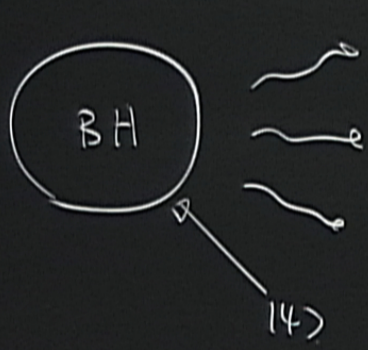
Title: IQC - Quantum Error Correction - Lecture 11

Date: Mar 22, 2018 02:00 PM

URL: http://pirsa.org/18030012

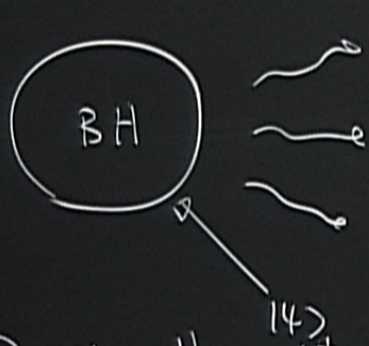
Abstract:

Black hole information problem & Scrambling



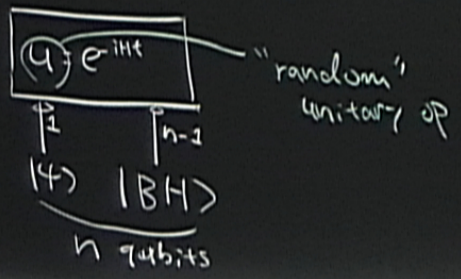
info is lost ?

Black hole information problem & Scrambling

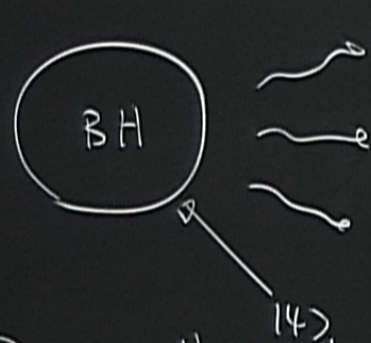


info is lost?

- Page's thought experiment.

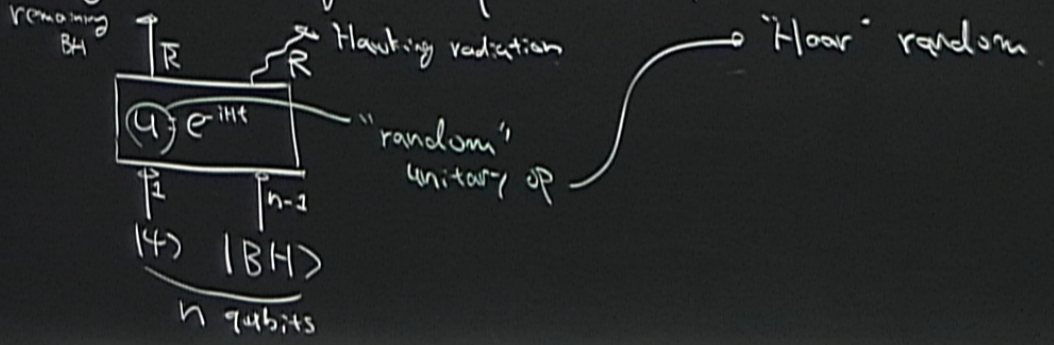


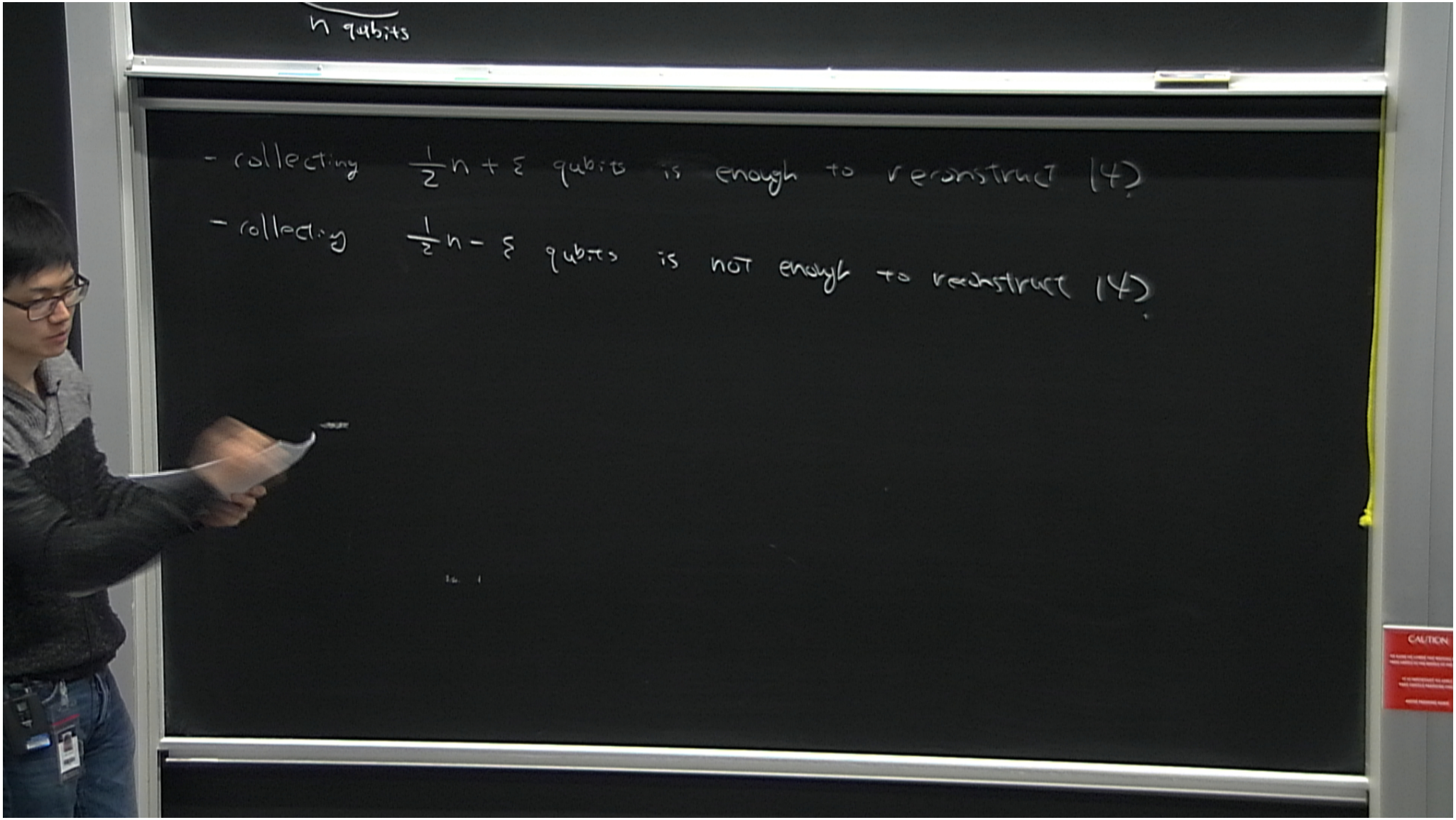
Black hole information problem & Scrambling

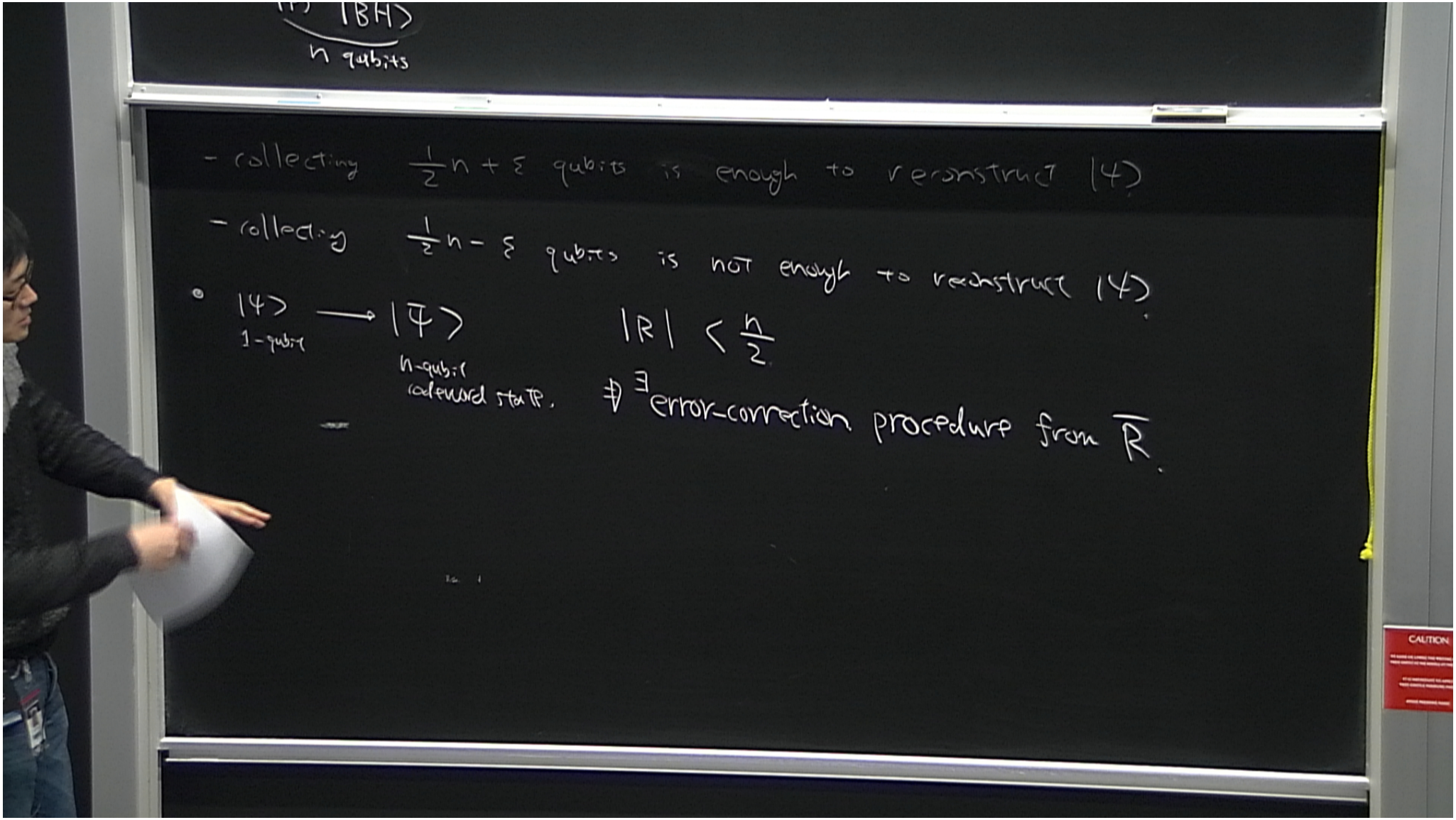


info is lost?

- Page's thought experiment.




n qubits
- collecting $\frac{1}{2}n + \varepsilon$ qubits is enough to reconstruct $|\psi\rangle$
- collecting $\frac{1}{2}n - \varepsilon$ qubits is not enough to reconstruct $|\psi\rangle$
CAUTION


$|\Phi\rangle$
n qubits
- collecting $\frac{1}{2}n + \varepsilon$ qubits is enough to reconstruct $|\Phi\rangle$
- collecting $\frac{1}{2}n - \varepsilon$ qubits is not enough to reconstruct $|\Phi\rangle$.
$|\psi\rangle$
1-qubit
$|\overline{\Psi}\rangle$
n-qubit
codeword state.
$|R| < \frac{n}{2}$
$\nexists$ error-correction procedure from $\overline{R}$.
CAUTION

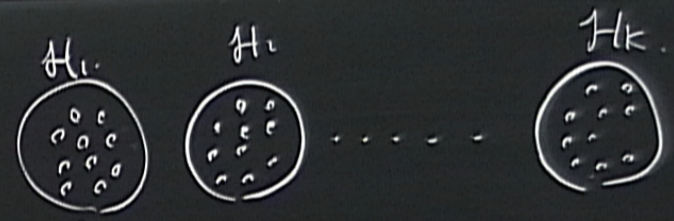
$|BH\rangle$
n qubits

- collecting $\frac{1}{2}n + \varepsilon$ qubits is enough to reconstruct $|\psi\rangle$
- collecting $\frac{1}{2}n - \varepsilon$ qubits is not enough to reconstruct $|\psi\rangle$.

- $|\psi\rangle \longrightarrow |\Psi\rangle$ $\quad |R| < \frac{n}{2}$
  1-qubit; n-qubit codeword state.
  $\nexists$ error-correction procedure from $\overline{R}$.
  — "Hawking radiation" R = loss of qubits.

