Title: PSI 2017/2018 - Quantum Gravity - Lecture 5

Date: Mar 23, 2018 10:15 AM

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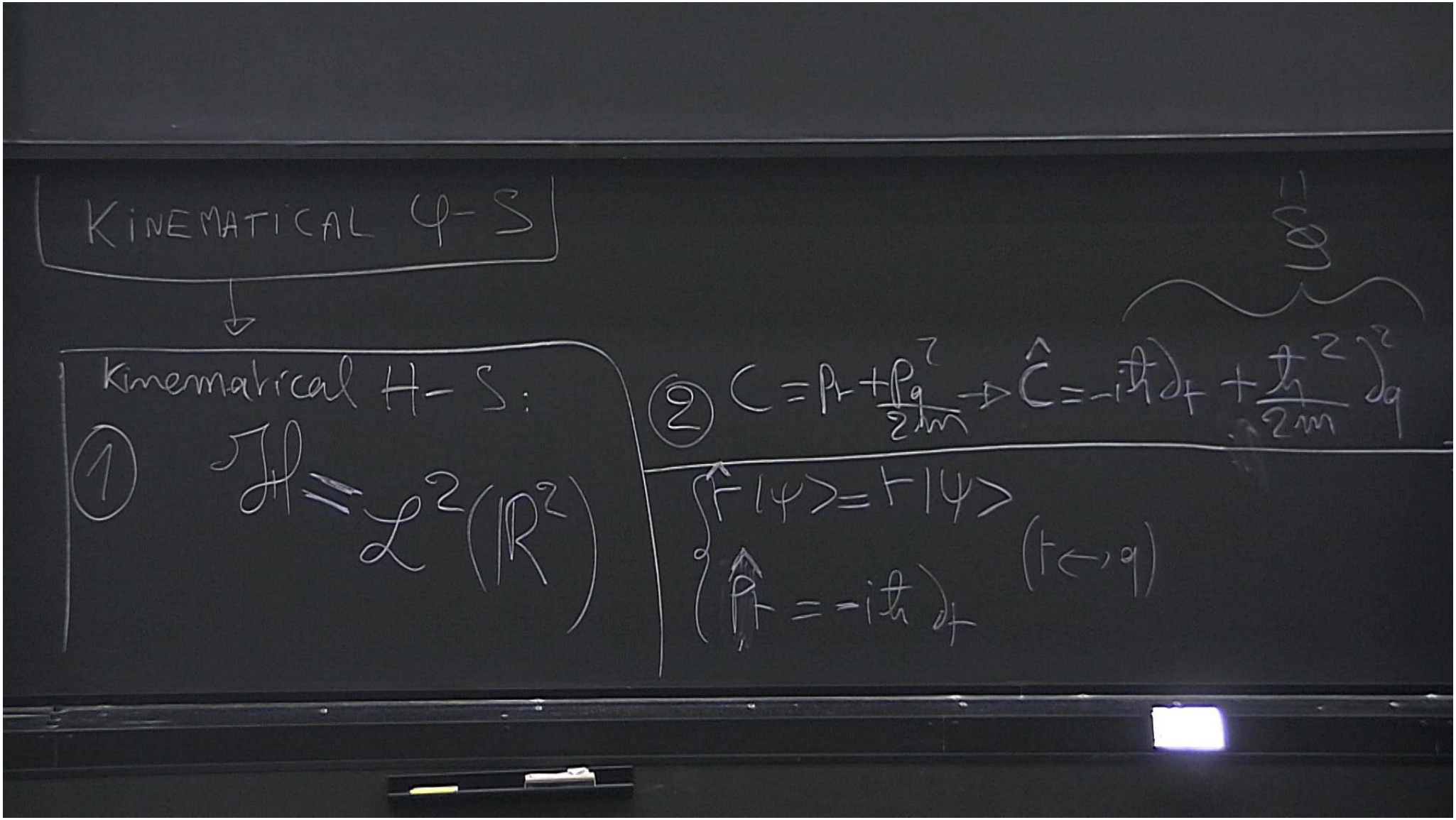
