Title: PSI 2017/2018 - Quantum Gravity - Lecture 4

Date: Mar 22, 2018 10:15 AM

URL: http://pirsa.org/18030041

Abstract:

Kinematical Phase Space

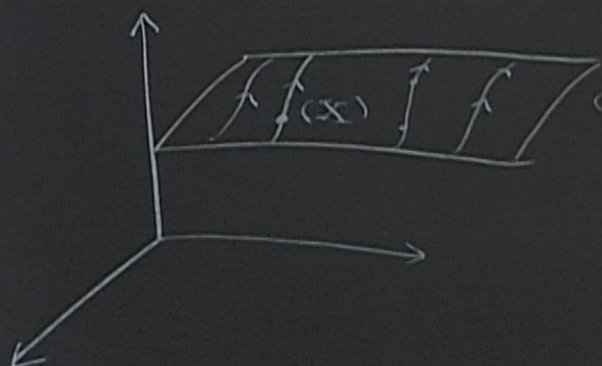
- constraints $\{\Phi^\alpha\} = \{\underbrace{1^{st}\text{ class}}_{C^a}\} \cup \{2^{nd}\text{ class}\}$

$$\{C^a, \Phi^\alpha\} \simeq 0 \longrightarrow \text{gauge} \neq \text{transformation}$$

redundancy

- phase space $\{X^a\}$ $\quad C^a = 0$

$$\frac{dX^a}{d\lambda} = \{X^a, C\}$$



constraint surface

$C^a(X) = 0$

sformation "
ndancy"

$\{X^a, C^a\}$

given initial conditions $\not\longrightarrow$ solutions

reasonable question : "relation" between dynamical variables

$\hookrightarrow$ "gauge fixing" $G^a(\underline{X}) = 0$

observables.

