Title: PSI 2017/2018 - Quantum Gravity - Lecture 5

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Abstract:

KINEMATICAL Φ-S

Kinematical H-S:

① $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$

② $C = p_t + \frac{p_q^2}{2m} \to \hat{C} = -i\hbar\partial_t + \frac{\hbar^2}{2m}\partial_q^2$

$$\begin{cases} \hat{t}|\psi\rangle = t|\psi\rangle \\ \hat{p}_t = -i\hbar\partial_t \end{cases} \quad (t \leftrightarrow q)$$

$$\underbrace{= -i\hbar\partial_t + \frac{\hbar^2}{2m}\partial_q^2}_{= 0}$$

$(\longleftrightarrow, q)$

3

$$\hat{C}|\Psi\rangle = 0 \Rightarrow \Psi(q,t) = e^{-\frac{iHt}{\hbar}}\psi(q)$$

Physical $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$

Inner Product: $\int \bar{\Psi}\Psi \, dq$

4 Gauge invariant OBS.

Dirac obs.

$\hat{F}_{\hat{q}}$, $\hat{F}_{\hat{p}_q}$

Gravity

E-H: $k\int \sqrt{g}\, R\, d^3X$.

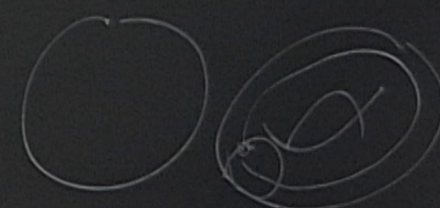
3D

$R_{\mu\nu}=0 \rightarrow R_{\mu\nu\rho\sigma}=0$.

No local d.o.f.