Schur–Weyl duality

- $\mathcal{H} = (\mathbb{C})^{\otimes d}$ $\qquad \mathcal{H}^{\otimes k} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{k}$



- Permutation ops π $\quad (1,2,\cdots k) \to (\pi(1), \pi(2), \cdots \pi(k))$

$1 \to 2$, $2 \to 1$, $3 \to 3$, $\cdots$, $k \to k$

$1 \to 2$, $2 \to 3$, $\cdots$, $k-1 \to k$, $k \to 1$

Cyclic permutation

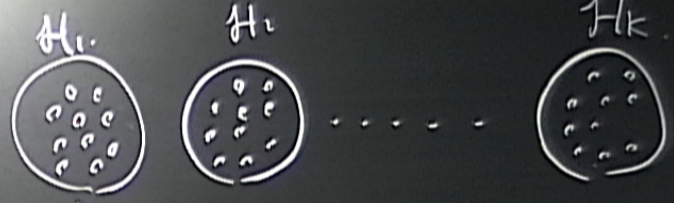
$$W_\pi |a_1, \cdots, a_k\rangle = |a_{\pi(1)}, a_{\pi(2)}, \cdots, a_{\pi(k)}\rangle$$

$a_j = 0, \cdots, d-1$

$|a_j\rangle \in \mathcal{H}_j$

Schur–Weyl duality

- $\mathcal{H} = (\mathbb{C})^{\otimes d}$. $\quad \mathcal{H}^{\otimes k} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{k}$



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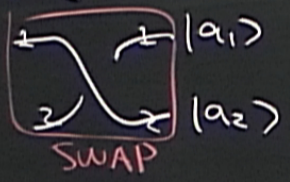
- $(k=2)$ SWAP $\pi: (1,2) \to (2,1)$

$\mathcal{H}_1 \otimes \mathcal{H}_2$

$|a_1\rangle$

$|a_2\rangle$

$|a_1, a_2\rangle$



$|a_1\rangle$

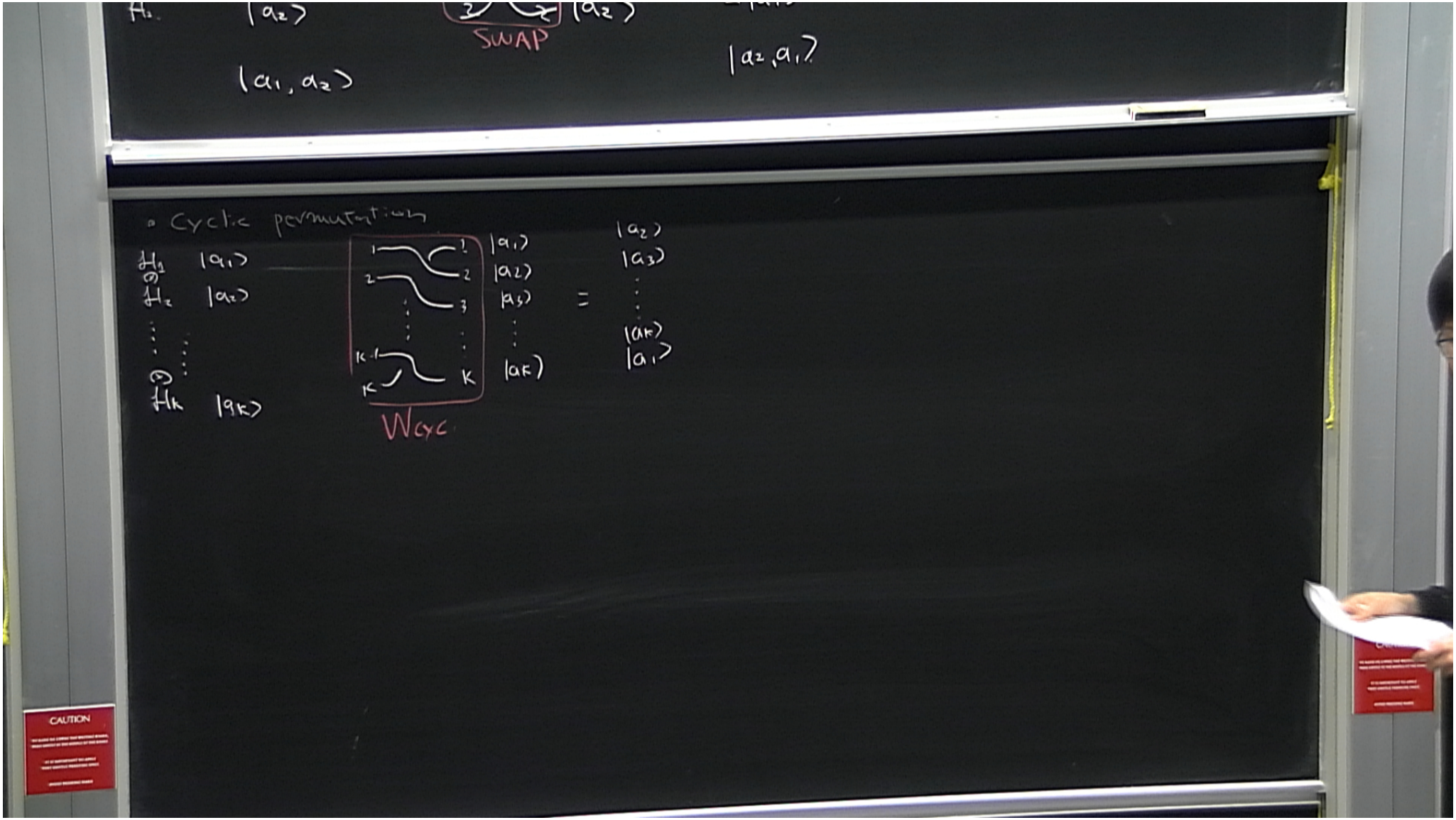
$|a_2\rangle$

$=$

$-|a_2\rangle$

$-|a_1\rangle$

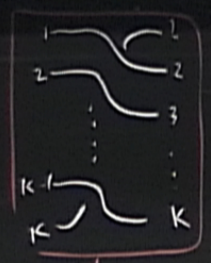
$|a_2, a_1\rangle$


SWAP
$|a_1, a_2\rangle$
$|a_2, a_1\rangle$
• Cyclic permutation
H_1 $|a_1\rangle$
H_2 $|a_2\rangle$
H_K $|a_K\rangle$
$|a_1\rangle$ $|a_2\rangle$ $|a_3\rangle$ $|a_K\rangle$
=
$|a_2\rangle$ $|a_3\rangle$ $|a_K\rangle$ $|a_1\rangle$
Wcyc
CAUTION

(a_1, a_2)

- Cyclic permutation

$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_K$ $|a_1\rangle |a_2\rangle \cdots |a_K\rangle$



$|a_1\rangle |a_2\rangle |a_3\rangle \cdots |a_K\rangle$ = $|a_2\rangle |a_3\rangle \cdots |a_K\rangle |a_1\rangle$

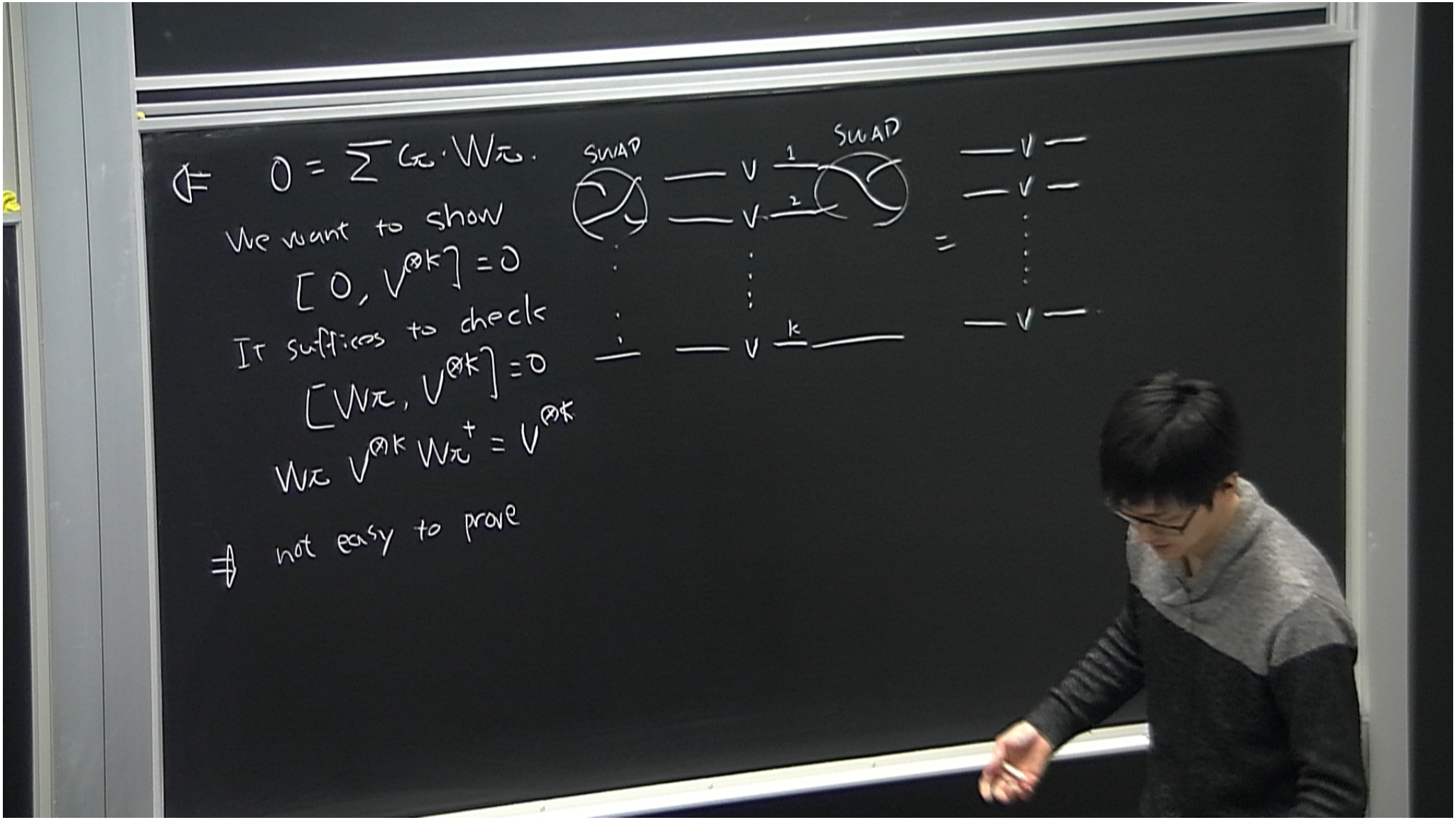
Theorem.

O : operator acting on $\mathcal{H}^{\otimes k}$

- $[O, V^{\otimes k}] = 0$ for all unitary ops V. (V acting on $\mathcal{H}$)

$\Leftrightarrow O = \sum_{\pi} c_\pi W_\pi$

permutation ops


$\mathbb{C}$ $O = \sum G_\pi \cdot W_\pi \cdot$
SWAP
SWAP
V
V
V
1
2
k
=
V
V
V
We want to show
$[O, V^{\otimes k}] = 0$
It suffices to check
$[W_\pi, V^{\otimes k}] = 0$
$W_\pi V^{\otimes k} W_\pi^\dagger = V^{\otimes k}$
$\Rightarrow$ not easy to prove

$\Leftarrow$ $O = \sum_\pi c_\pi \cdot W_\pi$.

