Title: PSI 2017/2018 - Quantum Gravity - Lecture 2

Date: Mar 20, 2018 10:15 AM

URL: http://pirsa.org/18030039

Abstract:

Canonical formulation of
constrained system

→ parametrized particle

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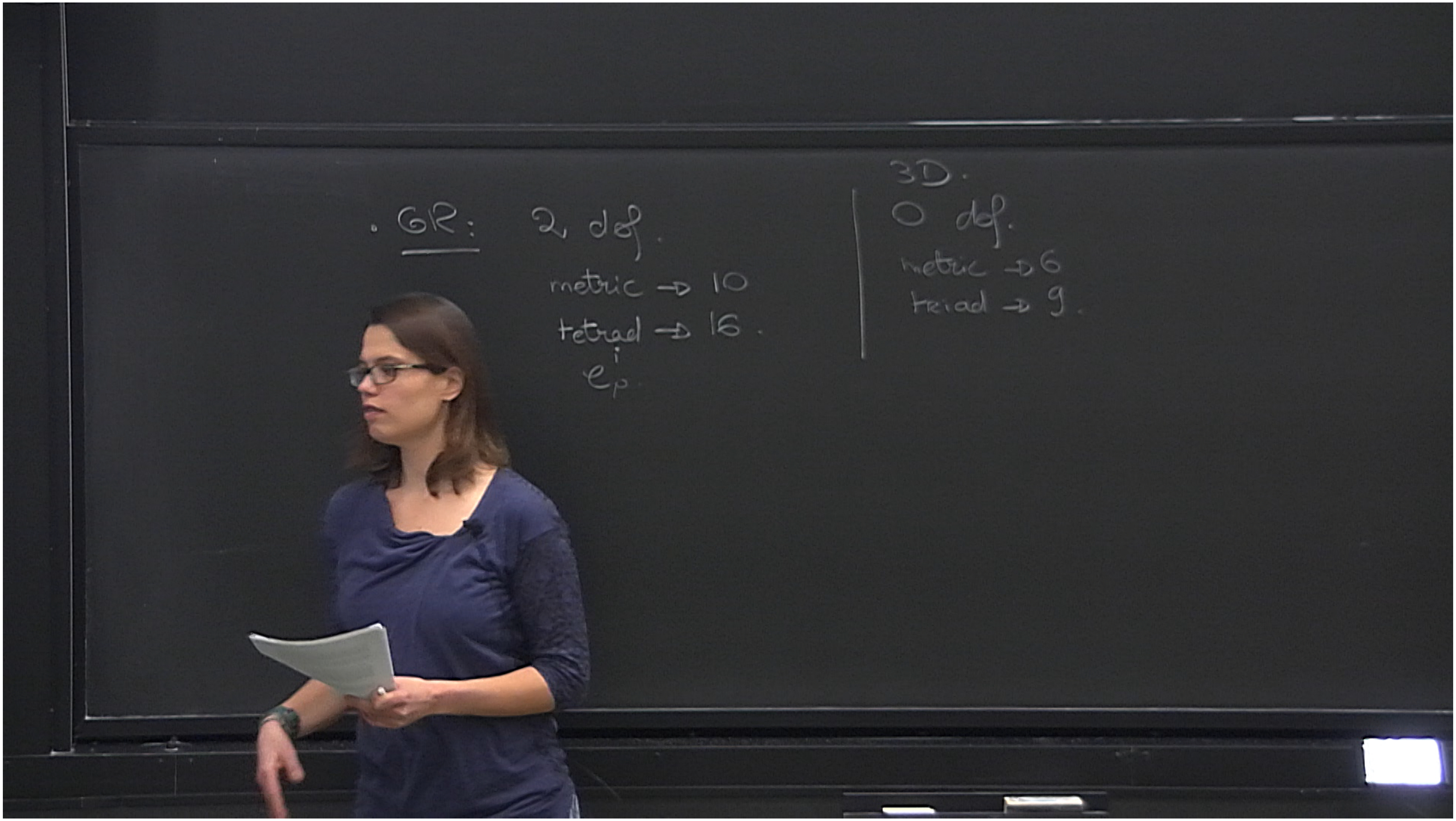
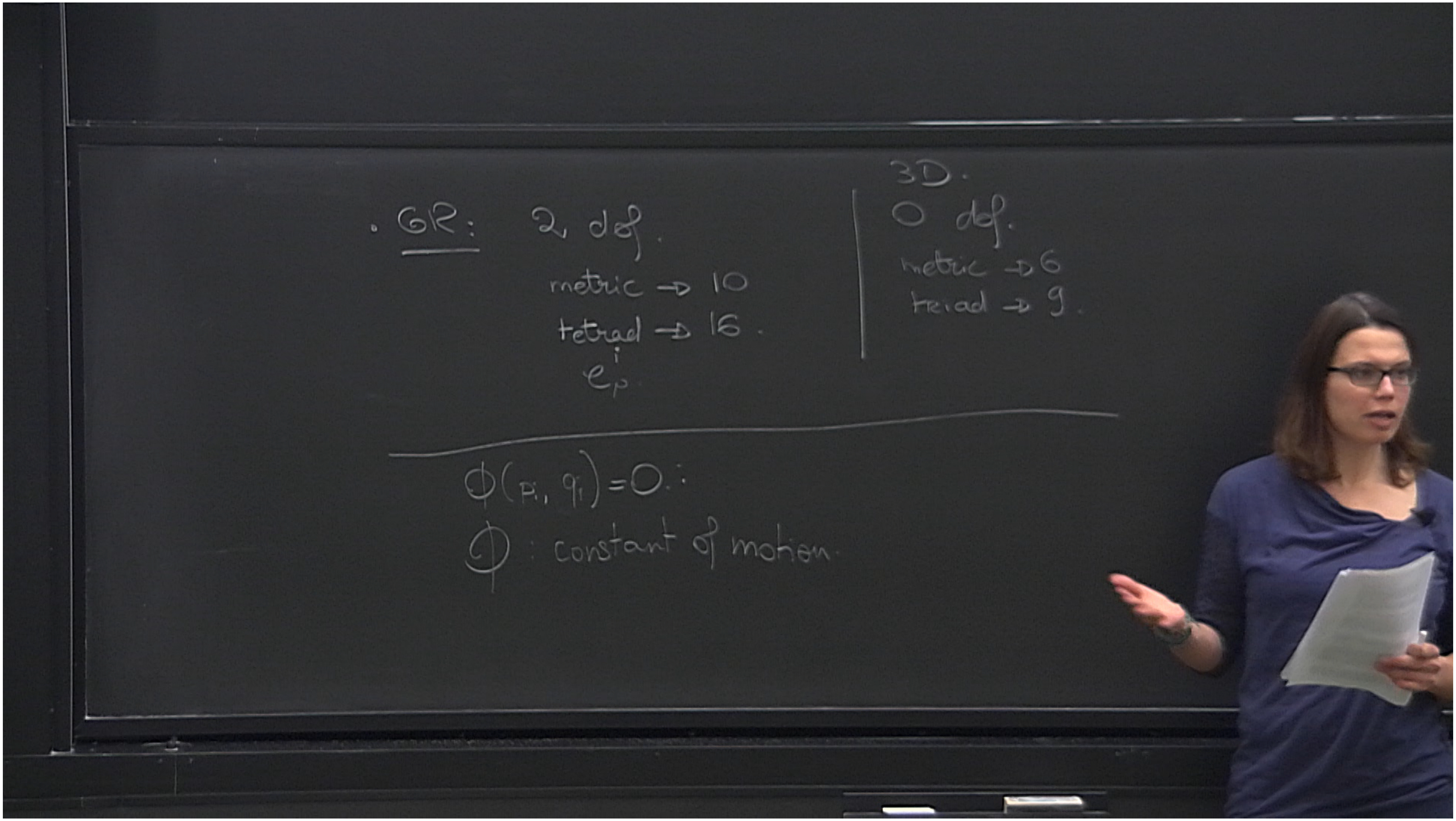
- Why are we interested in "constrained systems"?