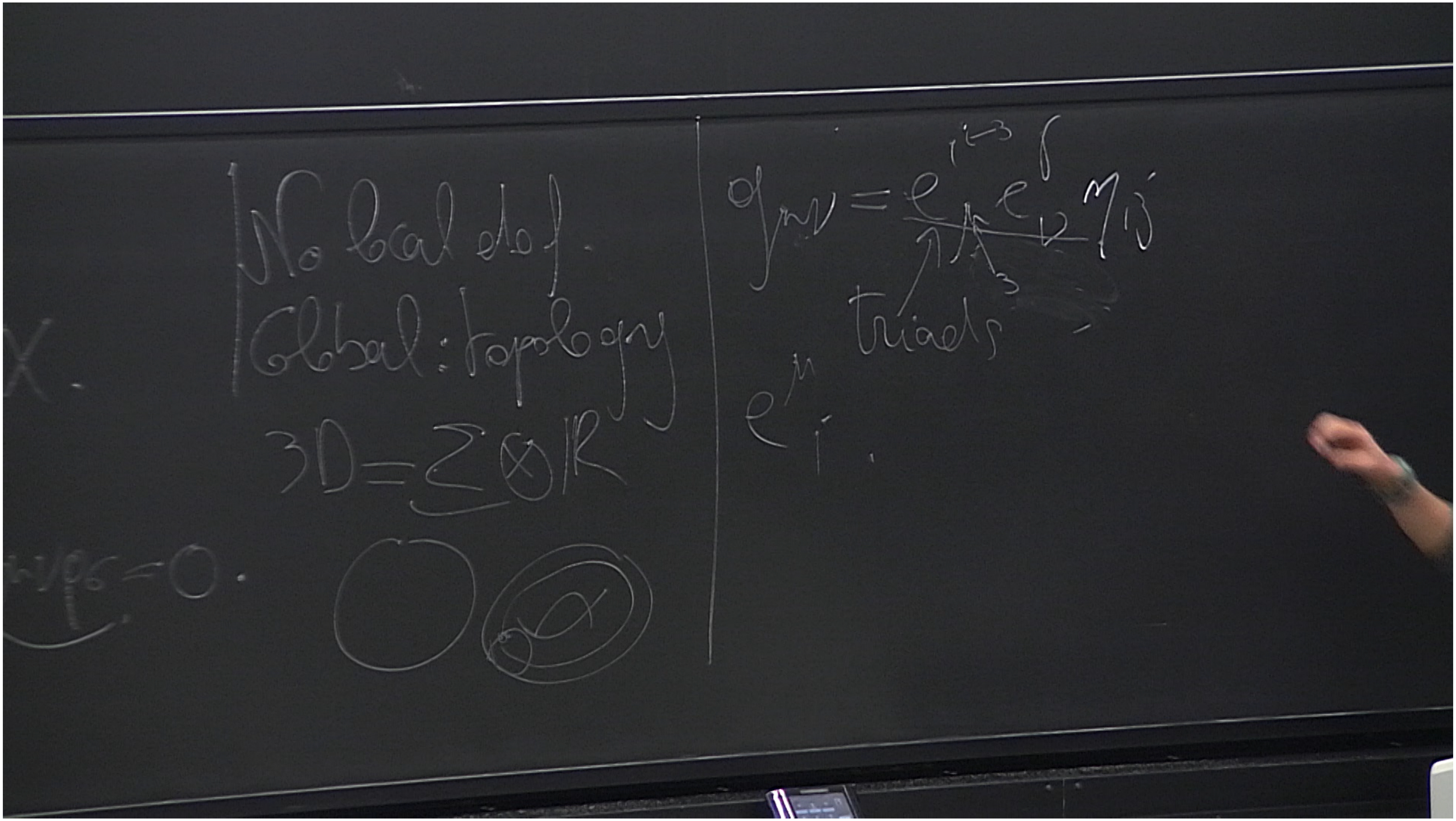
Global: topology

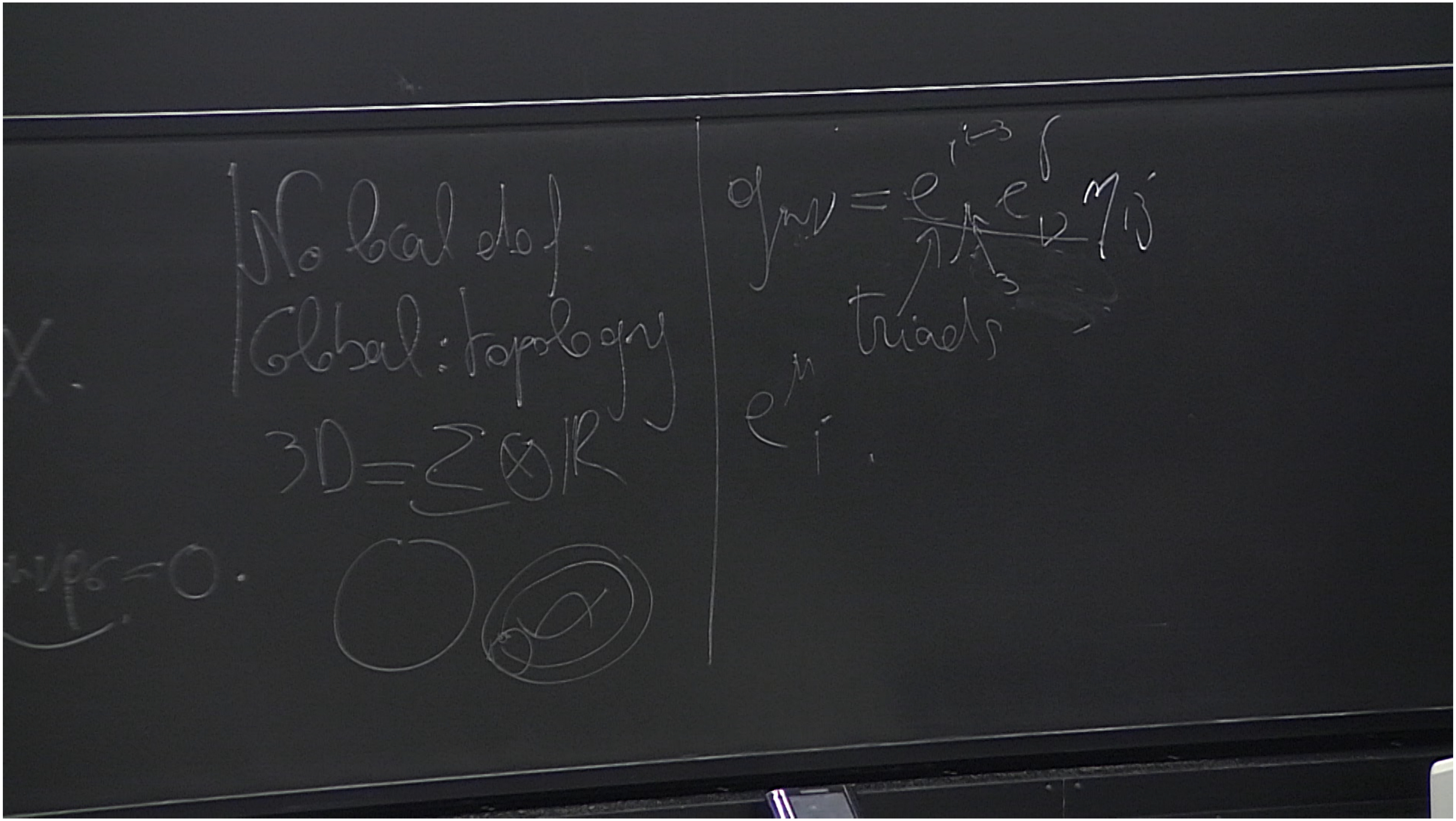
$3D = \Sigma \otimes \mathbb{R}$



$g_{\mu\nu}$

e^{μ}_{i}


No local d.o.f.
Global: topology
3D = Σ ⊗ R
triads


No local d.o.f.
Global: topology
3D = Σ ⊗ ℝ
$g_{\mu\nu} = e^{i}_{\mu} e^{j}_{\nu} \eta_{ij}$
i→3
triads
e^{μ}_{i}

$$g_{\mu\nu} = e^{i}_{\mu}\, e^{j}_{\nu}\, \boxed{\eta_{ij}}$$

$i \to 3$

triads $\qquad \delta_{ij}$

e^{μ}_{i}

connection $\omega^{j}_{\mu k} \longrightarrow D_\mu$

spacetime $\Gamma^{e}_{\mu\nu} \longrightarrow \nabla_\mu$

triads: e^{μ}_{i}

co-triads $e^{i}_{\mu} = e^{\nu}_{j}\, g_{\mu\nu}\, \delta^{ji}$

- compatibility condition $\boxed{D_\mu e^j_\nu = 0}$ $\qquad \omega^j_{\mu k} = - e^\nu_k \nabla_\mu e^j_\nu \quad (*)$

- $\tilde{\Gamma}$, from $(*)$: $\tilde{\Gamma}^e_{\mu\nu} = e^e_j \omega^j_{\mu k} e^k_\nu + e^e_j \partial_\mu e^j_\nu$

<u>conditions on ω such as $\tilde{\Gamma} = \Gamma$?</u>

