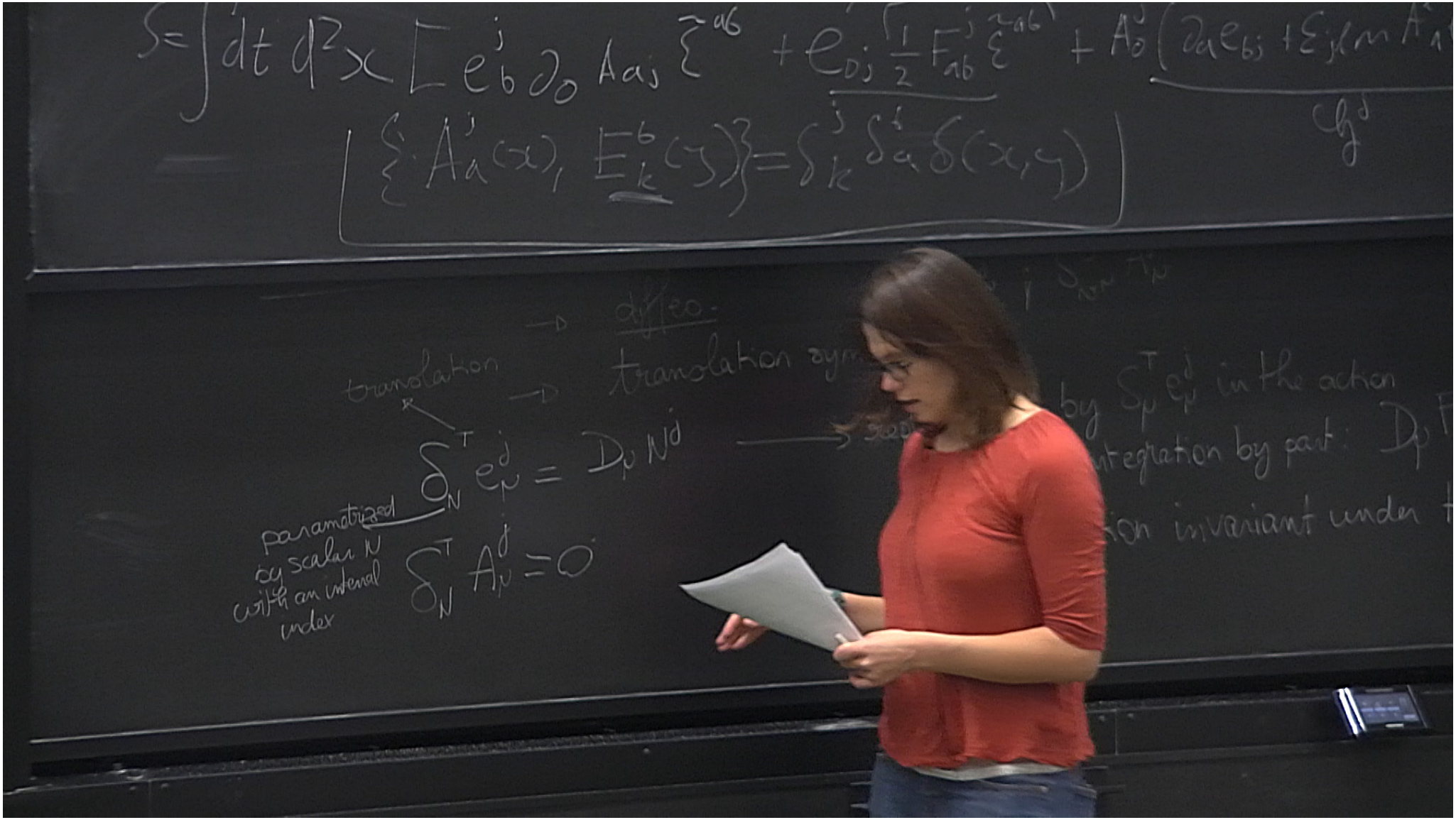
$\delta_\Lambda E$

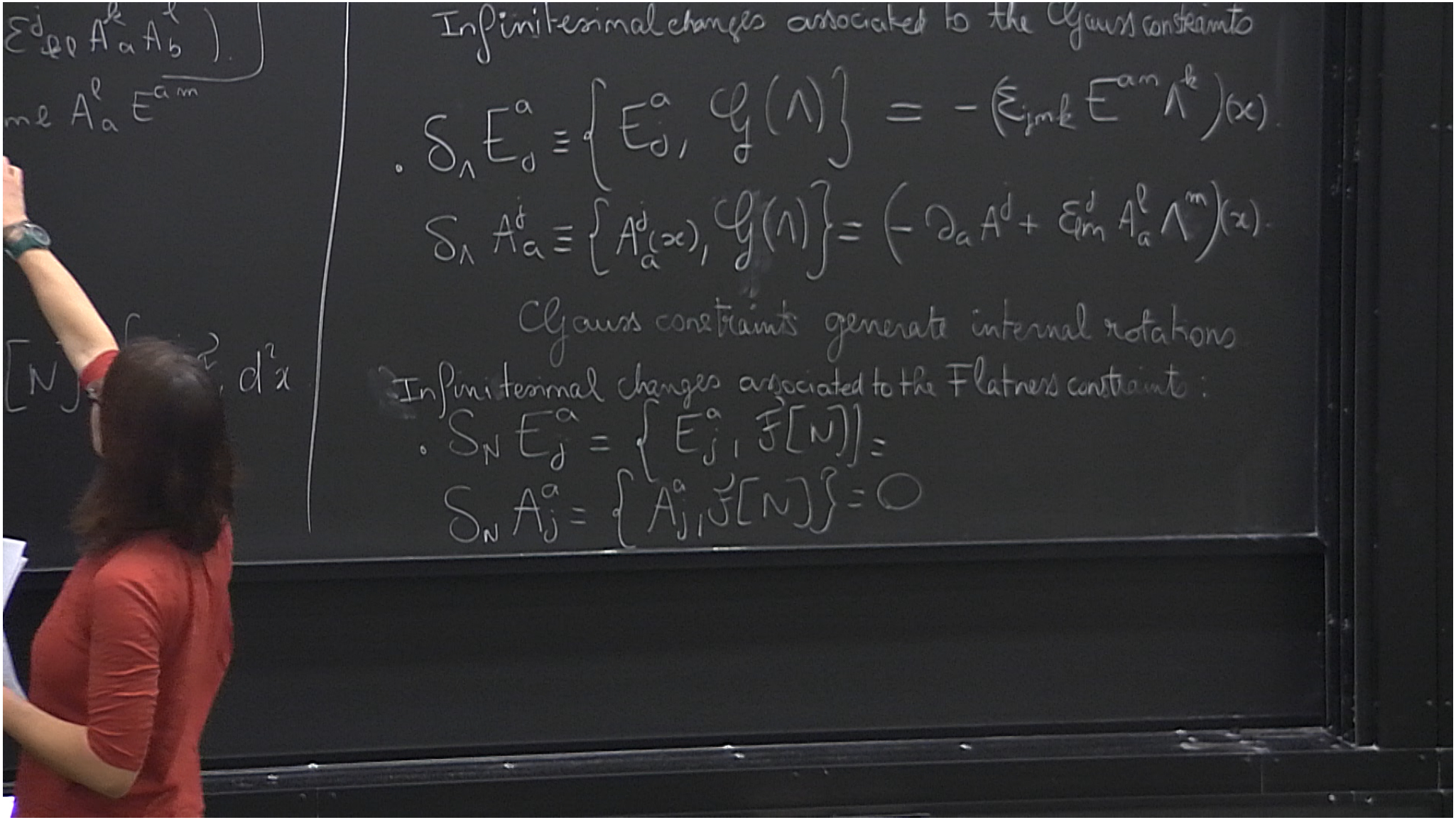
$\delta_\Lambda A$

Infinites

$\delta_N E$

$\delta_N A$


translation
diffeo.
translation
parametrized by scalar N with an internal index
by $S_\mu^T e_\mu^d$ in the action
integration by part
invariant under


Infinitesimal changes associated to the Gauss constraints
$\delta_\Lambda E^a_j \equiv \{E^a_j, \mathcal{G}(\Lambda)\} = -(\varepsilon_{jmk} E^{am} \Lambda^k)(x)$
$\delta_\Lambda A^d_a \equiv \{A^d_a(x), \mathcal{G}(\Lambda)\} = (-\partial_a A^d + \varepsilon^d_{lm} A^l_a \Lambda^m)(x)$
Gauss constraints generate internal rotations
Infinitesimal changes associated to the Flatness constraint:
$\delta_N E^a_j = \{E^a_j, \vec{\mathcal{F}}[N]\} =$
$\delta_N A^a_j = \{A^a_j, \vec{\mathcal{F}}[N]\} = 0$

$\varepsilon^{ijk} A^k_a A^l_b$)

$me\ A^l_a E^{am}$

m