$$\{F(\underline{X}), C^a\} = 0$$

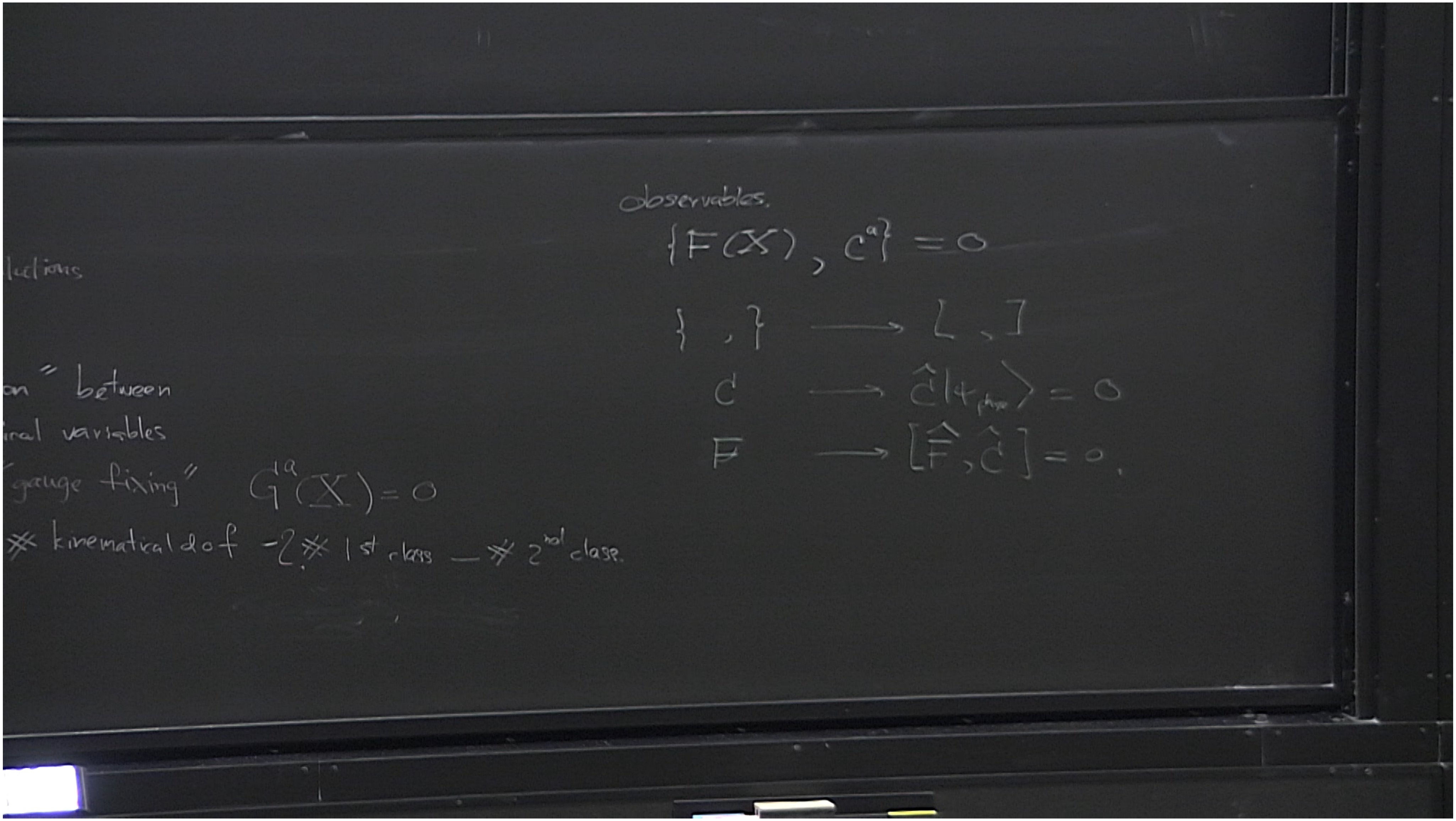
$$\{\ ,\ \} \longrightarrow [\ ,\ ]$$

$$C \longrightarrow \hat{C}|\psi_{phys}\rangle = 0$$

$$F \longrightarrow [\hat{F}, \hat{C}] = 0.$$

$$G^a(\underline{X}) = 0$$

dof $- 2 \#$ 1st class $- \#$ 2nd class.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kinematical dof − 2 # 1st class − # 2nd class.

phase space

a) $(q, t, p_q, p_t) \longrightarrow \mathcal{H}_{kin}$?

- $\mathcal{H}_{kin} = \mathcal{L}^2(\mathbb{R}^2)$

$\Psi(q,t)$ $\quad \langle \Psi, \phi \rangle = \int dq\,dt\, \overline{\Psi}(q,t)\, \phi(q,t).$

- $\left.\begin{matrix} q \\ t \end{matrix}\right) \longrightarrow \begin{matrix} \hat{q}|\Psi\rangle = q|\Psi\rangle \\ \hat{t}|\Psi\rangle = t|\Psi\rangle \end{matrix}$; $\left.\begin{matrix} p_q \\ p_t \end{matrix}\right) \longrightarrow \begin{matrix} \hat{p}_q = -i\hbar \dfrac{\partial}{\partial q} \\ \hat{p}_t = -i\hbar \dfrac{\partial}{\partial t} \end{matrix}$

phase space

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b) $C = p_t + \frac{p_q^2}{2m} \longrightarrow \hat{C} = -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2}$

$\hat{C}|\Psi\rangle = 0$: Schrödinger eq.

b) $C = p_t + \frac{p_q^2}{2m} \longrightarrow \hat{C} = -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2}$.

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$\hat{C}|\psi\rangle = 0$.

c) $\psi(q) = \psi(q, t = t_0)$.

solution Schrödinger eq: $\psi_{phys}(q, t) = \exp\left(-\frac{i}{\hbar} \hat{h}\, t\right) \psi($

$$\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}$$

$$\langle \Psi^{(1)}_{phy} , \Psi^{(2)}_{phys} \rangle = \underbrace{\int dt \underbrace{\left(\int dq \; \overline{\Psi^{(1)}(q)} \; \Psi^{(2)}(q) \right)}_{}}_{} = \infty$$

→ physical inner product?