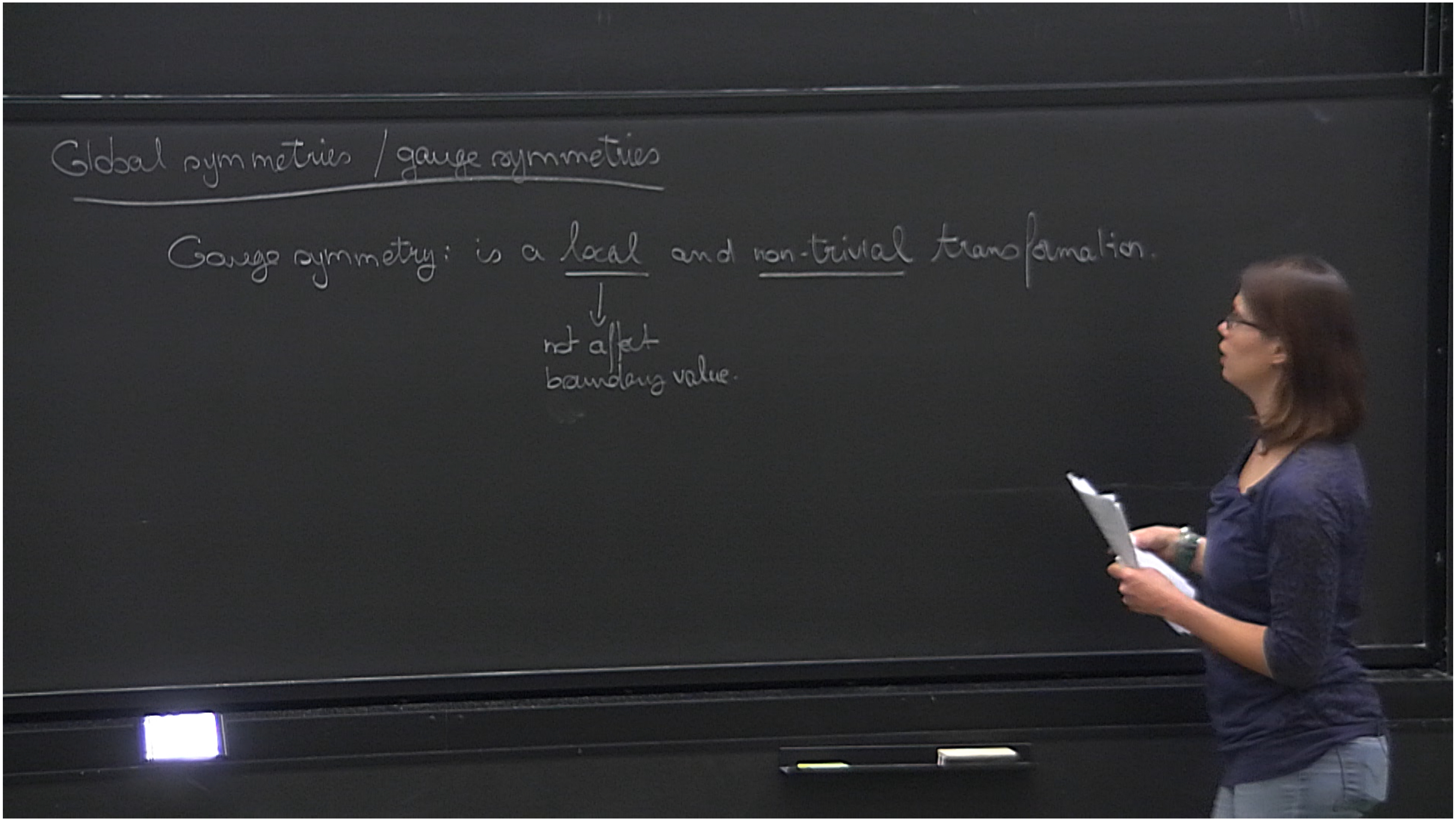
- Can we relate constraints/gauge symmetries?

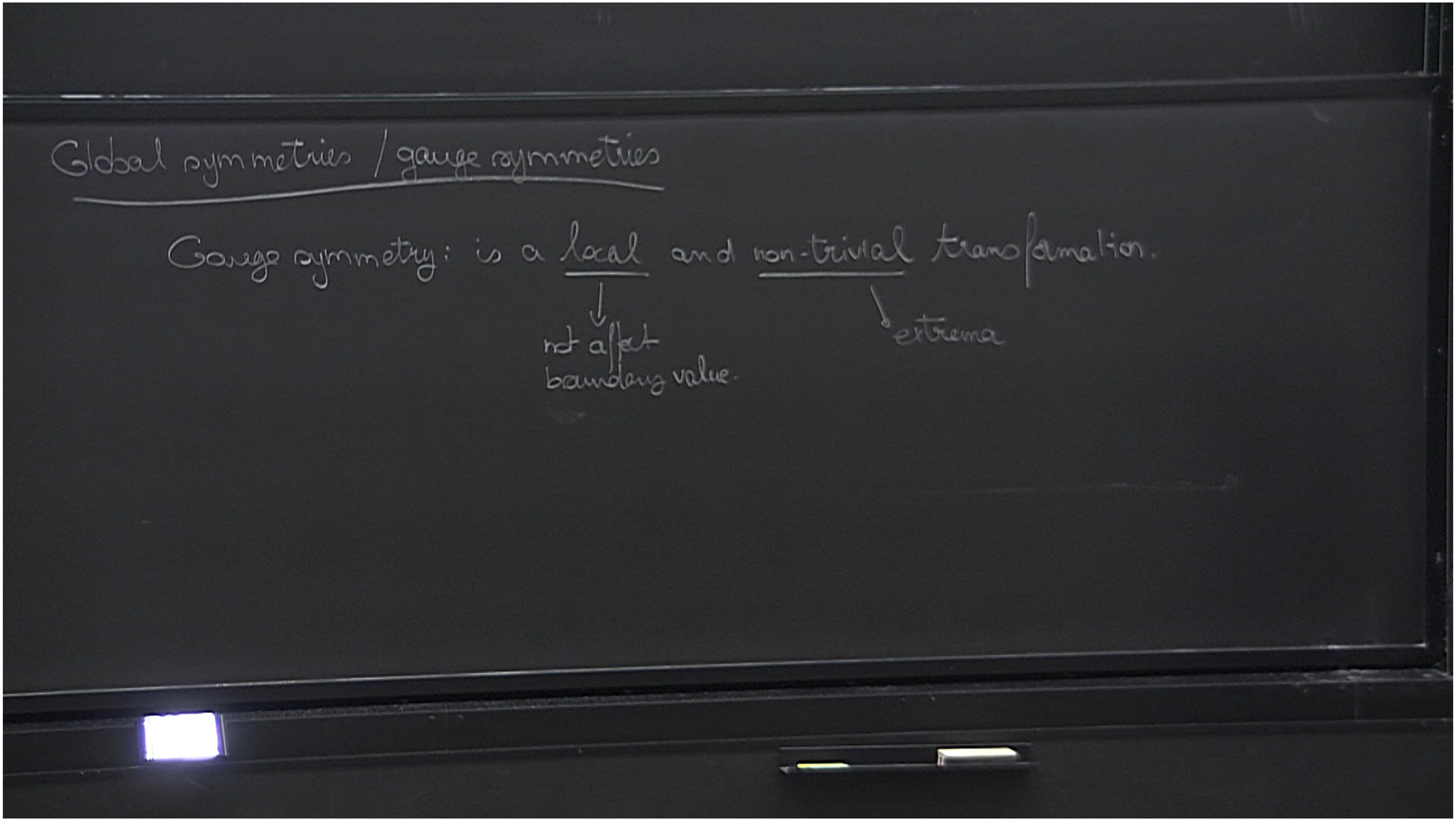


. GR: 2 dof.
metric → 10
tetrad → 16.
$e^i_\mu$
3D.
0 dof.
metric → 6
triad → 9.



. GR: 2 dof.
metric → 10
tetrad → 16.
$e^i_\mu$
3D.
0 dof.
metric → 6
triad → 9.
$\Phi(p_i, q_i) = 0$ :
$\Phi$ : constant of motion.


Global symmetries / gauge symmetries
Gauge symmetry: is a local and non-trivial transformation.
not affect
boundary value.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extrema

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Gauge symmetry: is a local and non-trivial transformation.