$$\nabla_\mu \big(\underbrace{e^\nu_k e^e_j g_{\nu e}}_{\delta_{kj}} \big) = 0 = -\omega_{\mu j k} - \omega_{\mu k j} \quad \text{where } \omega_{\mu j k} =$$

...ility
...ution $\boxed{D_\mu e^j_\nu = 0}$.

$$\omega^j_{\mu k} = - e^\nu_k \nabla_\mu e^j_\nu \quad (*)$$

...m $\circledast$. $\tilde{\Gamma}^e_{\mu\nu} = e^e_j \omega^j_{\mu k} e^k_\nu + e^e_j \partial_\mu e^j_\nu$

$\underline{\text{conditions on } \omega \text{ such as } \tilde{\Gamma} = \Gamma \text{ ?}}$

$$\nabla_\mu (\underbrace{e^\nu_k e^e_j g_{\nu e}}_{\delta_{kj}}) = 0 \underset{\substack{\uparrow \\ \text{if } \tilde{\Gamma} = \Gamma}}{=} - \omega_{\mu j k} - \omega_{\mu k j} \quad \text{where} \quad \omega_{\mu j k} = \omega_\mu{}^l{}_k \delta_{lj}$$

metric compatibility of $\Gamma \Rightarrow$ antisymme...

· Compatibility condition $\boxed{D_\mu e^j_\nu = 0}$ $\qquad \omega^j_{\mu k} = -e_k \nabla_\mu e^j_k$ ⊛

· $\tilde\Gamma$, from ⊛: $\tilde\Gamma^e_{\mu\nu} = e^e_j \omega^j_{\mu k} e^k_\nu + e^e_j \partial_\mu e^j_\nu$

conditions on ω such as $\tilde\Gamma = \Gamma$?