$$\mathcal{H}_{phys} = \mathcal{L}^2(\mathbb{R})$$

$$\langle \Psi^{(1)}_{phy} | \Psi^{(2)}_{phys} \rangle = \int dq \; \overline{\Psi}^{(1)}(q) \; \Psi^{(2)}(q)$$

c) $\hat{C}|\psi\rangle = 0.$

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d) $F \longrightarrow \hat{F}$ $\quad \hat{F}_1 = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t}-t_0)\,, \quad \hat{F}_2 = \hat{p}_q$ $\quad$ Exercise

$$= \exp\left(-\frac{i}{\hbar}\hat{h}t\right)\psi(q) \qquad \hat{h} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}$$

$$= \hat{p}_q$$

Exercise: Check that $[F_i, C] = 0$

→ physical inner p

$\mathcal{H}_{phys} = C$

$\langle \psi^{(1)}_{phy} | \psi$

-Hilbert action

$$S = \frac{1}{k} \int R \sqrt{g}\, d^4x \qquad k = \frac{16\pi G}{c^3} = 1.$$

eq of motion: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \iff R_{\mu\nu} = 0.$

y

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if $n=3$ $\quad R_{\mu\nu} = 0 \implies R_{\mu\nu\sigma\rho} = 0.$

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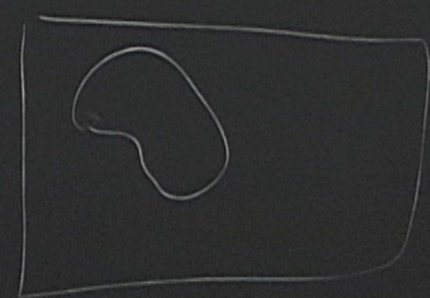
3 3

$= 1.$

$R_{\mu\nu} = 0.$

- No local degree of freedom
- Global DOF $\longrightarrow$ topology.

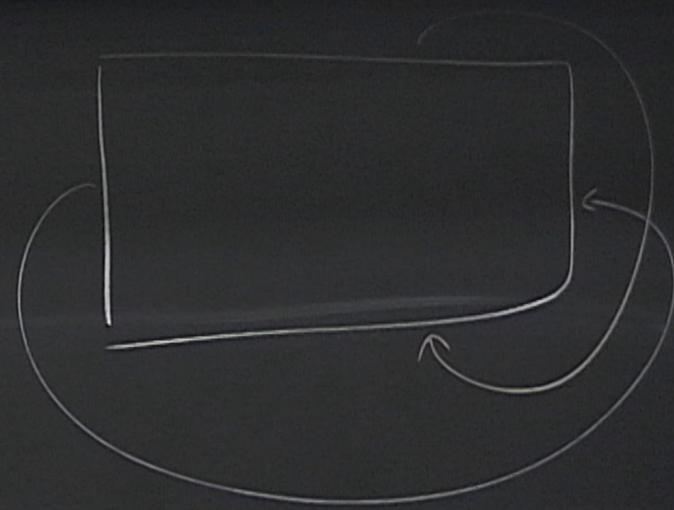
Topology characterized by the fundamental group



In 3D, $M = \Sigma \times \mathbb{R}$

↳ 2d surface

g: genus



d) $\bar{F} \longrightarrow \hat{F}$ $\quad \hat{F}_1 = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t} - t_0)$, $\hat{F}_2 = \hat{p}_q$ $\quad$ Exercise: Check that

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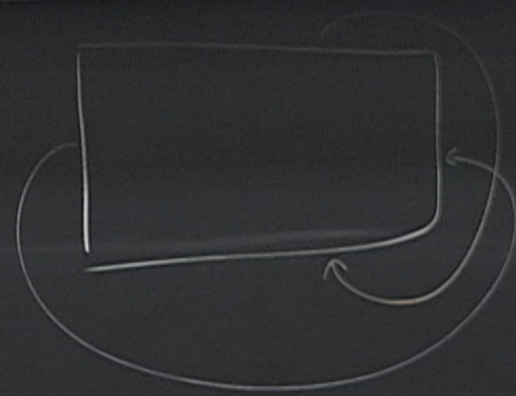
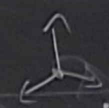


d) $F \longrightarrow \hat{F}$ $\qquad \hat{F}_1 = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t} - t_0)$, $\hat{F}_2 = \hat{p}_q$ $\qquad$ Exercise: Check