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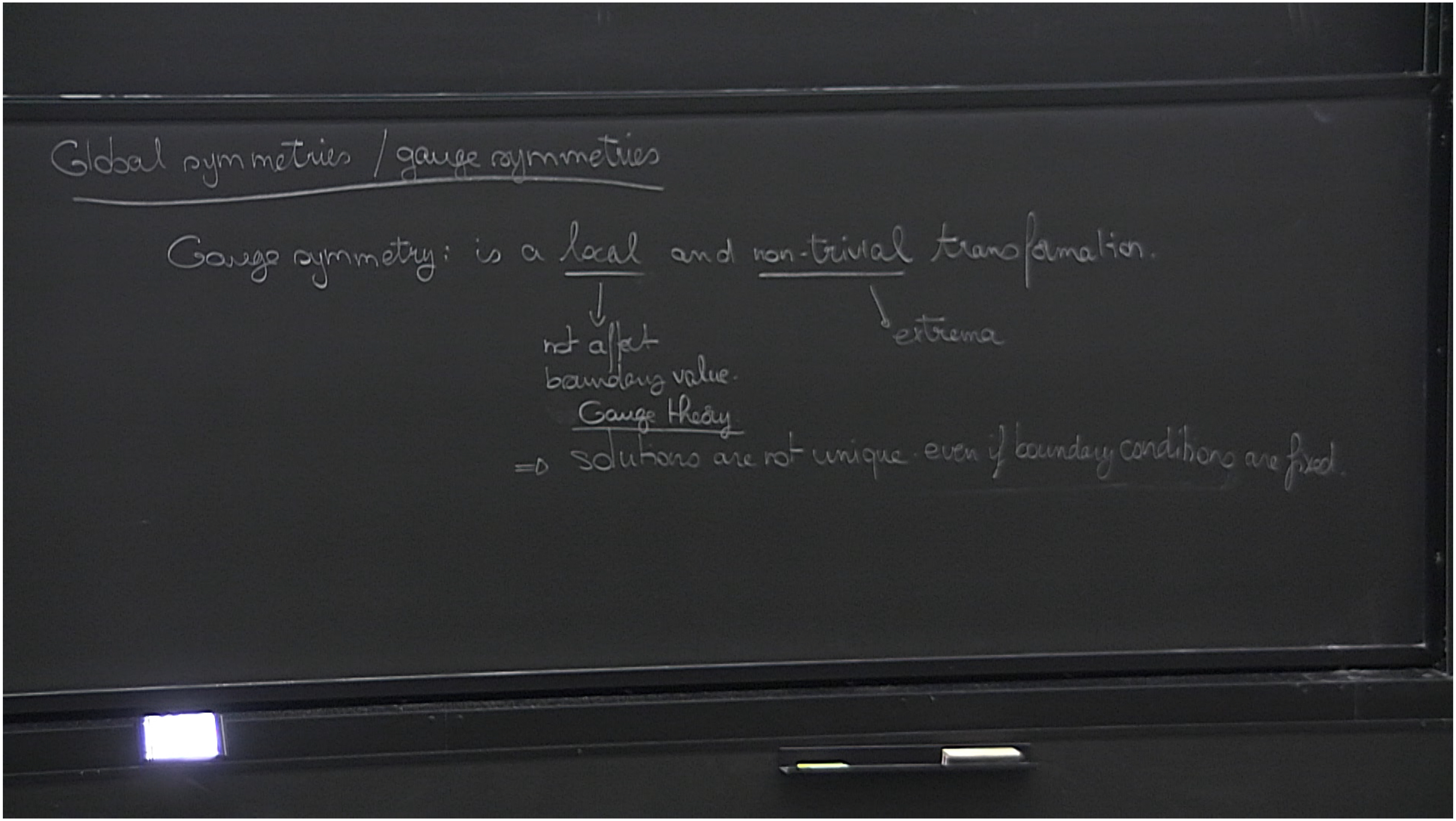
$\Rightarrow$ solutions are not unique even if boundary conditions are fixed.


Global symmetries / gauge symmetries
Gauge symmetry: is a local and non-trivial transformation.
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extrema
Gauge theory
⇒ solutions are not unique. even if boundary conditions are fixed.

$S = \int L(q_i, q_i)\,dt \xrightarrow[\text{transform}]{\text{Legendre}} p_i \equiv \dfrac{\partial L}{\partial \dot{q}_i} \qquad p_i(q_i, \dot{q}_i)$

when <u>not invertible</u> $\rightarrow$ primary constraint $\varphi_\alpha(p_i, q_i)$

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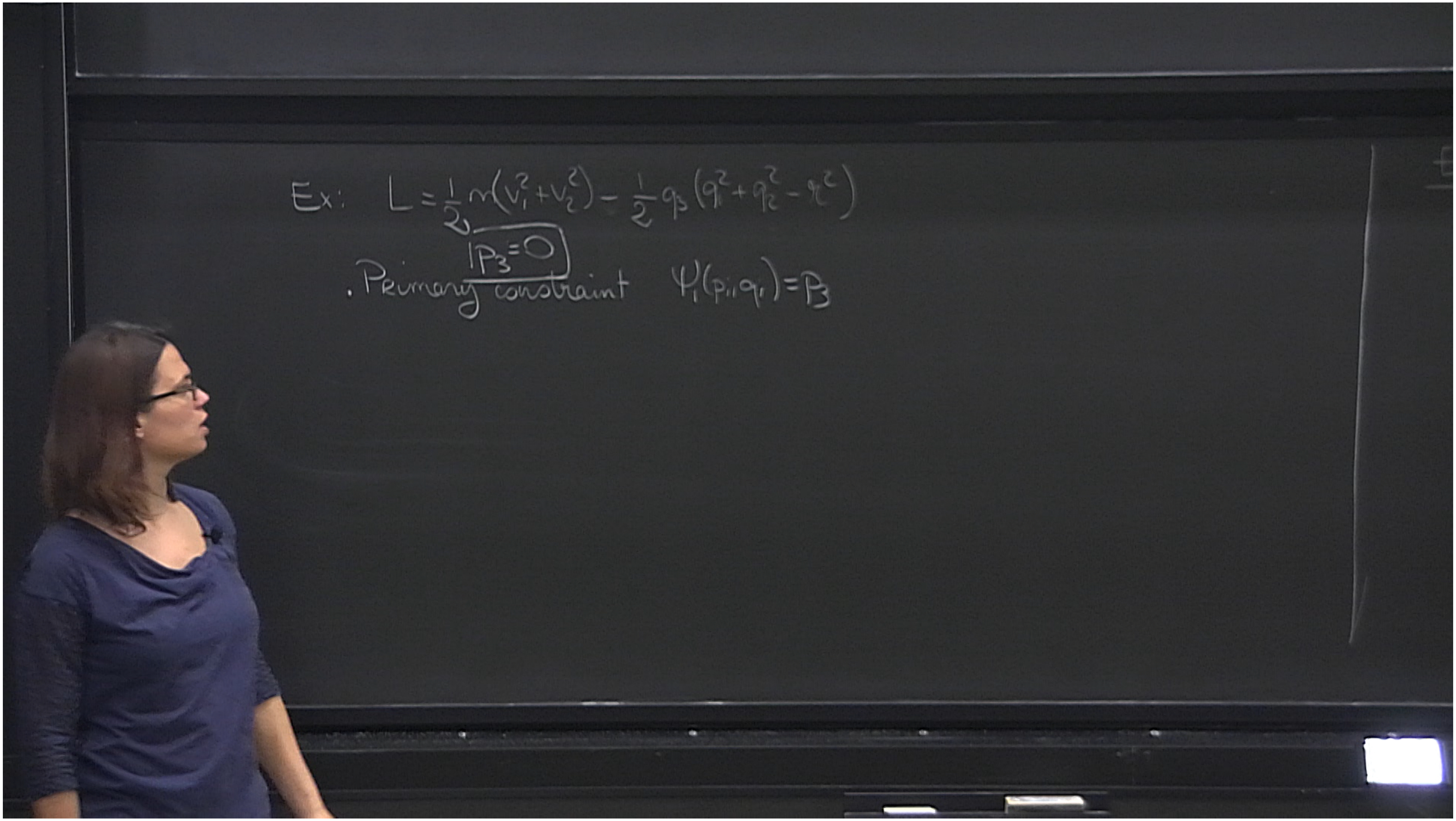
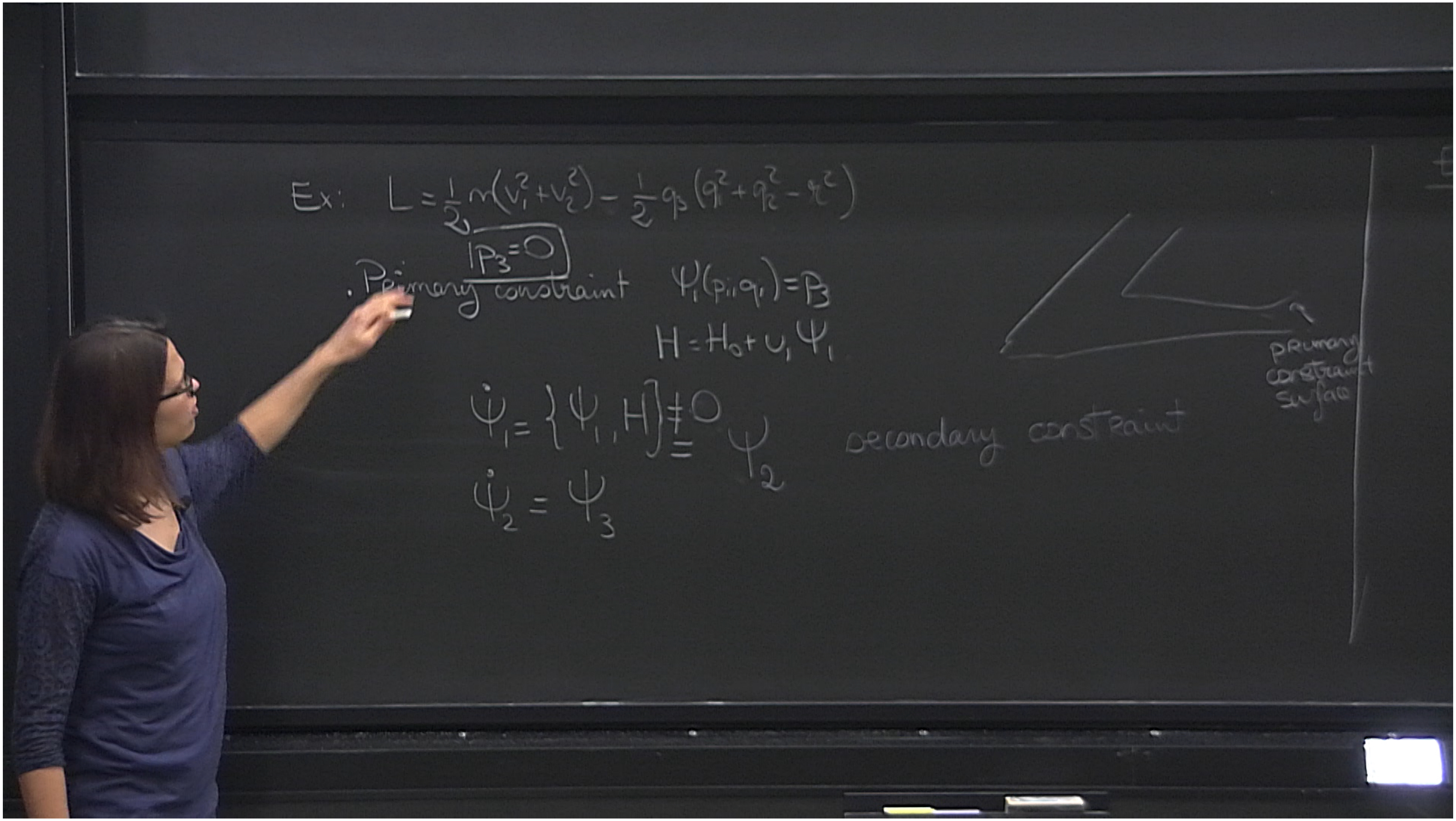
$$H = H_0 + u^\alpha \psi^\alpha$$