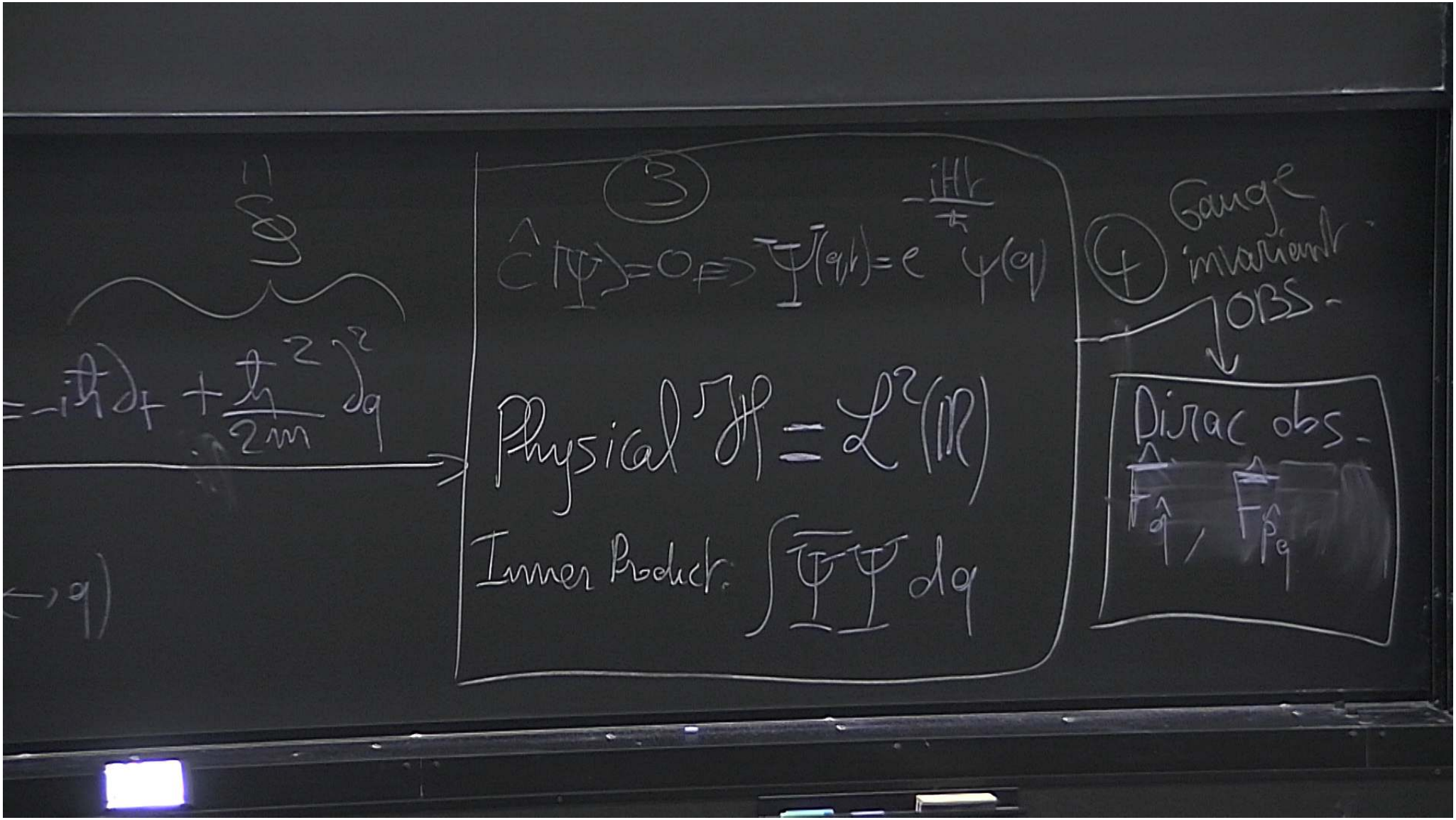
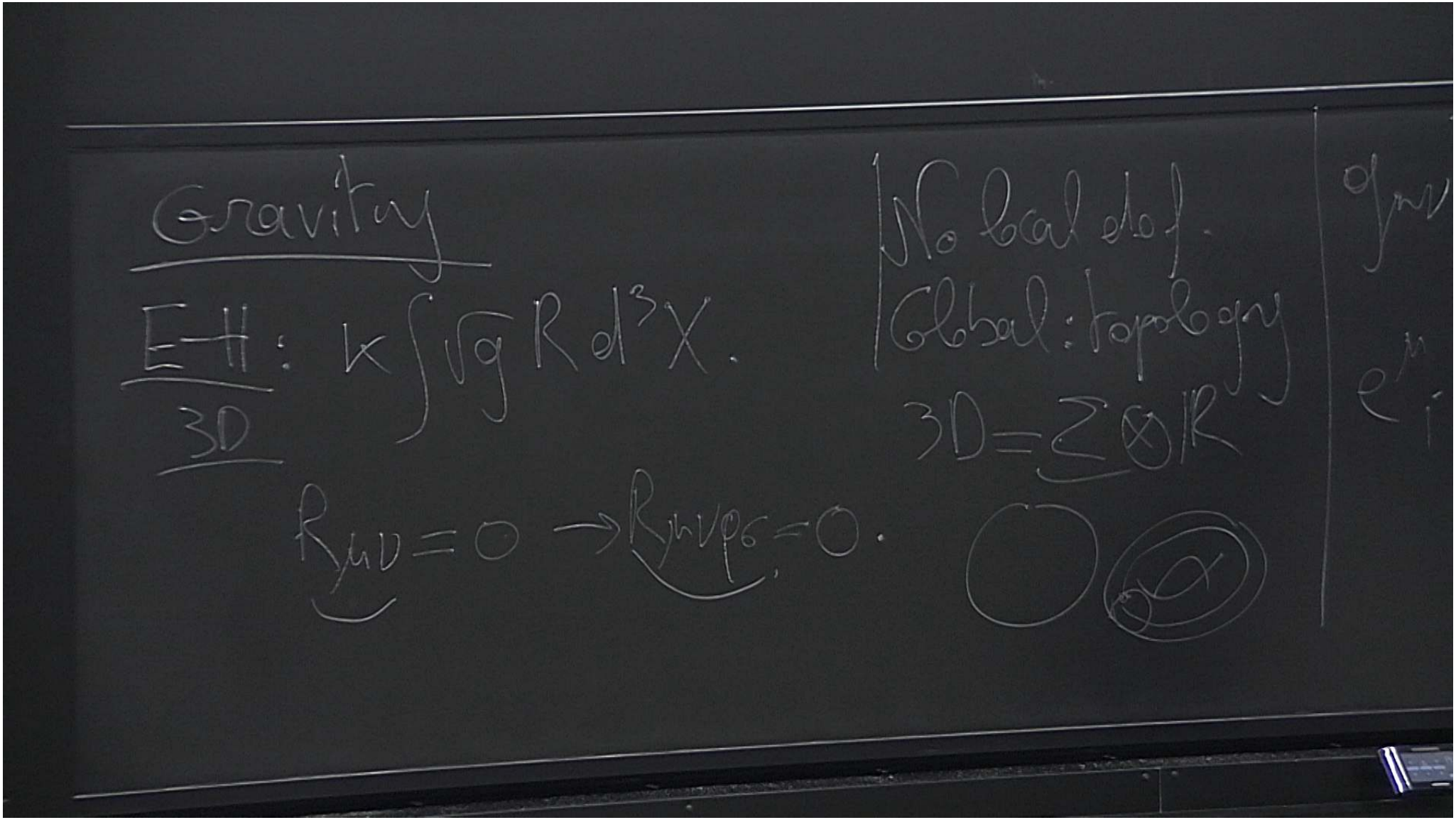
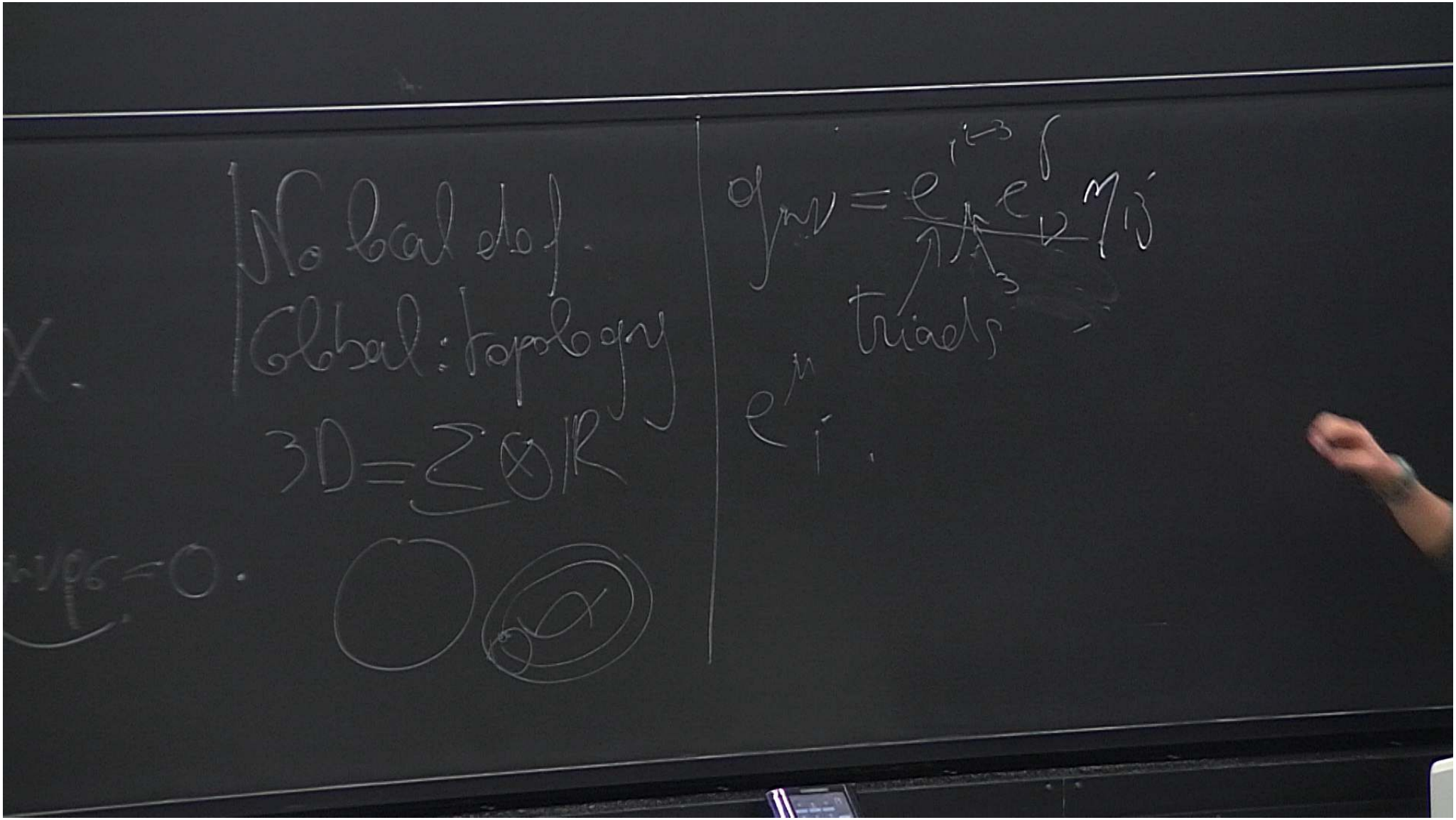
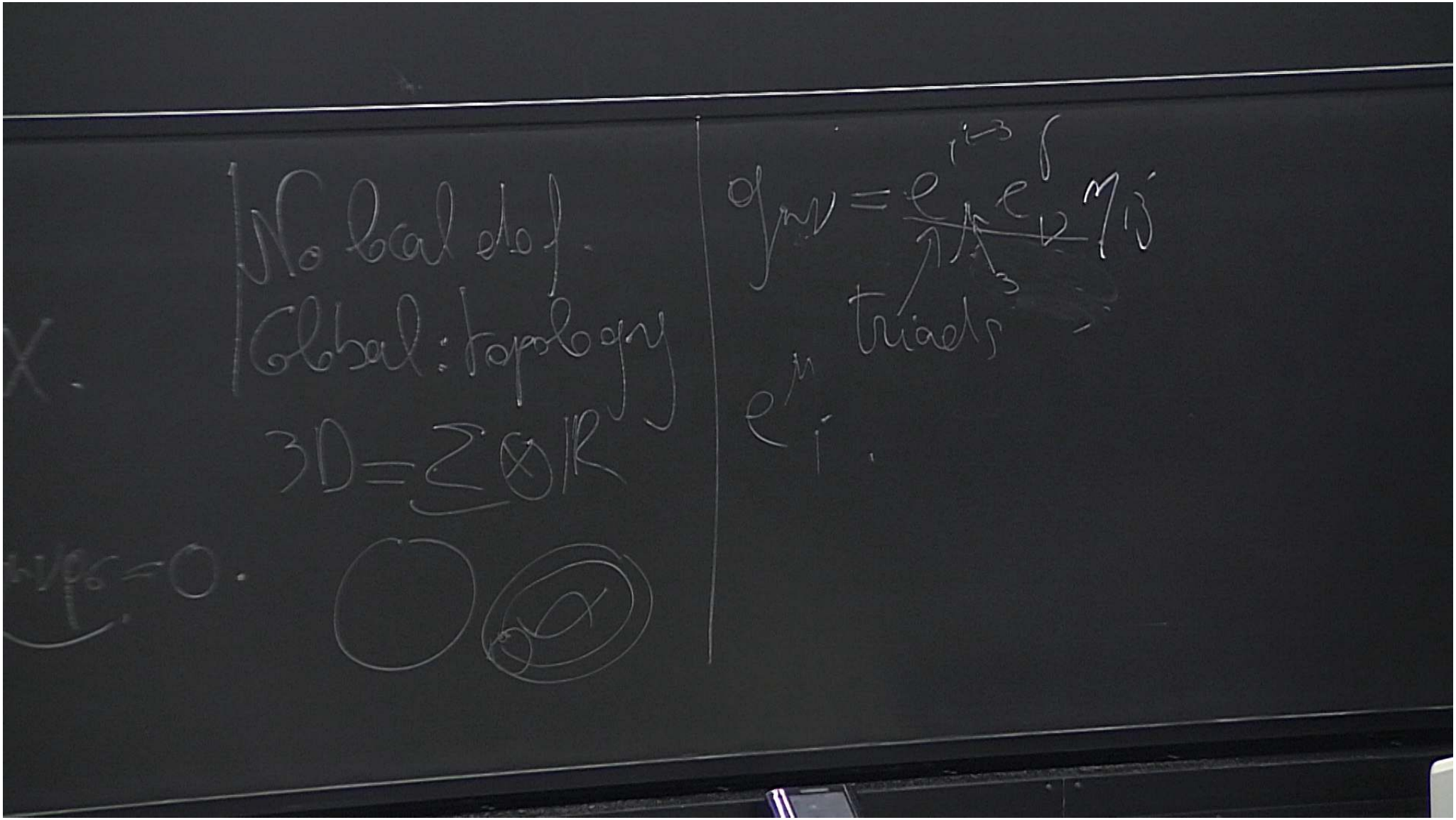
Abstract:

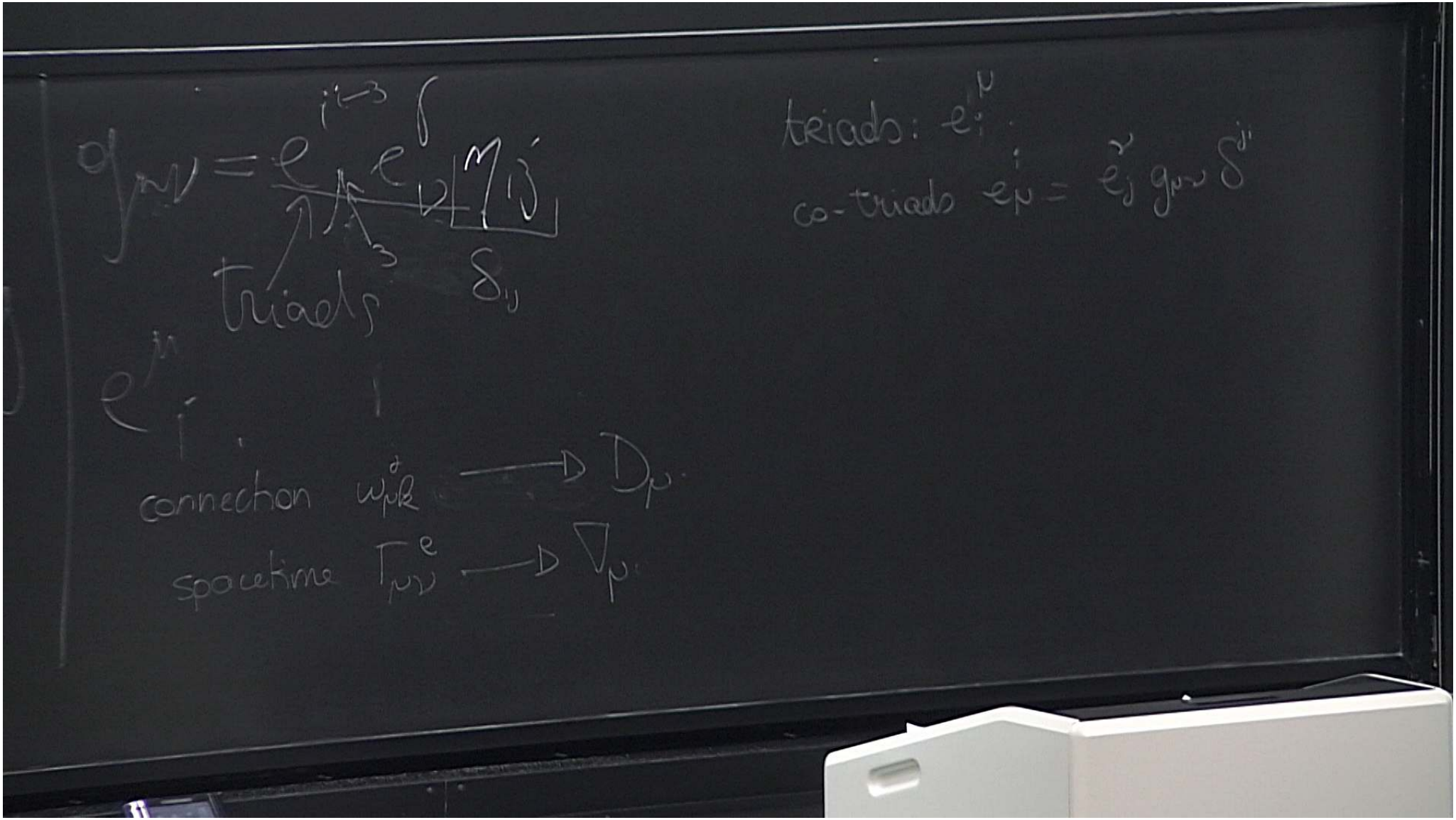

KINEMATICAL Ψ-S
Kinematical H-S:
① $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$
② $C = p_t + \frac{p_q^2}{2m} \to \hat{C} = -i\hbar\partial_t + \frac{\hbar^2}{2m}\partial_q^2$
$\hat{F}|\psi\rangle = f|\psi\rangle$
$(t \leftrightarrow q)$
$\hat{p}_t = -i\hbar\partial_t$


11
$= -i\hbar\partial_t + \frac{\hbar^2}{2m}\partial_q^2$
$\leftrightarrow, q)$
3
$\hat{C}|\Psi\rangle = 0 \Rightarrow \Psi(q,t) = e^{-\frac{iHt}{\hbar}}\psi(q)$
Physical $\mathcal{H} = L^2(\mathbb{R})$
Inner Product: $\int \bar{\Psi}\Psi\, dq$
4 Gauge invariant OBS.
Dirac obs.
$\hat{F}_q$, $\hat{F}_{p_q}$


Gravity
E-H: $k\int\sqrt{g}R\,d^3x$.
3D
$R_{\mu\nu}=0 \rightarrow R_{\mu\nu\rho\sigma}=0$.
No local d.o.f.
Global: topology
$3D=\Sigma\otimes R$
$g_{\mu\nu}$
e^{μ}_{i}


No local d.o.f.
Global: topology
3D = Σ ⊗ R
triads


No local dof.
Global: topology
3D = Σ ⊗ R
$g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}$
e in triads


triads: e^{μ}_{i}
co-triads $e^{i}_{\mu} = e^{\nu}_{j}\, g_{\mu\nu}\, \delta^{ij}$
triads
connection $\omega^{i}_{\mu k} \longrightarrow D_\mu$
spacetime $\Gamma^{\rho}_{\mu\nu} \longrightarrow \nabla_\mu$

- compatibility condition $\boxed{D_\mu e^a_\nu = 0}$ $\qquad \omega^a_{\mu k} = - e^\nu_k \nabla_\mu e^a_k$ (*)

- $\tilde{\Gamma}$, from (*). $\tilde{\Gamma}^\rho_{\mu\nu} = e^\rho_j \omega^a_{\mu k} e^k_\nu + e^\rho_j \partial_\mu e^a_\nu$

conditions on ω such as $\tilde{\Gamma} = \Gamma$?

$\nabla_\mu (\underbrace{e^\nu_k e^e_j g_{\nu e}}_{\delta_{kj}}) = 0 = - \omega_{\mu jk} - \omega_{\mu kj}$ where $\omega_{\mu jk} =$

$$\boxed{D_\mu e^a_\nu = 0} \qquad \omega^a_{\mu k} = -e^\nu_k \nabla_\mu e^a_k \quad (*)$$

$$\tilde{\Gamma}^\rho_{\mu\nu} = e^\rho_j \omega^a_{\mu k} e^k_\nu + e^\rho_j \partial_\mu e^a_\nu$$

conditions on ω such as $\tilde{\Gamma} = \Gamma$?

$$\nabla_\mu \underbrace{(e^\nu_k e^e_j g_{\nu e})}_{\delta_{kj}} = 0 = -\omega_{\mu jk} - \omega_{\mu kj} \quad \text{where } \omega_{\mu jk} = \omega_{\mu k}{}^l \delta_{lj}$$

$\uparrow$ if $\tilde{\Gamma} = \Gamma$ metric compatibility of $\tilde{\Gamma}$ $\Rightarrow$ antisymmet

- compatibility condition $D_\mu e^a_\nu = 0$ $\quad \omega^a_{\mu k} = -e_k \nabla_\mu e^a_k$ (*)

- $\tilde{\Gamma}$, from (*) $\tilde{\Gamma}^\rho_{\mu\nu} = e^\rho_a \omega^a_{\mu k} e^k_\nu + e^\rho_a \partial_\mu e^a_\nu$

conditions on ω such as $\tilde{\Gamma} = \Gamma$?

$D_\mu(\delta_{ij}) = 0 \iff \nabla_\mu \underbrace{(e^\nu_k e^e_j g_{\nu e})}_{\delta_{kj}} = 0 = -\omega_{\mu jk} - \omega_{\mu kj}$ where $\omega_{\mu jk} = \omega^a_{\mu k} e^a_j$

antisymmetry of ω in jk $\quad$ if $\tilde{\Gamma} = \Gamma$ $\quad$ metric compatibility of $\tilde{\Gamma}$

$\Rightarrow$ Covariant derivative D to metric

KINEMATICAL φ-S

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variant derivative D
to metric

Torsion free connection

$$D_\mu e^a_\nu - D_\nu e^a_\mu = 0 \quad \Rightarrow \quad T^a_{\mu\nu} = \underbrace{\left(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\right)}_{=0 \text{ if } \Gamma \text{ torsion free}} e^a_\rho$$

$$T^a_{\mu\nu} := \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\mu k} e^k_\nu - \omega^a_{\nu k} e^k_\mu$$

connection ω_{ijk} antisymmetric in its last 2 indices + torsion free $T^a_{\mu\nu}=0$ $\Rightarrow$ LC connection

variant derivative D
to metric

Torsion free connection

$$D_\mu e^a_\nu - D_\nu e^a_\mu = 0 \quad \Rightarrow \quad T^a_{\mu\nu} = \underbrace{\left(\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\right)}_{=0 \text{ if } \Gamma \text{ torsion free}} e^a_\rho$$

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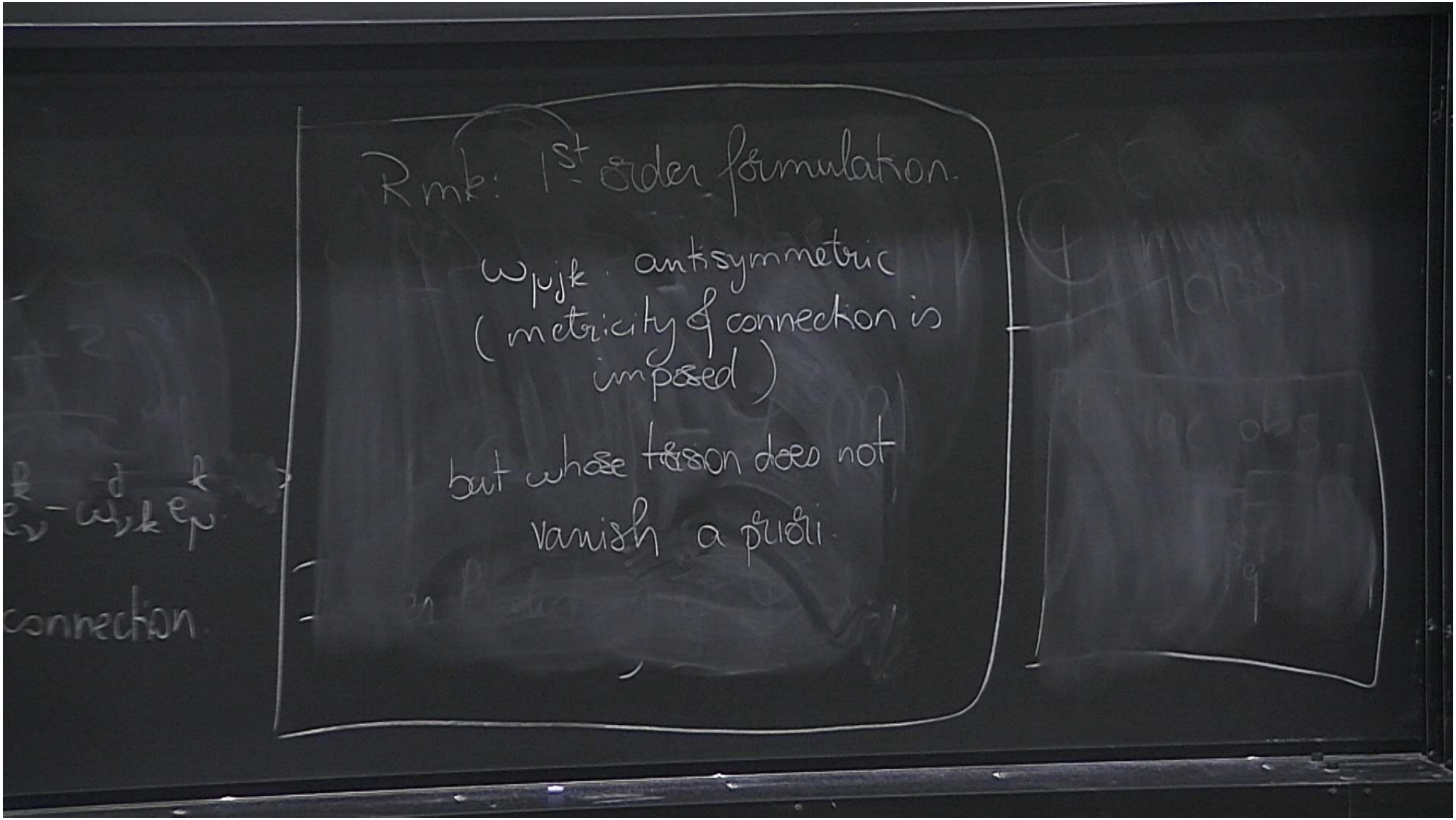
A connection ω_{ijk} antisymmetric in its last 2 indices + torsion free $T^a_{\mu\nu}=0$ $\Rightarrow$ LC connection