First order formulation



$$g_{\mu\nu} \longrightarrow \boxed{\underbrace{g_{\mu\nu}}_{6} = \underbrace{e^i_\mu}_{9}\, e^j_\nu\, \underbrace{\eta_{ij}}_{\delta_{ij}\ \text{Euclidean}}} \qquad i \in \{1,2,3\}$$

e^μ_i : triad

e^i_μ : co-triad

d) $\bar{F} \longrightarrow \hat{F}$ $\quad \hat{F}_1 = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t} - t_0)$, $\quad \hat{F}_2 = \hat{p}_q$ $\quad$ Exercise:

First order formulation

$$g_{\mu\nu} \longrightarrow \boxed{g_{\mu\nu}(x) = e^i_\mu(x)\, e^j_\nu(x)\, \underbrace{\eta_{ij}}_{\delta_{ij}\ \text{Euclidean}}} \qquad i \in \{1,2,3\}$$

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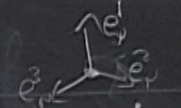
$6 \qquad 9$

d) $F \longrightarrow \hat{F}$ $\quad \hat{F}_1 = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t} - t_0)\ , \quad \hat{F}_2 = \hat{p}_q$

Exercise: Che

First order formulation



$$g_{\mu\nu} \longrightarrow \left| g_{\mu\nu}(x) = e^i_\mu(x)\, e^j_\nu(x)\, \underbrace{\eta_{ij}}_{\delta_{ij}\ \text{Euclidean}} \right| \qquad i \in \{1,2,3\}$$

e^μ_i : triad

e^i_μ : co-triad

$G \qquad g$

$$e'^i_\mu(x) = R^i_j(x)\, e^j_\mu(x) \qquad R \in SO(3)$$

$$e'^k_\mu\, e'^l_\nu\, \delta^{kl} = \underbrace{(R^k_i\, e^i_\mu)(R^l_j}_{R^T R = 1}\, e^j_\nu)\, \delta^{kl} = g_{\mu\nu}$$

d) $F \longrightarrow \hat{F}$ $\qquad \hat{F}_1 = \hat{q} - \frac{\hat{p}_q}{m}(\hat{t} - t_0)$, $\quad \hat{F}_2 = \hat{p}_q$ $\qquad$ Exercise:

First order formulation

a) Triads $\qquad g_{\mu\nu} \longrightarrow$ $\boxed{g_{\mu\nu}(x) = e^i_\mu(x) e^j_\nu(x) \eta_{ij}}$ $\quad i \in \{1,2,3\}$