$[N] = \int N^j \mathcal{F}_j \, d^2x$

Infinitesimal changes associated to the Gauss constraints

$$\delta_\Lambda E^a_j \equiv \{ E^a_j, \mathcal{G}(\Lambda) \} = -(\varepsilon_{jmk} E^{am} \Lambda^k)(x)$$

$$\delta_\Lambda A^d_a \equiv \{ A^d_a(x), \mathcal{G}(\Lambda) \} = (-\partial_a \Lambda^d + \varepsilon^d_{\ m} A^l_a \Lambda^m)(x)$$

Gauss constraints generate internal rotations

Infinitesimal changes associated to the Flatness constraint:

$$\delta_N E^a_j = \{ E^a_j, \vec{\mathcal{F}}[N] \} =$$

$$\delta_N A^a_j = \{ A^a_j, \vec{\mathcal{F}}[N] \} = 0$$

$\varepsilon^{d}{}_{ef} A^{k}_{a} A^{l}_{b}$)

me $A^{l}_{a} E^{am}$

m

$[N] = \int N^{j} \mathcal{F}_{j} \, d^{2}x$

Infinitesimal changes associated to the Gauss constraints

$\cdot \; \delta_{\Lambda} E^{a}_{j} \equiv \{ E^{a}_{j}, \mathcal{G}(\Lambda) \} = -(\varepsilon_{jmk} E^{am} \Lambda^{k})(x)$