Rmk: 1st order formulation.
$\omega_{\mu jk}$ antisymmetric
(metricity of connection is imposed)
but whose torsion does not vanish a priori.
connection

$e^e_j \, g_{ve}) = 0 \Rightarrow -\omega_{\mu jk} - \omega_{\mu kj}$ where $\omega_{\mu jk} = \omega_{\mu k}\, \delta_{lj}$

δ_{kj}

if $\tilde{\Gamma} = \Gamma$ | metric compatibility of $\tilde{\Gamma}$ $\Rightarrow$ antisymmetry of ω.

onnection

$\Rightarrow T^{\sigma}_{\mu\nu} = \left(\tilde{\Gamma}^{e}_{\mu\nu} - \tilde{\Gamma}^{e}_{\nu\mu}\right) e^{\sigma}_{e}$ $\quad \tilde{\Gamma} = \Gamma$

$= 0$ $\forall \Gamma$ torsion free

$T^{\sigma}_{\mu\nu} := \partial_\mu e^{\sigma}_{\nu} - \partial_\nu e^{\sigma}_{\mu} + \omega^{\sigma}_{\mu k} e^{k}_{\nu} - \omega^{\sigma}_{\nu k} e^{k}_{\mu}$

$\omega_{\mu jk}$: antisymmetric in its last 2 indices + torsion free $T^{\sigma}_{\mu\nu} = 0$ $\Rightarrow$ LC connection

Rmk: 1st order

$\omega_{\mu jk}$ antis

(metricity

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but whose torso

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metric compatibility of Γ $\Rightarrow$ a

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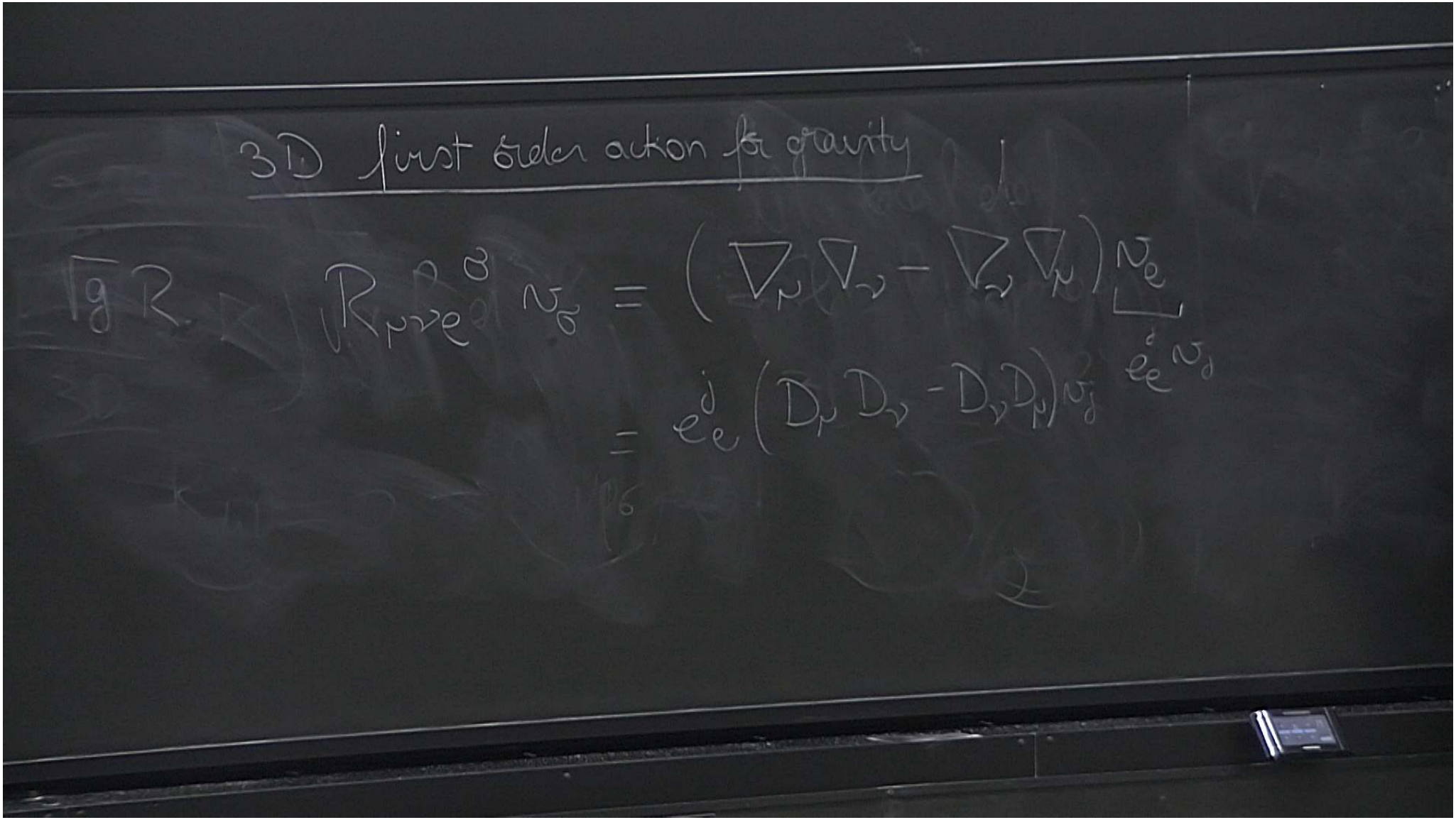
variant derivative D
to metric

Torsion free connection

$$D_\mu e^a_\nu - D_\nu e^a_\mu = 0 \quad \Rightarrow \quad T^a_{\mu\nu} = \underbrace{\left(\tilde{\Gamma}^\rho_{\mu\nu} - \tilde{\Gamma}^\rho_{\nu\mu}\right)}_{=0 \text{ if } \Gamma \text{ torsion free}} e^a_\rho \qquad \tilde{\Gamma} = \Gamma$$

$$T^a_{\mu\nu} := \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\mu k} e^k_\nu - \omega^a_{\nu k} e^k_\mu$$

A connection ω_{jk} antisymmetric in its last 2 indices + torsion free $T^a_{\mu\nu} = 0$ $\Rightarrow$ LC connection

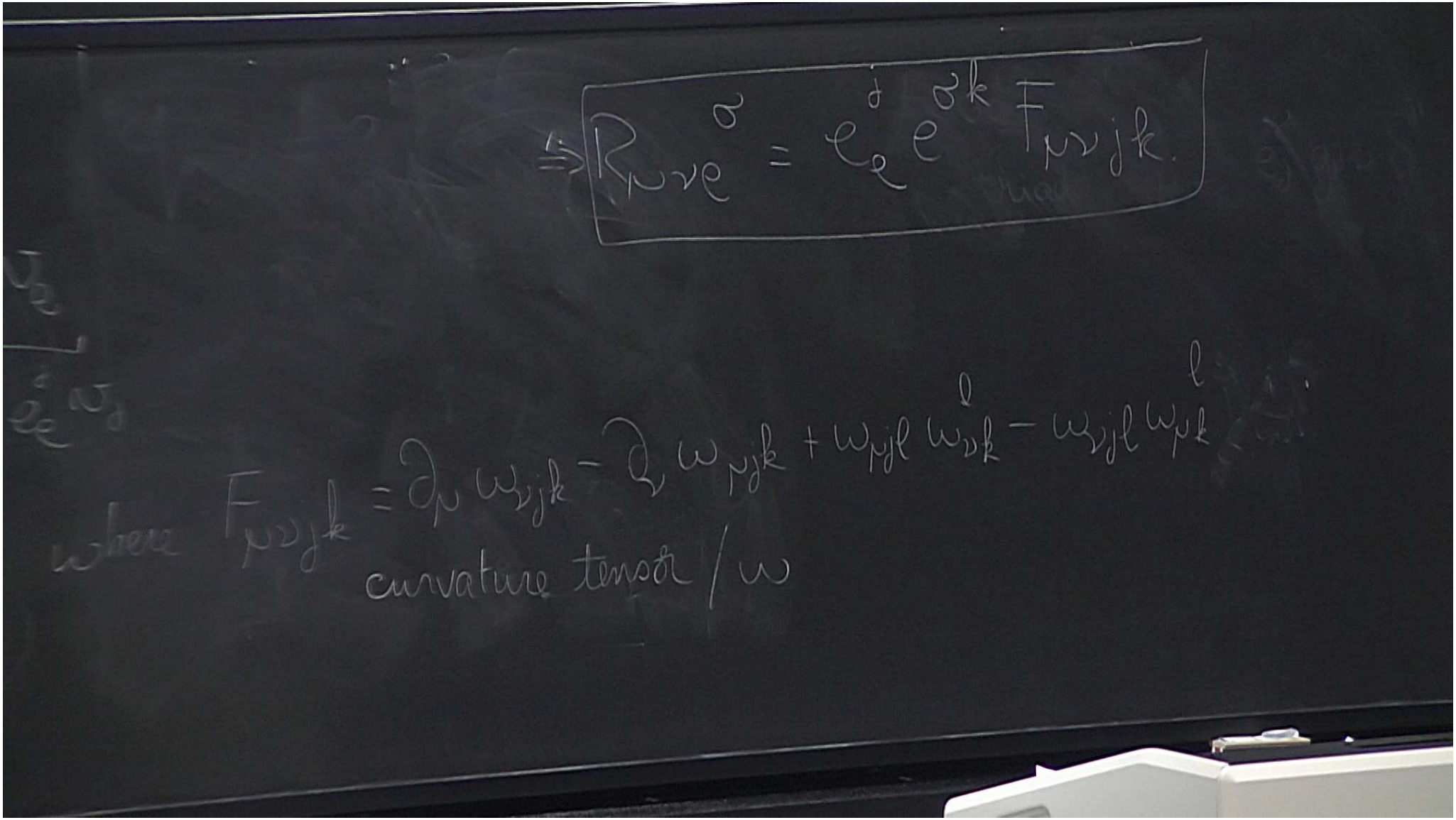

3D first order action for gravity
$\sqrt{g}R$
$R_{\mu\nu}{}^{\rho\sigma} v_\sigma = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) v_\rho$
$= e^j_\rho \left(D_\mu D_\nu - D_\nu D_\mu\right) v_j$
$e^\rho_i v_j$