$$\underbrace{\eta_{ij}}_{\delta_{ij}\ \text{Euclidean}}$$

$g_{\mu\nu}$: 6 $\qquad e^i_\mu$: 9

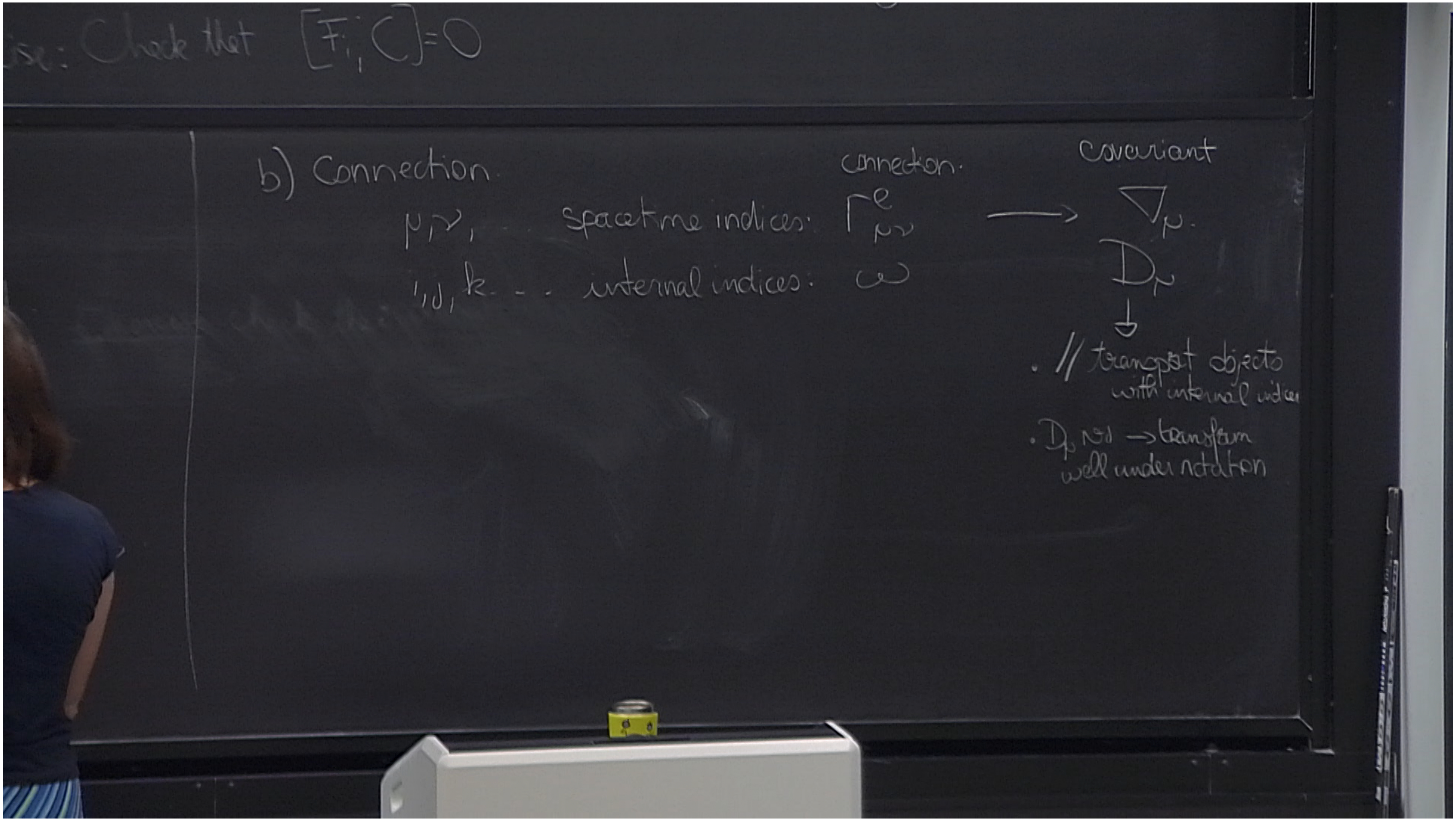
e_i^μ : triad

e^i_μ : co-triad

$$e'^i_\mu(x) = R^i_j(x)\, e^j_\mu(x) \qquad R \in SO(3)$$

$$e'^k_\mu e'^l_\nu \delta^{kl} = \left(R^k_i e^i_\mu\right)\left(R^l_j e^j_\nu\right)\delta^{kl} = g_{\mu\nu}$$

$$R^T R = \mathbb{1}$$


ise: Check that [F_i, C] = 0
b) Connection.
connection.
covariant
μ, ν, . . . spacetime indices: Γ^ρ_μν ⟶ ∇_μ.
i, j, k . . . internal indices: ω
D_ν
· // transport objects with internal indices
· D_μ ~ ⟶ transform well under rotation

$$D_\mu \phi^j = \partial_\mu \phi^j + \omega^j_{\mu k} \phi^k$$

$\omega_\mu{}^j{}_k$: spin connection

$$D_\mu \phi_j = \partial_\mu \phi_j - \omega^k_{\mu j} \phi_k$$

$$D_\mu \left(\phi^j \phi_j \right) = \partial_\mu \left(\phi^j \phi_j \right).$$

$$D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^e_{\mu\nu} v^j_e + \omega^j_{\mu k} \phi^k$$

$$D_\mu \phi^j = \partial_\mu \phi^j + \omega_{\mu}{}^{j}{}_{k} \phi^k$$

$\omega_{\mu}{}^{j}{}_{k}$: spin connection

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$$D_\mu v^j_\nu = \partial_\mu v^j_\nu - \Gamma^e_{\mu\nu} v^j_e + \omega_{\mu}{}^{j}{}_{k} v^k_\nu$$