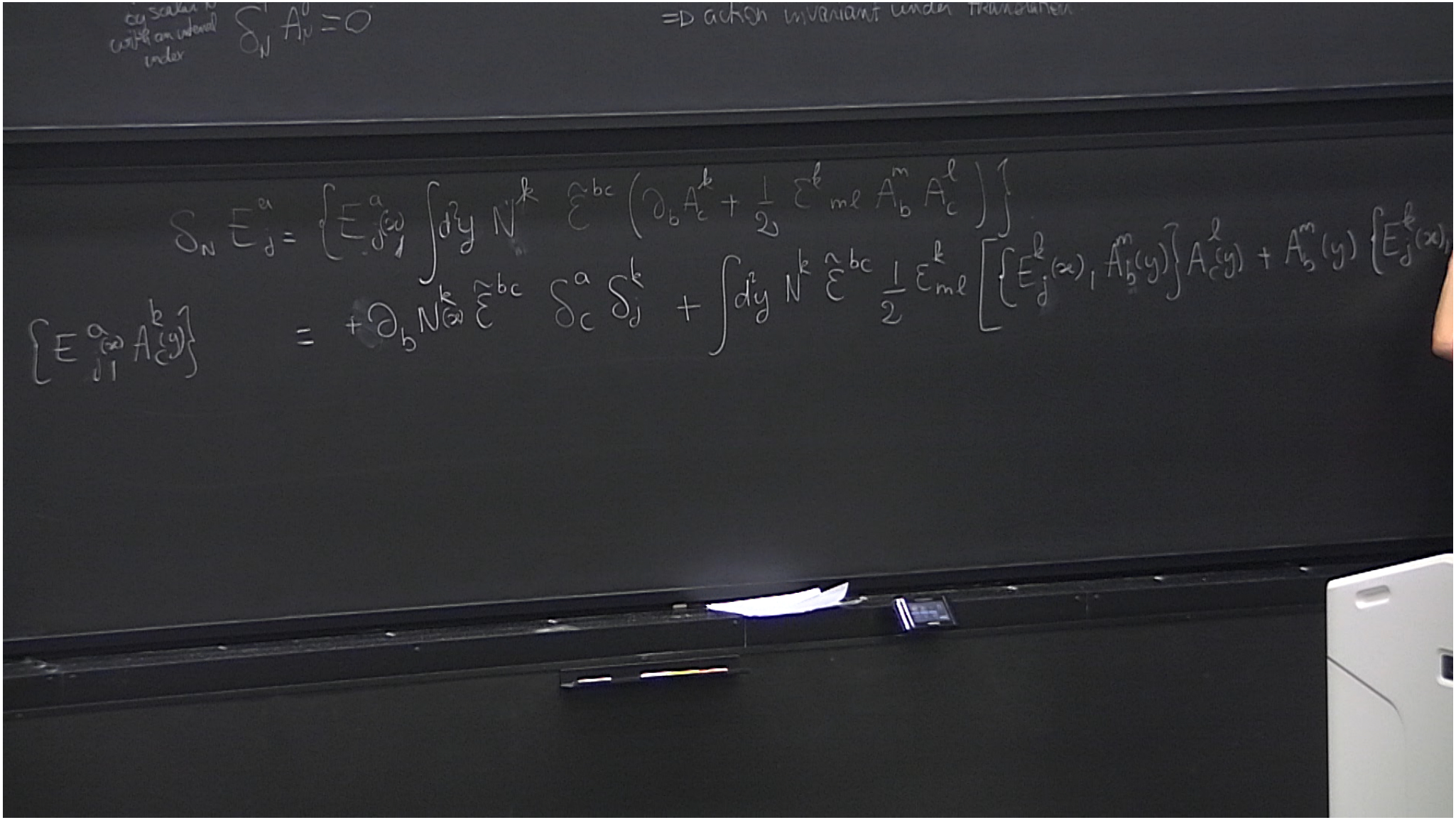
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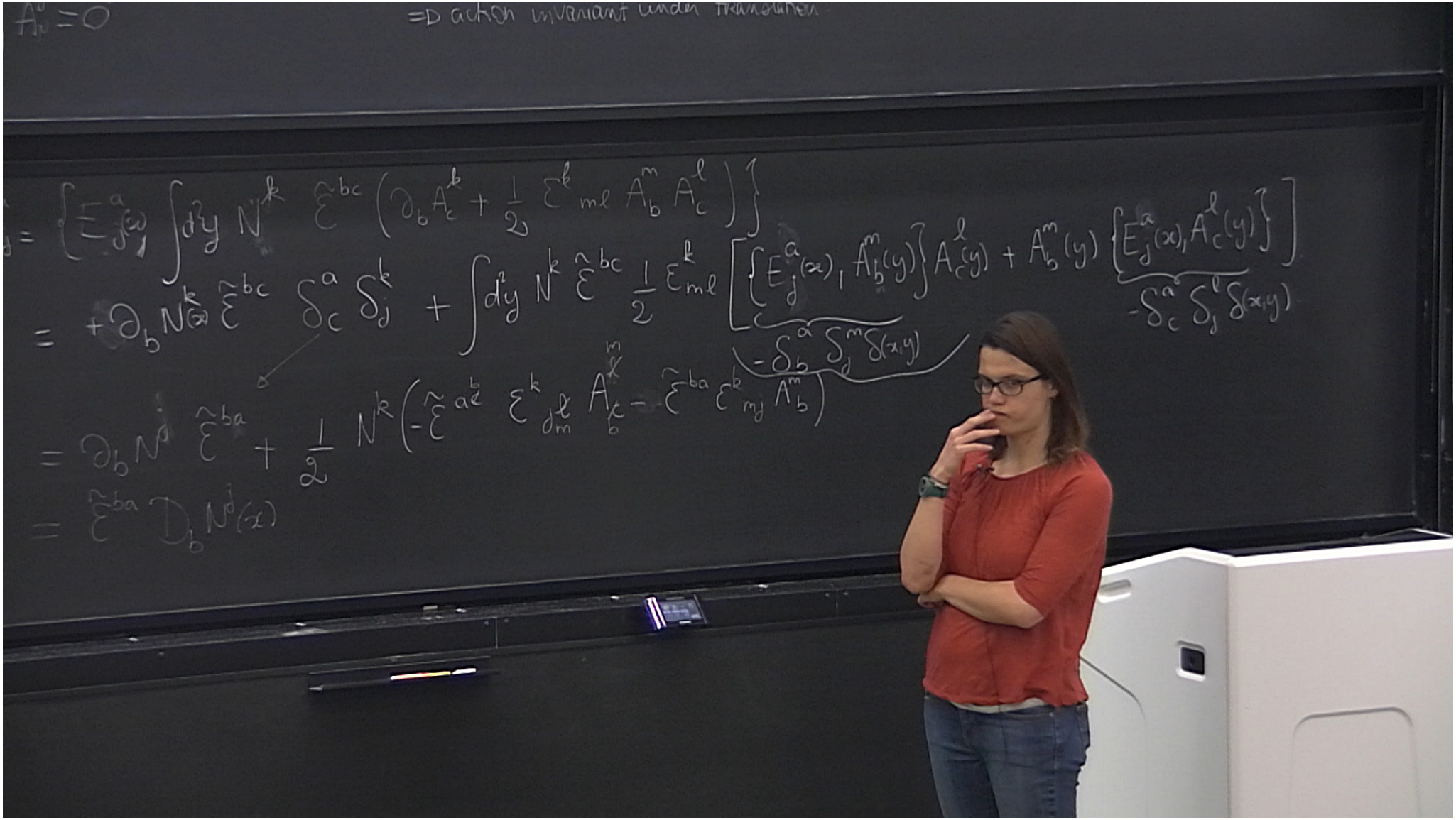
Gauss constraints generate internal rotations

