Title: PSI 2017/2018 - Quantum Gravity - Lecture 2

Date: Mar 20, 2018 10:15 AM

URL: http://pirsa.org/18030039

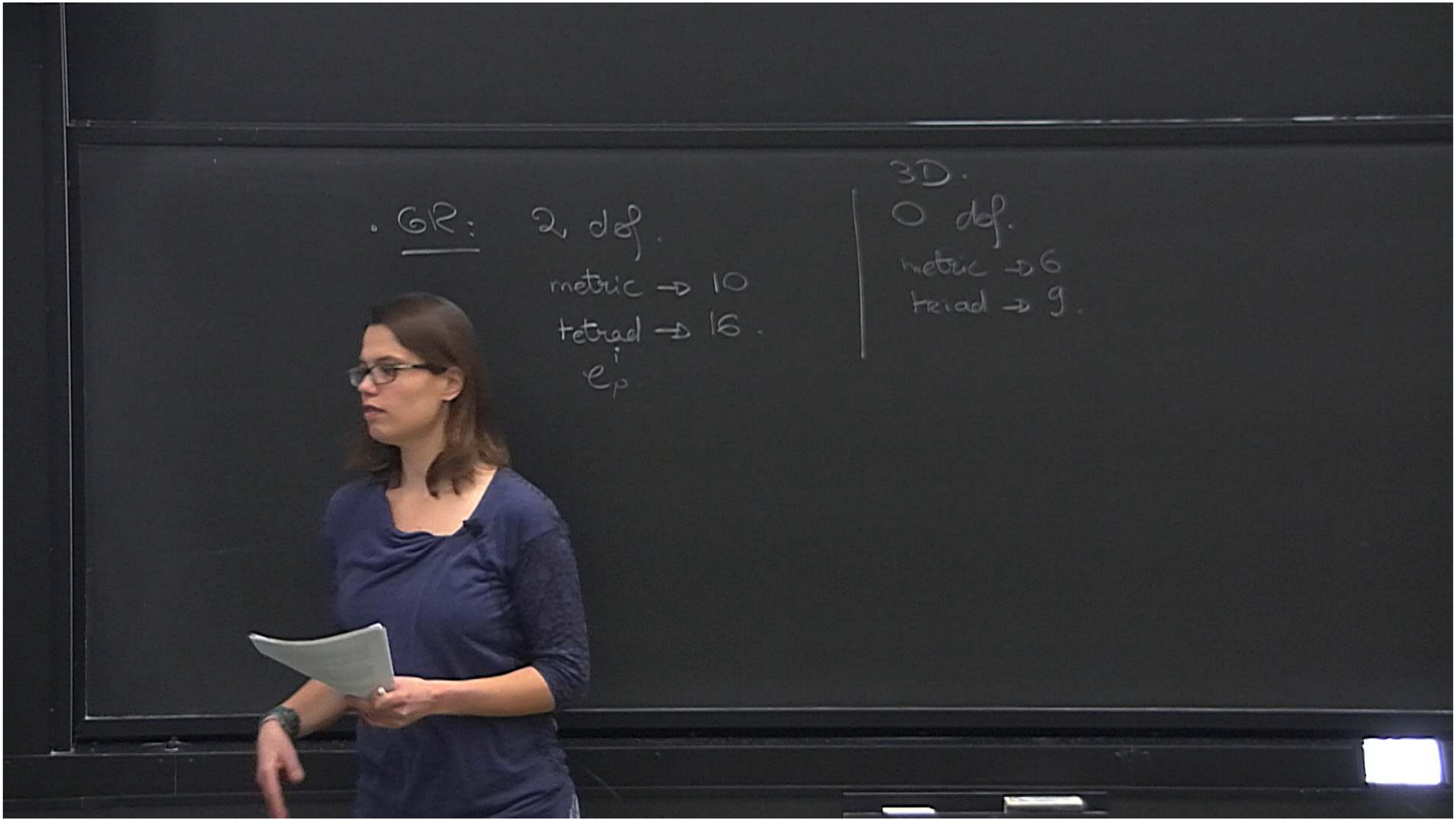
Abstract:

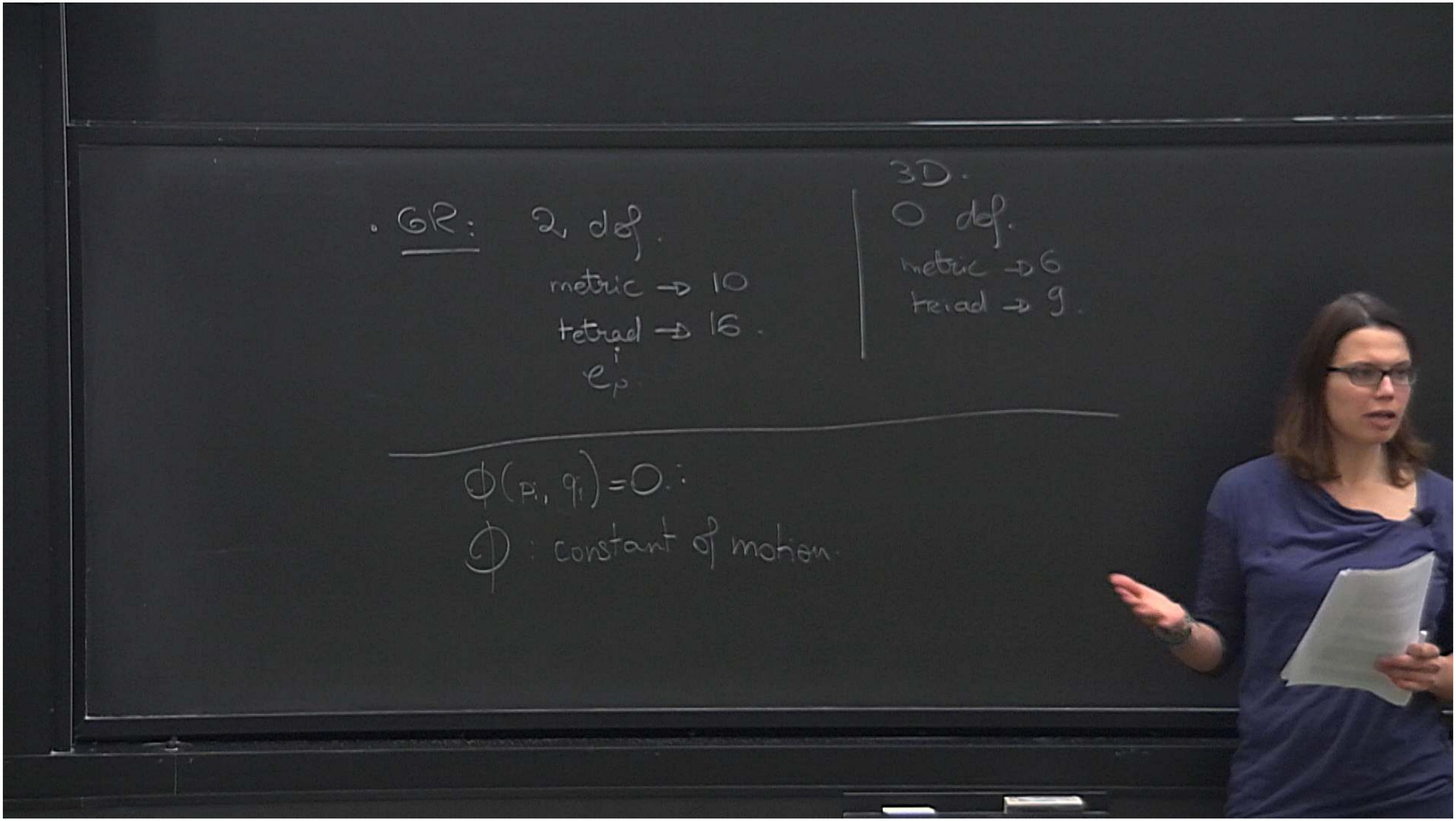
Canonical formulation of
constrained system

→ parametrized particle

- Why are we interested in "constrained systems"?

- Can we relate constraints/gauge symmetries?


GR: 2 dof.
metric → 10
tetrad → 16.
e_μ^i
3D.
0 dof.
metric → 6
triad → 9.