$\uparrow$ primary constraint


Ex: L = 1/2 m(v1^2 + v2^2) - 1/2 q3 (q1^2 + q2^2 - r^2)
p3 = 0
. Primary constraint φ1(pi, qi) = p3


Ex: $L = \frac{1}{2} m(v_1^2 + v_2^2) - \frac{1}{2} q_3 (q_1^2 + q_2^2 - r^2)$
$\boxed{p_3 = 0}$
. Primary constraint $\psi_1(p_i, q_i) = p_3$
$H = H_0 + u_1 \psi_1$
$\dot{\psi}_1 = \{\psi_1, H\} \neq 0$
$= \psi_2$
secondary constraint
$\dot{\psi}_2 = \psi_3$
primary constraint surface

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$$H = H_0 + u\,\psi_1$$

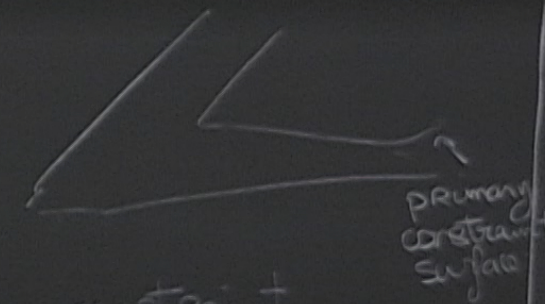
$$\dot{\psi}_1 = \{\psi_1, H\} \neq 0 = \psi_2$$ secondary constraint