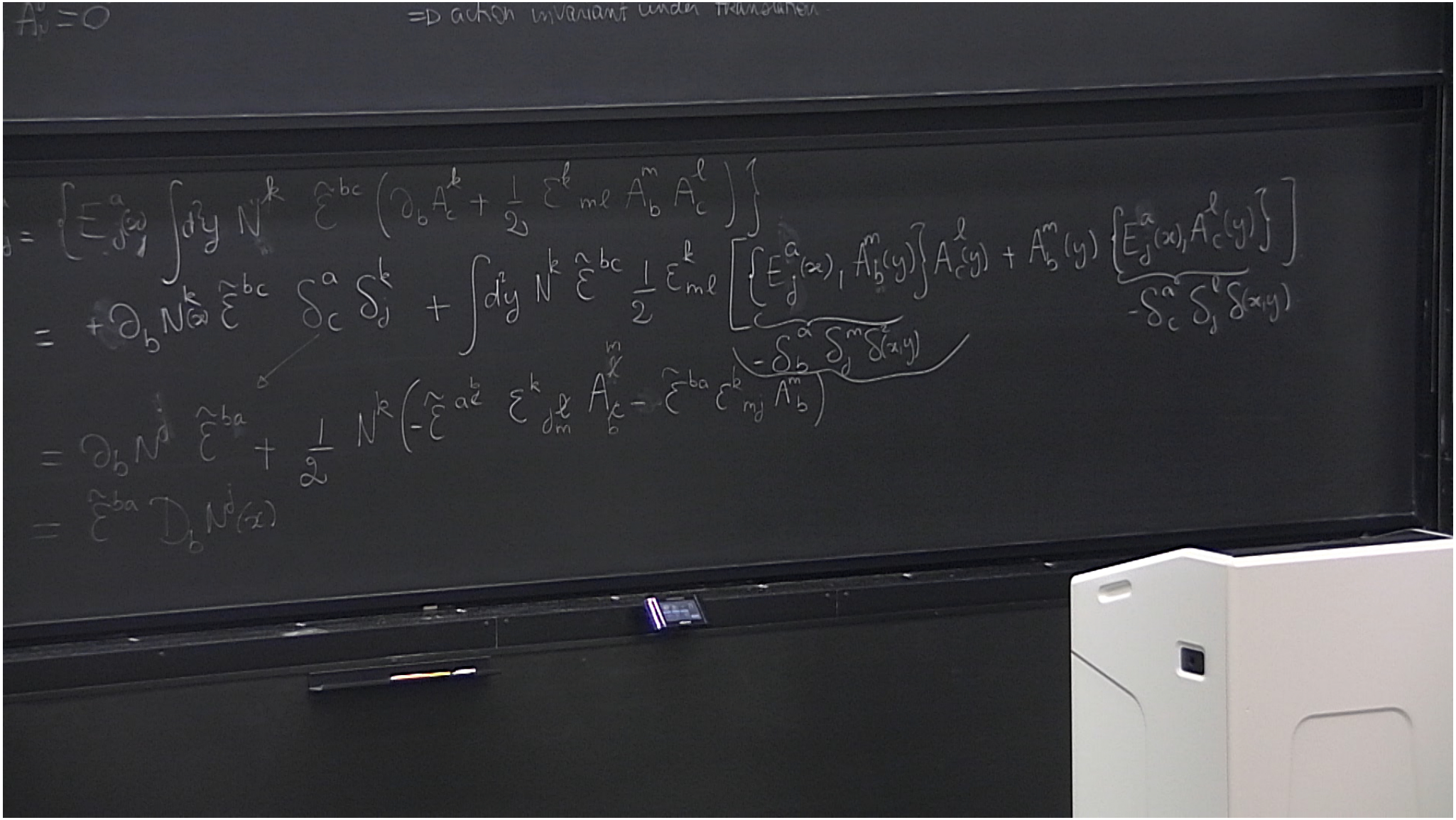
Infinitesimal changes associated to the Flatness constraint:

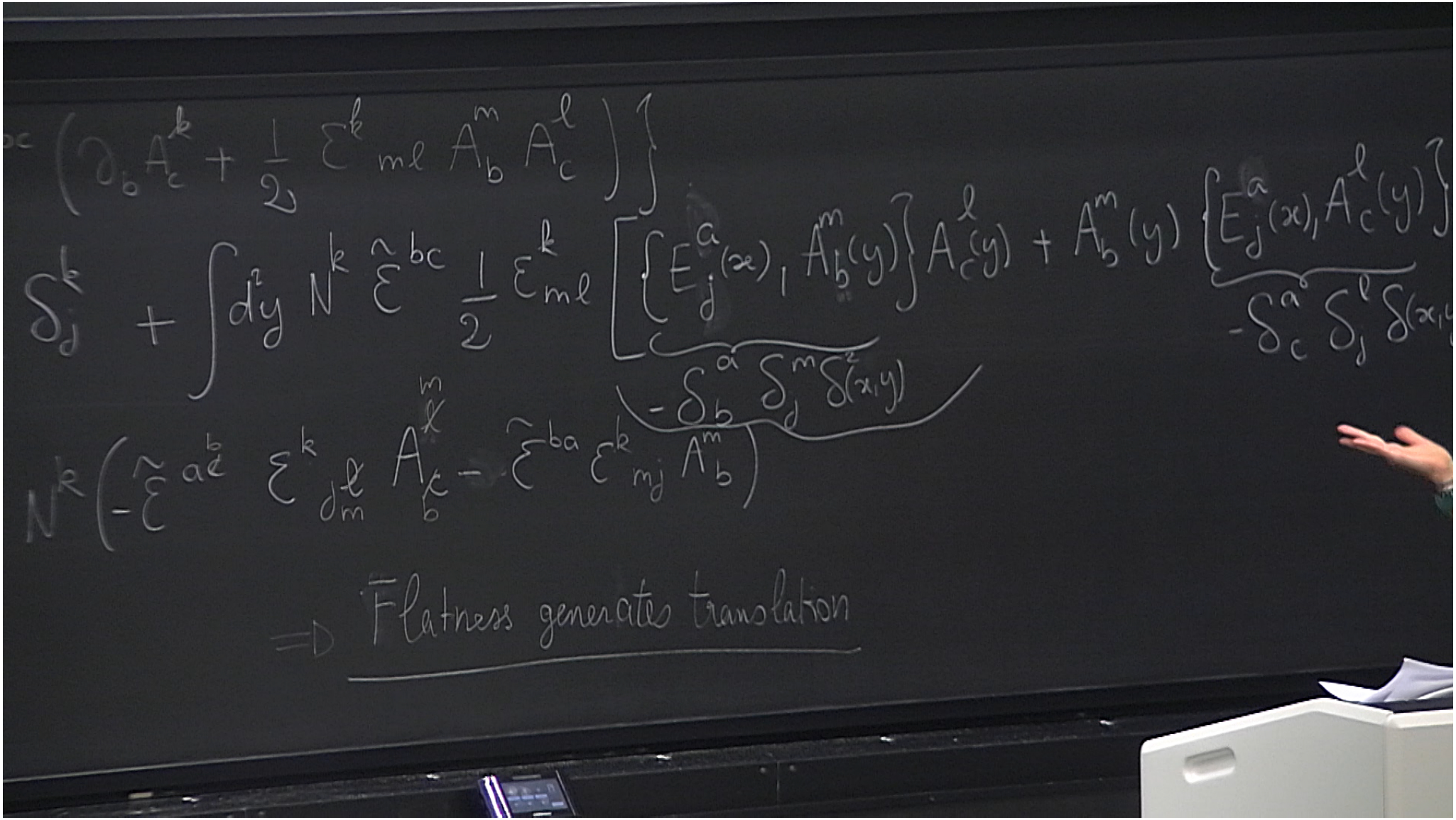
$\cdot \; \delta_{N} E^{a}_{j} = \{ E^{a}_{j}, \vec{\mathcal{F}}[N] \} = \circledast$