• GR: 2 dof.
metric → 10
tetrad → 16.
e^i_μ
3D.
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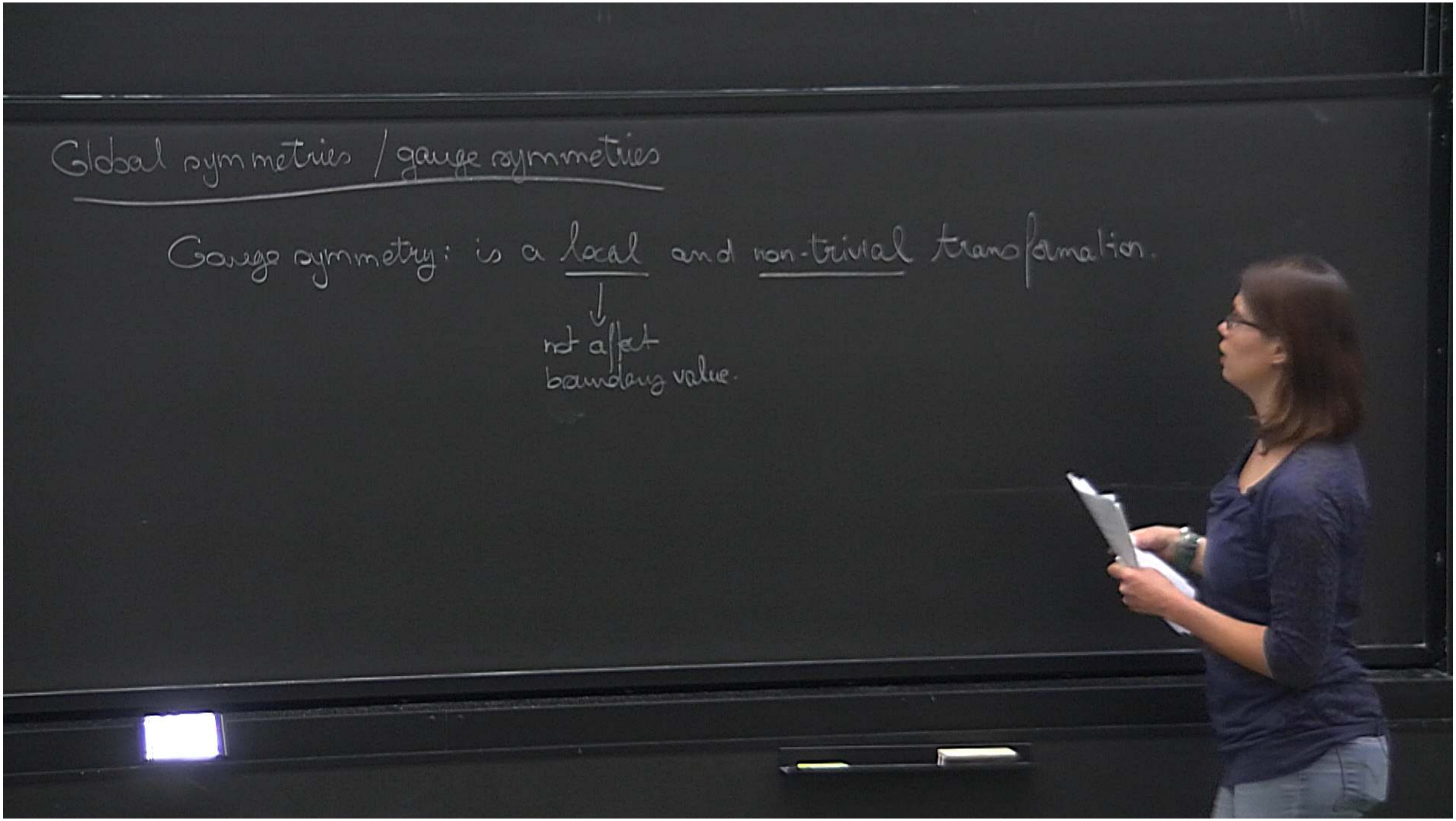
tetrad → 9.
$\Phi(p_i, q_i) = 0$:
Φ : constant of motion.


Global symmetries / gauge symmetries
Gauge symmetry: is a local and non-trivial transformation.
not affect
boundary value.

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local → not affect boundary value.

non-trivial → extrema

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$\Rightarrow$ solutions are not unique. even if boundary conditions are fixed.

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extrema

Gauge theory

$\Rightarrow$ solutions are not unique even if boundary conditions are fixed.

$$S = \int L(q_i, \dot{q}_i)\,dt \xrightarrow[\text{transform}]{\text{Legendre}} p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \qquad p_i(q_i, \dot{q}_i)$$

when <u>not invertible</u> $\rightarrow$ primary constraint $\varphi_\alpha(p_i, q_i)$

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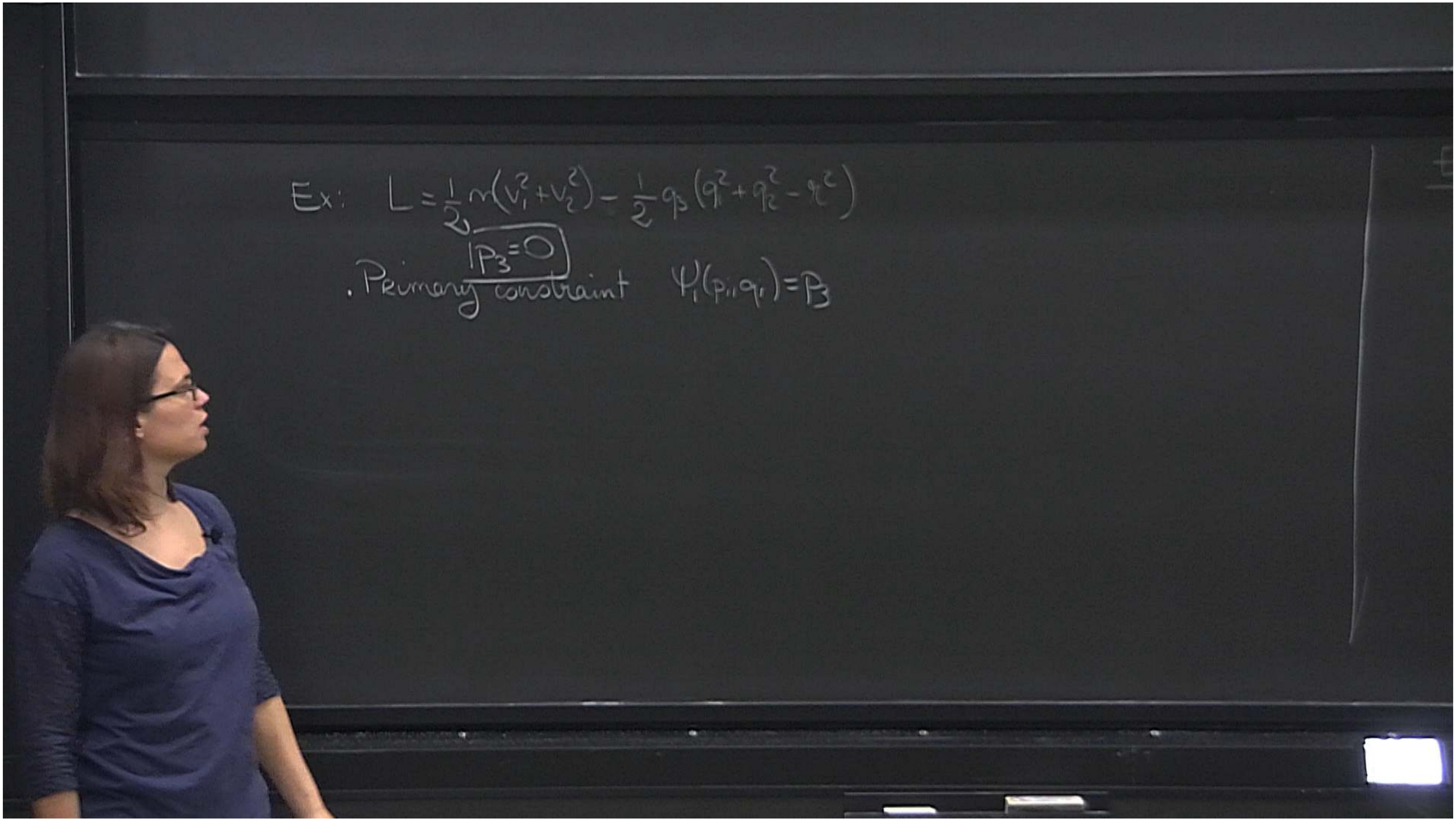
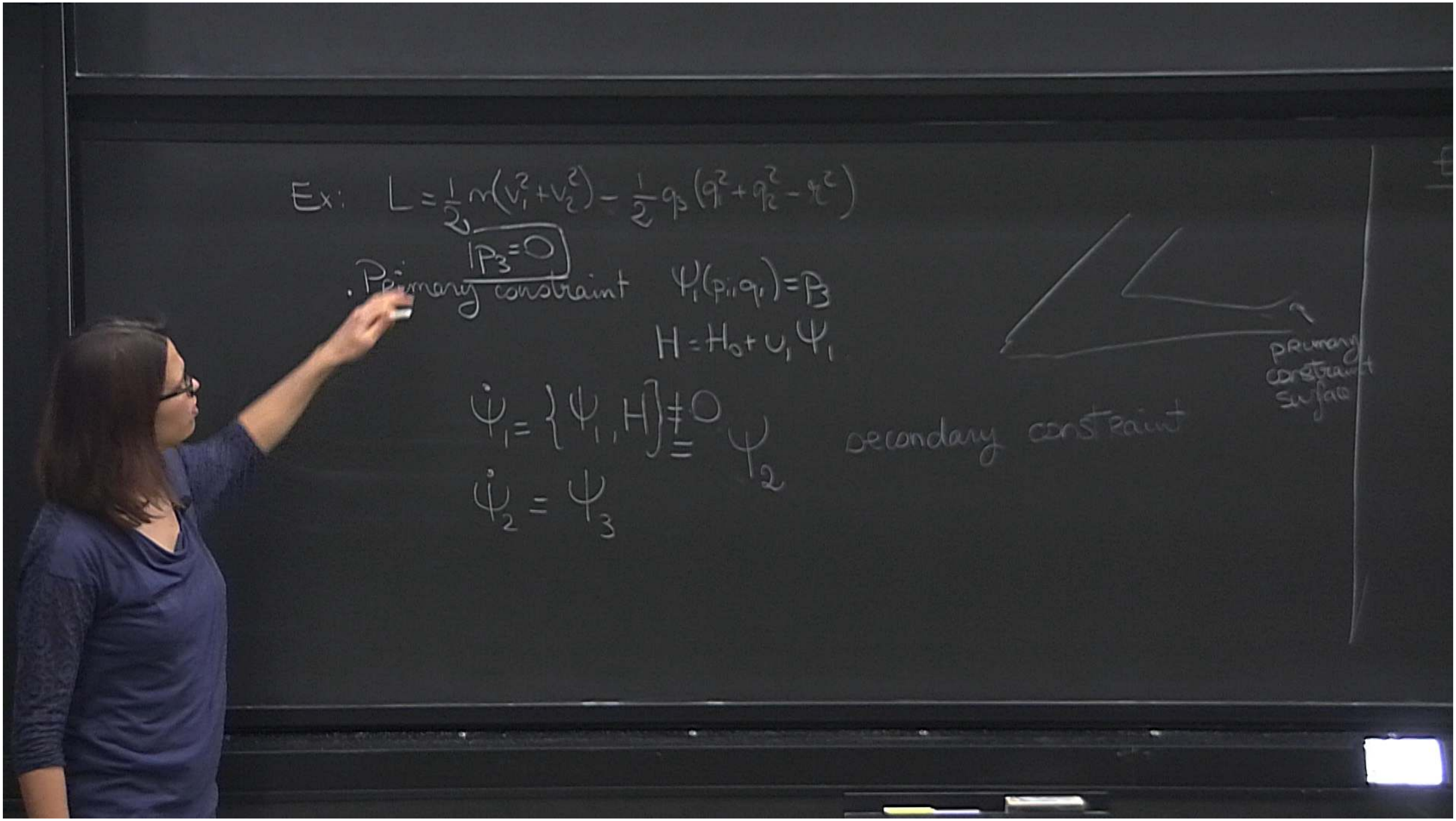
$$H = H_0 + u^\alpha \psi^\alpha$$