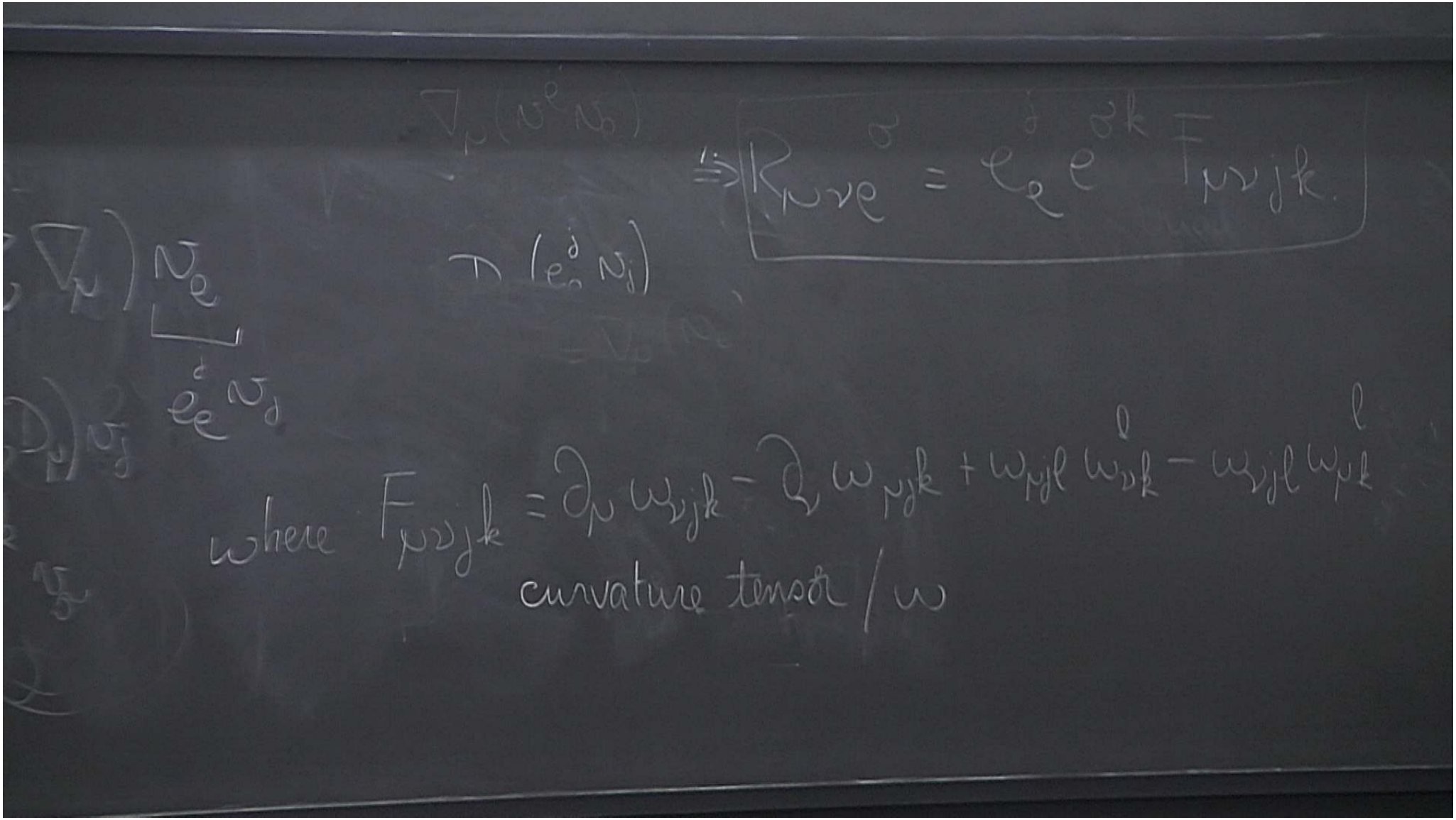
3D first order action for gravity

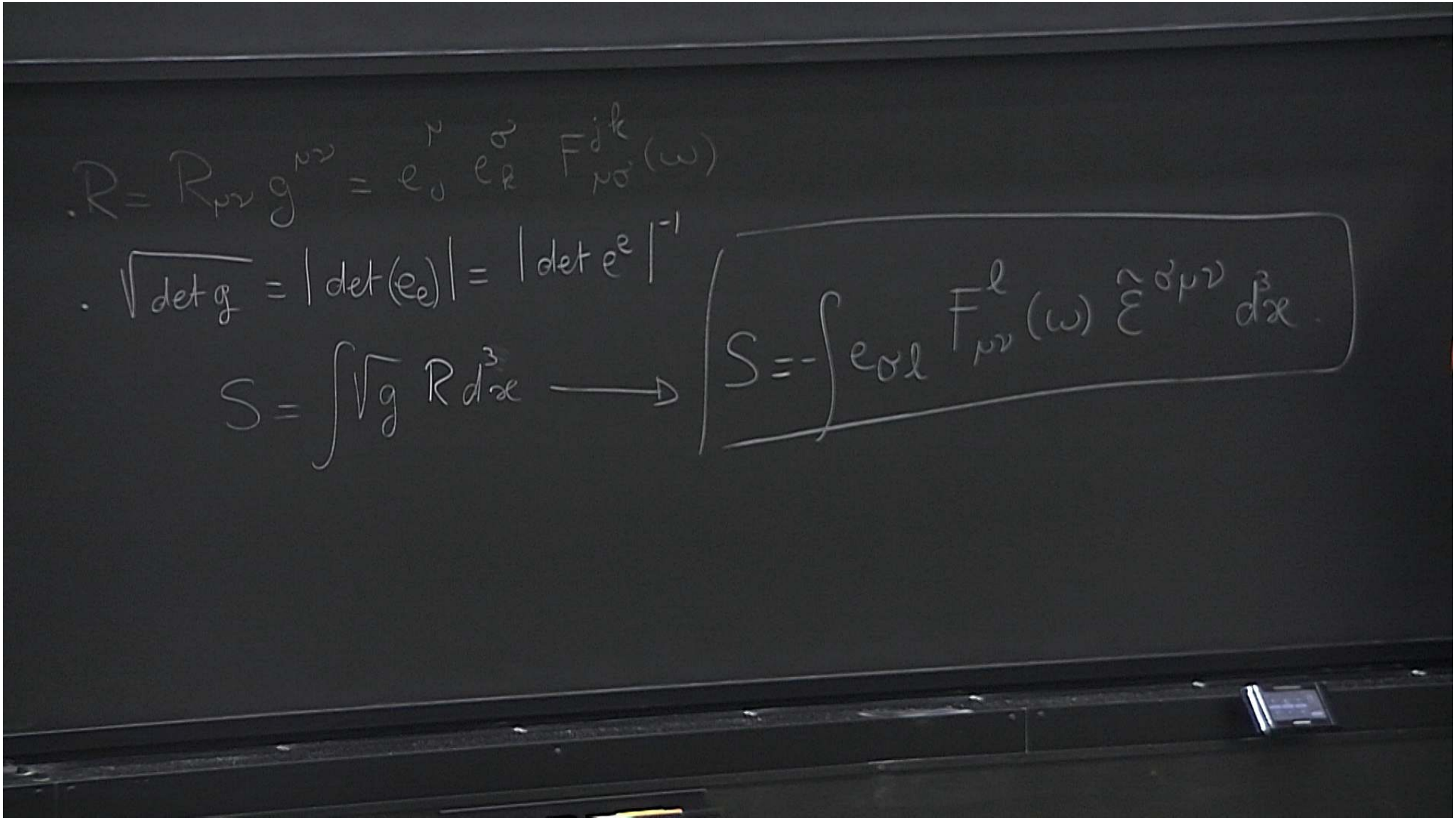
$$\sqrt{g}\,R \sim R_{\mu\nu e}{}^{\sigma}\, v_{\sigma} = \left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) \underline{v_e}$$

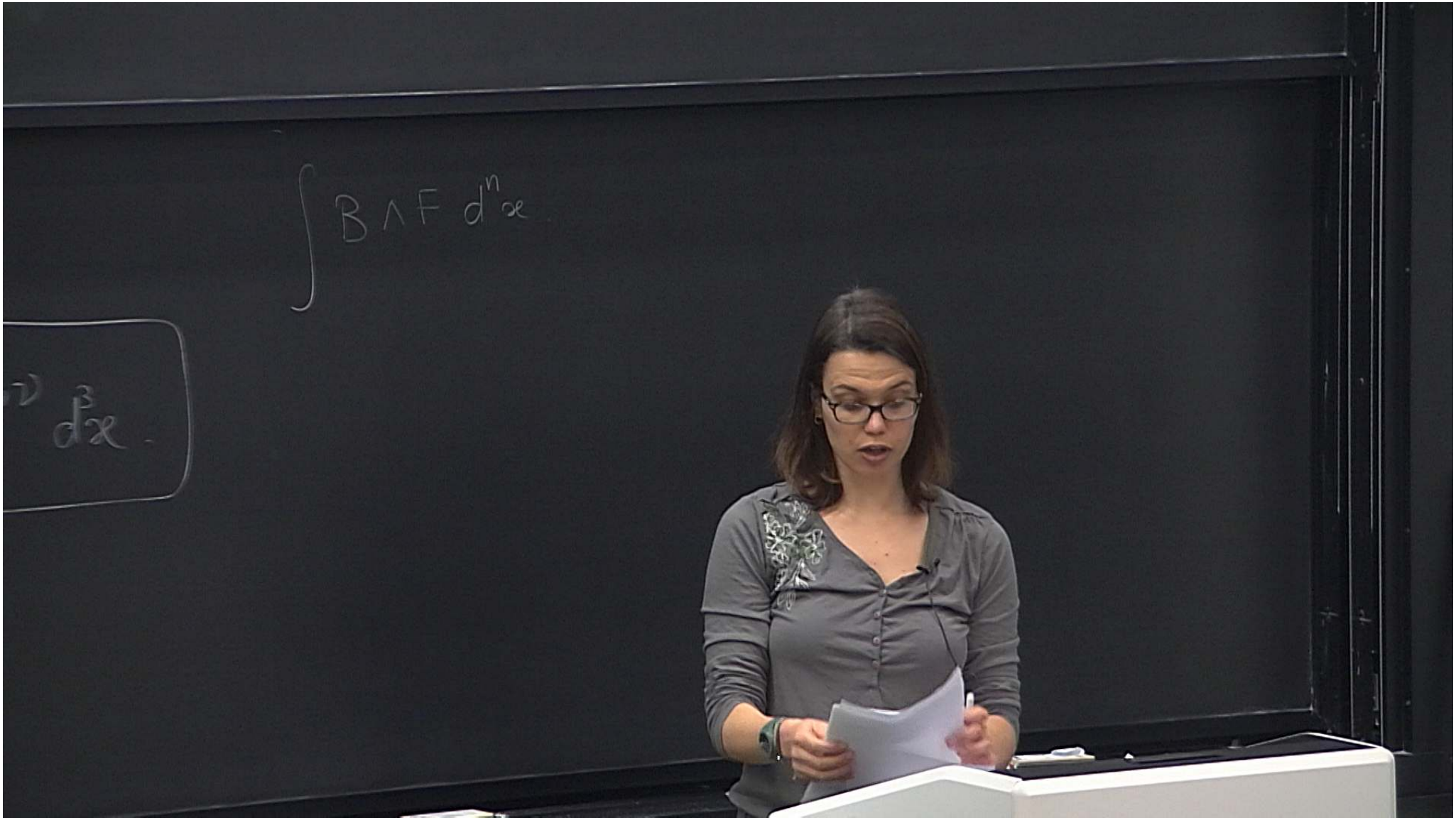
$$= e^{j}_{e}\left(D_\mu D_\nu - D_\nu D_\mu\right) v_j \qquad \underline{v_e} = e^{j}_{e}\, v_j$$

$$\equiv e^{j}_{e}\left(F_{\mu\nu jk}\right) e^{\sigma k}\, v_\sigma \qquad \text{where } F_{\mu\nu jk} = \partial$$