We want to show

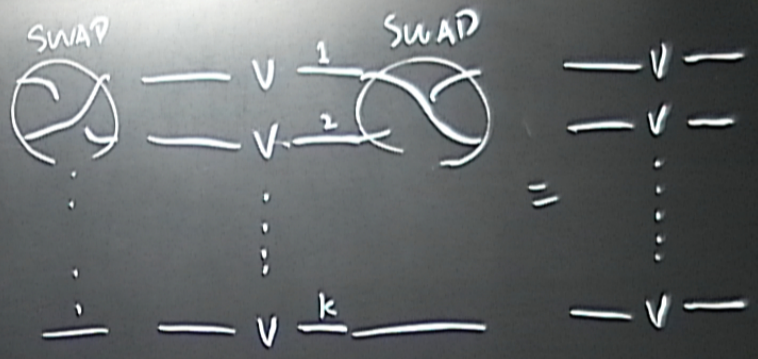
$$[O, V^{\otimes k}] = 0$$

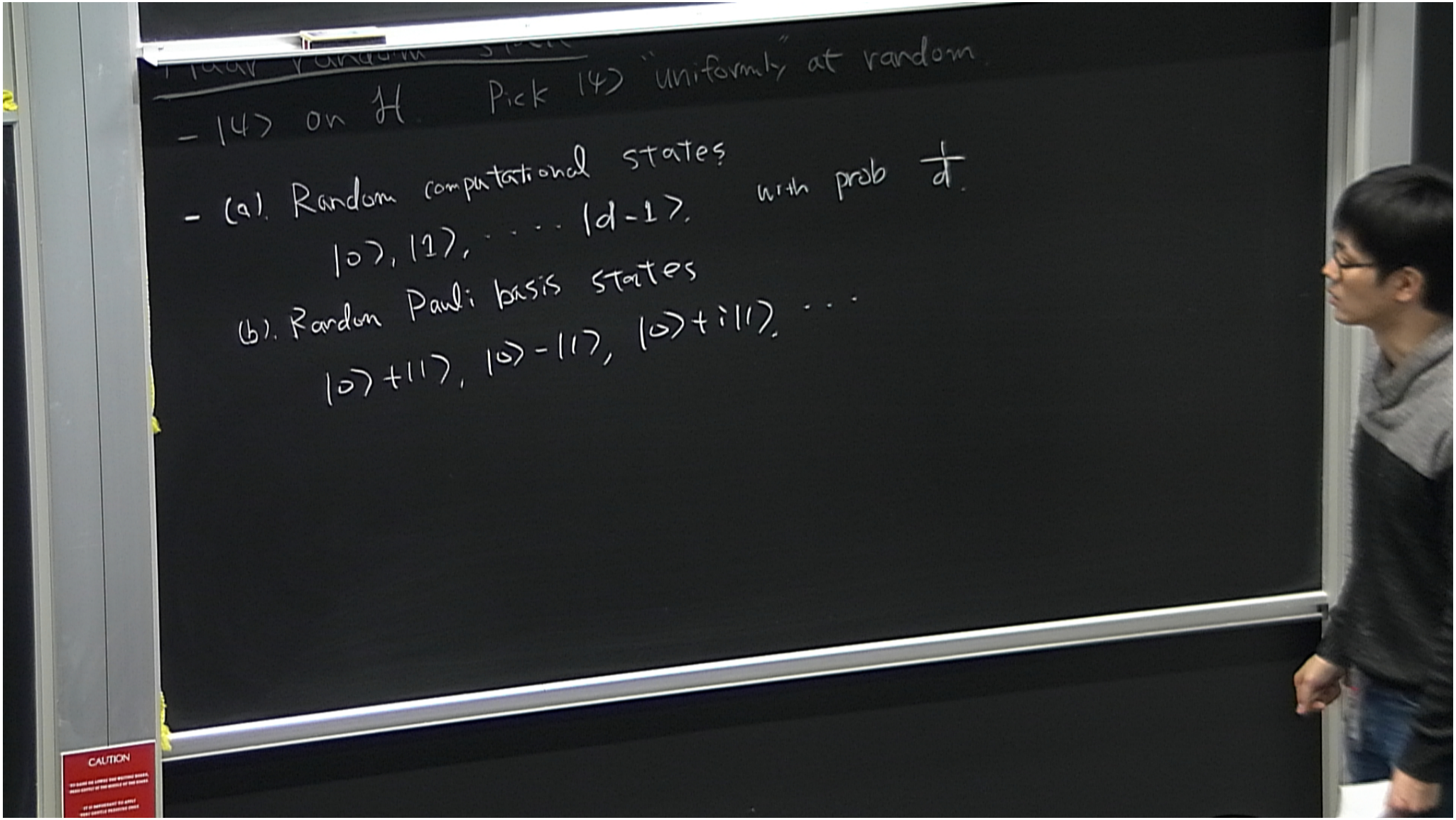
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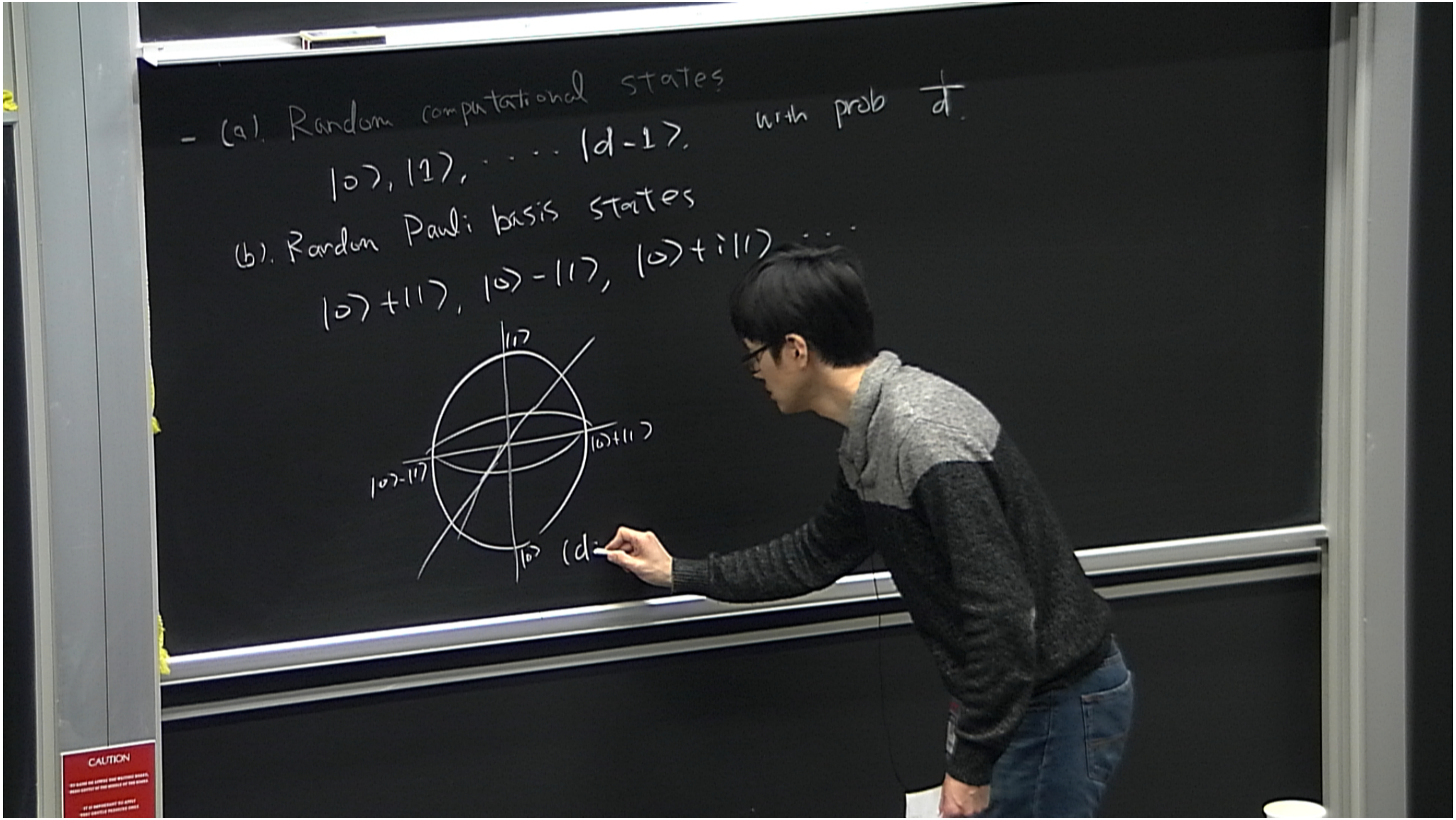
$$[W_\pi, V^{\otimes k}] = 0$$

$$W_\pi V^{\otimes k} W_\pi^\dagger = V^{\otimes k}$$

$\Rightarrow$ not easy to prove (Von Neumann's double commutant theorem)




- |4> on H. Pick |4> "uniformly" at random.
- (a). Random computational states
|0>, |1>, ... |d-1>. with prob 1/d.
(b). Random Pauli basis states
|0>+|1>, |0>-|1>, |0>+i|1>, ...
CAUTION

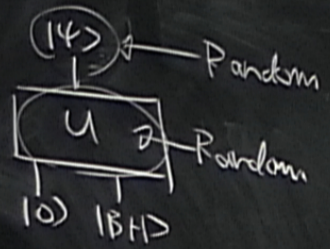

- (a). Random computational states
|0⟩, |1⟩, ... |d-1⟩. with prob 1/d.
(b). Random Pauli basis states
|0⟩+|1⟩, |0⟩-|1⟩, |0⟩+i|1⟩ ...
|1⟩
|0⟩+|1⟩
|0⟩-|1⟩
|0⟩
(d=
CAUTION

$|\psi\rangle$ $|BH\rangle$

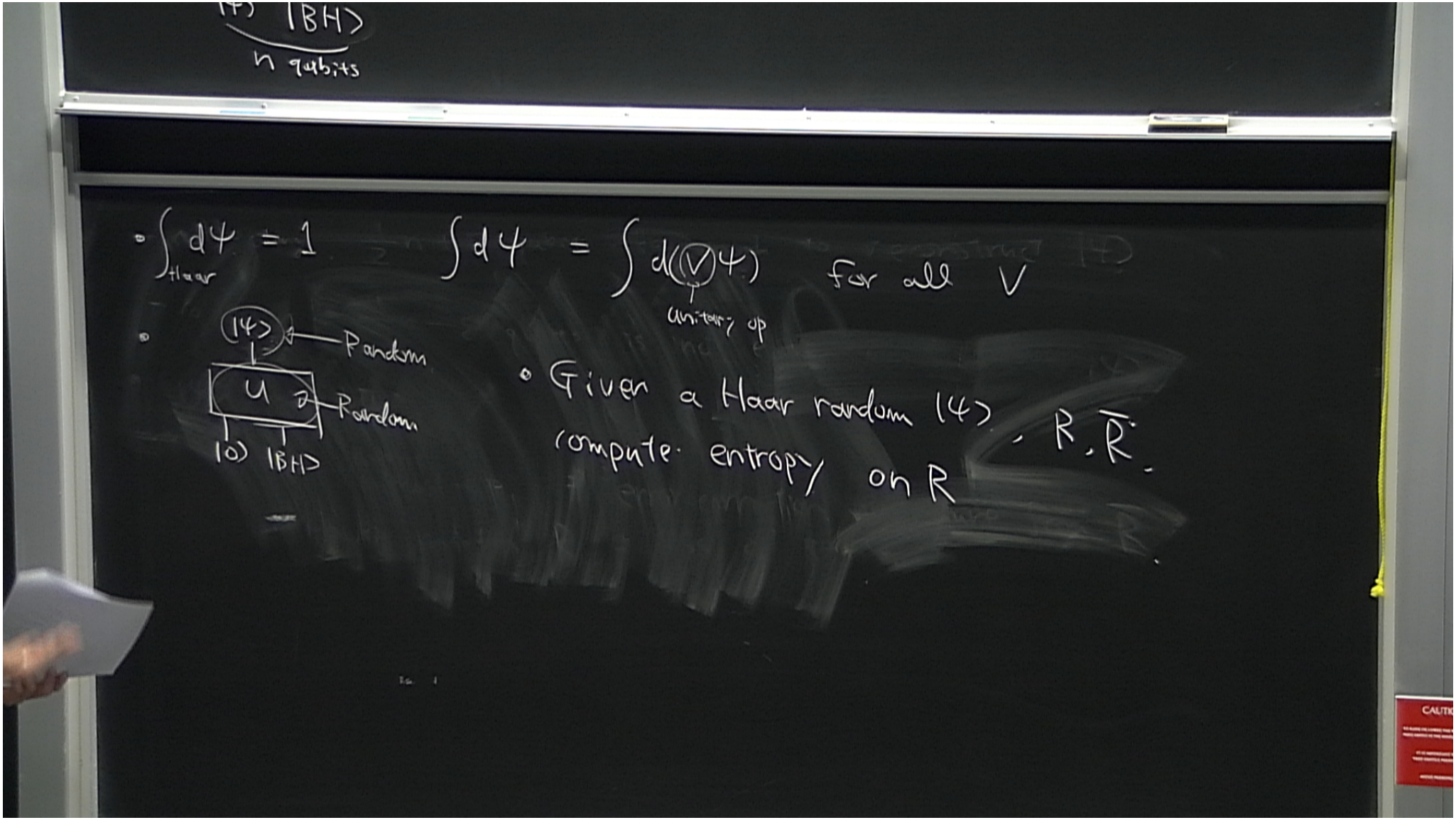
n qubits

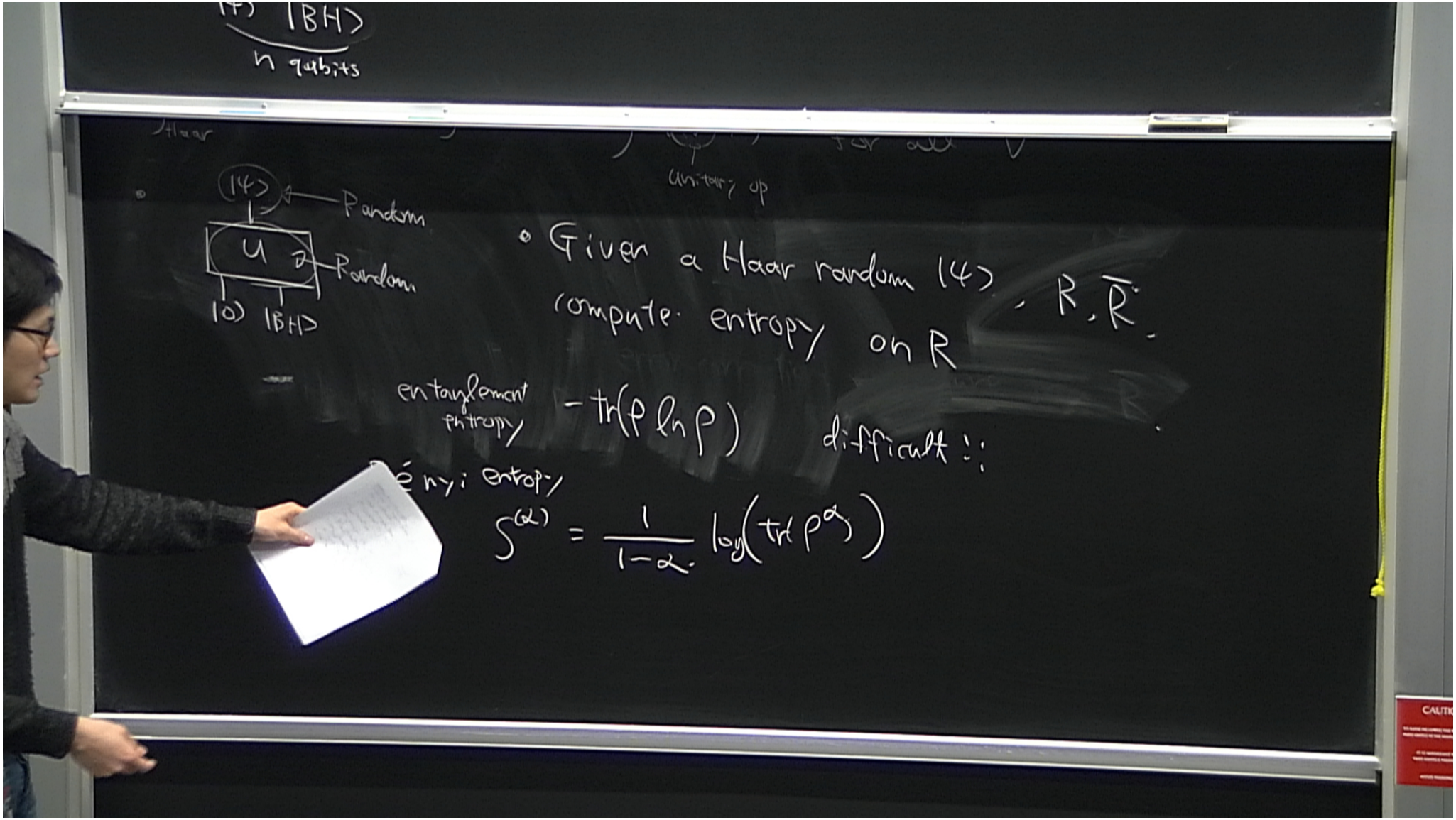
- $\int_{Haar} d\psi = 1$ $\int d\psi = \int d(V\psi)$ for all V