$\uparrow$ primary constraint


Ex: $L = \frac{1}{2} m (v_1^2 + v_2^2) - \frac{1}{2} q_3 (q_1^2 + q_2^2 - r^2)$
$p_3 = 0$
. Primary constraint $\varphi_1(p_i, q_i) = p_3$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$\boxed{p_3 = 0}$
. Primary constraint $\psi_1(p_i, q_i) = p_3$
$H = H_0 + u_1 \psi_1$
$\dot{\psi}_1 = \{\psi_1, H\} \neq 0$ ψ_2
secondary constraint
$\dot{\psi}_2 = \psi_3$
primary constraint surface

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. Primary constraint $\quad \Psi_1(p_i, q_i) = p_3$

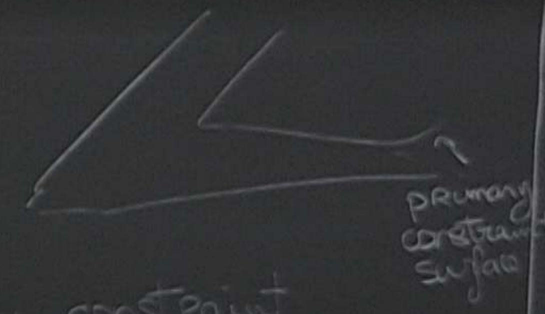
$H = H_0 + u_1 \Psi_1$

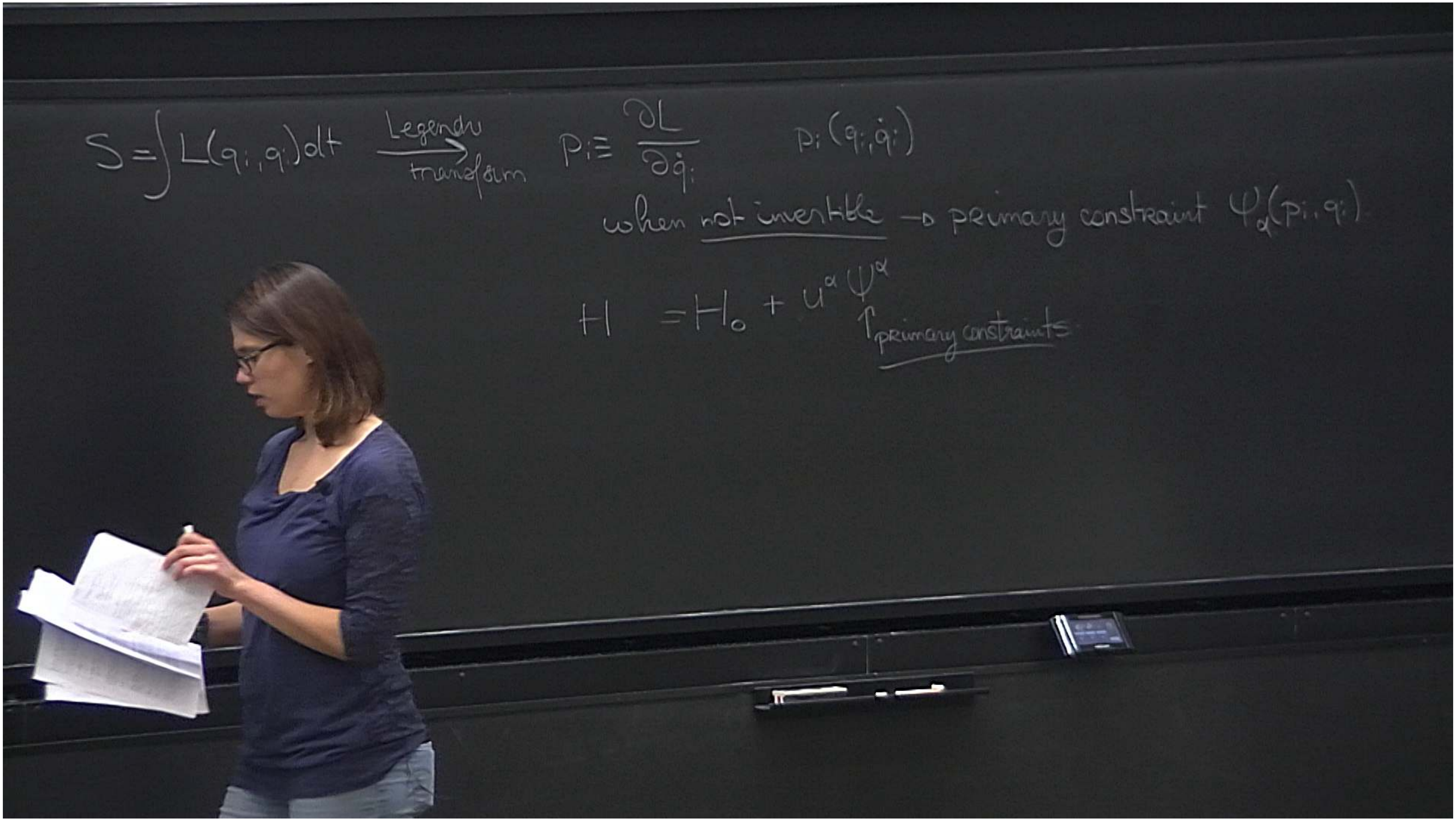
$\dot{\Psi}_1 = \{\Psi_1, H\} \neq 0 \;=\; \Psi_2$ secondary constraint