$\delta_{N} A^{a}_{j} = \{ A^{a}_{j}, \vec{\mathcal{F}}[N] \} = 0$


$\delta_N A^a_\nu = 0$
$\Rightarrow$ action invariant under
$\delta_N E^a_j = \{E^a_j(x), \int d^3y\, N^k\, \tilde{\varepsilon}^{bc} (\partial_b A^k_c + \frac{1}{2} \varepsilon^k{}_{ml} A^m_b A^l_c)\}$
$\{E^a_j(x), A^k_c(y)\} = +\partial_b N^k(x)\, \tilde{\varepsilon}^{bc}\, \delta^a_c \delta^k_j + \int d^3y\, N^k\, \tilde{\tilde{\varepsilon}}^{bc}\, \frac{1}{2} \varepsilon^k_{ml} \left[\{E^k_j(x), A^m_b(y)\} A^l_c(y) + A^m_b(y) \{E^k_j(x),$


$A_0^a = 0$
$\Rightarrow$ action invariant under
$= \{E_j^a(x), \int d^3y\, N^k\, \tilde{\varepsilon}^{bc} \left(\partial_b A_c^k + \frac{1}{2} \varepsilon^k{}_{ml} A_b^m A_c^l\right)\}$
$= + \partial_b N^k \tilde{\varepsilon}^{bc}\, \delta_c^a \delta_j^k + \int d^3y\, N^k \tilde{\varepsilon}^{bc} \frac{1}{2} \varepsilon^k{}_{ml} \left[\{E_j^a(x), A_b^m(y)\} A_c^l(y) + A_b^m(y) \{E_j^a(x), A_c^l(y)\}\right]$
$-\delta_b^a \delta_j^m \delta(x,y)$
$-\delta_c^a \delta_j^l \delta(x,y)$
$= \partial_b N^j \tilde{\varepsilon}^{ba} + \frac{1}{2} N^k \left(-\tilde{\varepsilon}^{ab} \varepsilon^k{}_{jm} A_b^m - \tilde{\varepsilon}^{ba} \varepsilon^k{}_{mj} A_b^m\right)$
$= \tilde{\varepsilon}^{ba} D_b N^j(x)$


$A_b^a = 0$
$\Rightarrow$ action invariant under translation
$= \{E_j^a(x), \int d^3y\, N^k\, \tilde{\varepsilon}^{bc} \left(\partial_b A_c^k + \frac{1}{2} \varepsilon^k{}_{ml} A_b^m A_c^l\right)\}$
$= + \partial_b N^k \tilde{\varepsilon}^{bc}\, \delta_c^a \delta_j^k + \int d^3y\, N^k \tilde{\varepsilon}^{bc} \frac{1}{2} \varepsilon^k{}_{ml} \left[\{E_j^a(x), A_b^m(y)\} A_c^l(y) + A_b^m(y) \{E_j^a(x), A_c^l(y)\}\right]$
$-\delta_b^a \delta_j^m \delta(x,y)$
$-\delta_c^a \delta_j^l \delta(x,y)$
$= \partial_b N^j \tilde{\varepsilon}^{ba} + \frac{1}{2} N^k \left(-\tilde{\varepsilon}^{ab} \varepsilon^k{}_{jm} A_b^m - \tilde{\varepsilon}^{ba} \varepsilon^k{}_{mj} A_b^m\right)$
$= \tilde{\varepsilon}^{ba} D_b N^j(x)$


$\partial_b A_c^k + \frac{1}{2} \varepsilon^k{}_{ml} A_b^m A_c^l \Big) \Big\}$
$\delta_j^k + \int d^2y \, N^k \tilde{\varepsilon}^{bc} \frac{1}{2} \varepsilon^k{}_{ml} \Big[\{ E_j^a(x), A_b^m(y) \} A_c^l(y) + A_b^m(y) \{ E_j^a(x), A_c^l(y) \}$
$- \delta_b^a \delta_j^m \delta^2(x,y)$
$- \delta_c^a \delta_j^l \delta(x,$
$N^k \Big(- \tilde{\varepsilon}^{ab} \varepsilon^k{}_{dm} \frac{\delta A_b^m}{\delta E_c^d} - \tilde{\varepsilon}^{ba} \varepsilon^k{}_{mj} A_b^m \Big)$
$\Rightarrow$ Flatness generates translation

pretation of the constraints

smeared constraints $\mathcal{G}[\Lambda] = \int \Lambda^j \mathcal{G}_j \, d^3x$, $\mathcal{F}[N] = \int N^a \mathcal{F}_a \, d^3x$

Gauss constraints gen

Infinitesimal changes associated

$\delta_N E^a_j = \{ E^a_j , \mathcal{F}[N] \}$

Where are the diffeo?

$\delta^D_v e^a_\mu = \mathcal{L}_v e^a_\mu = v^\nu \partial_\nu e^a_\mu + e^a_\nu \partial_\mu v^\nu$, $\delta^D_v A^a_\mu = \mathcal{L}_v A^a_\mu = v^\nu \partial_\nu A^a_\mu + A^a_\nu \partial_\mu v^\nu$

Interpretation of the constraints

smeared constraints $\mathcal{G}[\Lambda] = \int \Lambda^j \mathcal{G}_j \, d^3x$, $\mathcal{F}[N] = \int N^j \mathcal{F}_j \, d^3x$

Gauss constraints

In Pirsa terminal changes a...