$D_\mu(\delta_{ij}) = 0 \iff \nabla_\mu(\underbrace{e^\nu_k e^e_j}_{\delta_{kj}} g_{\nu e}) = 0 \overset{\uparrow}{=} -\omega_{\mu jk} - \omega_{\mu kj}$ where $\omega_{\mu jk}$

if $\tilde\Gamma = \Gamma$

antisymmetry of ω

⇒ covariant derivative D ∽ metric

$\boxed{\text{metric compatibility of } \Gamma}$ =

KINEMATICAL φ-S

...variant derivative D

...o metric

δ_{kj}

Torsion free connection

$$D_\mu e^j_\nu - D_\nu e^j_\mu = 0 \Rightarrow T^j_{\mu\nu} = \underbrace{\left(\Gamma^e_{\mu\nu} - \Gamma^e_{\nu\mu}\right)}_{=0 \text{ if } \Gamma \text{ torsion free}} e^j_e$$

$$T^j_{\mu\nu} := \partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu$$

(connection $\omega_{\mu jk}$ antisymmetric in its last 2 indices + torsion free $T^j_{\mu\nu} = 0$) $\Rightarrow$ L C connection

variant derivative D
to metric

Torsion free connection

$$D_\mu e^J_\nu - D_\nu e^J_\mu = 0 \Rightarrow T^J_{\mu\nu} = \underbrace{\left(\Gamma^e_{\mu\nu} - \Gamma^e_{\nu\mu}\right)}_{=0 \text{ if } \Gamma \text{ torsion free}} e^J_e$$

$$T^J_{\mu\nu} := \partial_\mu e^J_\nu - \partial_\nu e^J_\mu + \omega^J_{\mu k} e^k_\nu - \omega^J_{\nu k} e^k_\mu$$

A connection $\omega_{\mu jk}$ antisymmetric in its last 2 indices
+ torsion free $T^J_{\mu\nu} = 0$) $\Rightarrow$ L C connection

$e_\nu^k - \omega_{\nu}{}^{j}{}_{k}\, e_\mu^k$

connection

Rmk: 1^{st} order formulation.

$\omega_{\mu jk}$ antisymmetric
(metricity of connection is imposed)

but whose torsion does not vanish a priori.

$$\underbrace{e^e_j\, g_{\nu e}}_{\delta_{kj}}) = 0 \overset{\tilde\Gamma = \Gamma}{=} -\omega_{\mu j k} - \omega_{\mu k j} \quad \text{where} \quad \omega_{\mu j k} = \omega_{\mu}{}^{l}{}_{k}\, \delta_{lj}$$

if $\tilde\Gamma = \Gamma$

$\boxed{\text{metric compatibility of } \Gamma} \Rightarrow$ antisymmetry of ω.

onnection

$$e^j_\mu = 0 \Rightarrow T^j_{\mu\nu} = \underbrace{\left(\tilde\Gamma^e_{\mu\nu} - \tilde\Gamma^e_{\nu\mu}\right)}_{=0 \text{ if } \Gamma \text{ torsion free}} e^j_e \qquad \tilde\Gamma = \Gamma$$

$$T^j_{\mu\nu} := \partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega^j_{\mu k}\, e^k_\nu - \omega^j_{\nu k}\, e^k_\mu$$

$\omega_{\mu jk}$ antisymmetric in its last 2 indices
+ torsion free $T^j_{\mu\nu} = 0$) $\Rightarrow$ L C connection

Rmk: 1^{st} order

$\omega_{\mu jk}$ antis
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but whose torso
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...mmetry of ω δ_{kj} if $\Gamma = \Gamma$ $\boxed{\text{metric compatibility of } \Gamma} \Rightarrow D$ a...

...ovariant derivative D

... metric

Torsion free connection

$$D_\mu e^j_\nu - D_\nu e^j_\mu = 0 \quad \Rightarrow \quad T^j_{\mu\nu} = \underbrace{\left(\tilde{\Gamma}^e_{\mu\nu} - \tilde{\Gamma}^e_{\nu\mu}\right)}_{= 0 \text{ if } \Gamma \text{ torsion free}} e^j_e \qquad \tilde{\Gamma} = \Gamma$$

$$T^j_{\mu\nu} := \partial_\mu e^j_\nu - \partial_\nu e^j_\mu + \omega^j_{\mu k} e^k_\nu - \omega^j_{\nu k} e^k_\mu$$

A connection $\omega_{\mu jk}$ antisymmetric in its last 2 indices + torsion free $T^j_{\mu\nu} = 0$) $\Rightarrow$ L C connection