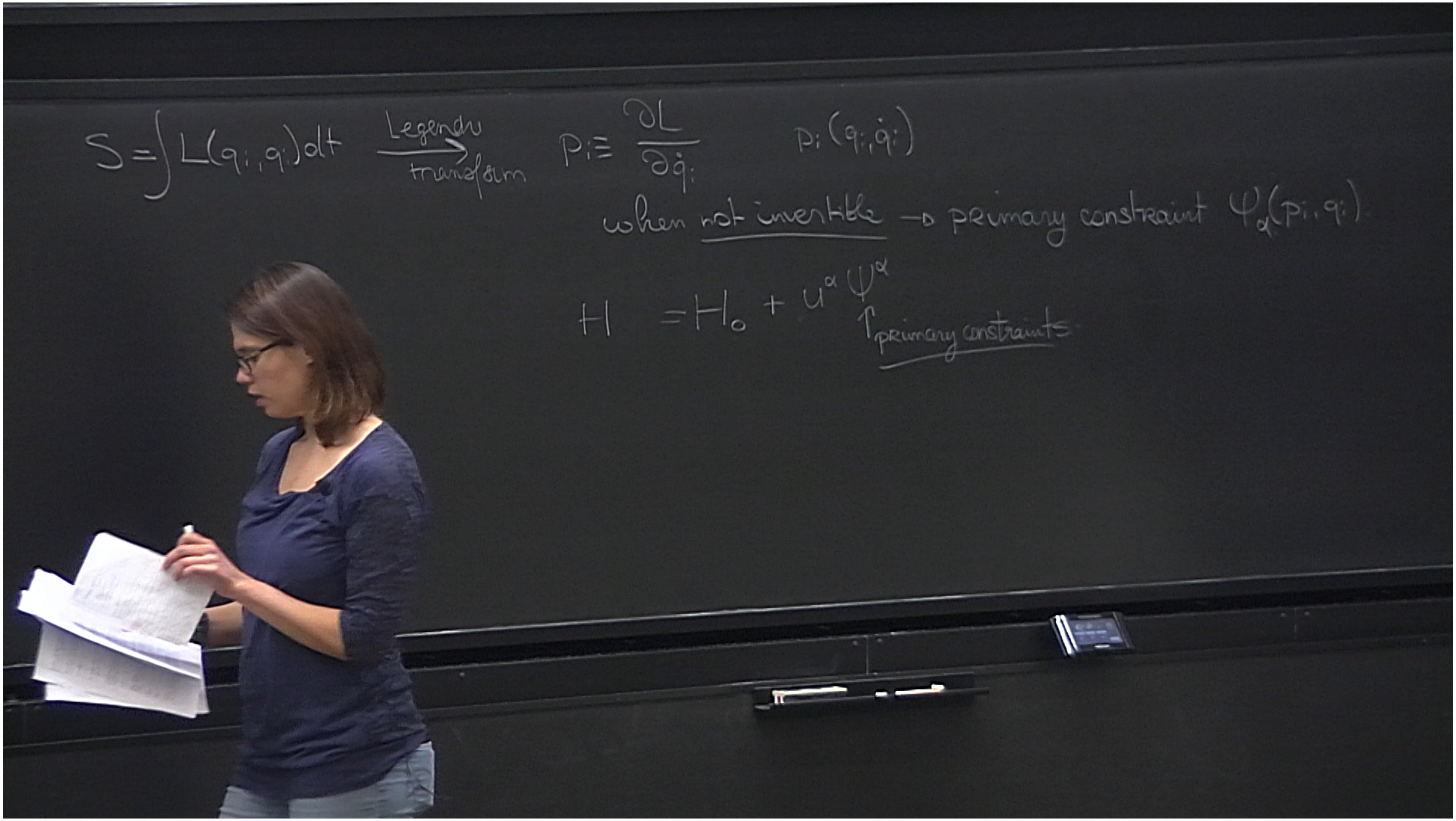
$$\dot{\psi}_2 = \psi_3$$

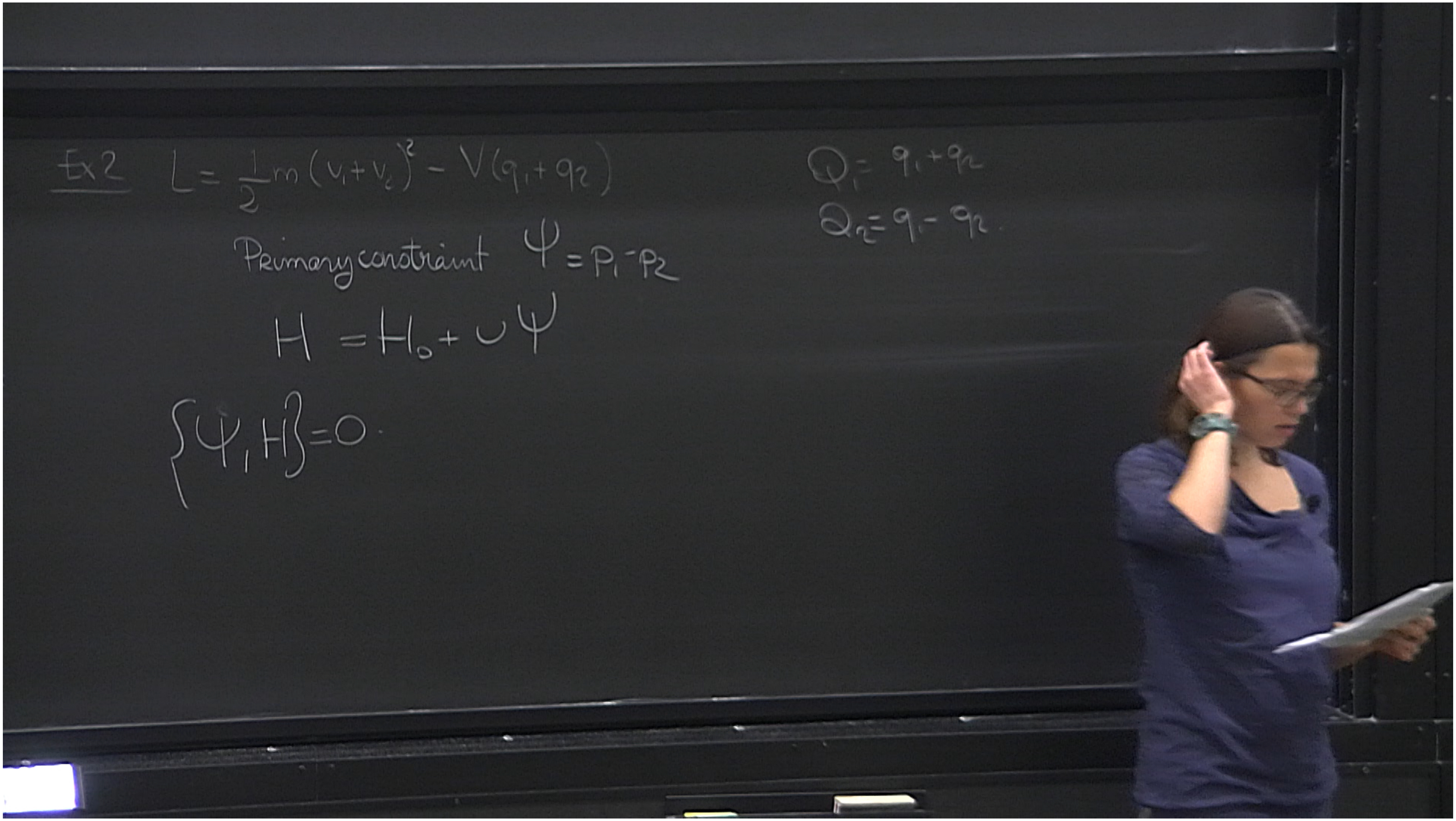
$$\dot{\psi}_3 = \psi_4$$

$$\dot{\psi}_4 = \alpha \psi_3 + u \approx 0$$

impose that $u = 0$.




S = ∫ L(qᵢ, q̇ᵢ) dt
Legendre transform
pᵢ ≡ ∂L/∂q̇ᵢ
pᵢ(qᵢ, q̇ᵢ)
when not invertible → primary constraint φ_α(pᵢ, qᵢ)
H = H₀ + u^α ψ^α
primary constraints


Ex 2 $L = \frac{1}{2} m (v_1 + v_2)^2 - V(q_1 + q_2)$
$Q_1 = q_1 + q_2$
$Q_2 = q_1 - q_2$.
Primary constraint $\psi = p_1 - p_2$
$H = H_0 + u\psi$
$\{\psi, H\} = 0$.

Ex 2 $L = \frac{1}{2}m(v_1+v_2)^2 - V(q_1+q_2)$

Primary constraint $\psi = p_1 - p_2$

$$H = H_0 + u\psi$$

$$\{\psi, H\} = 0.$$

$$\{Q_2, H\} = 2u$$

$Q_1 = q_1 + q_2$

$Q_2 = q_1 - q_2.$

What is the role of the constraint $\psi$ in this case?

$$\delta q_1 = u\{q_1, \psi\} = u.$$

$$\delta p_1 = u\{p_1, \psi\} = 0$$

$$\delta q_2 = u\{q_2, \psi\} = -u$$

$$\delta p_2 = u\{p_2, \psi\} = 0$$

Ex2 $L = \frac{1}{2} m (v_1 + v_2)^2 - V(q_1 + q_2)$

Primary constraint $\Psi = p_1 - p_2$

$$H = H_0 + u\Psi$$

$$\{\Psi, H\} = 0.$$

$$\{Q_2, H\} = 2u$$

$$\delta q_1 = \{q_1, u\Psi\} \simeq u\{q_1, \Psi\}$$

$Q_1 = q_1 + q_2$

$Q_2 = q_1 - q_2.$

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Primary constraint $\Psi = p_1 - p_2$

$$H = H_0 + u\Psi$$

$$\{\Psi, H\} = 0.$$

$$\{Q_2, H\} = 2u$$

$$\delta q_1 = \{q_1, u\Psi\} \approx u\{q_1, \Psi\} + \underbrace{\cancel{\{q_1, u\}}}_{\approx 0}\Psi$$

$Q_1 = q_1 + q_2$

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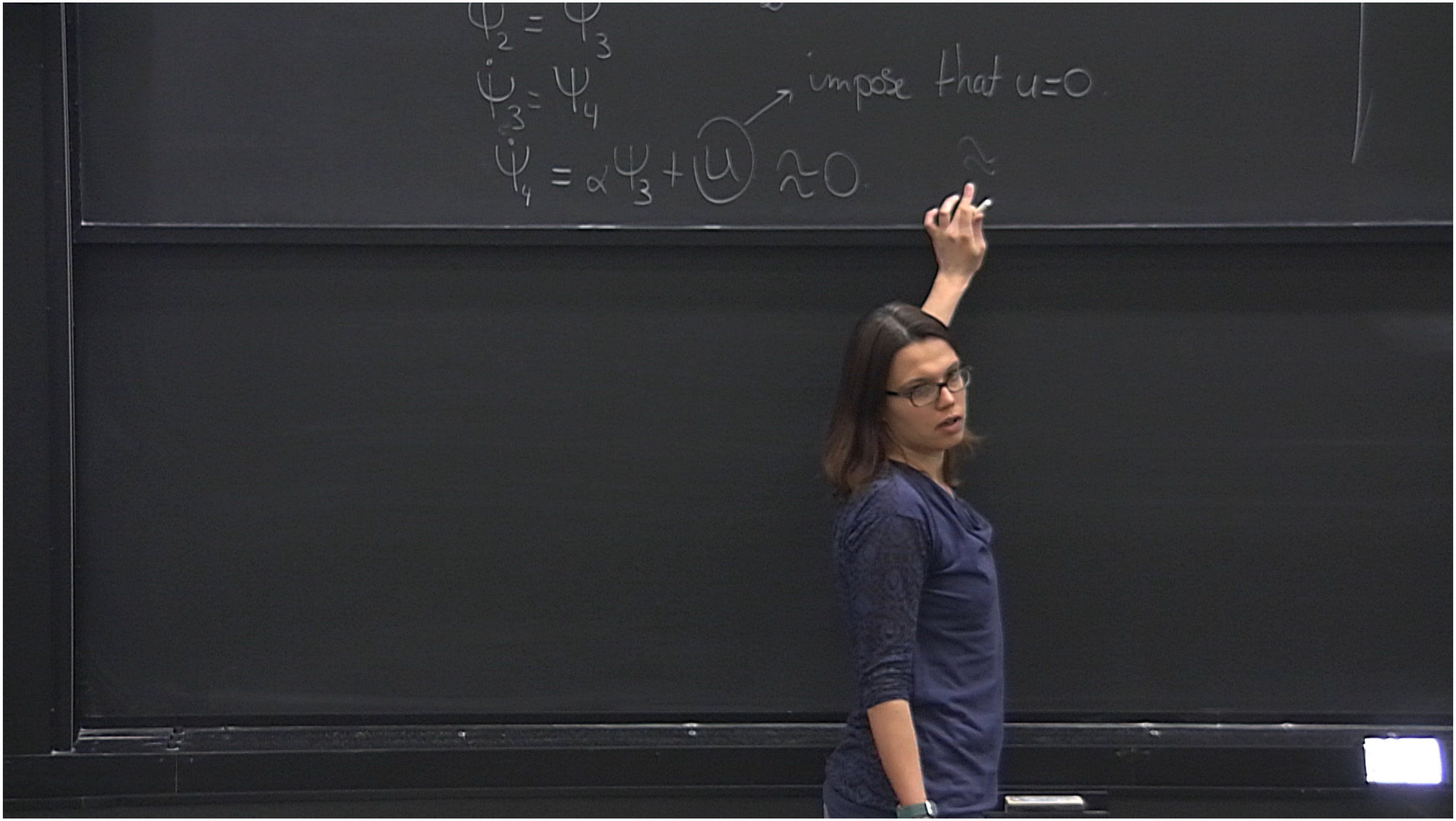
$\dot{\psi}_2 = \psi_3$
$\dot{\psi}_3 = \psi_4$
impose that u=0
$\dot{\psi}_4 = \alpha\psi_3 + u \approx 0$