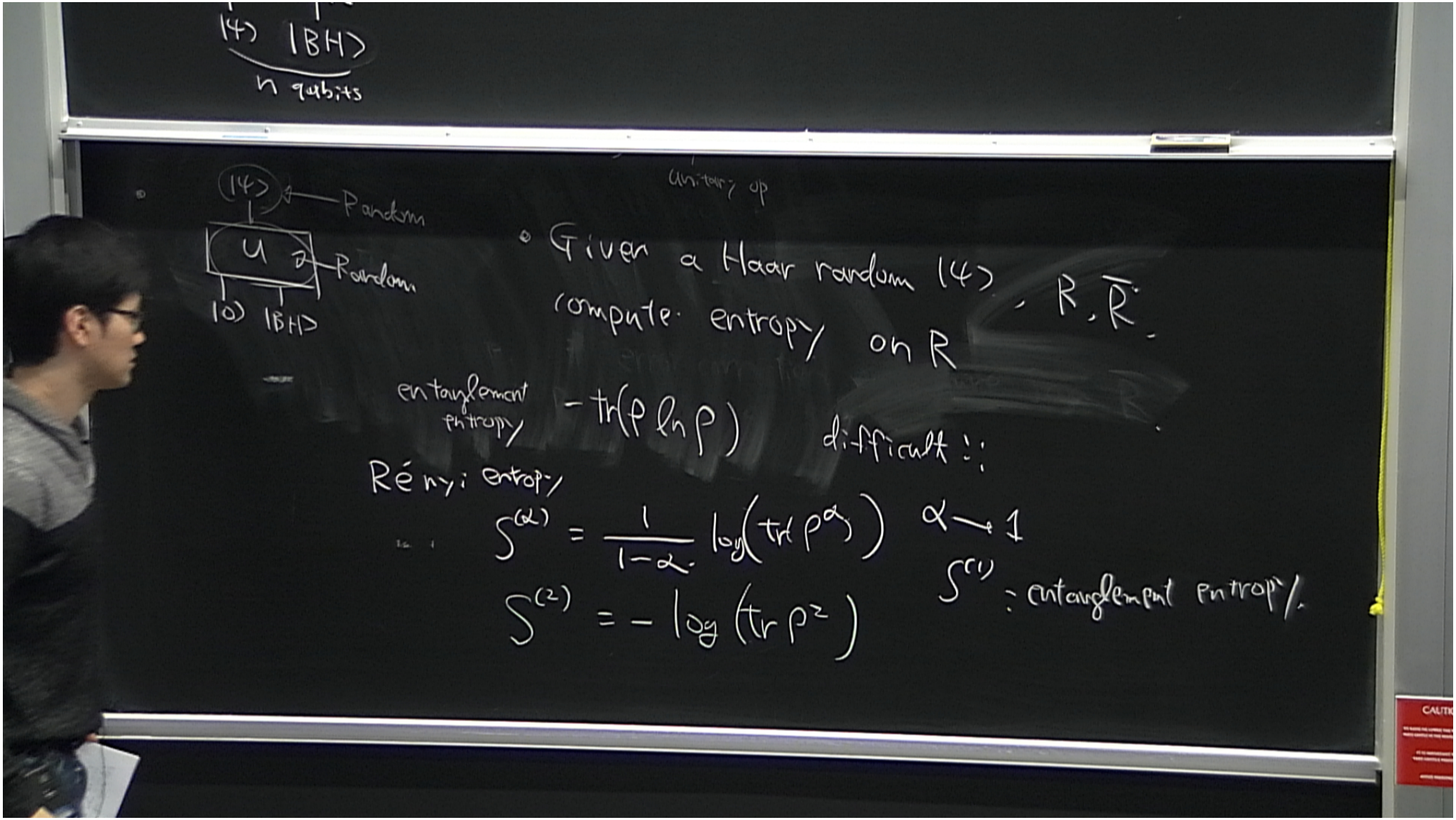
unitary op

- 

- Given a Haar random $|\psi\rangle$ - $R, \bar{R}$.
compute entropy on R

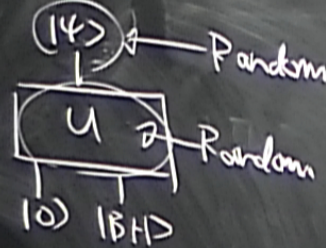

|BH⟩
n qubits
• $\int_{Haar} d\psi = 1$
$\int d\psi = \int d((V)\psi)$ for all V
unitary op
|ψ⟩
Random
U
Random
|0⟩ |BH⟩
• Given a Haar random |ψ⟩ - R, $\bar{R}$.
compute entropy on R


|BH⟩
n qubits
Haar
|ψ⟩
Random
U
Random
|0⟩
|BH⟩
unitary op
• Given a Haar random |ψ⟩ - R, R̄.
compute entropy on R
entanglement entropy
- tr(ρ ln ρ)
difficult!!
Rényi entropy
$S^{(\alpha)} = \frac{1}{1-\alpha} \log(\mathrm{tr}\,\rho^{\alpha})$


|ψ⟩ |BH⟩
n qubits
|ψ⟩
Random
U
Random
|0⟩ |BH⟩
unitary op
• Given a Haar random |ψ⟩ - R, R̄.
compute entropy on R
entanglement entropy
$-\mathrm{tr}(\rho \ln \rho)$ difficult!!
Rényi entropy
$S^{(\alpha)} = \frac{1}{1-\alpha} \log(\mathrm{tr}\,\rho^{\alpha})$ $\alpha \to 1$
$S^{(1)}$: entanglement entropy.
$S^{(2)} = -\log(\mathrm{tr}\,\rho^{2})$

$\int_{Haar} d\psi = 1$ $\int d\psi = \int d(V\psi)$ for all V

unitary op



- Given a Haar random $|\psi\rangle$. $R, \bar{R}$.
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entanglement entropy $-\mathrm{tr}(\rho \ln \rho)$ difficult!!

Rényi entropy

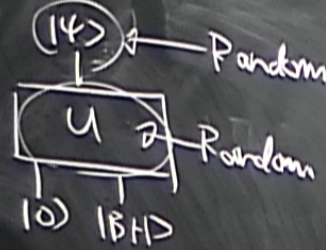
$$S^{(\alpha)} = \frac{1}{1-\alpha} \log(\mathrm{tr}\,\rho^\alpha) \quad \alpha \to 1$$

$S^{(1)}$ = entanglement entropy.

$$S^{(2)} = -\log(\mathrm{tr}\,\rho^2)$$

$\int_{\text{Haar}} d\psi = 1$ $\int d\psi = \int d(V\psi)$ for all V

unitary op



- Given a Haar random $|\psi\rangle$. R . $\bar{R}$.

compute entropy on R

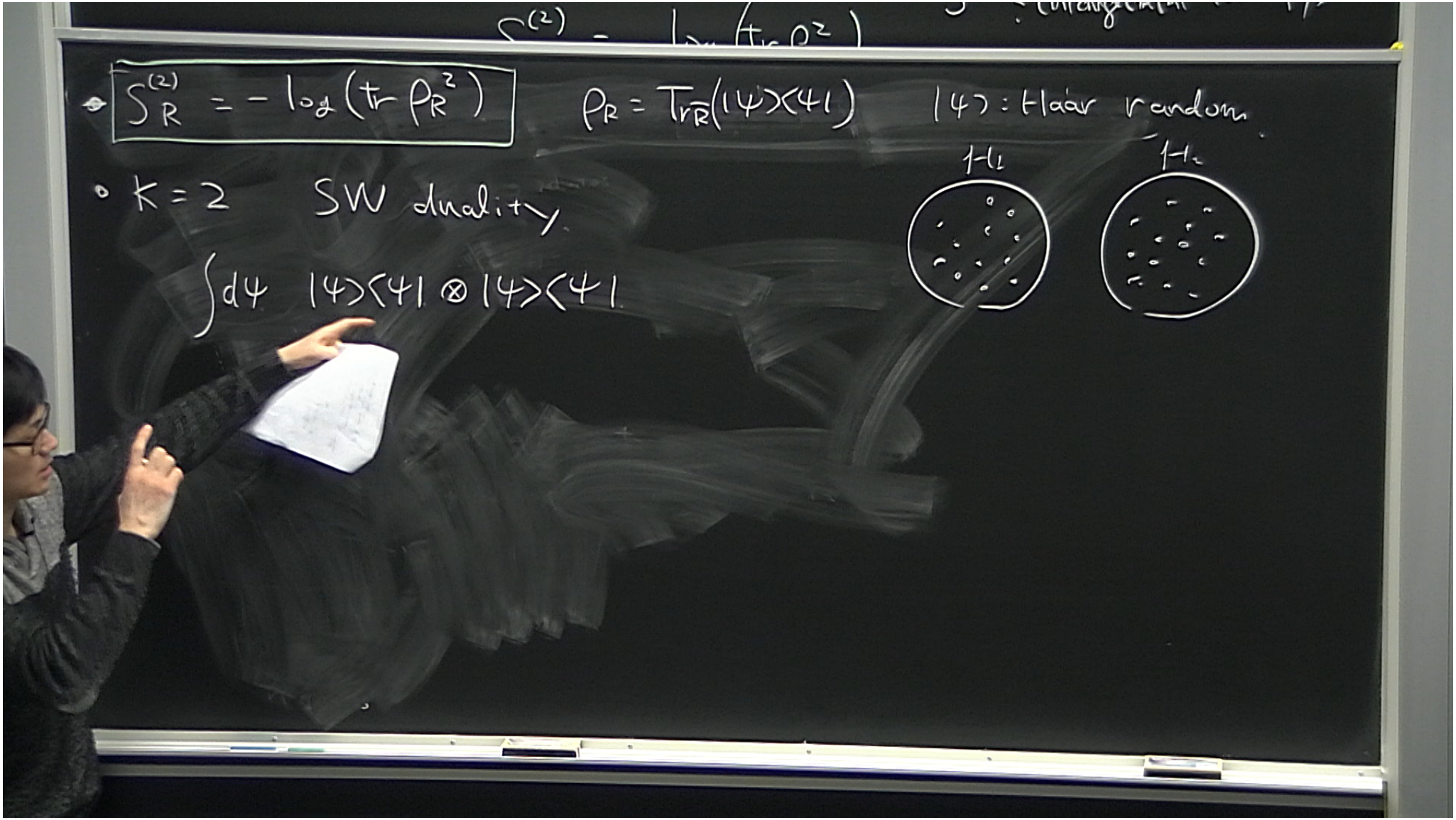
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Rényi entropy

$$S^{(\alpha)} = \frac{1}{1-\alpha} \log(\mathrm{tr}\,\rho^{\alpha}) \qquad \alpha \to 1$$

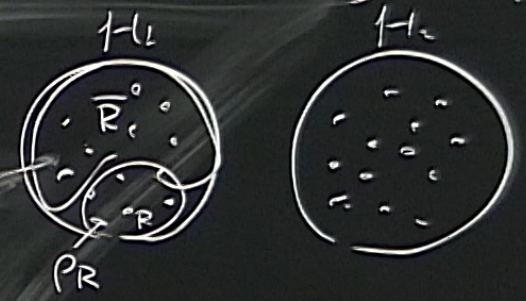
$S^{(1)}$: entanglement entropy.

$$S^{(2)} = -\log(\mathrm{tr}\,\rho^2)$$


$S_R^{(2)} = -\log(\mathrm{tr}\,\rho_R^2)$
$\rho_R = \mathrm{Tr}_{\bar{R}}(|\psi\rangle\langle\psi|)$
$|\psi\rangle$: Haar random
$k=2$ SW duality
$\int d\psi\ |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|$
$\mathcal{H}_1$
$\mathcal{H}_2$

$$S^{(2)} = -\log(\mathrm{tr}\,\rho^2)$$

- $\boxed{S_R^{(2)} = -\log(\mathrm{tr}\,\rho_R^2)}$ $\quad \rho_R = \mathrm{Tr}_{\bar{R}}(|\psi\rangle\langle\psi|)$ $\quad |\psi\rangle$: Haar random.



- $k=2$ SW duality.

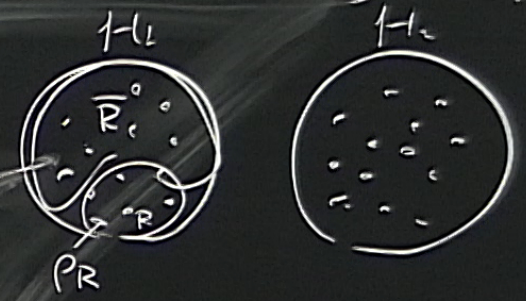
$$f \equiv \int d\psi \; |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|$$

(a) $[V, f] = 0$. ⟸ def of Haar measure.

$$= \int d\psi \; V|\psi\rangle\langle\psi|V^\dagger \otimes V|\psi\rangle\langle\psi|V^\dagger$$

$$S^{(2)} = -\log(\mathrm{tr}\,\rho^2)$$

- $\boxed{S_R^{(2)} = -\log(\mathrm{tr}\,\rho_R^2)}$ $\quad \rho_R = \mathrm{Tr}_{\bar{R}}(|\psi\rangle\langle\psi|)$ $\quad |\psi\rangle$: Haar random.



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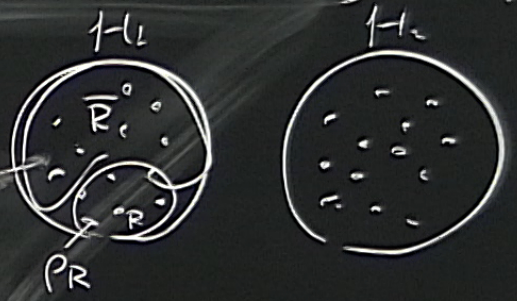
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- $k = 2$ SW duality.