$\cdot \, \delta_N E^a_j = \{ E^a_j, \mathcal{F}[$

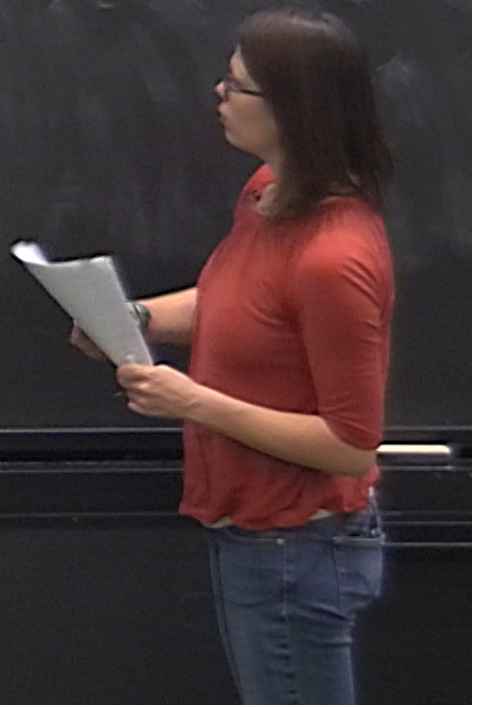
Where are the diffeo?

$$\delta^D_v e^d_\mu = \mathcal{L}_v e^d_\mu = v^\nu \partial_\nu e^d_\mu + e^d_\nu \partial_\mu v^\nu, \quad \delta^D_v A^d_\mu = \mathcal{L}_v A^d_\mu = v^\nu \partial_\nu A^d_\mu + A^d_\nu \partial_\mu v^\nu$$

$$\delta^R_{A(v)} e^d_\mu + \delta^T_{e(v)} e^d_\mu = \delta^D_v e^d_\mu + \underbrace{T^d_{\mu\nu} v^\nu}_{=0 \text{ on shell}}$$

$A(v) = A_\nu v^\nu$

$e(v) = e^d_\nu v^\nu$



Interpretation of the constraints