$\Rightarrow R_{\mu\nu e}{}^{\sigma} = e_e^{\dot{\sigma}} e^{\sigma k} F_{\mu\nu jk}$
$F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu j l}\, \omega_{\nu k}^{l} - \omega_{\nu j l}\, \omega_{\mu k}^{l}$
where $F_{\mu\nu jk}$
curvature tensor / ω


$\Rightarrow R_{\mu\nu\rho}{}^{\sigma} = e_i^{\sigma} e^{j}_{\rho} F_{\mu\nu jk}$
where $F_{\mu\nu jk} = \partial_\mu \omega_{\nu jk} - \partial_\nu \omega_{\mu jk} + \omega_{\mu jl}\omega_{\nu k}^{l} - \omega_{\nu jl}\omega_{\mu k}^{l}$
curvature tensor / ω


$R = R_{\mu\nu} g^{\mu\nu} = e_j^{\ \nu} e_k^{\ \sigma} F_{\mu\sigma}^{jk}(\omega)$
$\sqrt{\det g} = |\det(e_e)| = |\det e^e|^{-1}$
$S = \int \sqrt{g}\, R\, d^3x \longrightarrow$
$S = -\int e_{\sigma l}\, F_{\mu\nu}^{l}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3x$


$\int B \wedge F \, d^n x$
$d^3 x$

$$R = R_{\mu\nu} g^{\mu\nu} = e^{\mu}_{j}\, e^{\nu}_{k}\, F^{jk}_{\mu\nu}(\omega)$$

$$\sqrt{\det g} = |\det(e^{a}_{\mu})| = |\det e^{a}|^{-1}$$

$$S = \int \sqrt{g}\, R\, d^3x \longrightarrow \boxed{S = -\int e_{al}\, F^{al}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\mu\nu\sigma}\, d^3x}$$

(see tutorial 2)
$\tilde{\varepsilon}$ tensor density

curvature tenso

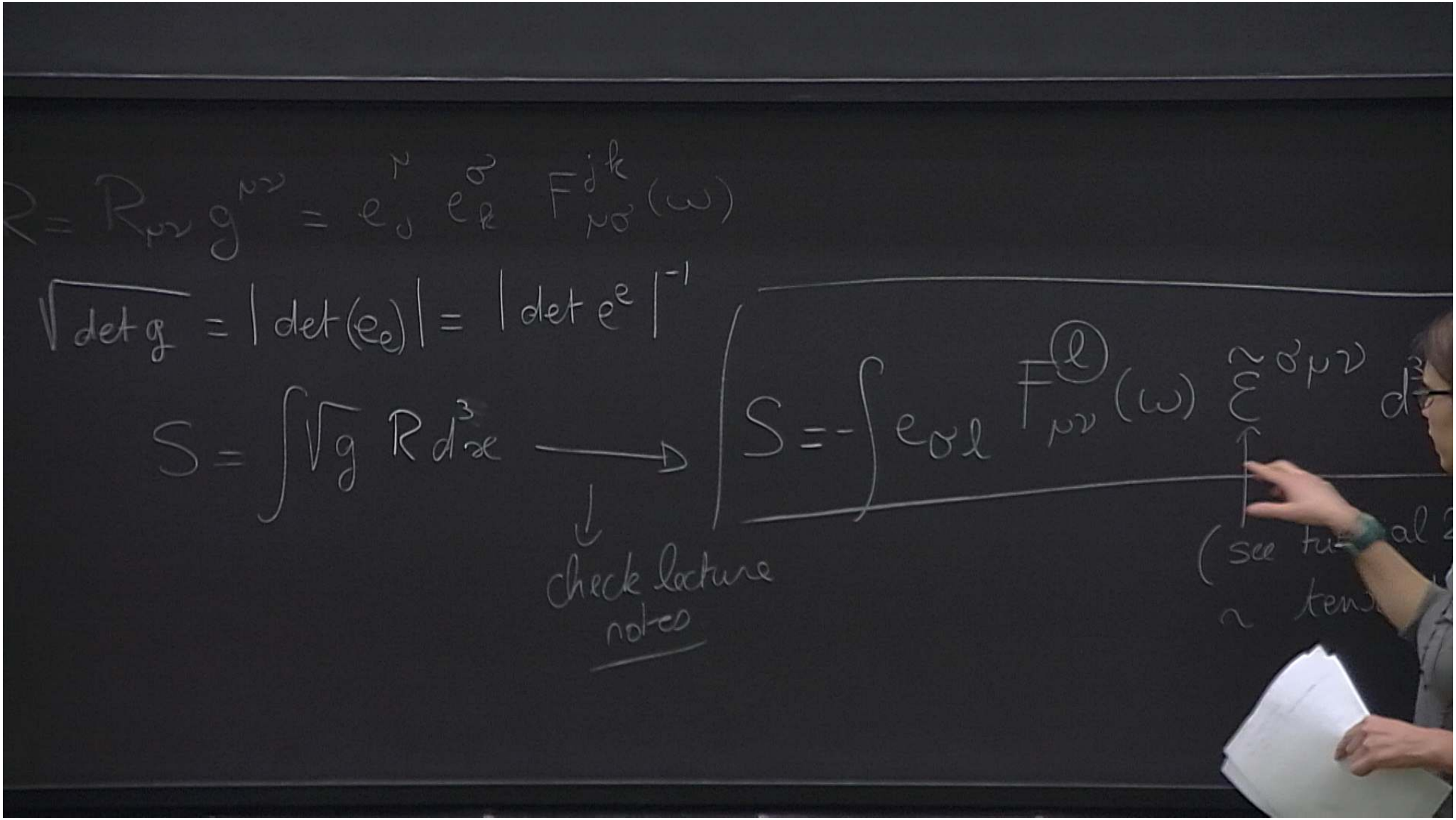
$$\int B \wedge F\, d^n x$$

$$\int B \wedge F \, d^n x$$

(su(2)-valued object)

$$\omega_{\mu jk} = \varepsilon_{jlk} \omega_\mu^{la}$$

$$F_{\mu\nu jk} = \varepsilon_{jlk} F_{\mu\nu}^{l}$$


$R = R_{\mu\nu} g^{\mu\nu} = e^{\mu}_{j} e^{\sigma}_{k} F^{jk}_{\mu\sigma}(\omega)$
$\sqrt{\det g} = |\det(e_e)| = |\det e^e|^{-1}$
$S = \int \sqrt{g}\, R\, d^3x$
check lecture notes
$S = -\int e_{\sigma l} F^{l}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\sigma\mu\nu}\, d^3$
(see tu al 2
~ tens

$$g^{\mu\nu} = e^{\mu}_{j}\, e^{\nu}_{k}\, F^{jk}_{\mu\nu}(\omega)$$

$$= |\det(e^{a}_{\mu})| = |\det e^{a}_{\mu}|^{-1}$$

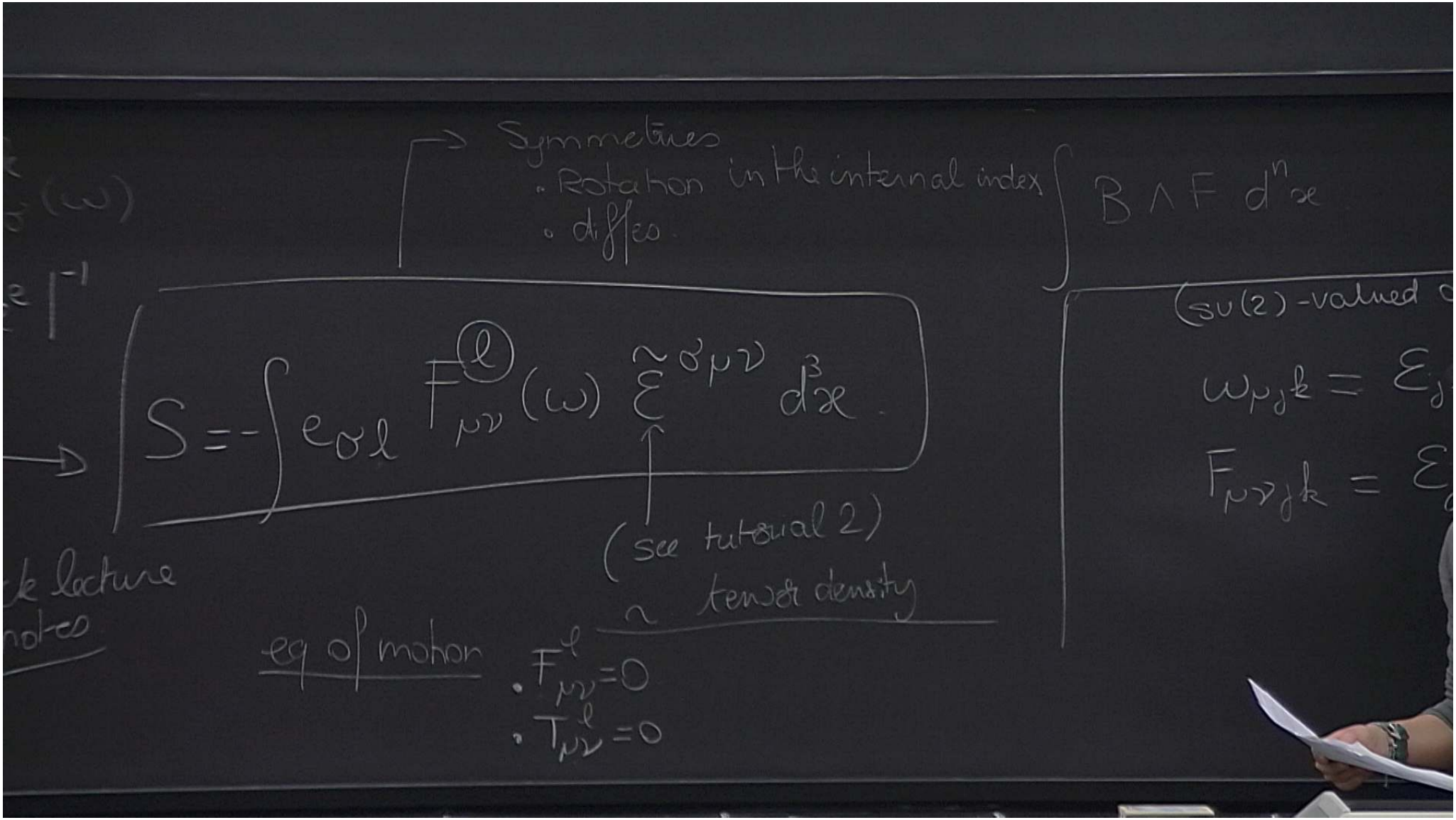
$$= \int \sqrt{g}\, R\, d^3x \longrightarrow \boxed{S = -\int e_{\alpha\ell}\, F^{(\ell)}_{\mu\nu}(\omega)\, \tilde{\varepsilon}^{\alpha\mu\nu}\, d^3x}$$