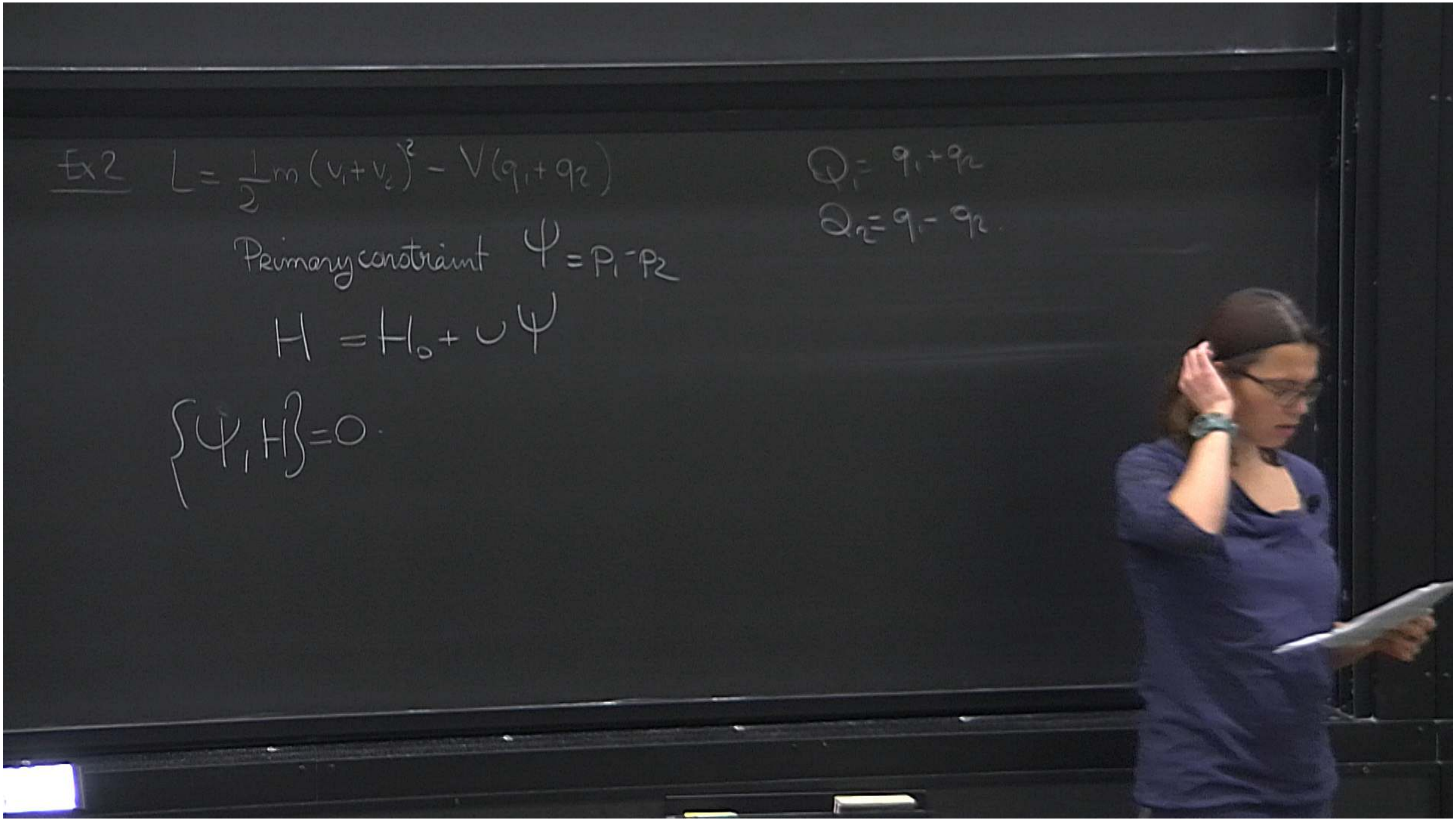
$\dot{\Psi}_2 = \Psi_3$

$\dot{\Psi}_3 = \Psi_4$ impose that $u=0$

$\dot{\Psi}_4 = \alpha \Psi_3 + u \approx 0$




$S = \int L(q_i, \dot{q}_i) dt \xrightarrow[\text{transform}]{\text{Legendre}} p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$
$p_i(q_i, \dot{q}_i)$
when not invertible → primary constraint $\varphi_\alpha(p_i, q_i)$
$H = H_0 + u^\alpha \psi^\alpha$
primary constraints


Ex 2 $L = \frac{1}{2} m (v_1 + v_2)^2 - V(q_1 + q_2)$
$Q_1 = q_1 + q_2$
$Q_2 = q_1 - q_2$.
Primary constraint $\psi = p_1 - p_2$
$H = H_0 + u\psi$
$\{\psi, H\} = 0$.

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$$H = H_0 + u\psi$$

$$\{\psi, H\} = 0$$

$$\{Q_2, H\} = 2u$$

$Q_1 = q_1 + q_2$

$Q_2 = q_1 - q_2$

What is the role of the constraint ψ in this case?

$$\delta q_1 = u\{q_1, \psi\} = u$$

$$\delta p_1 = u\{p_1, \psi\} = 0$$

$$\delta q_2 = u\{q_2, \psi\} = -u$$

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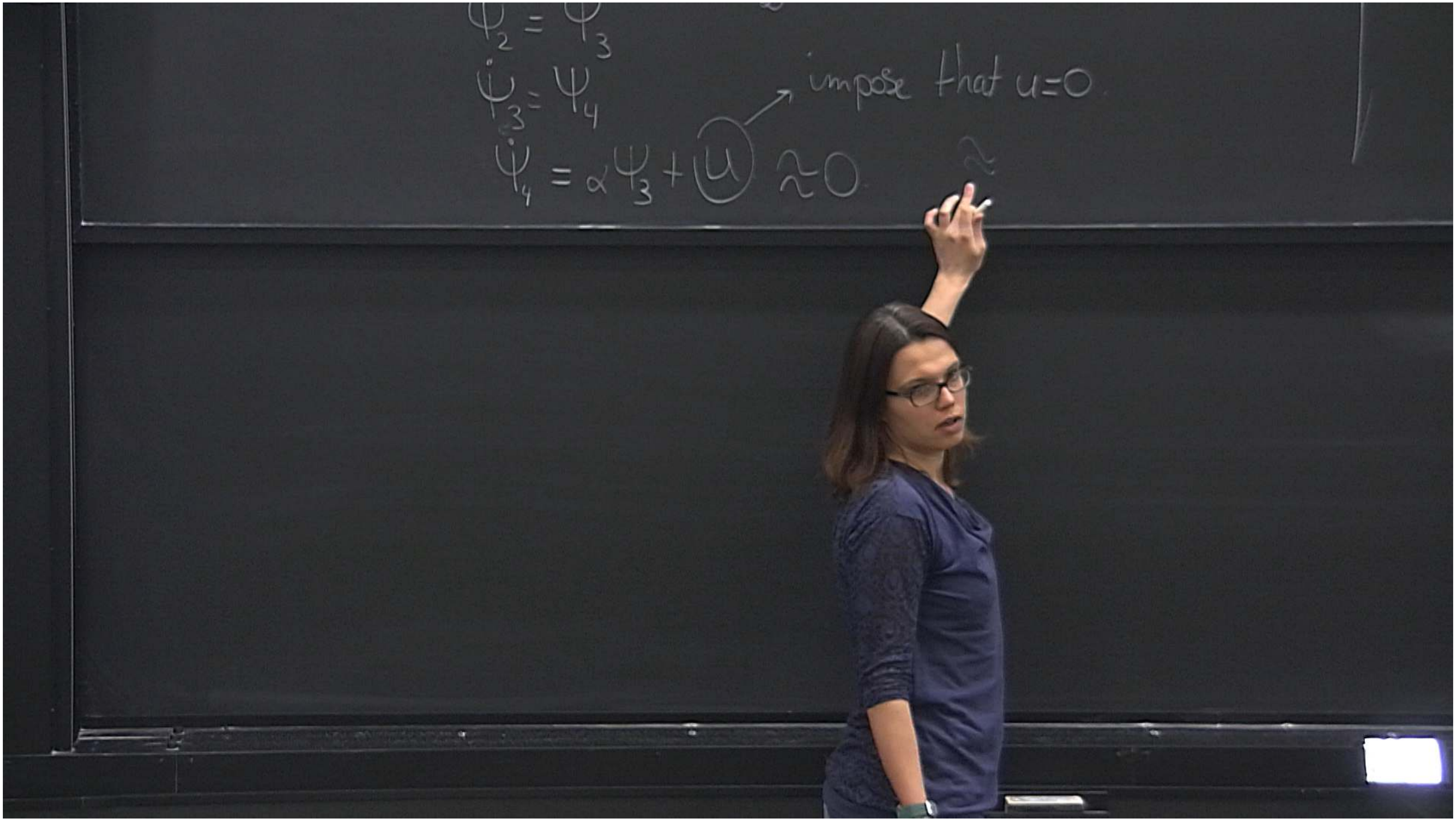
$\dot{\psi}_2 = \psi_3$
$\dot{\psi}_3 = \psi_4$
$\dot{\psi}_4 = \alpha \psi_3 + u \approx 0$
impose that $u=0$