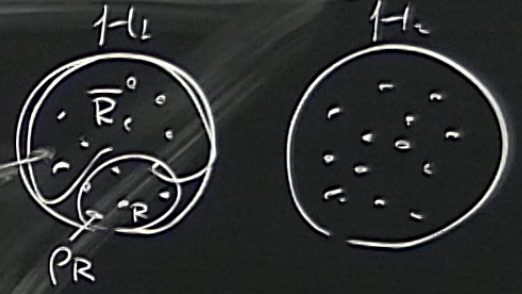
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- $\boxed{S_R^{(2)} = -\log(\mathrm{tr}\,\rho_R^2)}$ $\quad \rho_R = \mathrm{Tr}_{\bar{R}}(|\psi\rangle\langle\psi|)$ $\quad |\psi\rangle$: Haar random.



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(a) $[f, V \otimes V] = 0$ $\Leftarrow$ def of Haar measure.

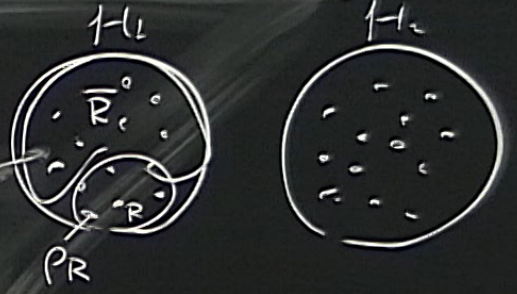
$$f \equiv \int d\psi \; V|\psi\rangle\langle\psi|V^\dagger \otimes V|\psi\rangle\langle\psi|V^\dagger$$

(b) SW duality

$$f = \sum_\pi C_\pi \cdot W_\pi$$

$S = -\log(\mathrm{tr}\,\rho^2)$

- $\boxed{S_R^{(2)} = -\log(\mathrm{tr}\,\rho_R^2)}$ $\quad \rho_R = \mathrm{Tr}_{\bar{R}}(|\psi\rangle\langle\psi|)$ $\quad |\psi\rangle$: Haar random.



- $k = 2$ SW duality.

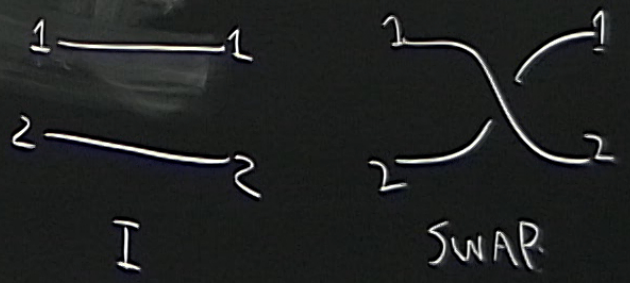
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(b) SW duality

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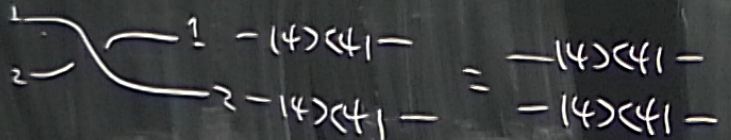


$$\int d\psi \; |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| = C_I \cdot I + C_{SWAP} \cdot SWAP$$

compute C_I, C_{SWAP}

(c). $C_I = C_{SWAP}$.

$SWAP \cdot f = f$



RHS $= C_I \cdot SWAP + C_{SWAP} \cdot I$

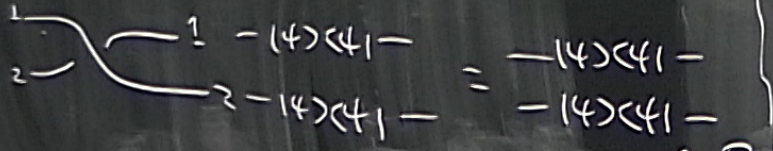
$\Leftrightarrow C_I = C_{SWAP}$.

$$\int d\psi \; |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| = C_I \cdot I + C_{SWAP} \cdot SWAP$$

compute C_I, C_{SWAP}

(c). $C_I = C_{SWAP}$.

$SWAP \cdot f = f$



RHS $= C_I \cdot SWAP + C_{SWAP} \cdot I$

$\Leftrightarrow C_I = C_{SWAP}$.

(d). Take Trace of LHS & RHS.

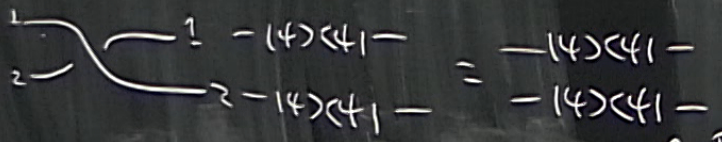
$tr(LHS)$

$$\int d\psi \; |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| = C_I \cdot I + C_{SWAP} \cdot SWAP \qquad C = \frac{1}{d(d+1)}$$

compute C_I, C_{SWAP}

(c). $C_I = C_{SWAP}$.

$SWAP \cdot f = f$



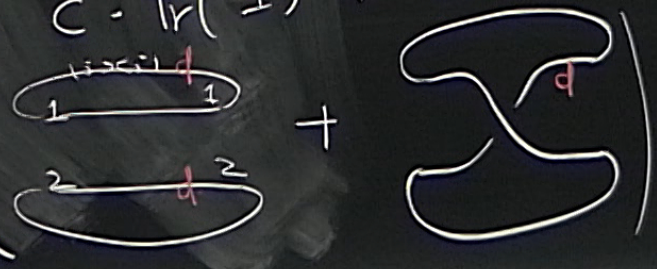
RHS $= C_I \cdot SWAP + C_{SWAP} \cdot I$

$\Leftrightarrow C_I = C_{SWAP}$.

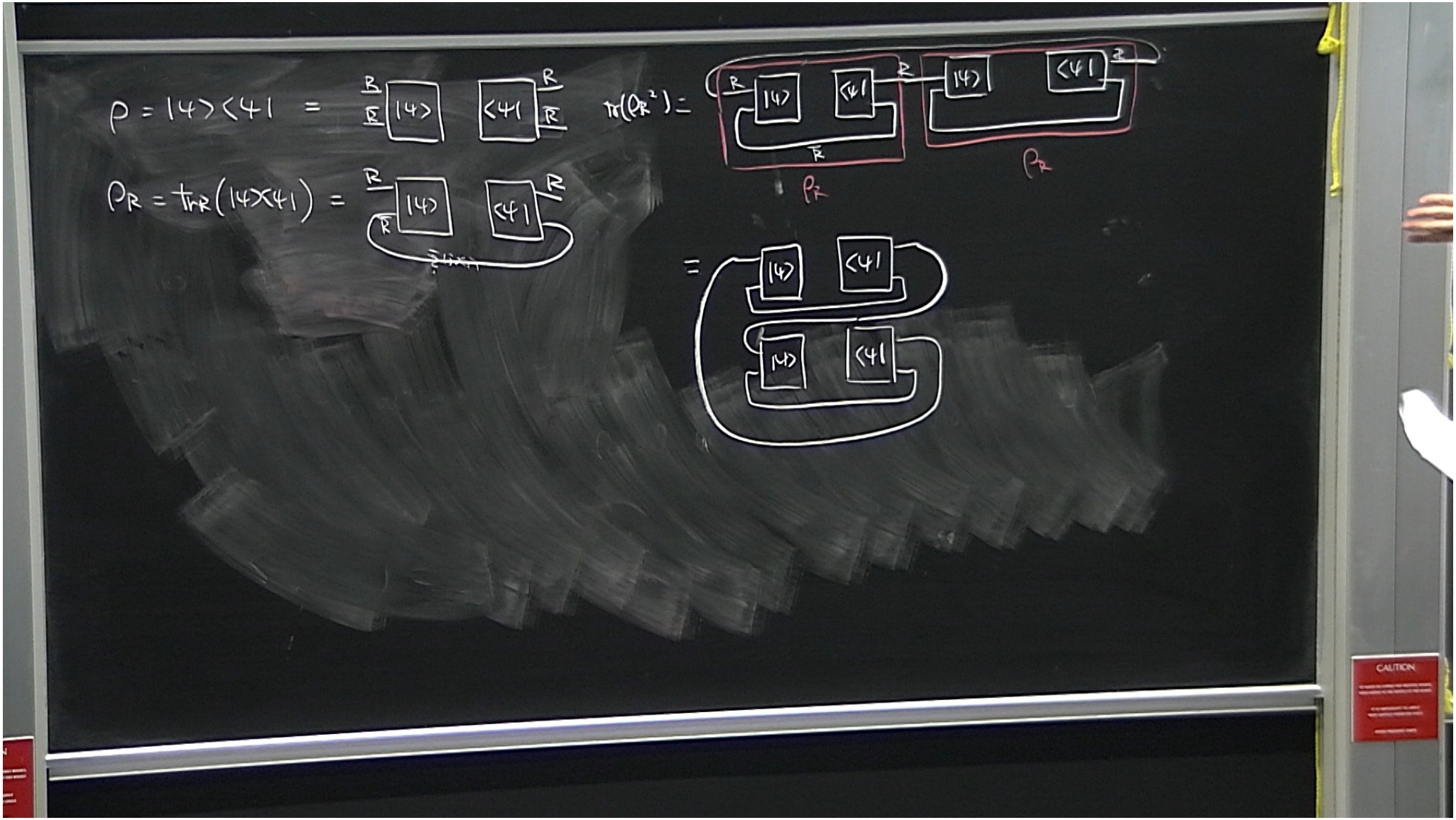
(d). Take Trace of LHS & RHS.