Compatibility condition

$$\boxed{D_\mu e^j_\nu = 0}$$

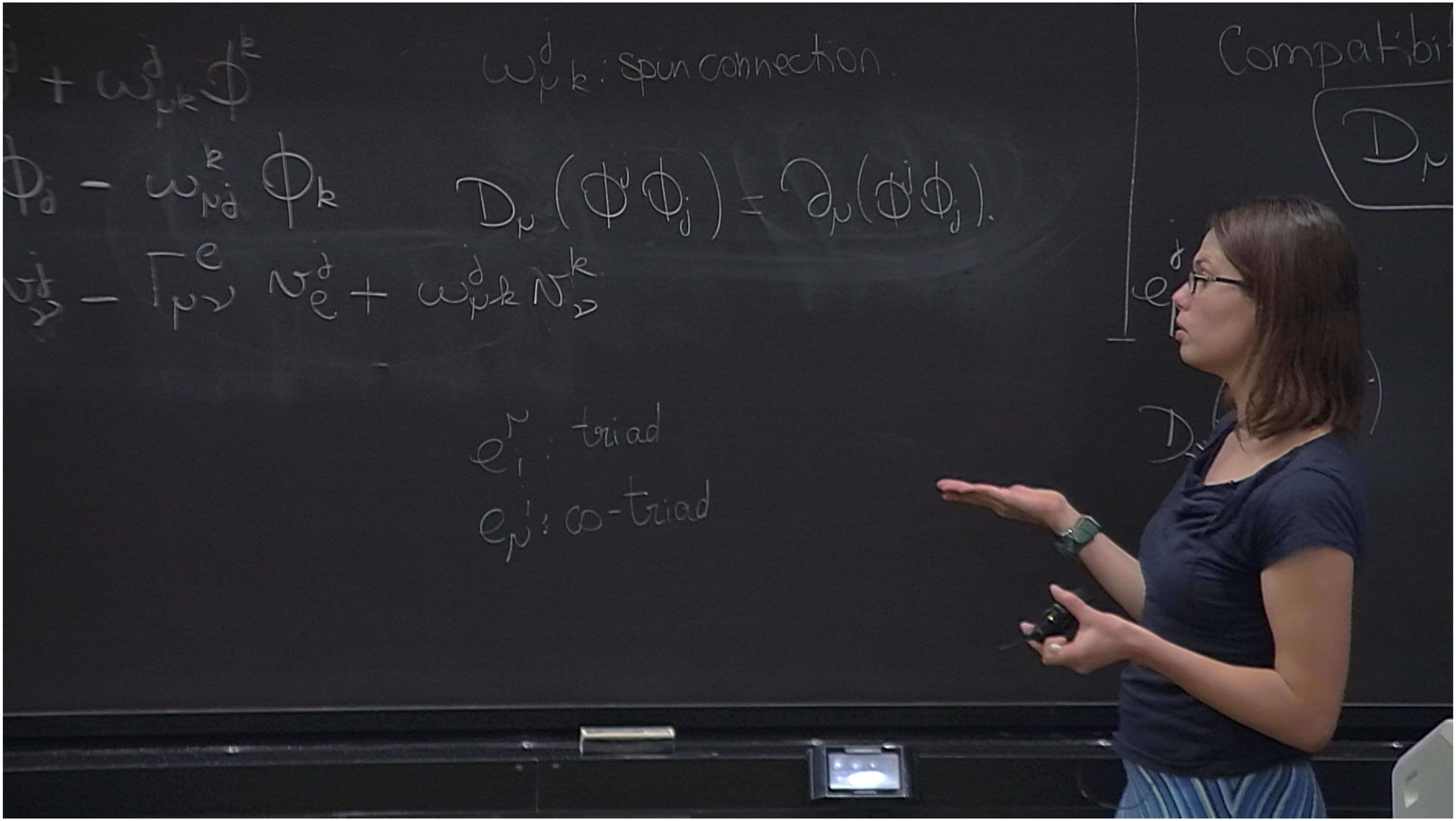
$\Rightarrow$ spin connection completly determined by the Levi-Civita connect° and triad

$$e^j_\mu D_\nu N^\mu$$

$$\downarrow$$

$$D_\nu (e^j_\mu N^\mu)$$

$$\boxed{\omega_\mu{}^j{}_k = - e^\nu_k \nabla_\mu e^j_\nu = e^\nu_k \Gamma^\rho_{\mu\nu} e^j_\rho - e^\nu_k \partial_\mu e^j_\nu}$$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