$\{ Q_2, H \} = \ldots$

$\delta q_1 = \{ q_1, u\psi \} \simeq u \{ q_1, \psi \} + \{ q_1, \cancel{u} \} \psi$

$\delta p_1 = u \{ p_1, \psi \} = 0$

$\delta q_2 = u \{ q_2, \psi \} = -u$

$\delta p_2 = -u \{ p_2, \psi \} = 0$

"physical quantities" Q_1, P_1] are invariant

"interal" degree of freedom Q_2: changed by $u\psi$

$\{uQ_2, H\} = \ldots$

$\delta q_1 = \{q_1, u\psi\} \approx u\{q_1, \psi\} + \underbrace{\{q_1, \cancel{u}\}}_{\approx 0}\psi$

$\delta p_1 = u\{p_1, \psi\} = 0$

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- "physical quantities" $\left.\begin{matrix} Q_1 \\ P_1 \end{matrix}\right]$ are invariant
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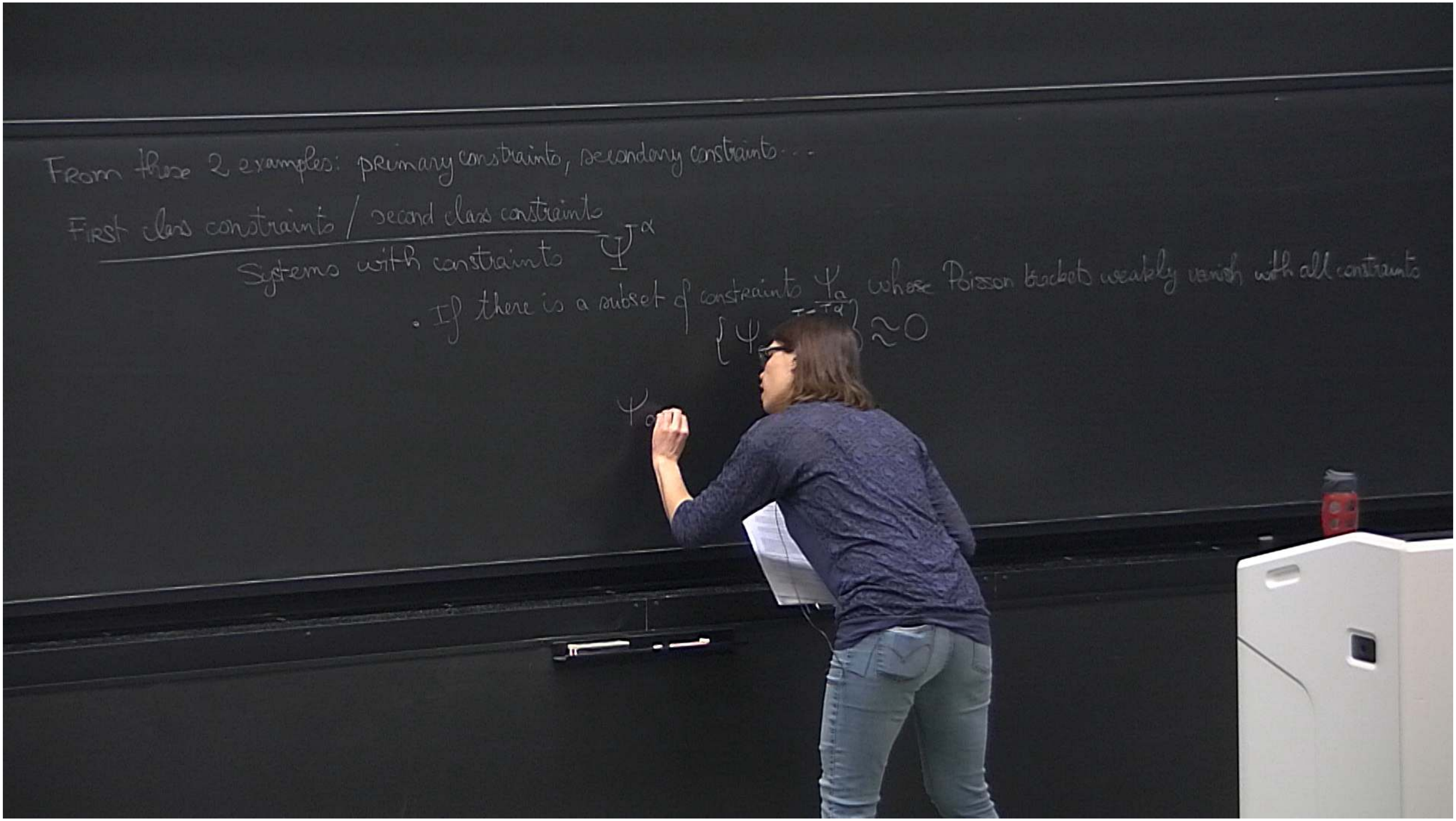
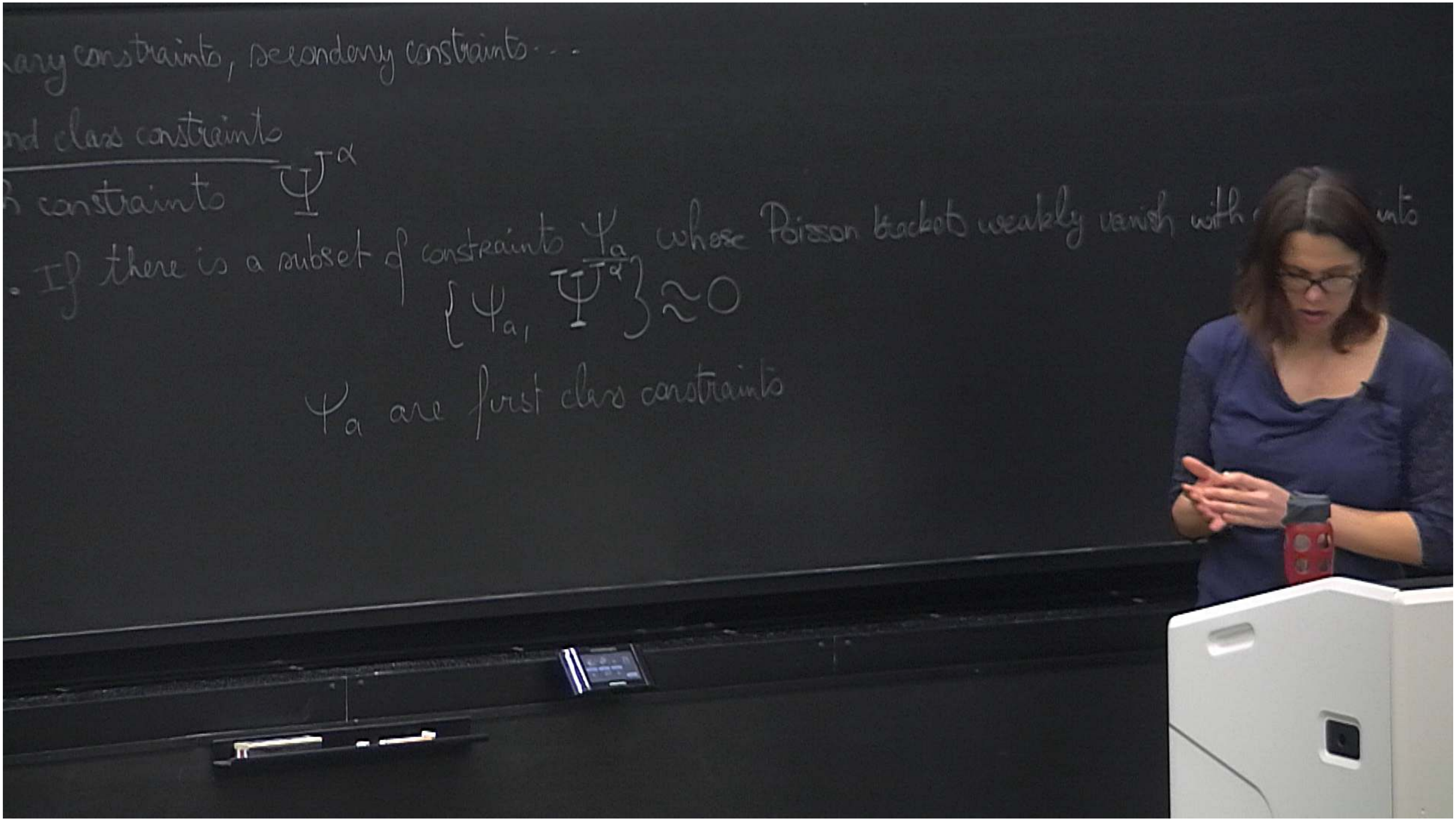
From these 2 examples: primary constraints, secondary constraints...