3D first order action for gravity

$\sqrt{g}\, R$

$$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \underbrace{v_\rho}_{e_\rho^i v_i}$$

$$= e_\rho^i (D_\mu D_\nu - D_\nu D_\mu) v_i$$

3D first order action for gravity

$\sqrt{g}\, R$

$$R_{\mu\nu\rho}{}^{\sigma} v_\sigma = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \underbrace{v_\rho}_{e_\rho^{j} v_j}$$

$$= e_\rho^{j} (D_\mu D_\nu - D_\nu D_\mu) v_j$$

$$= e_\rho^{j} (F_{\mu\nu jk})\, e^{\sigma k} v_\sigma$$

where $F_{\mu\nu jk} =$

$$\Rightarrow \boxed{R_{\mu\nu\rho}{}^{\sigma} = e_{\rho}^{\ j} e^{\sigma k} F_{\mu\nu jk}.}$$

where $F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j l}\, \omega_{\nu}{}^{l}{}_{k} - \omega_{\nu j l}\, \omega_{\mu}{}^{l}{}_{k}$

curvature tensor / ω

$$\Rightarrow \boxed{R_{\mu\nu\rho}{}^{\sigma} = e_\rho{}^{j}\, e^{\sigma k}\, F_{\mu\nu jk}}$$

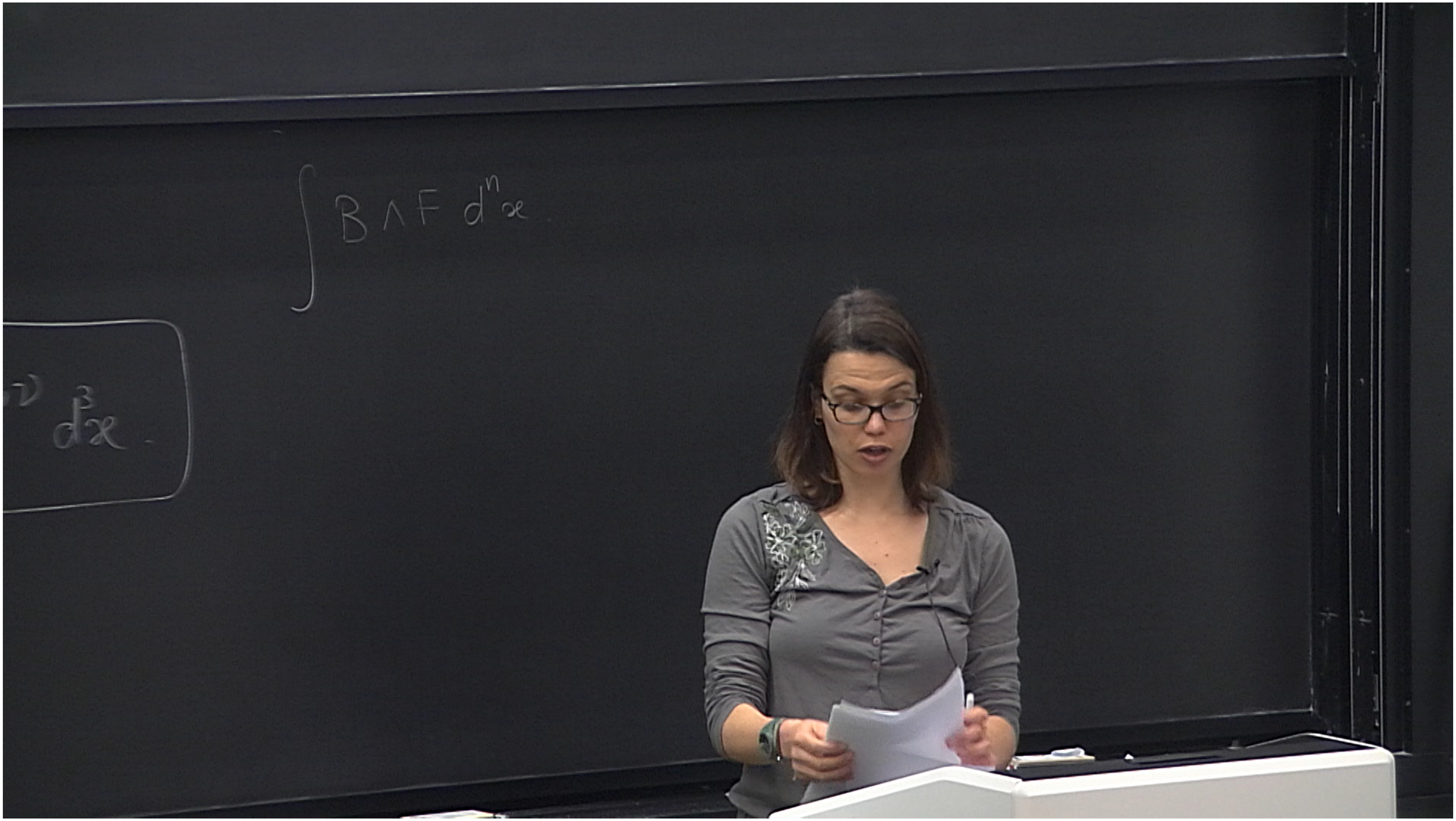
where $F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu jl}\, \omega_\nu{}^{l}{}_{k} - \omega_{\nu jl}\, \omega_\mu{}^{l}{}_{k}$