$\{u\varphi_2, H\} = \ldots$

$\delta q_1 = \{q_1, u\psi\} \approx u\{q_1, \psi\}$
$+ \{q_1, \cancel{u}\}\psi$

$\delta p_1 = u\{p_1, \psi\} = 0$

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· physical quantities " $\left.\begin{matrix} Q_1 \\ P_1 \end{matrix}\right]$ are invariant

· interal "degree of freedom $Q_2$: changed by $u\psi$

$$\delta q_1 = \{q_1, u\psi\} \simeq u\{q_1,\psi\} + \{q_1, \cancel{u}\}\psi$$

$$\delta p_1 = u\{p_1, \psi\} = 0$$

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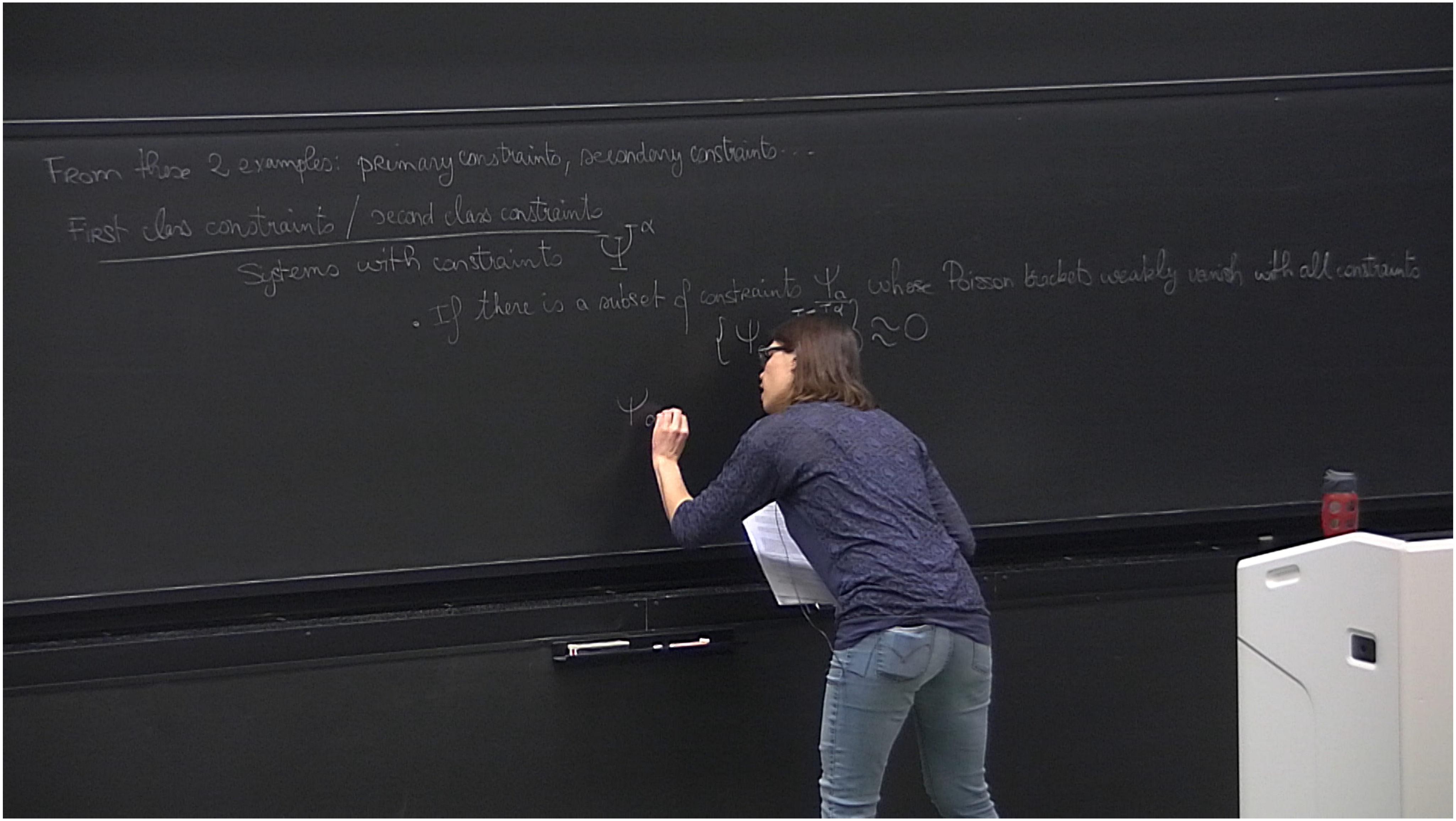
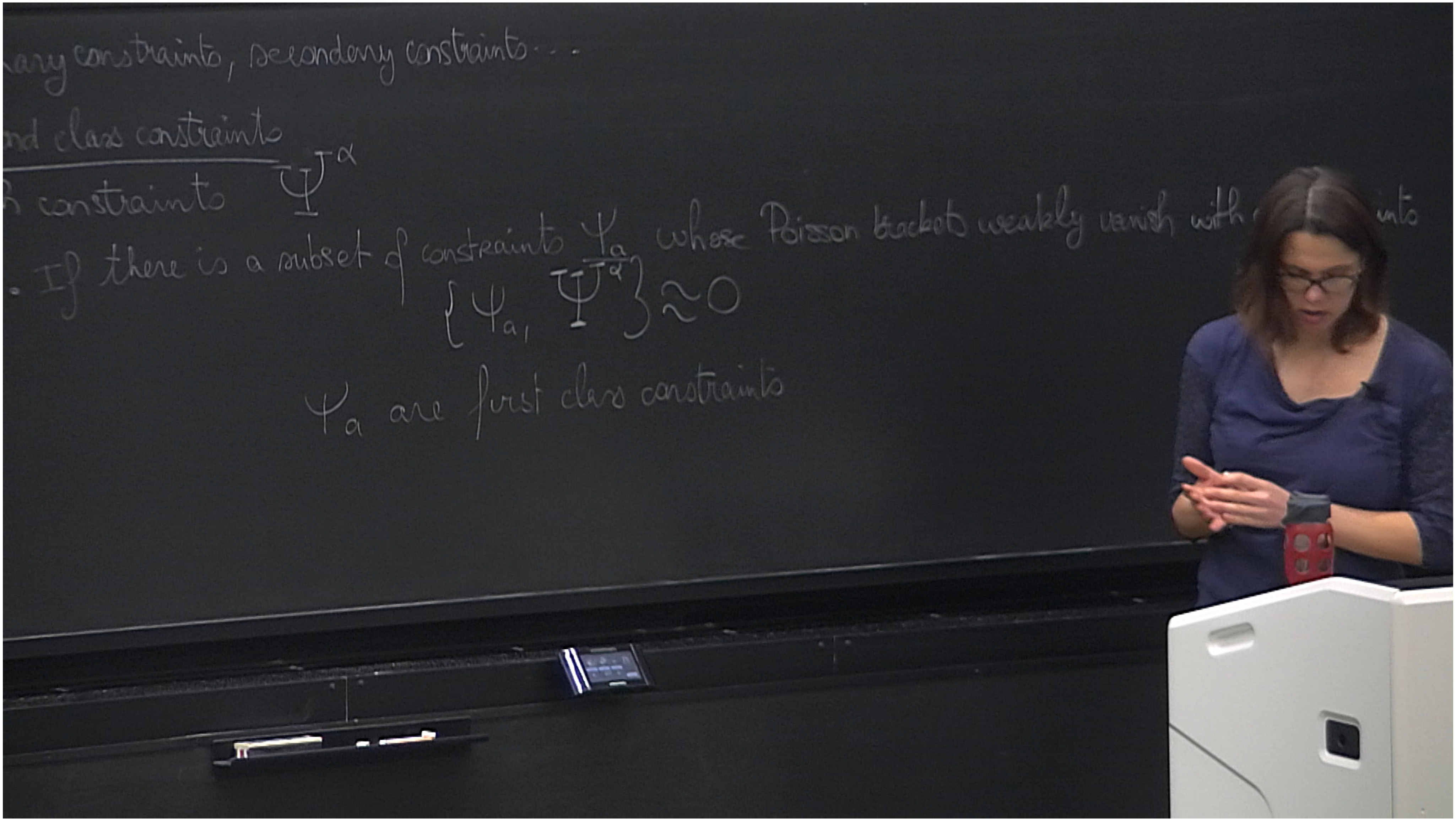
"interal" degree of freedom $Q_2$: changed by $u\psi$

From these 2 examples: primary constraints, secondary constraints...