From these 2 examples: primary constraints, secondary constraints...
First class constraints / second class constraints
Systems with constraints Ψ^α
• If there is a subset of constraints Ψ_a whose Poisson brackets weakly vanish with all constraints
{Ψ_a, Ψ^α} ≈ 0
Ψ_a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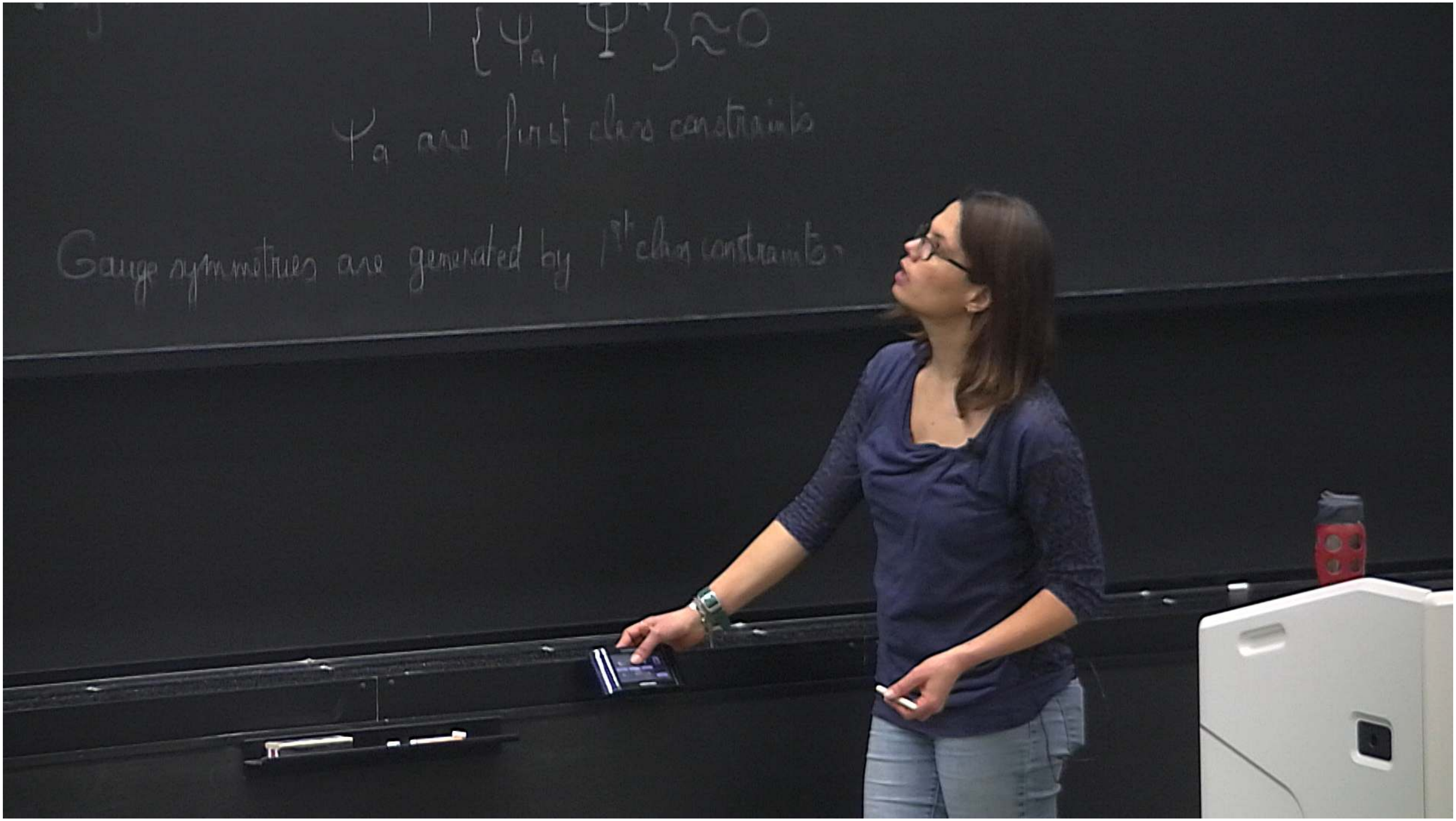
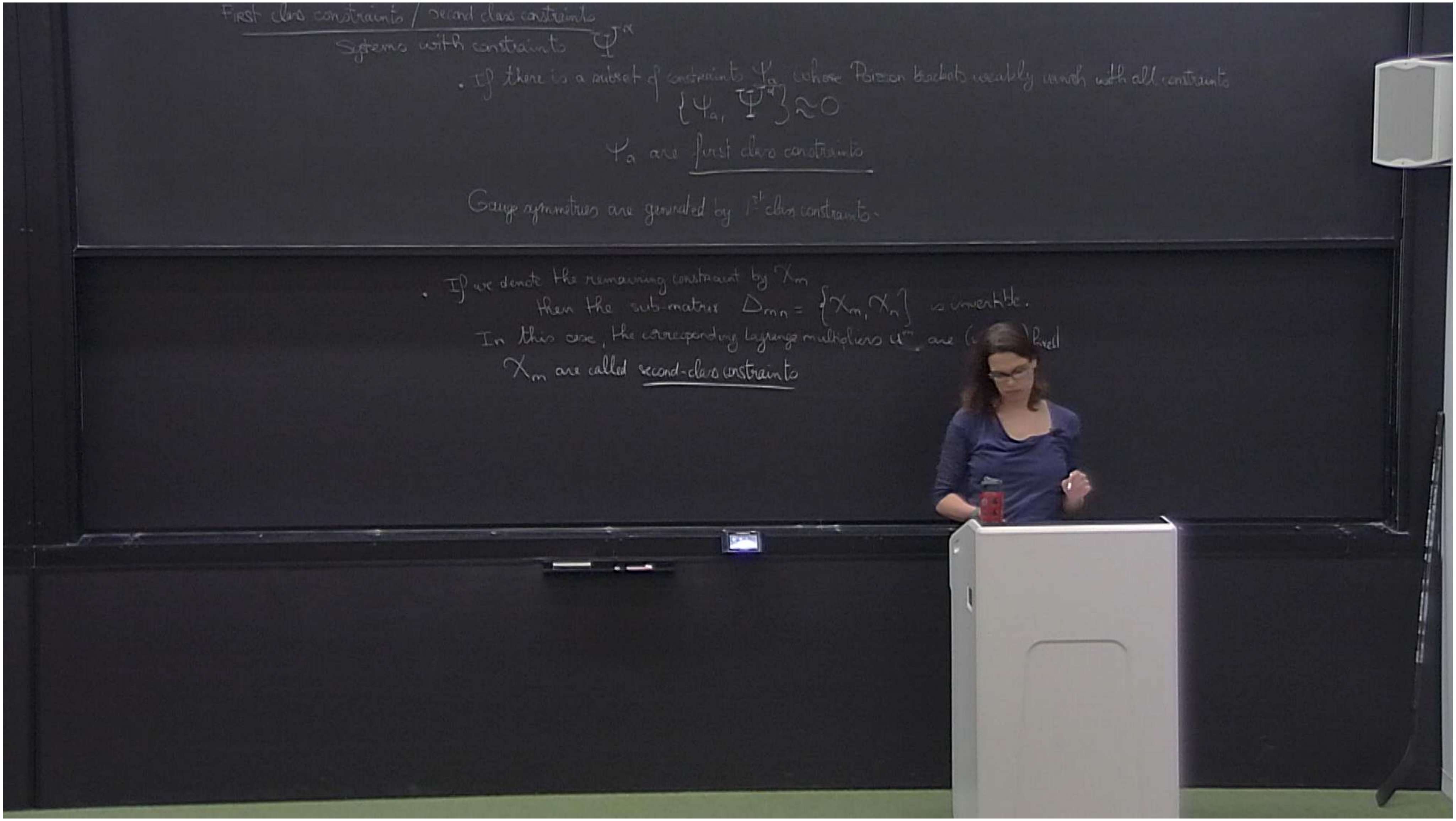
Ψ_a are first class constraints