curvature tensor / ω

$\cdot\ R = R_{\mu\nu}\, g^{\mu\nu} = e^{\mu}_{j}\, e^{\sigma}_{k}\, F^{jk}_{\mu\sigma}(\omega)$

$\cdot\ \sqrt{\det g} = |\det(e_e)| = |\det e^e|^{-1}$

$$S = \int \sqrt{g}\, R\, d^3x \longrightarrow \boxed{S = -\int e_{\sigma l}\, F^{l}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x}$$


$\int B \wedge F \, d^n x$
d^3x

$\overset{?}{=} e_e^{\sigma} (\Gamma_{\mu\nu jk}) e \ldots$ curvature tensor

$\cdot\, R = R_{\mu\nu} g^{\mu\nu} = e_j^{\mu} e_k^{\sigma} F_{\mu\sigma}^{jk}(\omega)$

$\cdot\, \sqrt{\det g} = |\det(e_e)| = |\det e^e|^{-1}$

$$S = \int \sqrt{g}\, R\, d^3x \longrightarrow \boxed{S = -\int e_{\sigma l} F_{\mu\nu}^{l}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu} d^3x.}$$

(see tutorial 2)

$\sim$ tensor density

$\int B \wedge F\, d^n x$

$$\int B \wedge F \, d^n x$$

(su(2)-valued object)

$$\omega_{\mu j k} = \varepsilon_{j l k} \, \omega_\mu^{l}$$

$$F_{\mu\nu j k} = \varepsilon_{j l k} \, F_{\mu\nu}^{l}$$

$$R = R_{\mu\nu} g^{\mu\nu} = e^{\mu}_{j}\, e^{\sigma}_{k}\, F^{jk}_{\mu\sigma}(\omega)$$

$$\sqrt{\det g} = |\det(e_e)| = |\det e^{e}|^{-1}$$

$$S = \int \sqrt{g}\, R\, d^3x \longrightarrow \boxed{S = -\int e_{\sigma l}\, F^{l}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3}$$

$\downarrow$

check lecture notes

$$g^{\mu\nu} = e^{\mu}_{j}\, e^{\sigma}_{k}\, F^{jk}_{\mu\sigma}(\omega)$$

$$= |\det(e_e)| = |\det e^e|^{-1}$$

$$= \int \sqrt{g}\, R\, d^3x \longrightarrow \boxed{S = -\int e_{\sigma l}\, F^{l}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x\,.}$$

check lecture notes

(see tutorial 2)

$\sim$ tensor density

eq of motion

- $F^{l}_{\mu\nu} = 0$
- $T^{l}_{\mu\nu} = 0$

Symmetries
- Rotation in the internal index
- diffeo.

$$\int B \wedge F \, d^n x$$

$$S = -\int e_{\sigma l} \, F^{l}_{\mu\nu}(\omega) \, \tilde{\varepsilon}^{\sigma\mu\nu} \, d^3x .$$

(see tutorial 2)

~ tensor density

(su(2)-valued

$\omega_{\mu j k} = \varepsilon_j$

$F_{\mu\nu j k} = \varepsilon$

lecture notes

eq of motion
- $F^{l}_{\mu\nu} = 0$
- $T^{l}_{\mu\nu} = 0$

Canonical analysis 3D gravity first order action

$\mathcal{M} = \Sigma \times \mathbb{R}$ $\qquad \mu = (0, a)$ $\hookrightarrow$ space coordinates