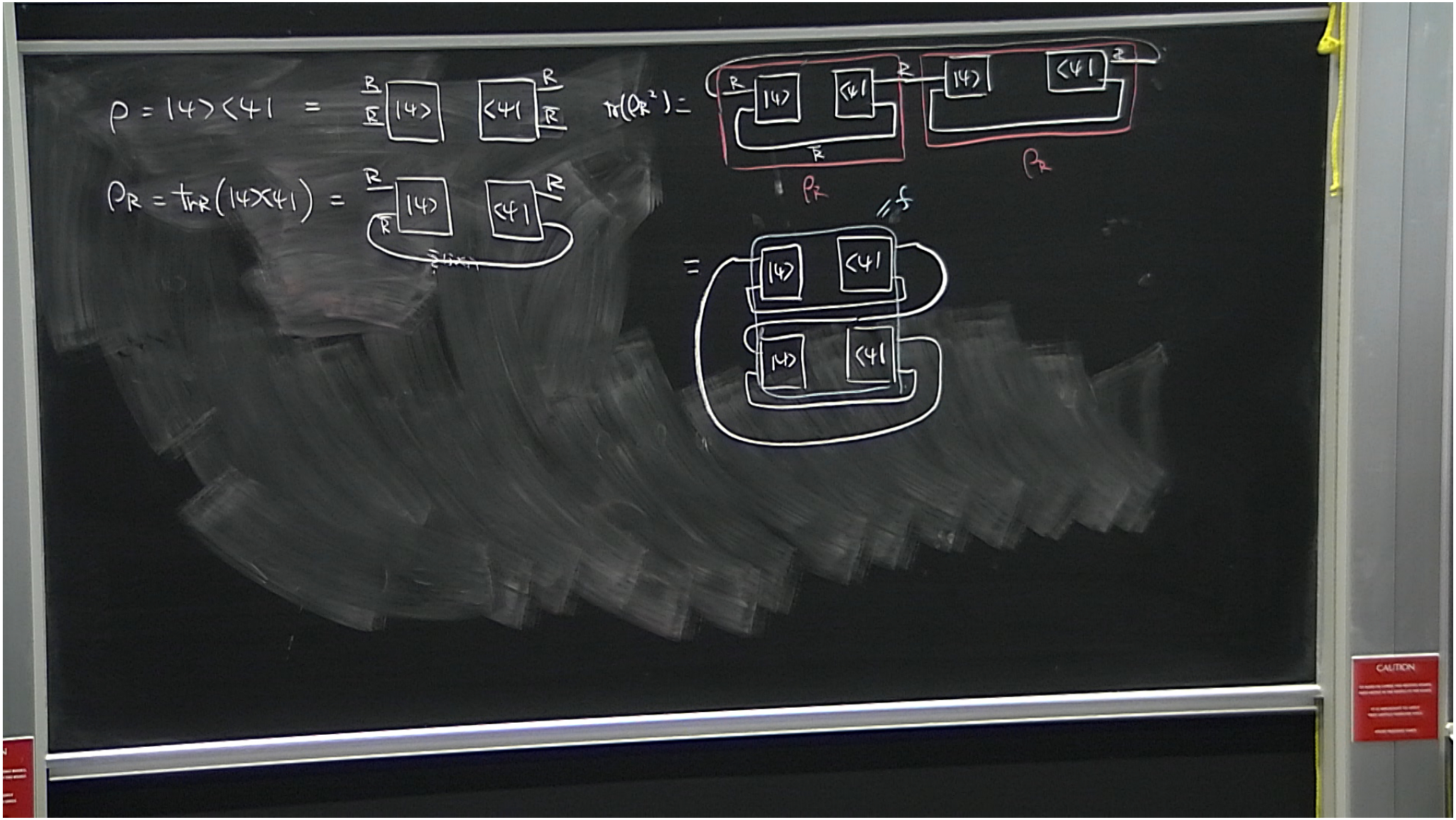
$tr(LHS) = 1$

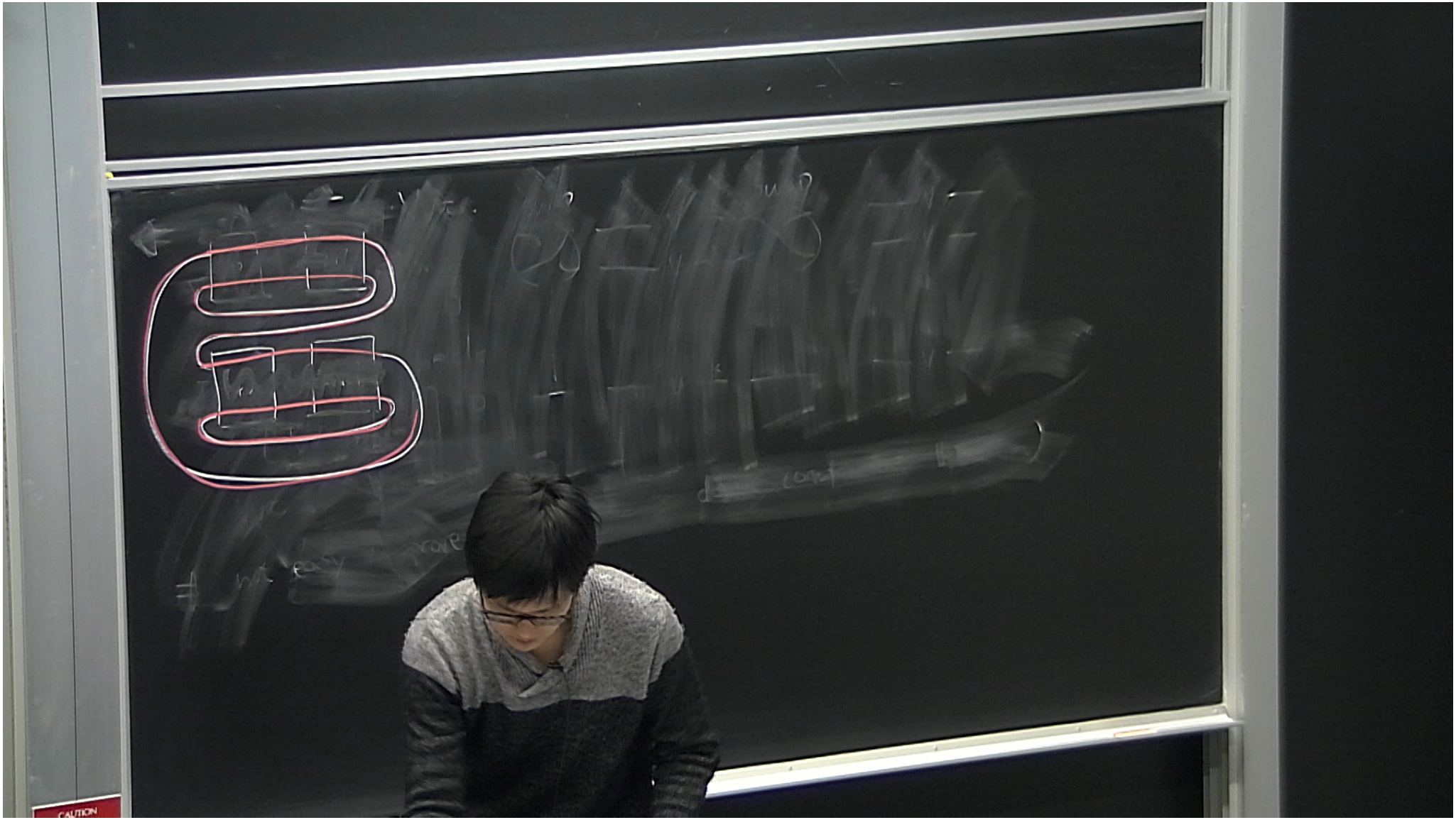
$tr(RHS) = C \cdot tr(I) + C \cdot Tr(SWAP)$

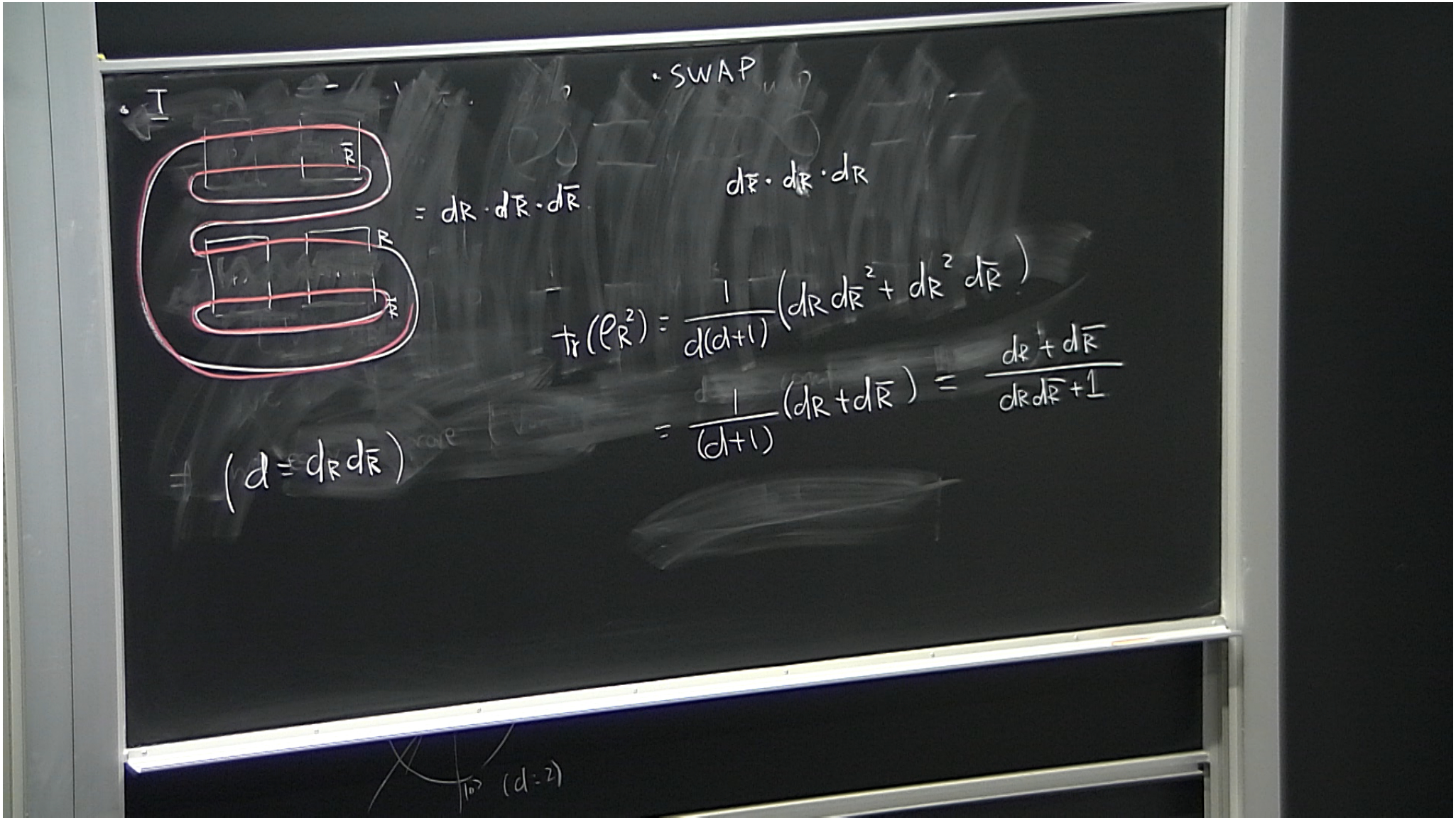
$= C \cdot$ 

$= c(d^2+d)$

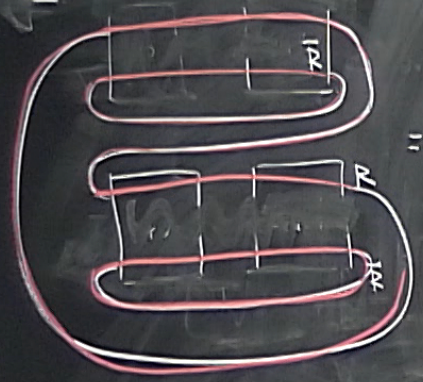

ρ = |ψ⟩⟨ψ| =
|ψ⟩
⟨ψ|
$\rho_R = \mathrm{tr}_{\bar{R}}(|\psi\rangle\langle\psi|) =$
$\mathrm{tr}(\rho_R^2) =$
ρ_R
ρ_R
=


ρ = |4><4| =
ρ_R = tr_R(|4><4|) =
tr(ρ_R²) =
|4>
<4|
R
R̄
ρ_R
=
=




I
SWAP
$\bar{R}$
R
$\bar{R}$
$= d_R \cdot d_{\bar{R}} \cdot d_{\bar{R}}$
$d_{\bar{R}} \cdot d_R \cdot d_R$
$\mathrm{Tr}(\rho_R^2) = \frac{1}{d(d+1)}\left(d_R d_{\bar{R}}^2 + d_R^2 d_{\bar{R}}\right)$
$= \frac{1}{(d+1)}(d_R + d_{\bar{R}}) = \frac{d_R + d_{\bar{R}}}{d_R d_{\bar{R}} + 1}$
$= \left(d = d_R d_{\bar{R}}\right)$
(d=2)

• I



$= d_R \cdot d_{\bar{R}} \cdot d_{\bar{R}}$

• SWAP

$d_{\bar{R}} \cdot d_R \cdot d_R$

$$\int d\psi \, \mathrm{tr}(\rho_R^2) = \frac{1}{d(d+1)}\left(d_R d_{\bar{R}}^2 + d_R^2 d_{\bar{R}}\right)$$

$$= \frac{1}{(d+1)}\left(d_R + d_{\bar{R}}\right) = \frac{d_R + d_{\bar{R}}}{d_R d_{\bar{R}} + 1}$$

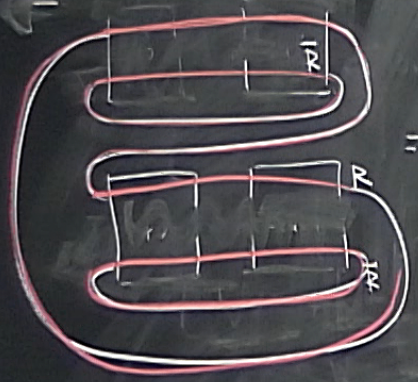
$\left(d = d_R d_{\bar{R}}\right)$

• $d_R \ll d_{\bar{R}}$ $\int d\psi \, \mathrm{tr}(\rho_R^2) \approx \frac{d_{\bar{R}}}{d_R d_{\bar{R}}} = \frac{1}{d_R}$

(b). Random Pauli basis states

$|0\rangle + i|1\rangle$

• I



$= d_R \cdot d_{\bar{R}} \cdot d_{\bar{R}}$

• SWAP

$d_{\bar{R}} \cdot d_{\bar{R}} \cdot d_R$

$$\int d\psi \; \mathrm{Tr}(\rho_R^2) = \frac{1}{d(d+1)}\left(d_R d_{\bar{R}}^2 + d_R^2 d_{\bar{R}}\right)$$

$$= \frac{1}{(d+1)}(d_R + d_{\bar{R}}) = \frac{d_R + d_{\bar{R}}}{d_R d_{\bar{R}} + 1}$$