{Ψa, Φ}≈0
Ψa are first class constraints
Gauge symmetries are generated by 1st class constraints

• If we denote the remaining constraint by χ_m
then the sub-matrix $\Delta_{mn} = \{\chi_m, \chi_n\}$ is invertible.



First class constraints / second class constraints
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In this case, the corresponding Lagrange multipliers u^m are fixed
χ_m are called second-class constraints

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χ_m are called <u>second-class constraints</u>

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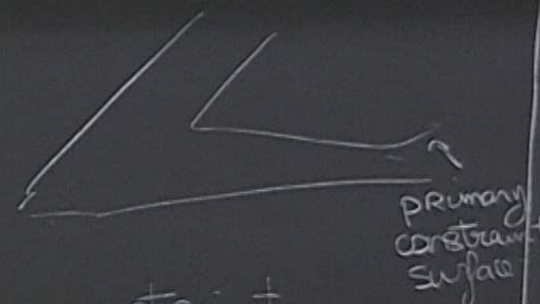
impose that $u=0$ on the constraint surface

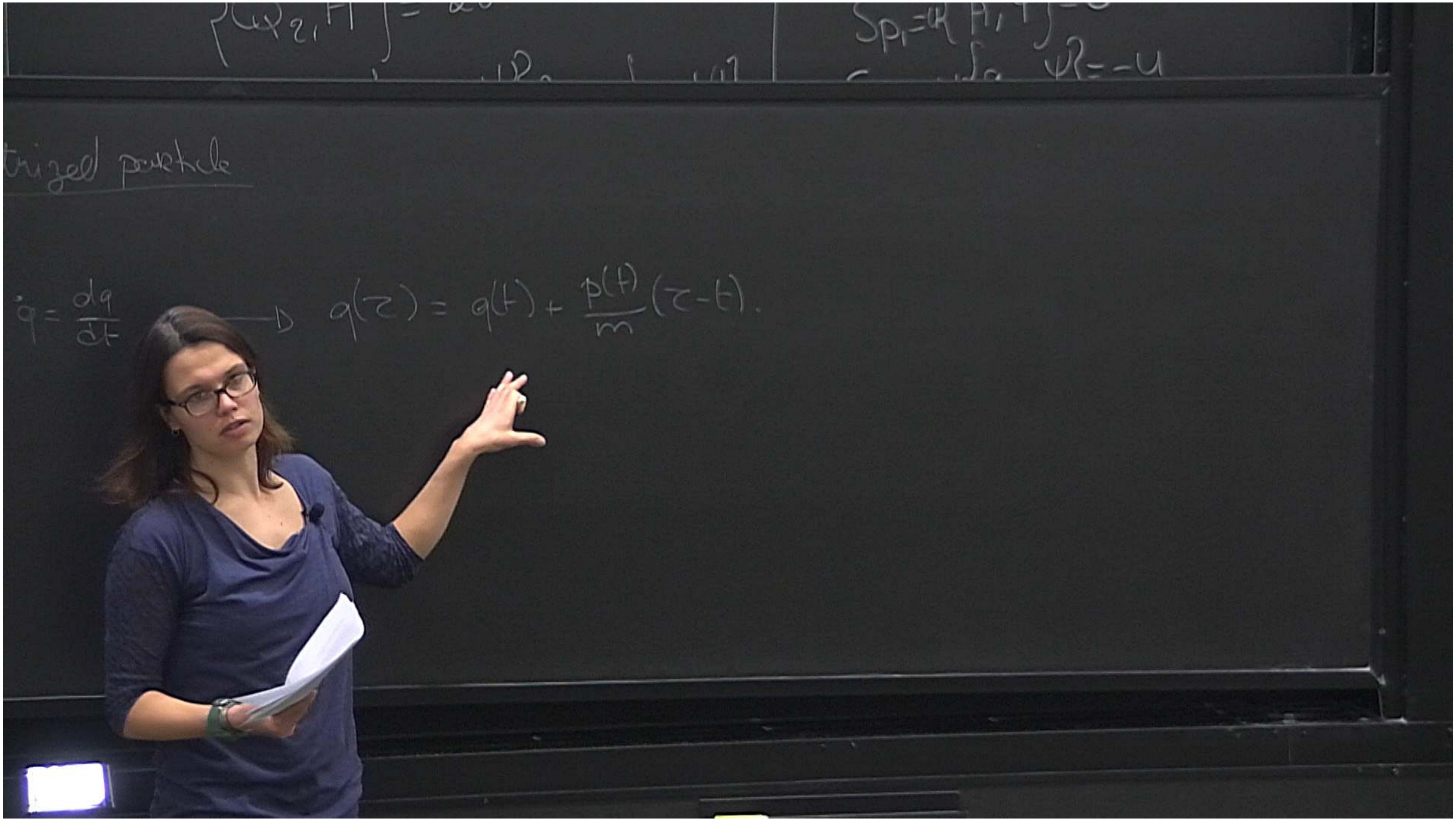


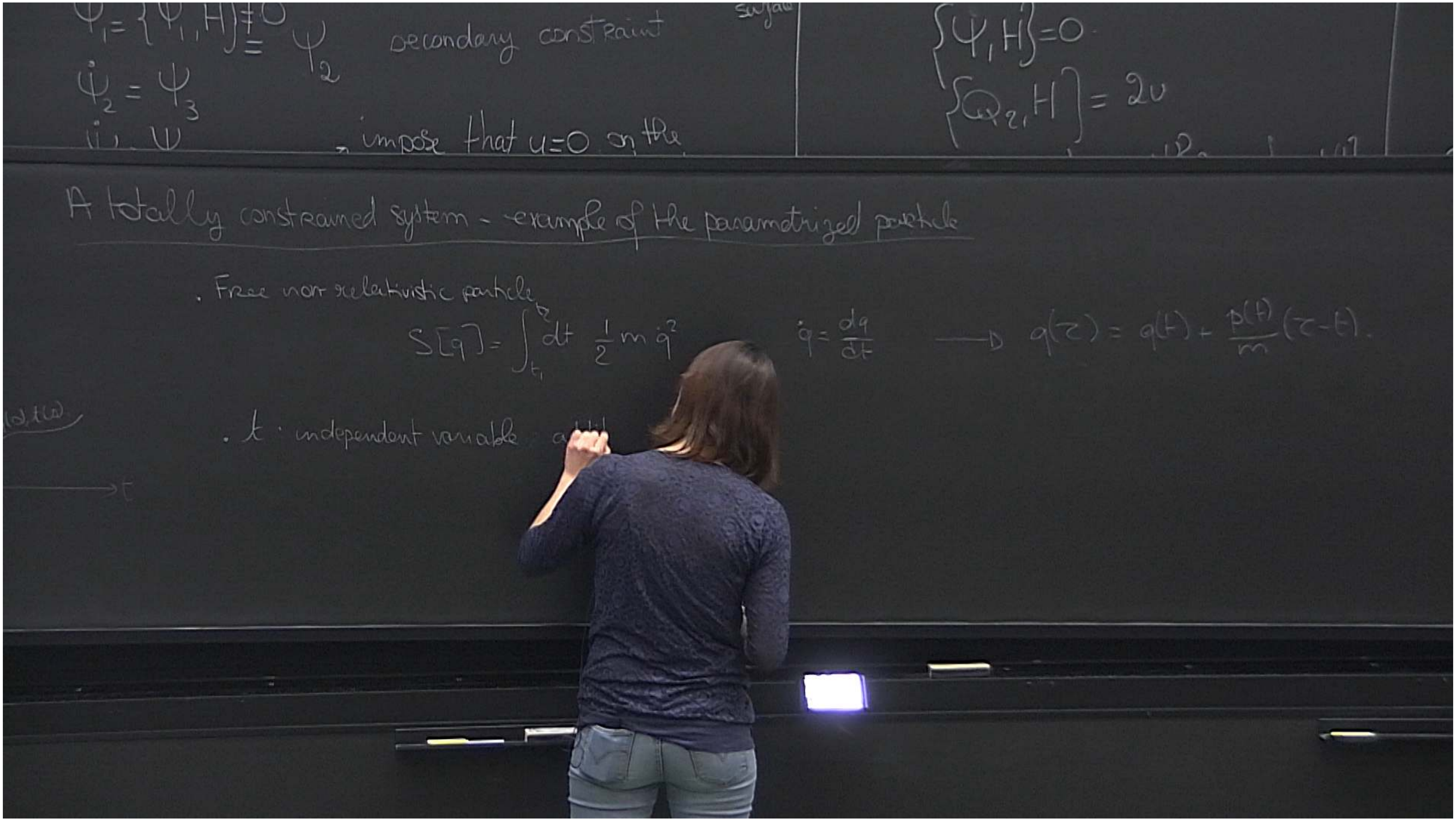
(i) Ψ impose that $u=0$ on the

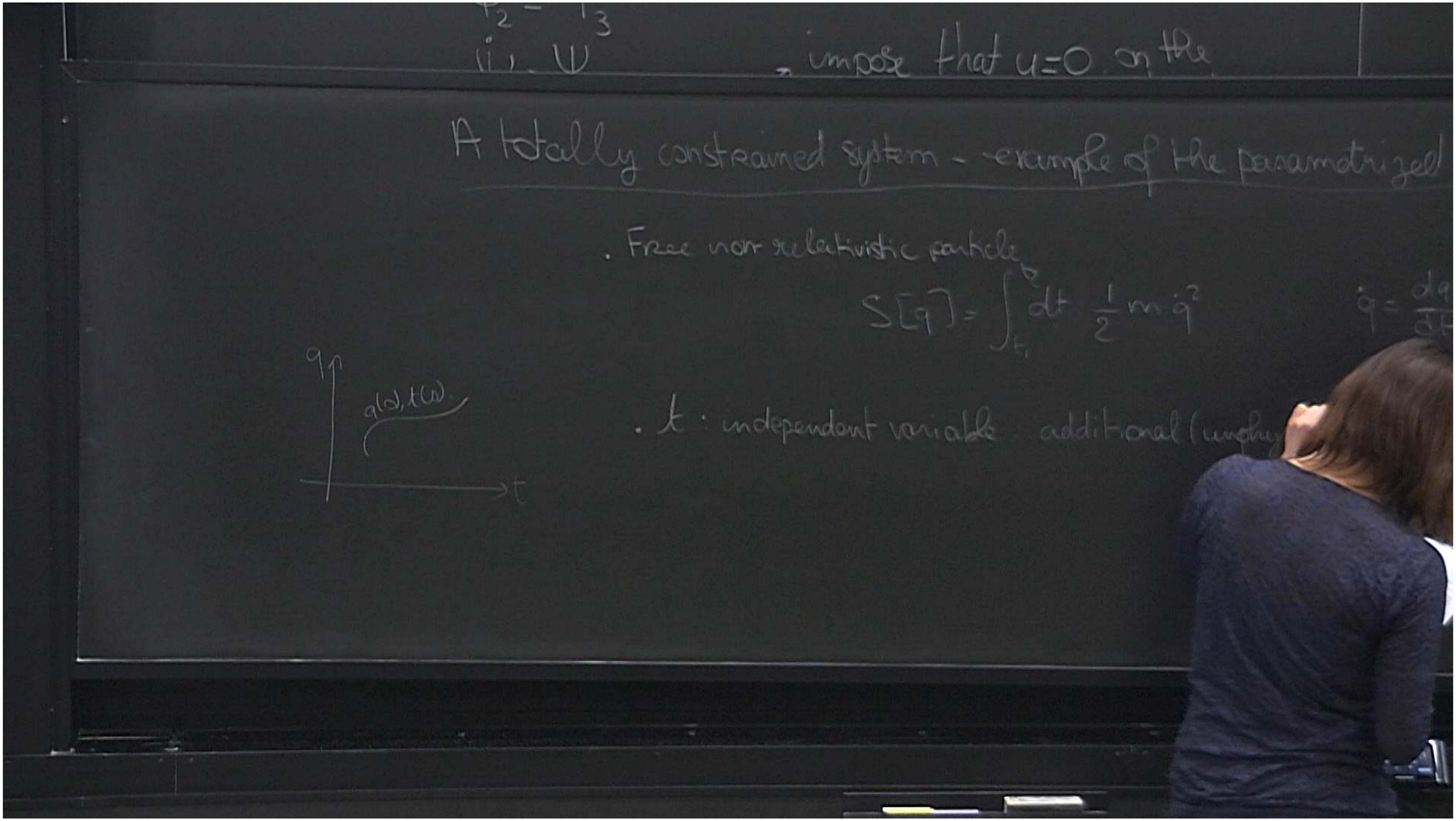
A totally constrained system - example of the parametrized particle

Free non relativistic particle

$$S[q] = \int_{t_1}^{t_2} dt \; \frac{1}{2} m \dot{q}^2 \qquad \dot{q} = \frac{dq}{dt} \longrightarrow q(\tau) = q$$


trized particle
$\dot{q} = \frac{dq}{dt}$
$q(\tau) = q(t) + \frac{p(t)}{m}(\tau - t).$


secondary constraint
impose that u=0 on the
{Ψ1, H} = 0
A totally constrained system - example of the parametrized particle
. Free non relativistic particle
S[q] = ∫ dt 1/2 m q̇²
q̇ = dq/dt
q(τ) = q(t) + p(t)/m (τ - t).
. t : independent variable



impose that u=0 on the
A totally constrained system - example of the parametrized
. Free non relativistic particle
$S[q] = \int_{t_1}^{t_2} dt \, \frac{1}{2} m \dot{q}^2$
q(λ), t(λ)
. t : independent variable : additional