$\boxed{\omega = A}$

$$S = \int e^j_b\, \partial_0 A_{aj}\, \tilde{\varepsilon}^{ab} + e_{0j}\, \frac{1}{2} F^j_{ab}\, \tilde{\varepsilon}^{ab} + A^j_0 \left(\partial_a e_{bj} + \varepsilon_{jlm} A^l_a\, e^m_b\right) \tilde{\varepsilon}^{ab}\, d^2x\, dt$$

Canonical analysis 3D gravity first order action

$\mathcal{M} = \Sigma \times \mathbb{R}$ $\qquad \mu = (0, a)$ $\hookrightarrow$ space coordinates $\qquad \tilde{\varepsilon}^{0ab} = \tilde{\varepsilon}^{ab}$

$\boxed{\omega = A}$

$$S = \frac{1}{2}\int e_\ell F^\ell_{\mu\nu} \tilde{\varepsilon}^{\rho\mu\nu} \Rightarrow S = \int e^j_b \, \partial_0 A_{aj} \, \tilde{\varepsilon}^{ab} + e_{0j} \frac{1}{2} F^j_{ab} \, \tilde{\varepsilon}^{ab} + A^j_0 \left(\partial_a e_{bj} + \varepsilon_{j\ell m} A^\ell_a \, e^m_b \right) \tilde{\varepsilon}^{ab} \, d^2x \, dt$$

ab.

$$) \tilde{\varepsilon}^{ab} \, d^2x \, dt$$

$$\{A_a^d(x), E_k^b(y)\} = \delta_k^d \, \delta_a^b \, \delta(x,y)$$

$$\boxed{E_j^a = \frac{\delta S}{\delta(\partial_0 A_a^d)} = \tilde{\varepsilon}^{ab} e_{bj}}$$

Canonical analysis 3D gravity first order action

$$\mathcal{M} = \Sigma \times \mathbb{R} \qquad \mu = (0, a) \qquad \tilde{\varepsilon}^{0ab} = \tilde{\varepsilon}^{ab}$$