Smeared constraints $\mathcal{G}[\Lambda] = \int \Lambda^j \mathcal{G}_j \, d^3x$, $\mathcal{F}[N] = \int N^j \mathcal{F}_j \, d^3x$

Gauss constraints

Infinitesimal changes a...

$\cdot \delta_N E^a_j = \{ E^a_j, \mathcal{F}[$

Where are the diffeo?

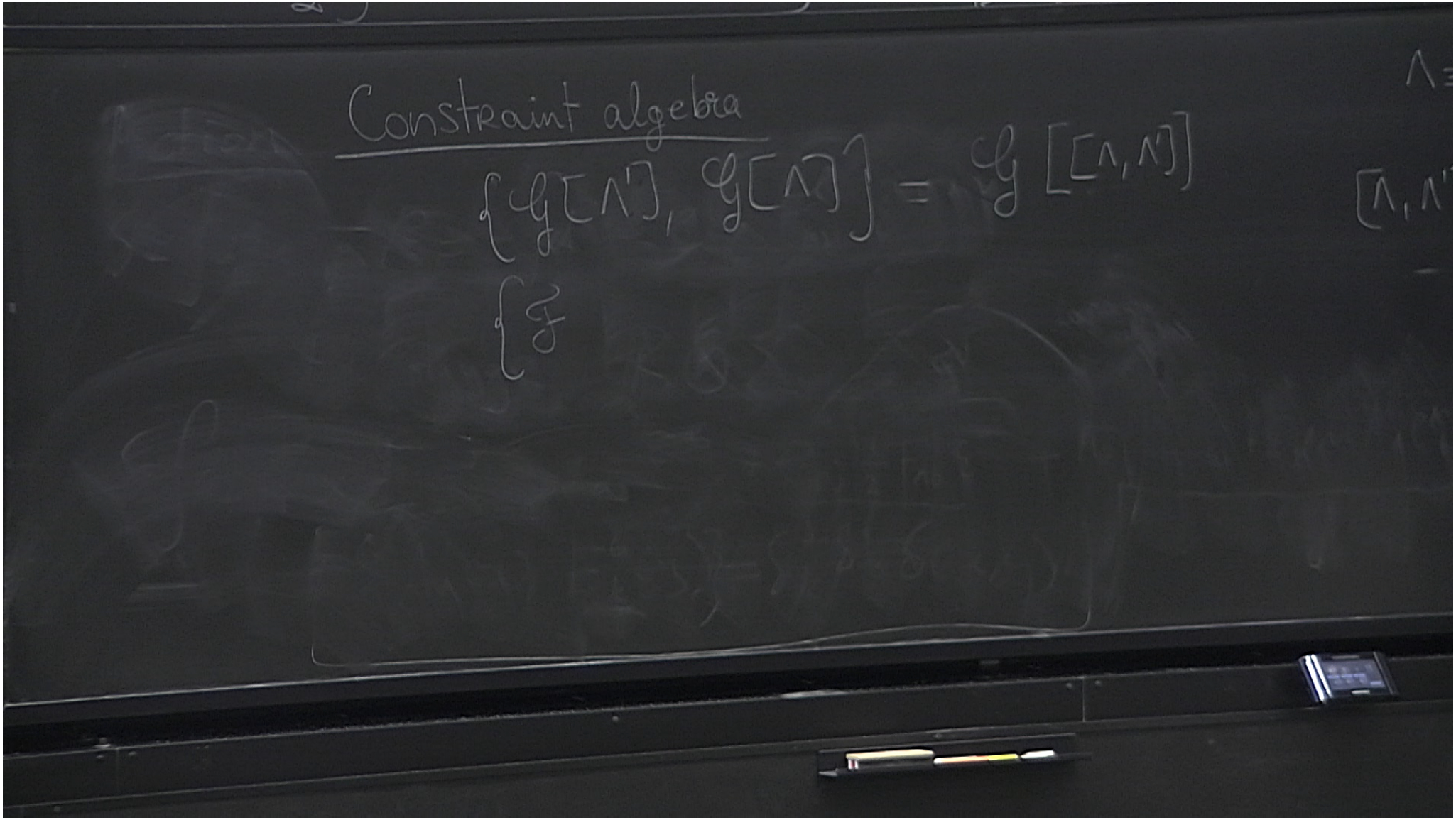
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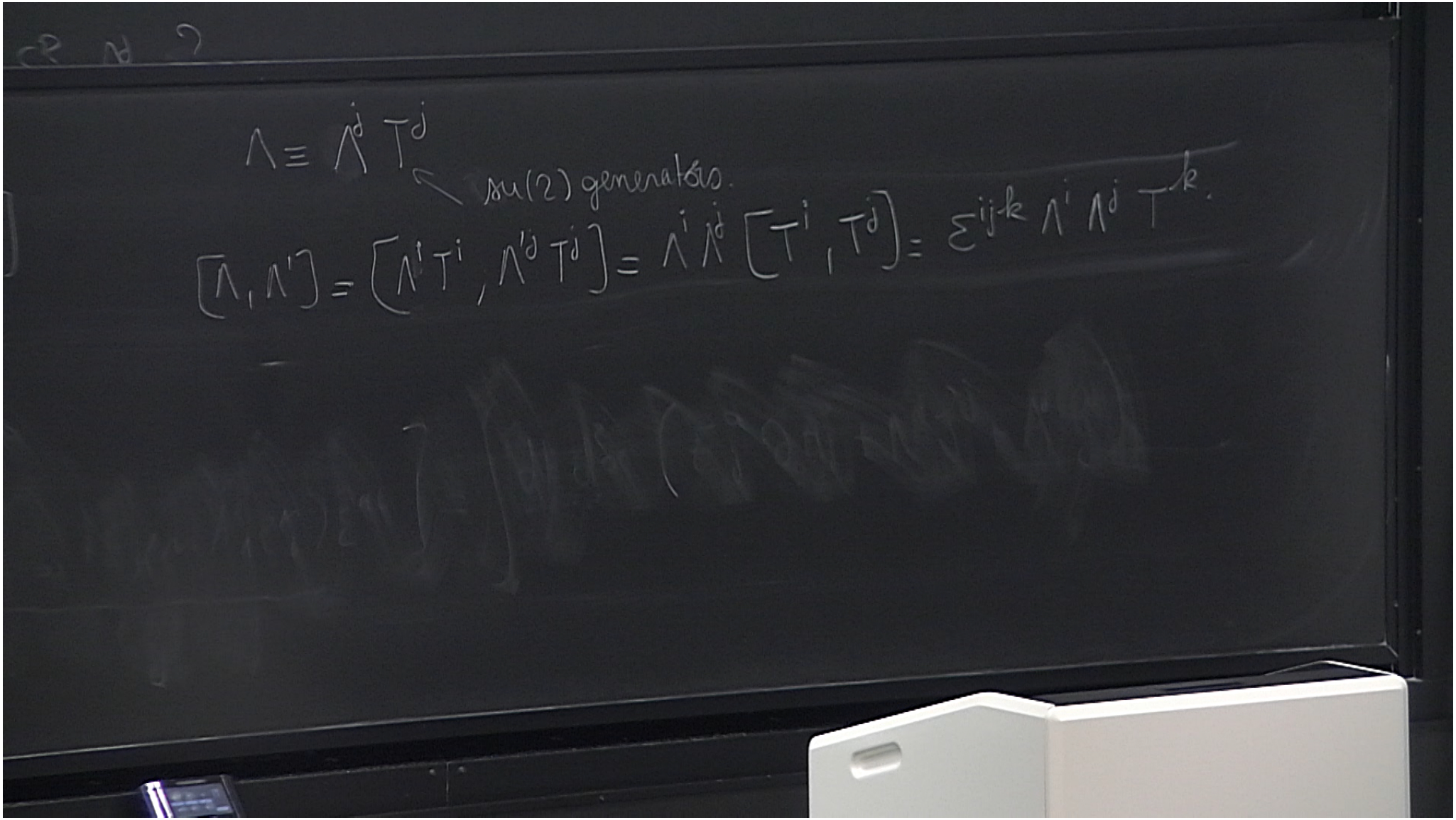
$$\delta^R_{A(v)} e^d_\mu + \delta^T_{e(v)} e^d_\mu = \delta^D_v e^d_\mu + \underbrace{T^d_{\mu\nu} v^\nu}_{=0 \text{ on shell}}$$