$\{Q_2, H\} = 2u$

impose that $u=0$ on the

ly constrained system - example of the parametrized particle

. Free non relativistic particle

$$S[q] = \int_{t_1}^{t_2} dt \ \frac{1}{2} m \dot{q}^2 \qquad \dot{q} = \frac{dq}{dt} \longrightarrow q(\tau) = q(t) + \frac{p(t)}{m}(\tau - t).$$

. t : independent variable : additional (unphysical) parameter s.

$$S_p[q,t] = \int_{s'}^{s^2} \frac{1}{2} m \frac{q'^2}{t'} ds.$$

the prime denotes the derivative with respect to s

trized particle

$\dot{q} = \frac{dq}{dt}$ $\longrightarrow$ $q(\tau) = q(t) + \frac{p(t)}{m}(\tau - t)$.

) parameter s.

ds. The prime denotes the derivative with respect to s.

Exercise: Check that the new action is invariant under reparametrization $s \longrightarrow \tilde{s} = f(s)$

trized particle

$\dot{q} = \frac{dq}{dt} \longrightarrow q(\tau) = q(t) + \frac{p(t)}{m}(\tau - t).$

) parameter s.

ds

the prime denotes the derivative with respect to s

. Canonical analysis?

. Observables

$\{O, C\} = 0$

Exercise: Check that the new action is invariant under reparametrization

$s \longrightarrow \tilde{s} = f(s)$

$\{\psi_2, H\} = \ldots$ $\quad$ $\delta p_1 = \epsilon \{H, \ldots\}$

trized particle

$\dot{q} = \frac{dq}{dt}$ $\longrightarrow$ $q(\tau) = q(t) + \frac{p(t)}{m}(\tau - t)$.

) parameter s.

ds $\quad$ the prime denotes the derivative with respect to s

. Canonical analysis?

. Observables (Gauge fixing).

$\{O, C\} = 0 \rightarrow$ 2 observ

. Dirac program.

. Gravity

Exercise: Check that the new action is invariant under reparametrization

$s \longrightarrow \tilde{s} = f(s)$