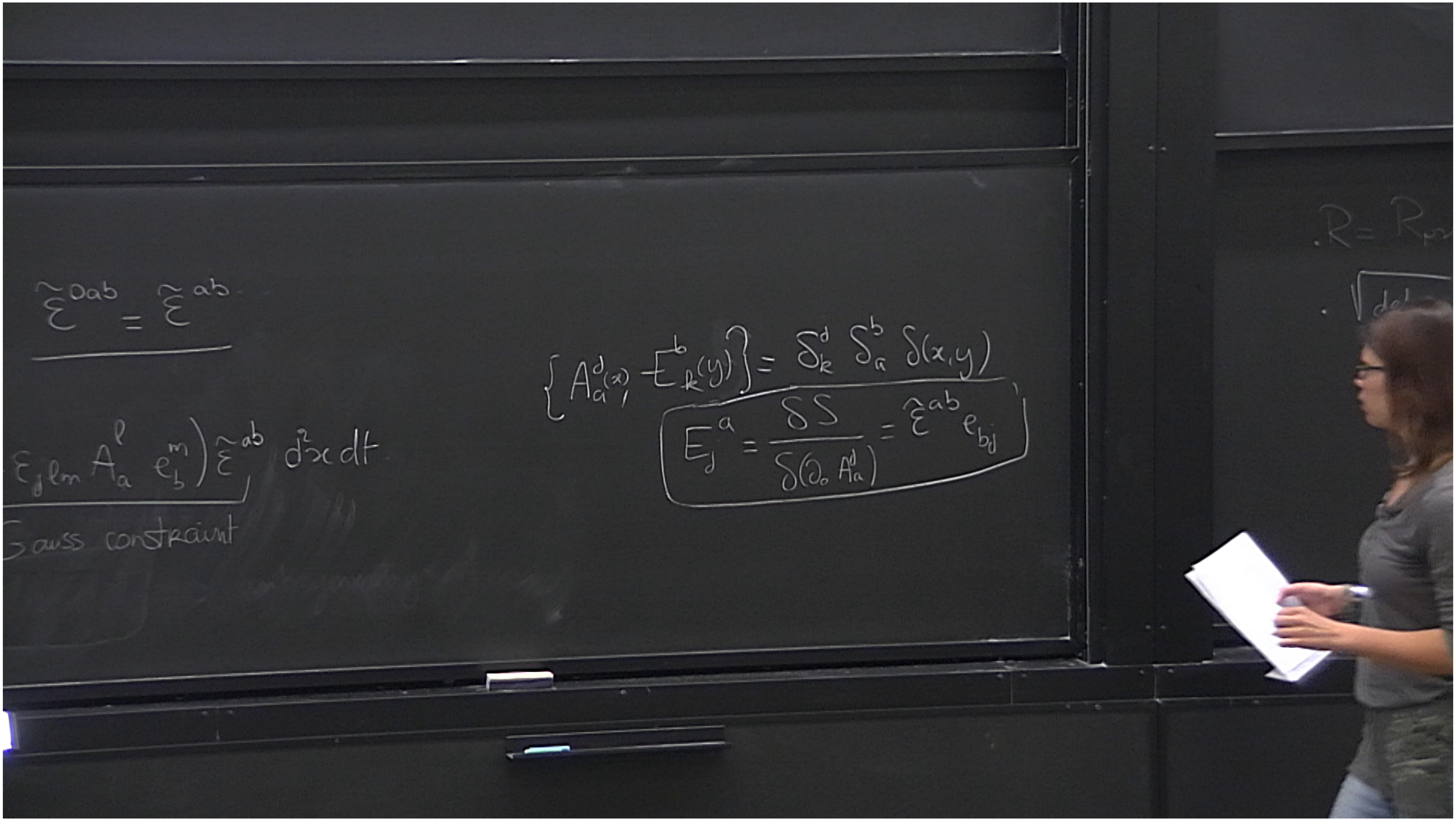
a ↳ space coordinates

$$\boxed{\omega = A}$$

Flatness constraint

$$S = \frac{1}{2}\int \delta_{IL}\, e^I_\rho F^L_{\mu\nu} \tilde{\varepsilon}^{\rho\mu\nu} \Rightarrow S = \int e_b^J \partial_0 A_{aJ} \tilde{\varepsilon}^{ab} + e_{0J} \frac{1}{2} F_{ab}^J \tilde{\varepsilon}^{ab} + A_0^J \underbrace{\left(\partial_a e_{bJ} + \varepsilon_{Jlm} A_a^l e_b^m\right)\tilde{\varepsilon}^{ab}}_{\text{Gauss constraint}} d^2x\, dt$$

Lagrange multipliers


$\tilde{\varepsilon}^{0ab} = \tilde{\varepsilon}^{ab}$
$\varepsilon_{jlm} A^l_a \, e^m_b) \tilde{\varepsilon}^{ab} \, d^2x \, dt$
Gauss constraint
$\{ A^d_a(x), E^b_k(y) \} = \delta^d_k \, \delta^b_a \, \delta(x,y)$
$E^a_j = \frac{\delta S}{\delta(\partial_0 A^j_a)} = \tilde{\varepsilon}^{ab} e_{bj}$
$R = R_{\mu\nu}$

$= \tilde{\varepsilon}^{ab}.$

$e_b^m) \tilde{\varepsilon}^{ab} \, d^2x \, dt.$

onstraint

$\underbrace{(\text{primary} + \text{secondary} + \dots) \text{ constraints}}$

1st class 2nd class.

$$\{A_a^j(x), E_k^b(y)\} = \delta_k^j \, \delta_a^b \, \delta(x,y)$$

$$\boxed{E_j^a = \frac{\delta S}{\delta(\partial_0 A_a^j)} = \tilde{\varepsilon}^{ab} e_{bj}}$$

$= \tilde{\varepsilon}^{ab}$

$e^m_b) \tilde{\varepsilon}^{ab} \, d^2x \, dt$

raint

$\underbrace{(\text{primary} + \text{secondary} + \dots)\ \text{constraints}}$

1st class   2nd class

$$\boxed{\{A^d_a(x), E^b_k(y)\} = \delta^d_k \, \delta^b_a \, \delta(x,y)}$$

$$\boxed{E^a_j = \frac{\delta S}{\delta(\partial_0 A^d_a)} = \tilde{\varepsilon}^{ab} e_{bj}}$$