$(d = d_R d_{\bar{R}})$

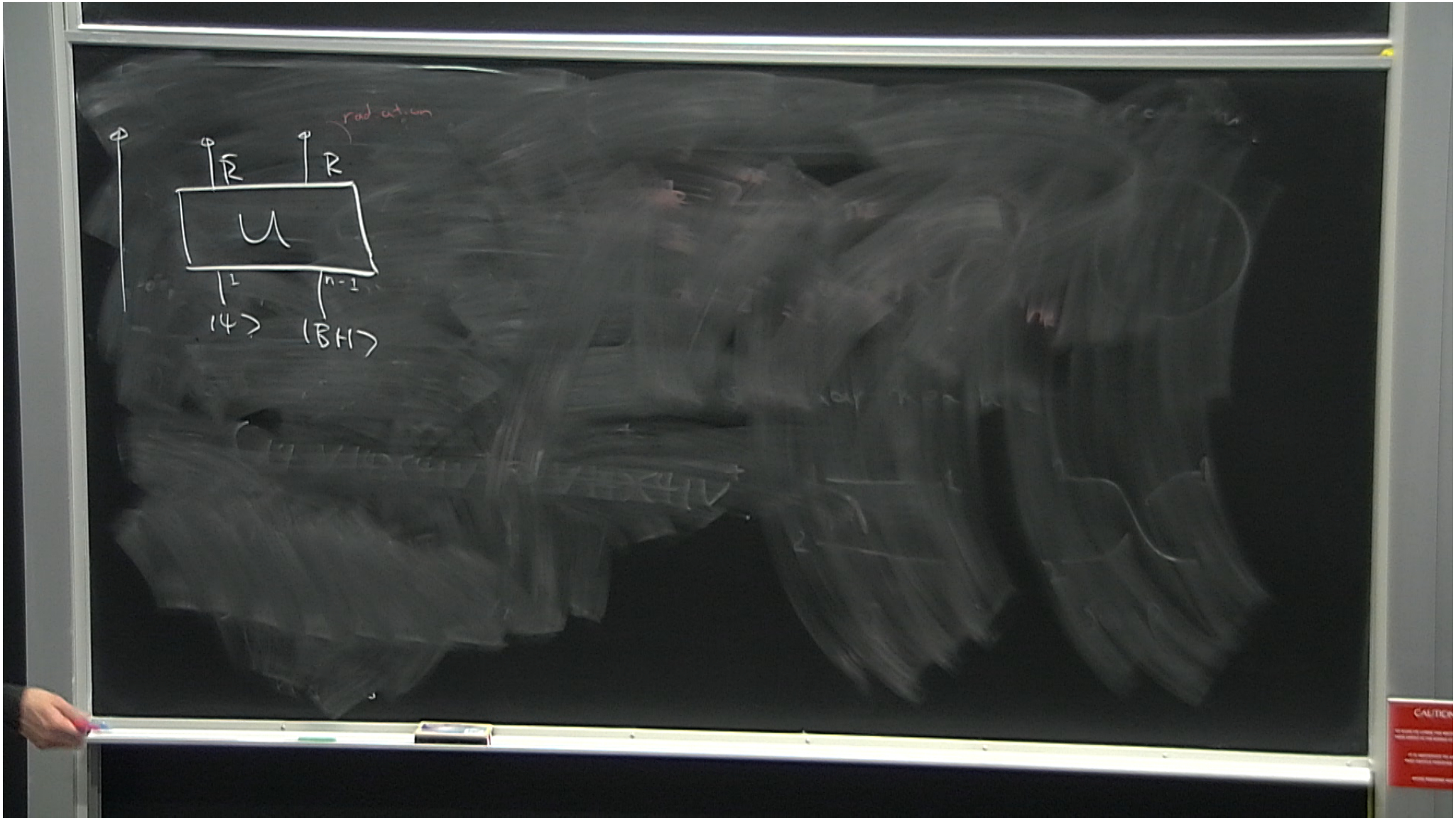
• $d_R \ll d_{\bar{R}}$

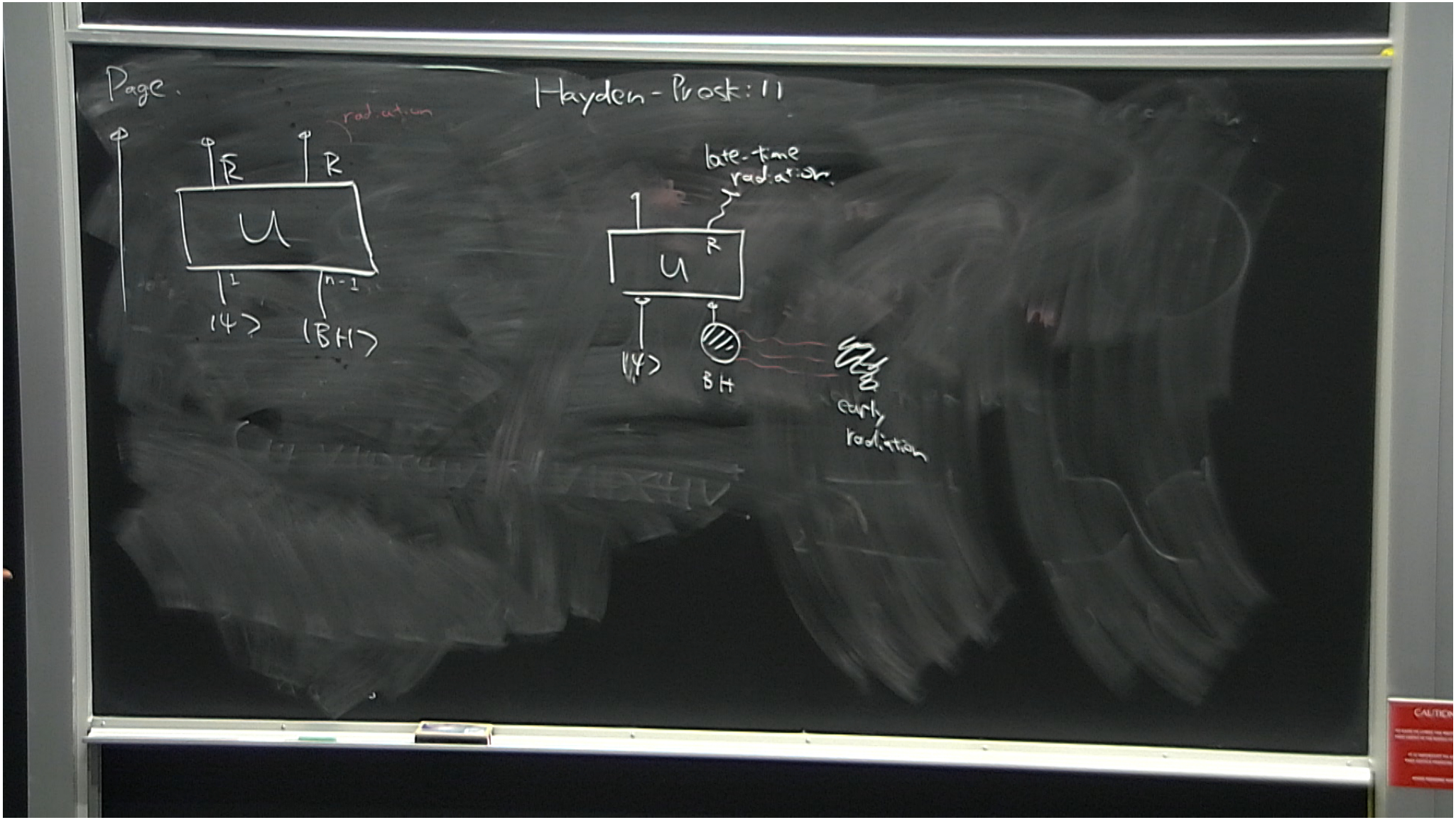
$$\int d\psi \; \mathrm{Tr}(\rho_R^2) \simeq \frac{d_{\bar{R}}}{d_R d_{\bar{R}}} = \frac{1}{d_R}$$