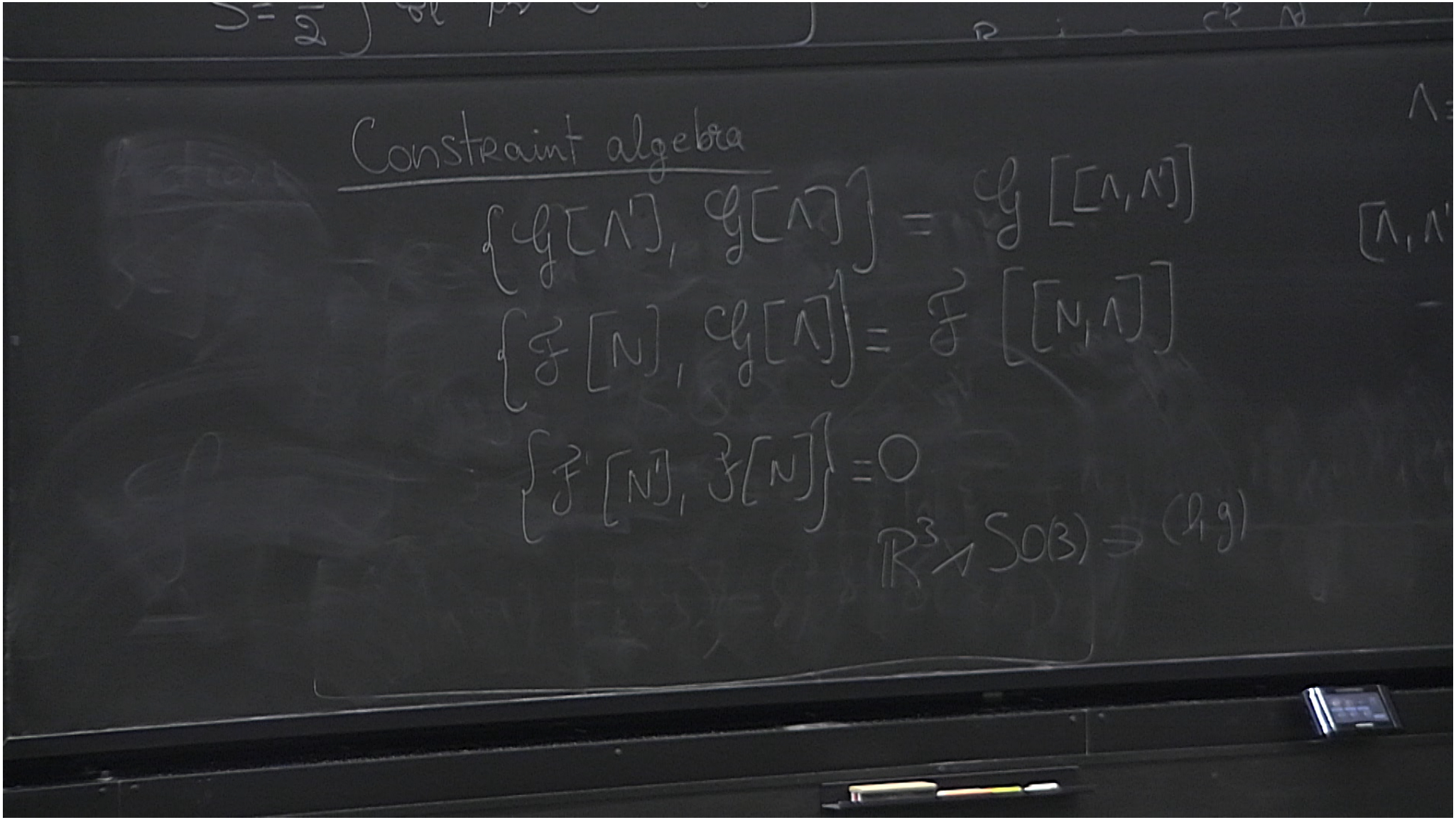
$A(v) = A_\nu v^\nu$

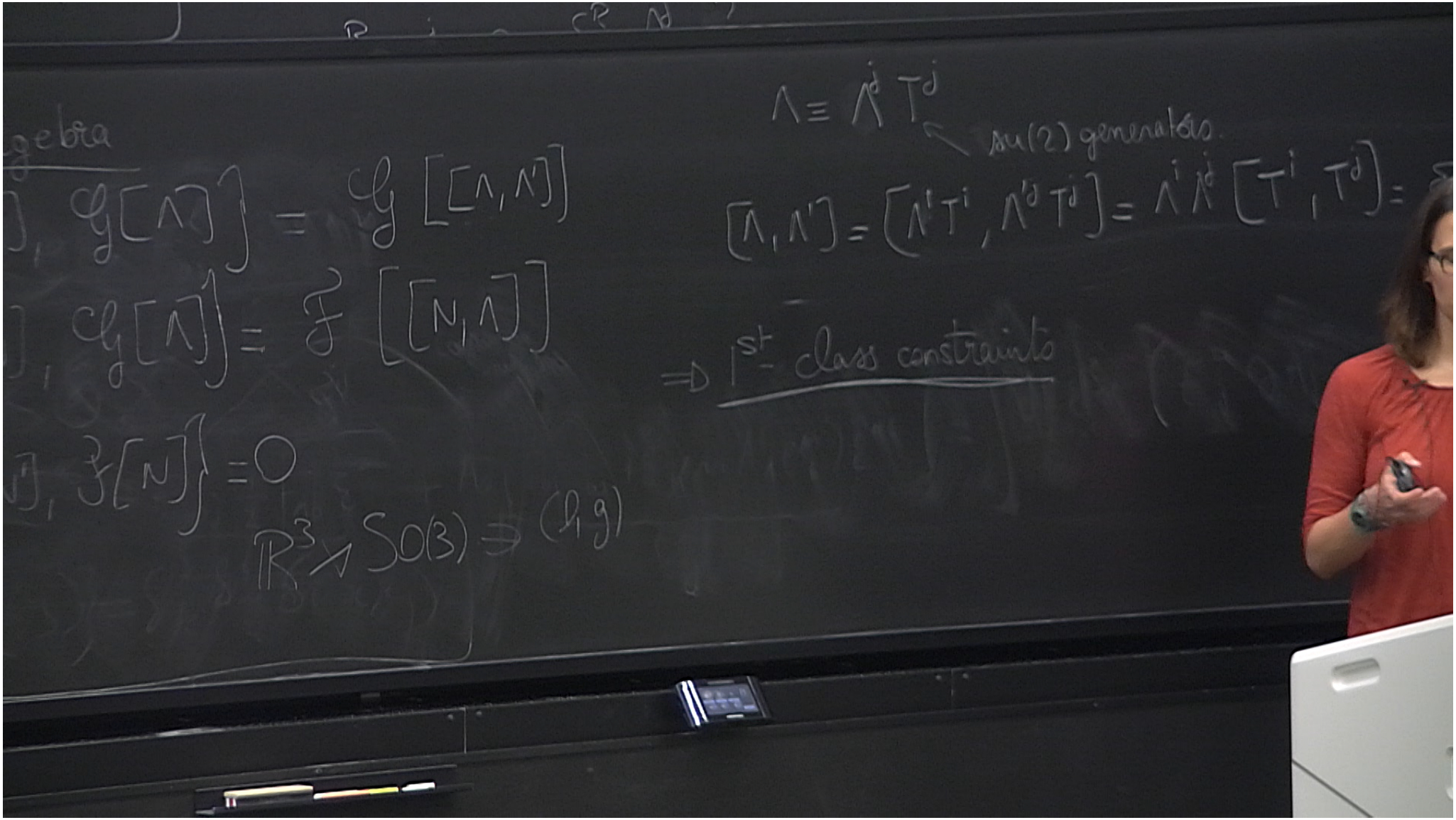
$e(v) = e^d_\nu v^\nu$

$\Rightarrow$ The symmetries are not independent on shell


Constraint algebra
$\{\mathcal{G}[\Lambda'], \mathcal{G}[\Lambda]\} = \mathcal{G}[[\Lambda, \Lambda]$


$\Lambda \equiv \Lambda^j T^j$
su(2) generators.
$[\Lambda, \Lambda'] = [\Lambda^i T^i, \Lambda'^j T^j] = \Lambda^i \Lambda^j [T^i, T^j] = \varepsilon^{ijk} \Lambda^i \Lambda^j T^k$.


Constraint algebra
$\{\mathcal{G}[\Lambda'], \mathcal{G}[\Lambda]\} = \mathcal{G}[[\Lambda,\Lambda]]$
$\{\mathcal{F}[N], \mathcal{G}[\Lambda]\} = \mathcal{F}[[N,\Lambda]]$
$\{\mathcal{F}[N'], \mathcal{F}[N]\} = 0$
$R^3 \rightarrow SO(3) \Rightarrow (h,g)$


gebra
$\Lambda \equiv \Lambda^a T^a$
su(2) generators.
$[\Lambda, \Lambda'] = [\Lambda^i T^i, \Lambda'^j T^j] = \Lambda^i \Lambda'^j [T^i, T^j] =$
$\Rightarrow$ 1st class constraints
$R^3 \to SO(3) \Rightarrow (1,1)$