check lecture notes

$\tilde{\varepsilon}^{\alpha\mu\nu}$ (see tutorial 2) ~ tensor density

eq of motion

- $F^{\ell}_{\mu\nu} = 0$
- $T^{\ell}_{\mu\nu} = 0$


Symmetries
. Rotation in the internal index
. diffeo
$\int B \wedge F \, d^n x$
$S = -\int e_{\alpha l} \, F^{l}_{\mu\nu}(\omega) \, \tilde{\varepsilon}^{\alpha\mu\nu} \, d^3x$
(see tutorial 2)
$\tilde{\varepsilon}$ tensor density
eq of motion
. $F^{l}_{\mu\nu} = 0$
. $T^{l}_{\mu\nu} = 0$
(su(2)-valued
$\omega_{\mu jk} = \varepsilon_j$
$F_{\mu\nu jk} = \varepsilon$
k lecture
notes

Canonical analysis 3D gravity first order action

$\mathcal{M} = \Sigma \times \mathbb{R}$ $\quad \mu = (0, a)$ ↳ space coordinates

$\boxed{\omega = A}$

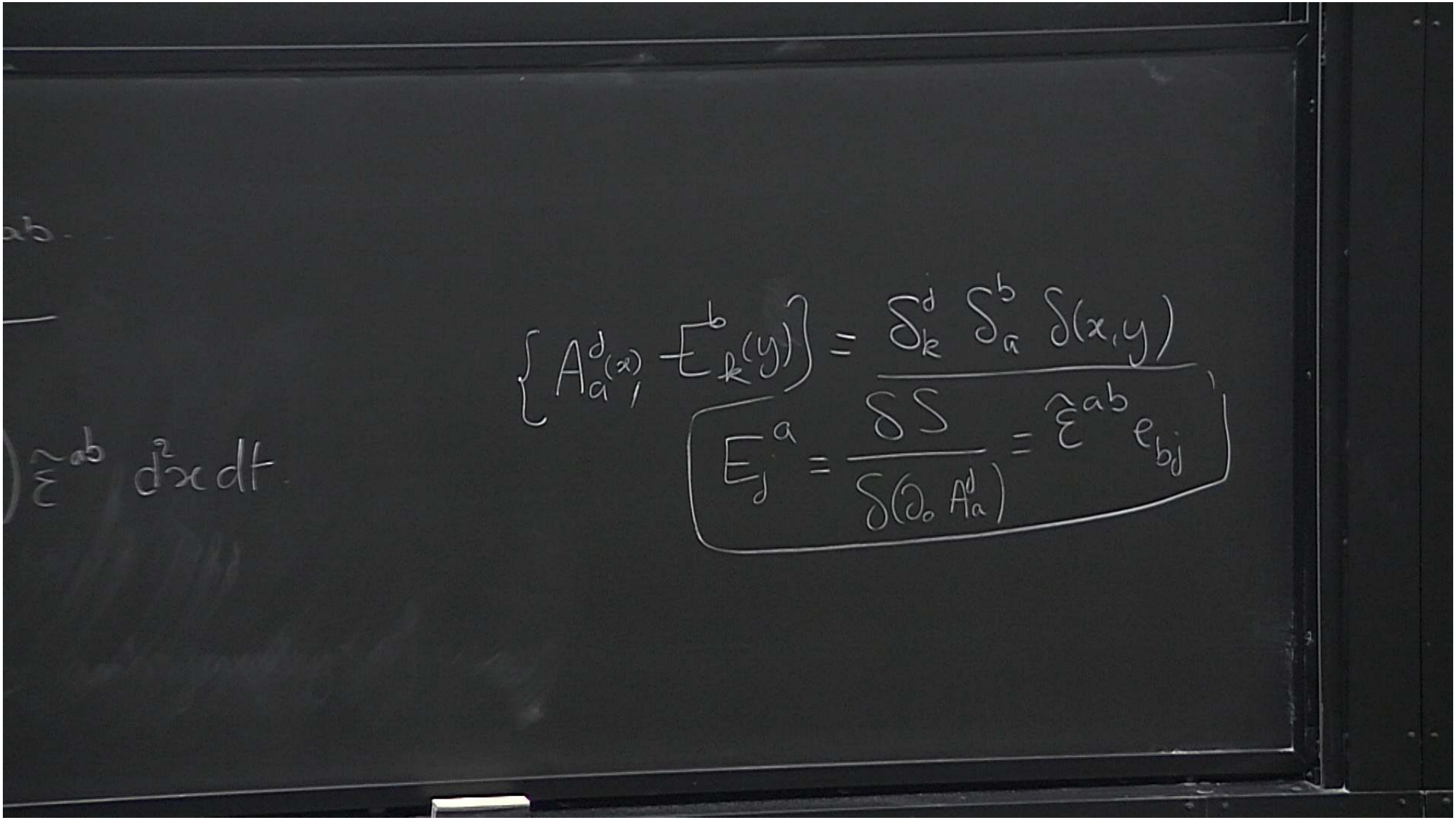
$$S = \int e^a_b \, \partial_0 A_{aj} \, \tilde{E}^{ab} + e_{0j} \frac{1}{2} F^d_{ab} \, \tilde{E}^{ab} + A^j_0 \left(\partial_a e_{bj} + \varepsilon_{jlm} A^l_a \, e^m_b \right) \tilde{E}^{ab} \, d^2x \, dt$$

Canonical analysis 3D gravity first order action

$\mathcal{M} = \Sigma \times \mathbb{R}$ $\quad \mu = (0, a)$ ↳ space coordinates $\quad \tilde{\varepsilon}^{0ab} = \tilde{\varepsilon}^{ab}$

$\boxed{\omega = A}$

$$S = \frac{1}{2}\int e\, e^{\mu}_{I} F^{I}_{\mu\nu} \tilde{\varepsilon}^{\mu\nu\rho} \Rightarrow \quad S = \int e^{d}_{b}\, \partial_0 A_{aj}\, \tilde{\varepsilon}^{ab} + e_{0j}\, \frac{1}{2} F^{d}_{ab}\, \tilde{\varepsilon}^{ab} + A^{j}_{0}\left(\partial_a e_{bj} + \varepsilon_{jlm} A^{l}_{a}\, e^{m}_{b}\right)\tilde{\varepsilon}^{ab}\, d^2x\, dt$$


$\{A_a^d(x), E_k^b(y)\} = \delta_k^d \delta_a^b \delta(x,y)$
$E_j^a = \frac{\delta S}{\delta(\partial_0 A_a^d)} = \tilde{\varepsilon}^{ab} e_{bj}$
$\tilde{\varepsilon}^{ab} d^2x dt$

Canonical analysis 3D gravity first-order action

$\mathcal{M} = \Sigma \times \mathbb{R}$ $\quad \mu = (0, a)$ → space coordinates