From these 2 examples: primary constraints, secondary constraints . . .
First class constraints / second class constraints
Systems with constraints Ψ^α
• If there is a subset of constraints Ψ_a whose Poisson brackets weakly vanish with all constraints
{Ψ ≈ 0
Ψ_a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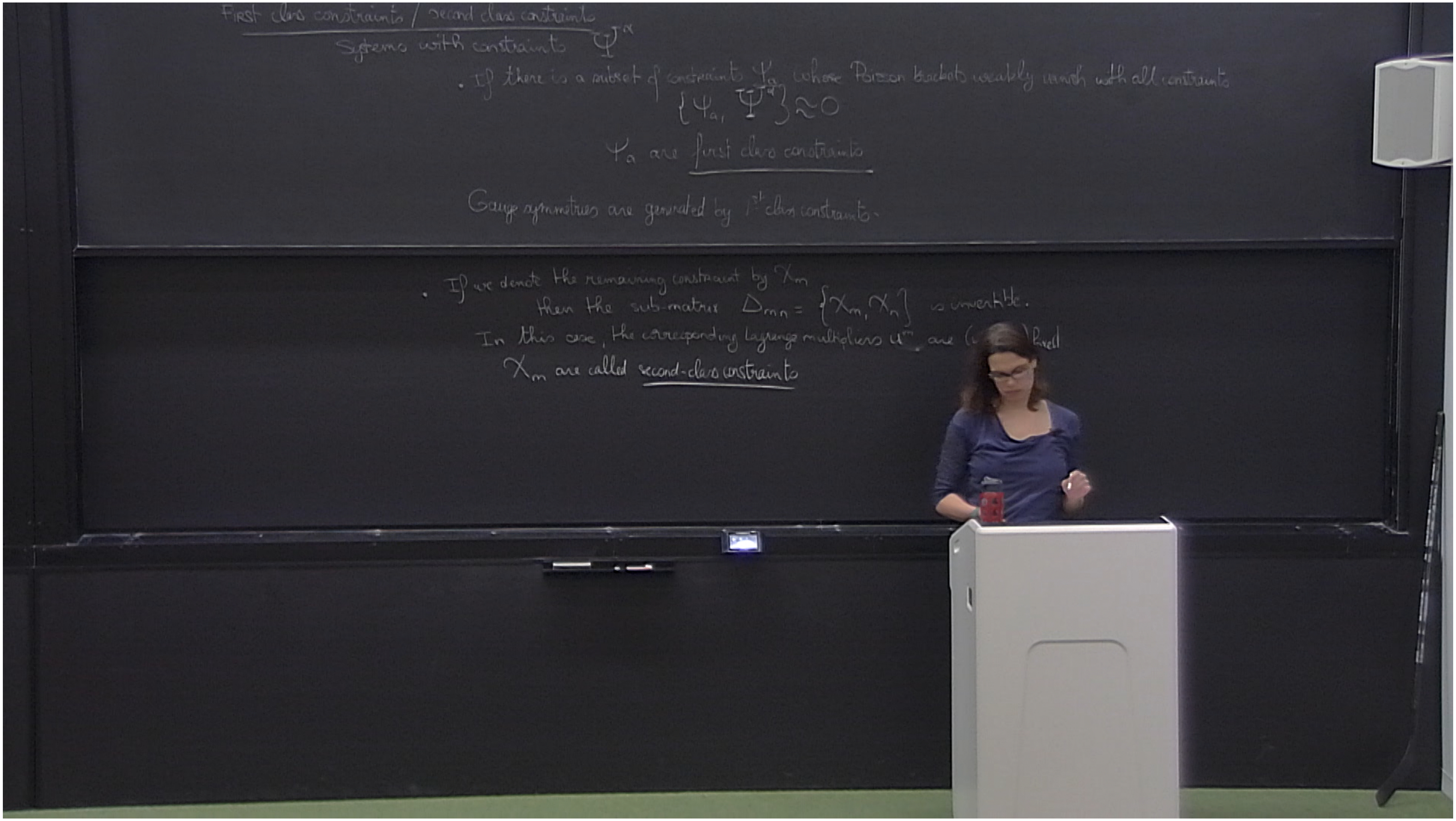
nd class constraints
constraints $\Psi^\alpha$
If there is a subset of constraints $\psi_a$ whose Poisson brackets weakly vanish with
nts
$\{\psi_a, \Psi^\alpha\} \approx 0$
$\psi_a$ are first class constraints


{Ψa, Φ} ≈ 0
Ψa are first class constraints
Gauge symmetries are generated by 1st class constraints

• If we denote the remaining constraint by $\chi_m$
then the sub-matrix $\Delta_{mn} = \{\chi_m, \chi_n\}$ is invertible.


First class constraints / second class constraints
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In this case, the corresponding Lagrange multipliers $u^m$ are fixed
$\chi_m$ are called second-class constraints

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$\chi_m$ are called <u>second-class constraints</u>

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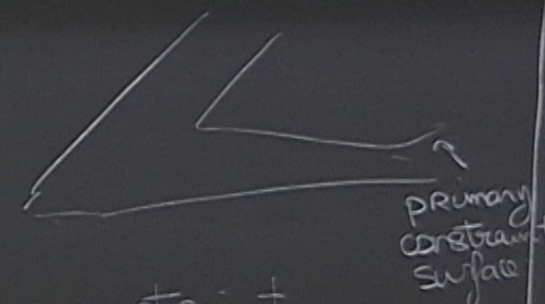
impose that $u = 0$ on the constraint surface