$\tilde{\varepsilon}^{0ab} = \tilde{\varepsilon}^{ab}$

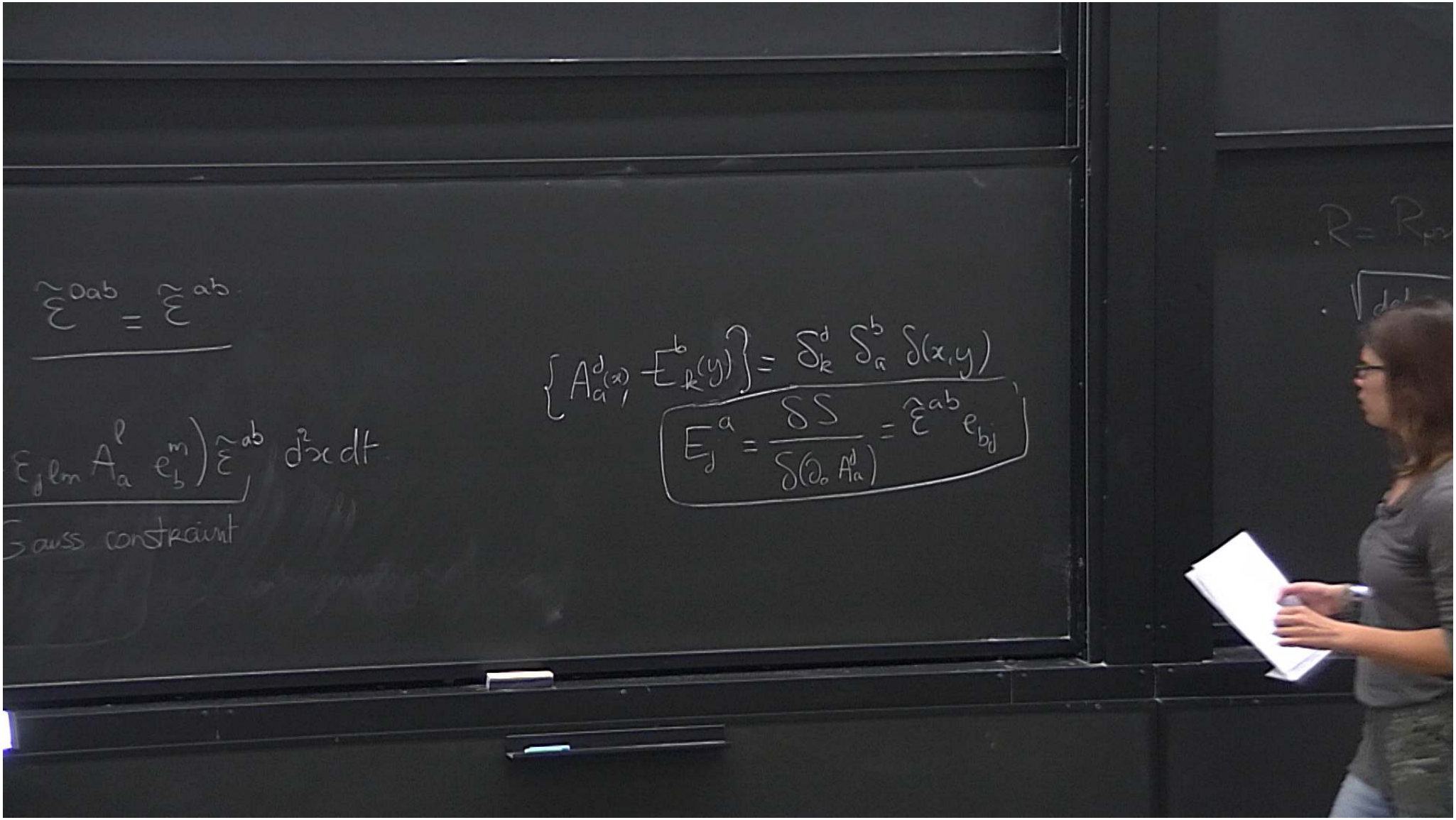
$\omega = A$

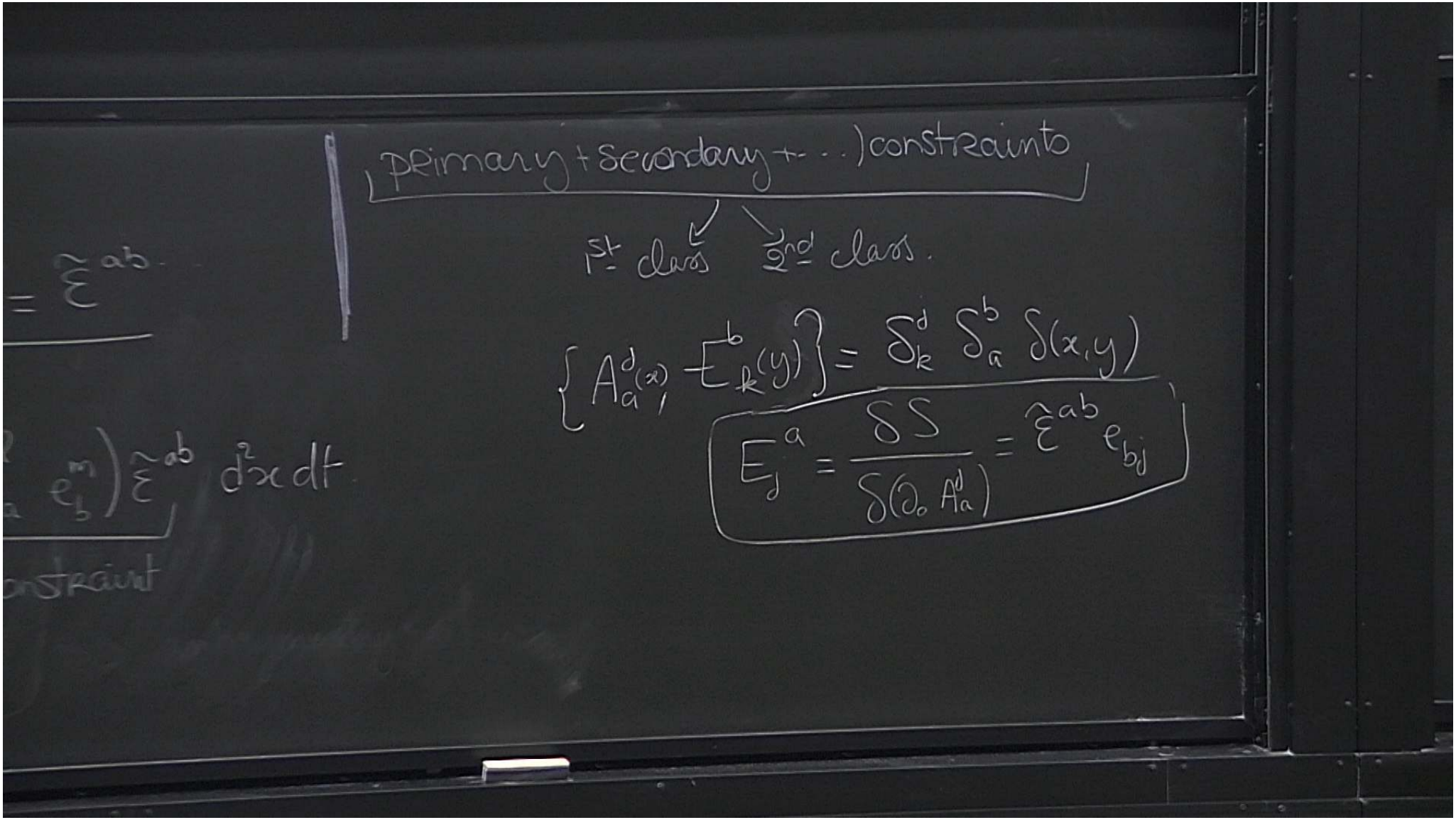
$$S = \frac{1}{2}\int e\, e^{\mu}_{I} F^{IJ}_{\mu\nu} \tilde{\varepsilon}^{\mu\nu\rho} \Rightarrow S = \int e^{d}_{b}\, \partial_0 A_{aj}\, \tilde{\varepsilon}^{ab} + e_{0j}\, \frac{1}{2} F^{j}_{ab}\, \tilde{\varepsilon}^{ab} + A^{j}_{0} \left(\partial_a e_{bj} + \varepsilon_{jlm} A^{l}_{a} e^{m}_{b}\right) \tilde{\varepsilon}^{ab}\, d^2x\, dt$$

Flatness constraint

Gauss constraint

Lagrange multipliers


$\tilde{\varepsilon}^{0ab} = \tilde{\varepsilon}^{ab}$
$\{A_a^d(x), E_k^b(y)\} = \delta_k^d \delta_a^b \delta(x,y)$
$E_j^a = \frac{\delta S}{\delta(\partial_0 A_a^d)} = \tilde{\varepsilon}^{ab} e_{bj}$
$\varepsilon_{jlm} A_a^l e_b^m \tilde{\varepsilon}^{ab} d^3x dt$
Gauss constraint
$R = R_{\mu\nu}$



$= \tilde{\varepsilon}^{ab}$

$e^m_b) \tilde{\varepsilon}^{ab} \, d^2x \, dt$

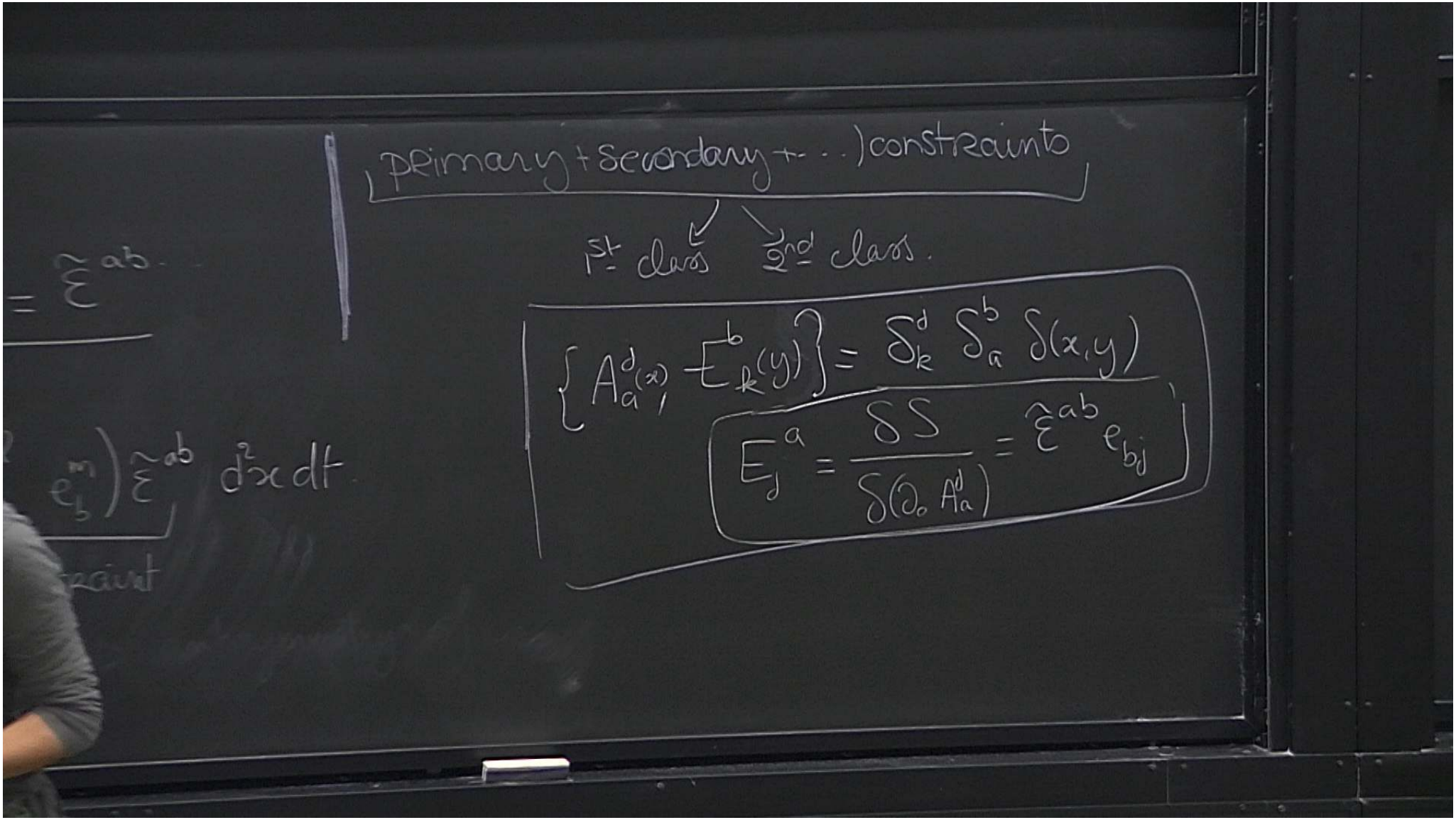
onstraint

$\underbrace{\text{primary + secondary + ...) constraints}}$

1st class 2nd class.

$$\{A^j_a(x), E^b_k(y)\} = \delta^j_k \, \delta^b_a \, \delta(x,y)$$

$$\boxed{E^a_j = \frac{\delta S}{\delta(\partial_0 A^j_a)} = \tilde{\varepsilon}^{ab} e_{bj}}$$


Primary + Secondary + . . .) constraints
1st class
2nd class.
$\{A_a^d(x), E_k^b(y)\} = \delta_k^d \delta_a^b \delta(x,y)$
$E_j^a = \frac{\delta S}{\delta(\partial_0 A_a^d)} = \tilde{\varepsilon}^{ab} e_{bj}$
$= \tilde{\varepsilon}^{ab}$
$e_b^m) \tilde{\varepsilon}^{ab} d^2x dt$
straint