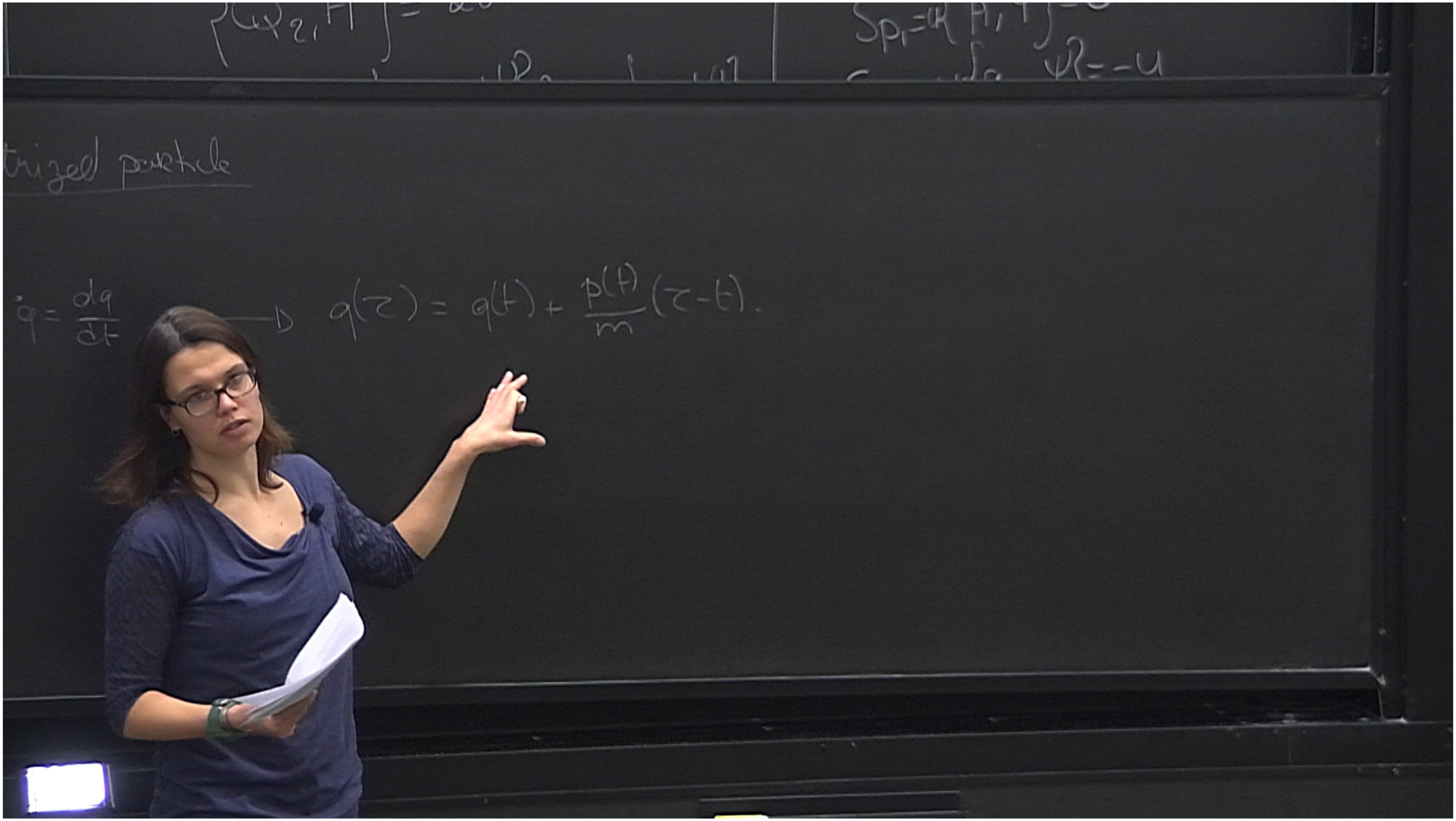


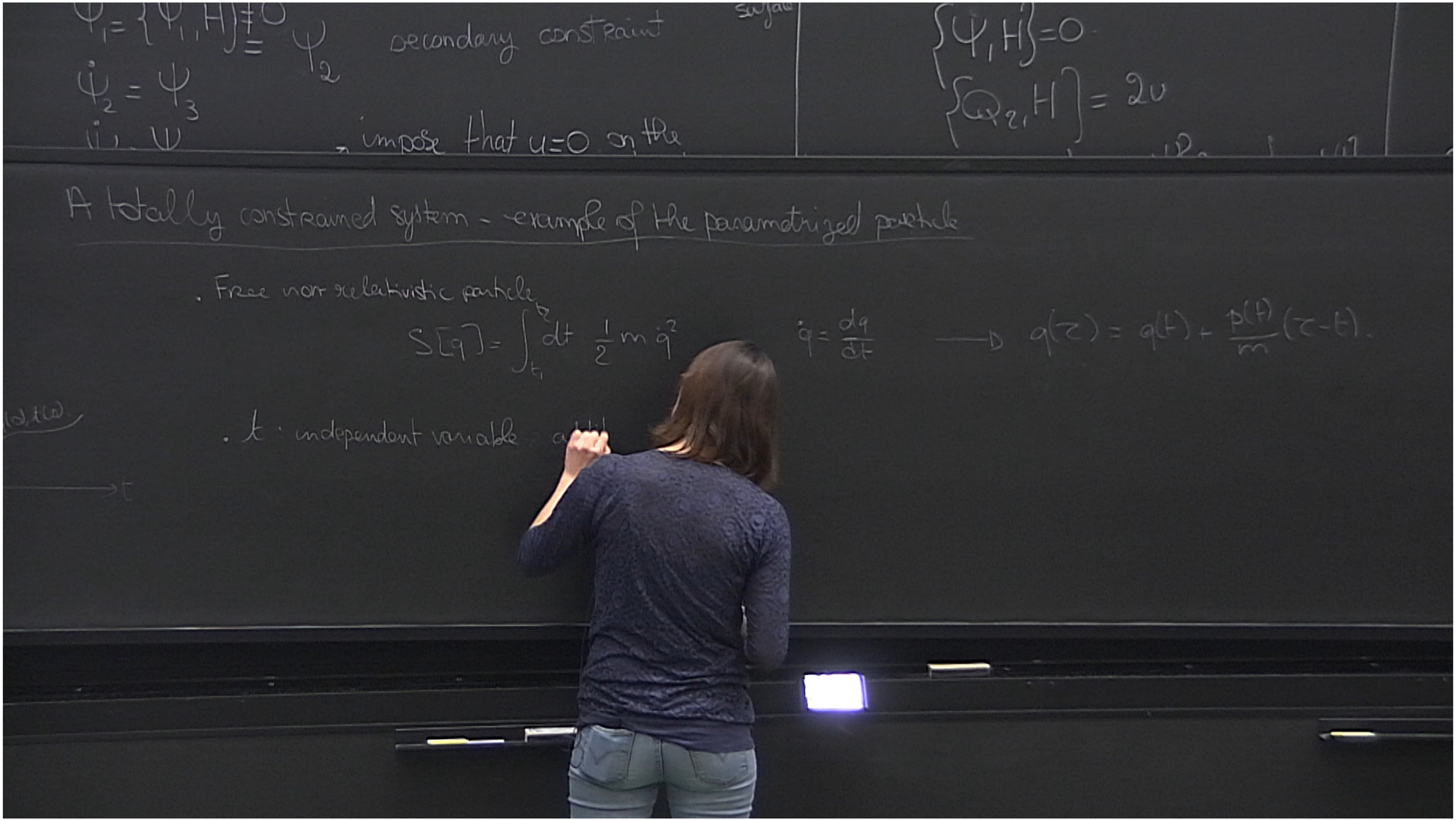
impose that $u=0$ on the

A totally constrained system - example of the parametrized particle

Free non relativistic particle

$$S[q] = \int_{t_1}^{t_2} dt \; \frac{1}{2} m \dot{q}^2 \qquad \dot{q} = \frac{dq}{dt} \longrightarrow q(\tau) = q$$


trized particle
q̇ = dq/dt ⟶ q(τ) = q(t) + p(t)/m (τ − t).


secondary constraint
impose that u=0 on the
A totally constrained system - example of the parametrized particle
. Free non relativistic particle
. t : independent variable

impose that $u=0$ on the

## A totally constrained system - example of the parametrized

. Free non relativistic particle

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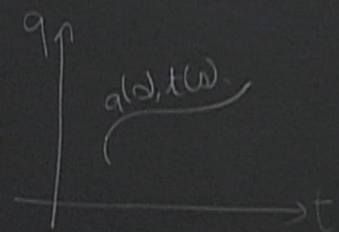


. $t$ : independent variable additional (unphy

$q_2$

impose that $u=0$ on the

$\{Q_2, H\} = 2u$

$\delta p_1 =$

ly constrained system - example of the parametrized particle

. Free non relativistic particle

$$S[q] = \int_{t_1}^{t_2} dt \; \frac{1}{2} m \dot{q}^2 \qquad \dot{q} = \frac{dq}{dt} \longrightarrow q(\tau) = q(t) + \frac{p(t)}{m}(\tau - t).$$

. $t$ : independent variable. additional (unphysical) parameter $s$.

$$S_p[q,t] = \int_{s'}^{s''} \frac{1}{2} m \frac{q'^2}{t'} \, ds.$$

The prime denotes the derivative with respect to $s$

trized particle

$\dot{q} = \frac{dq}{dt}$ $\longrightarrow$ $q(\tau) = q(t) + \frac{p(t)}{m}(\tau - t)$.

) parameter s.

ds.

The prime denotes the derivative with respect to s

Exercise: Check that the new action is invariant under reparametrization

$s \longrightarrow \tilde{s} = f(s)$

$\{ \psi_2, H \} = \ldots$ | $\delta p_1 = \epsilon \{ H, \ldots \}$

trized particle

$\dot{q} = \frac{dq}{dt}$ $\longrightarrow$ $q(\tau) = q(t) + \frac{p(t)}{m}(\tau - t)$.

) parameter s.

ds

the prime denotes the derivative with respect to s

. Canonical analysis?

. Observables

$\{O, C\} = 0$

Exercise: Check that the new action is invariant under reparametrization

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) parameter s.

ds

the prime denotes the derivative with respect to s

. Canonical analysis?

. Observables (Gauge fixing).

$\{O, C\} = 0 \rightarrow$ 2 observ...

. Dirac program.

. Gravity

Exercise: Check that the new action is invariant under reparametrization

$s \longrightarrow \tilde{s} = f(s)$