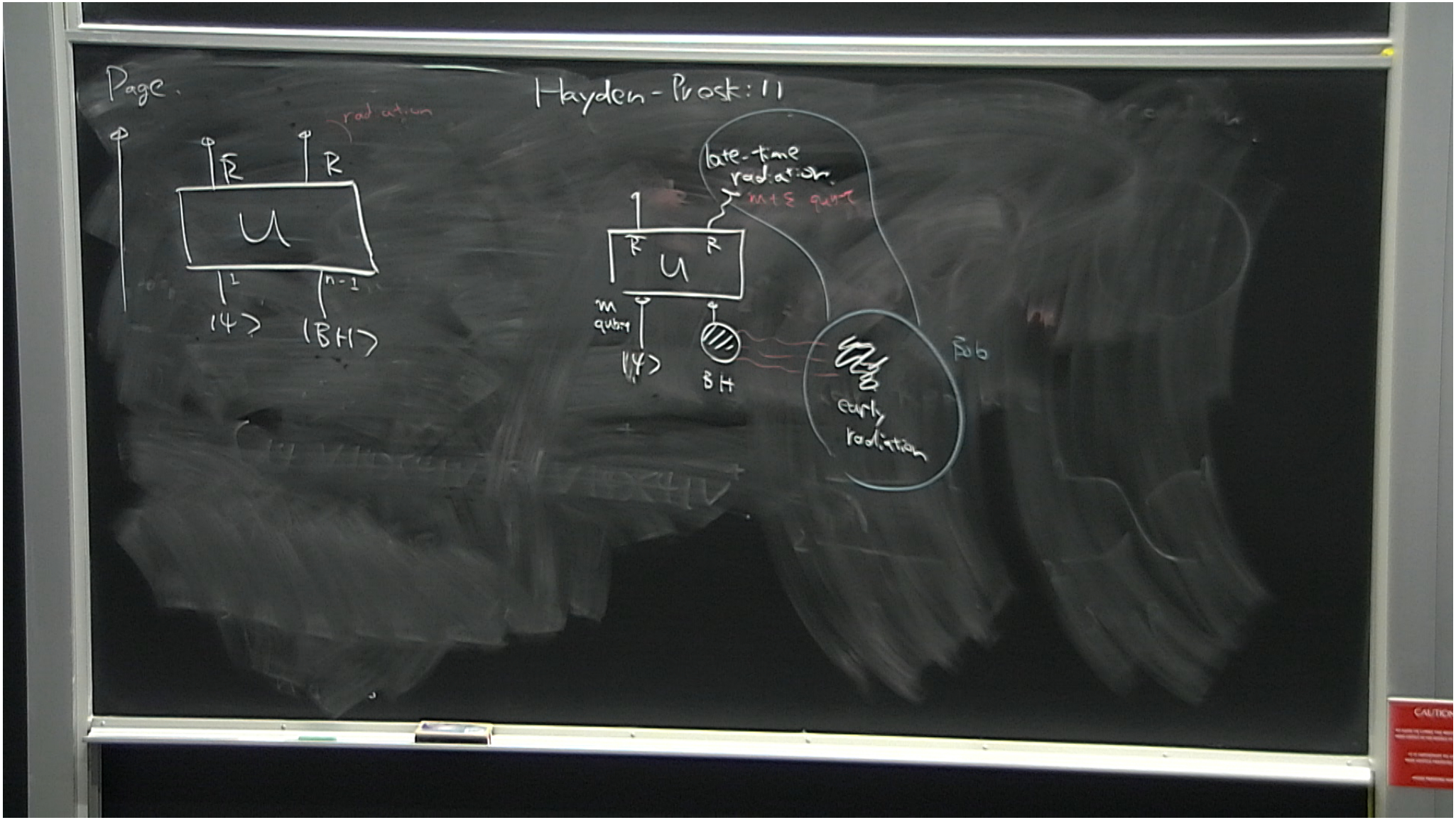
$\rho_R \simeq \frac{I_R}{d_R}$

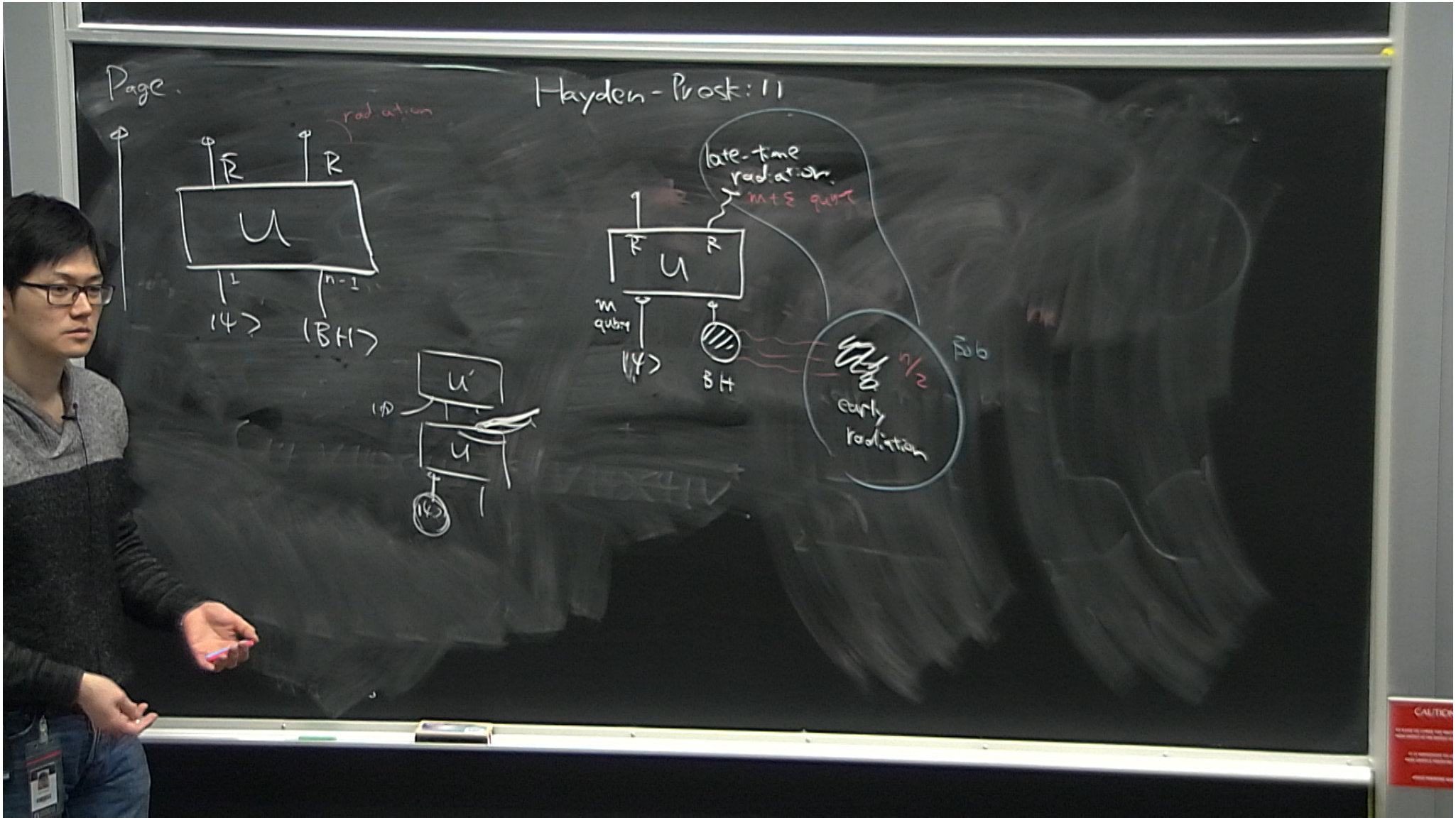
$\mathrm{Tr}(\rho_R^2) = \frac{1}{d_R}$

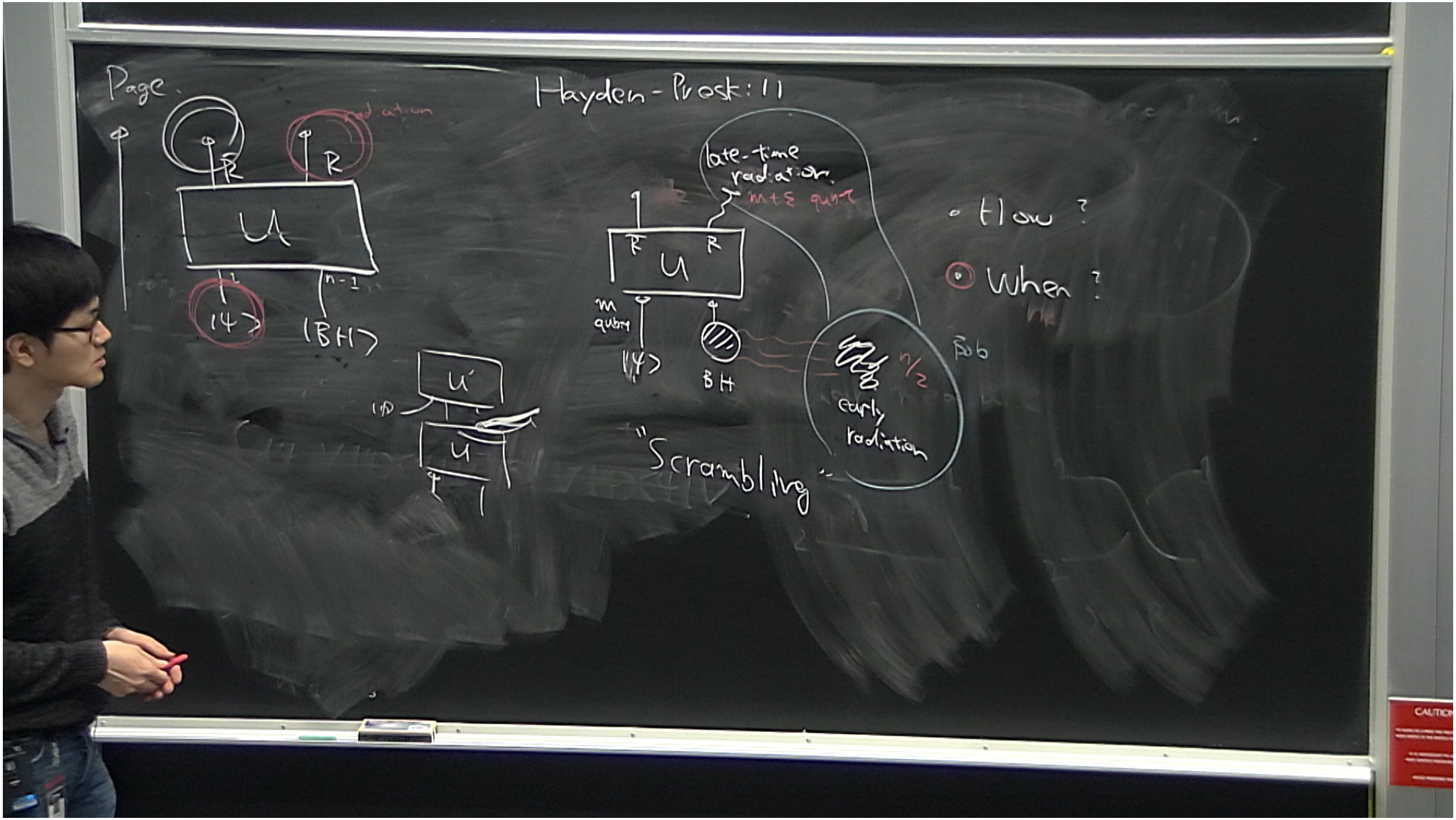
If ρ_R is $\frac{1_R}{d_R}$ (maximally mixed state)


R
R
U
1
n-1
|4>
|BH>

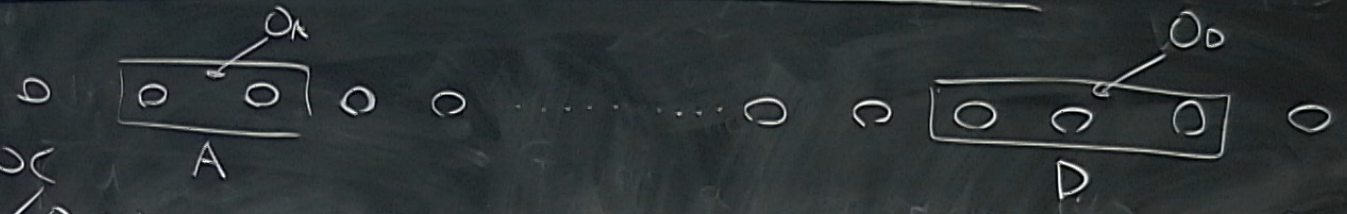

Page.
Hayden-Preskill
radiation
R
R
U
1
n-1
|ψ⟩
|BH⟩
late-time radiation
R
U
|ψ⟩
BH
early radiation


Page.
Hayden-Preskill
radiation
R
R
U
1
n-1
|ψ⟩
|BH⟩
late-time radiation
m+ε qubits
R
R
U
m qubits
|ψ⟩
BH
early radiation
ρab


Page.
Hayden-Preskill
radiation
R
R
U
1
n-1
|ψ⟩
|BH⟩
late-time radiation
m+ε qubits
R
R
U
m qubits
|ψ⟩
BH
n/2
early radiation
U'
U
|ψ⟩


Page.
Hayden-Preskill
R
R
U
n-1
|ψ⟩
|BH⟩
late-time radiation
m+ε qubits
R
R
U
m qubits
|ψ⟩
BH
n/2
early radiation
How?
When?
Bob
U'
|0⟩
U
t
"Scrambling"

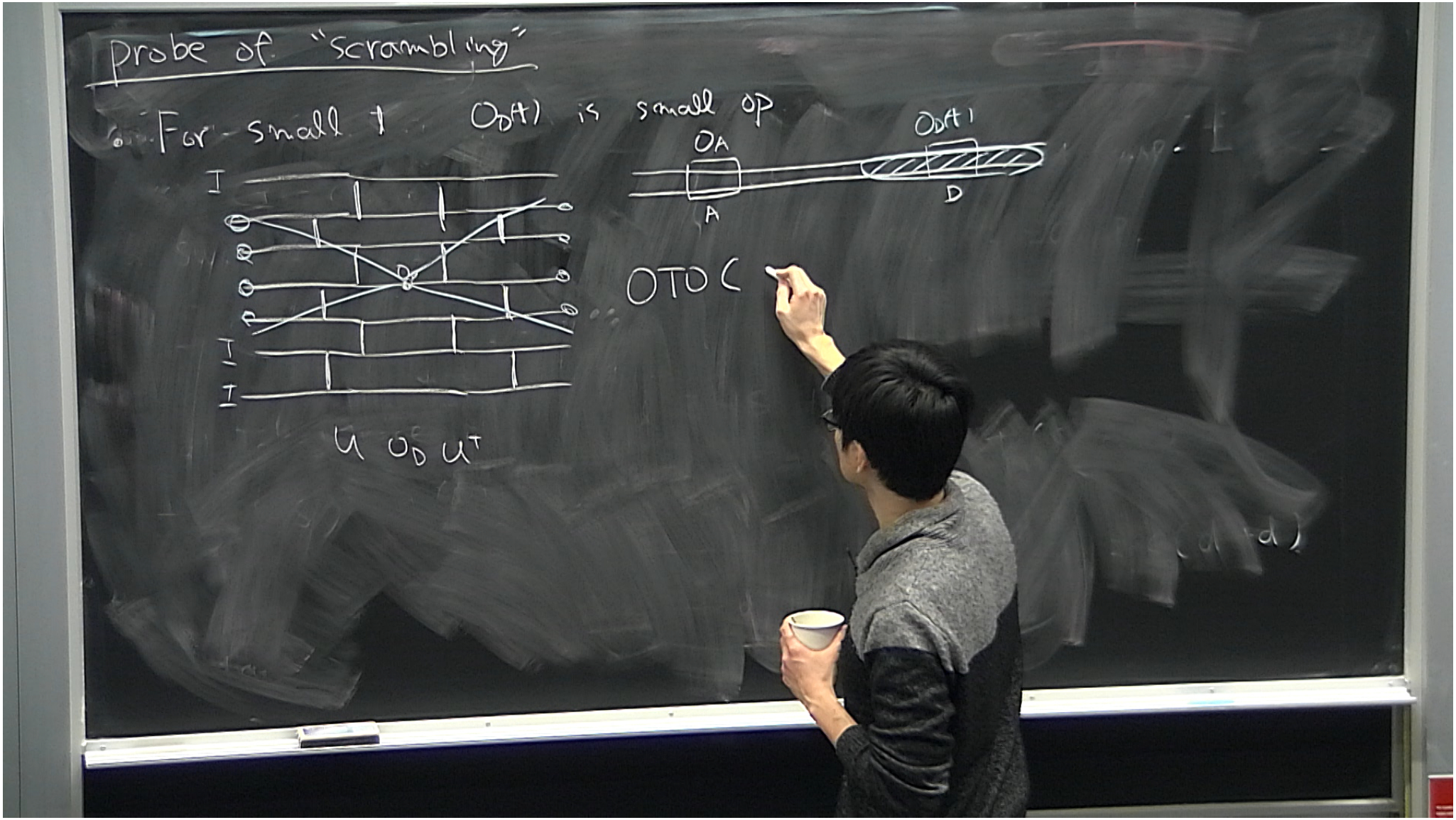
Out-of-time order correlation function



OTOC

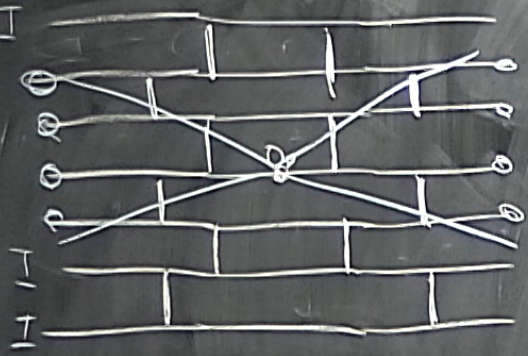
$$= \langle O_A(0)\, O_D(t)\, O_A(0)\, O_D(t) \rangle = \mathrm{Tr}\left(O_A\, O_D(t)\, O_A\, O_D(t)\, \rho\right)$$

$O_D(t) = U\, O_D\, U^\dagger$ $\quad$ U : time-evolution op.


Probe of "scrambling"
For small t, $O_D(t)$ is small op
O_A
$O_D(t)$
A
D
OTOC
I
I
I
$U\ O_D\ U^\dagger$

Probe of "scrambling"

For small t, $O_D(t)$ is small op



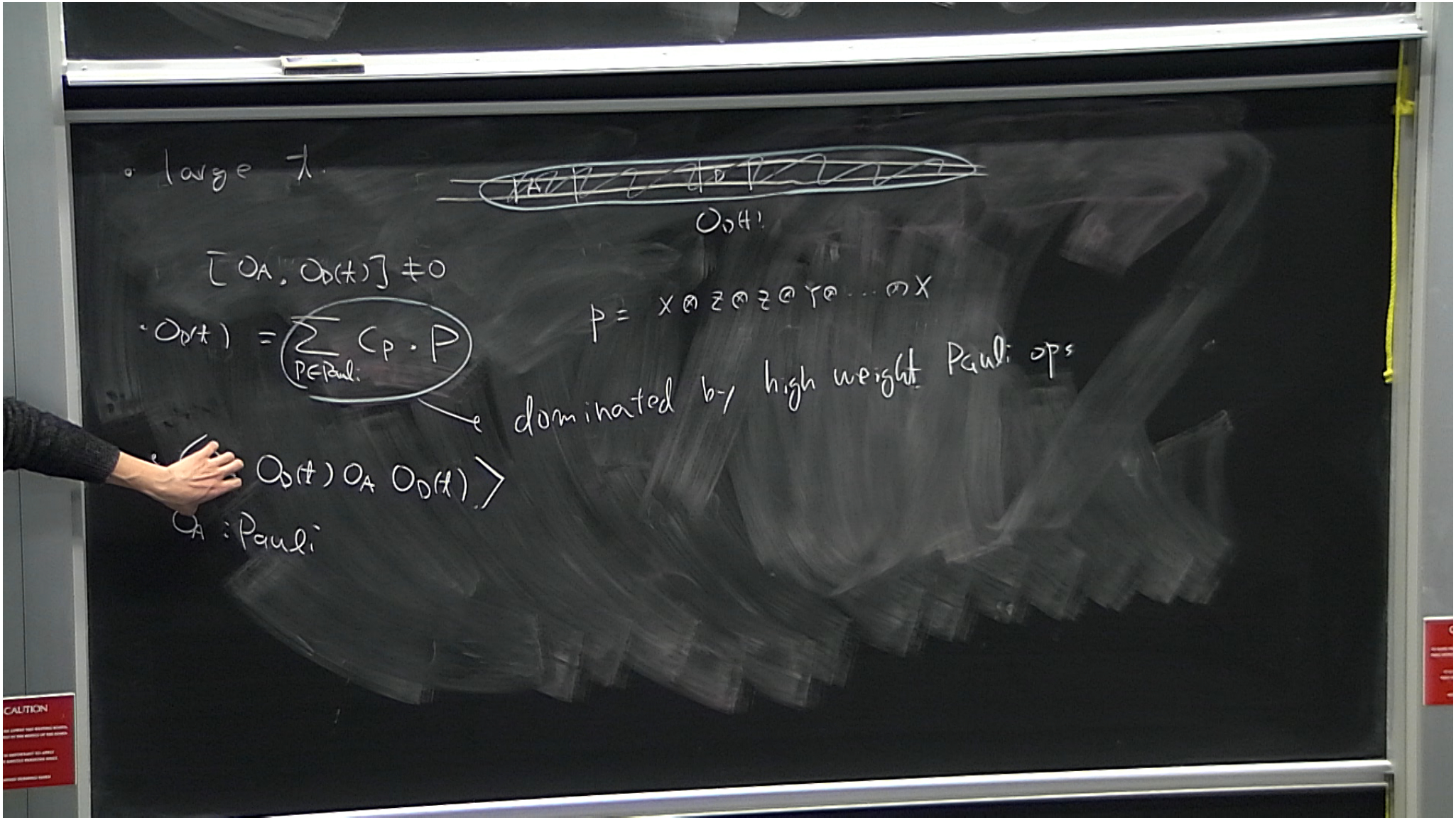
$U\, O_D\, U^\dagger$

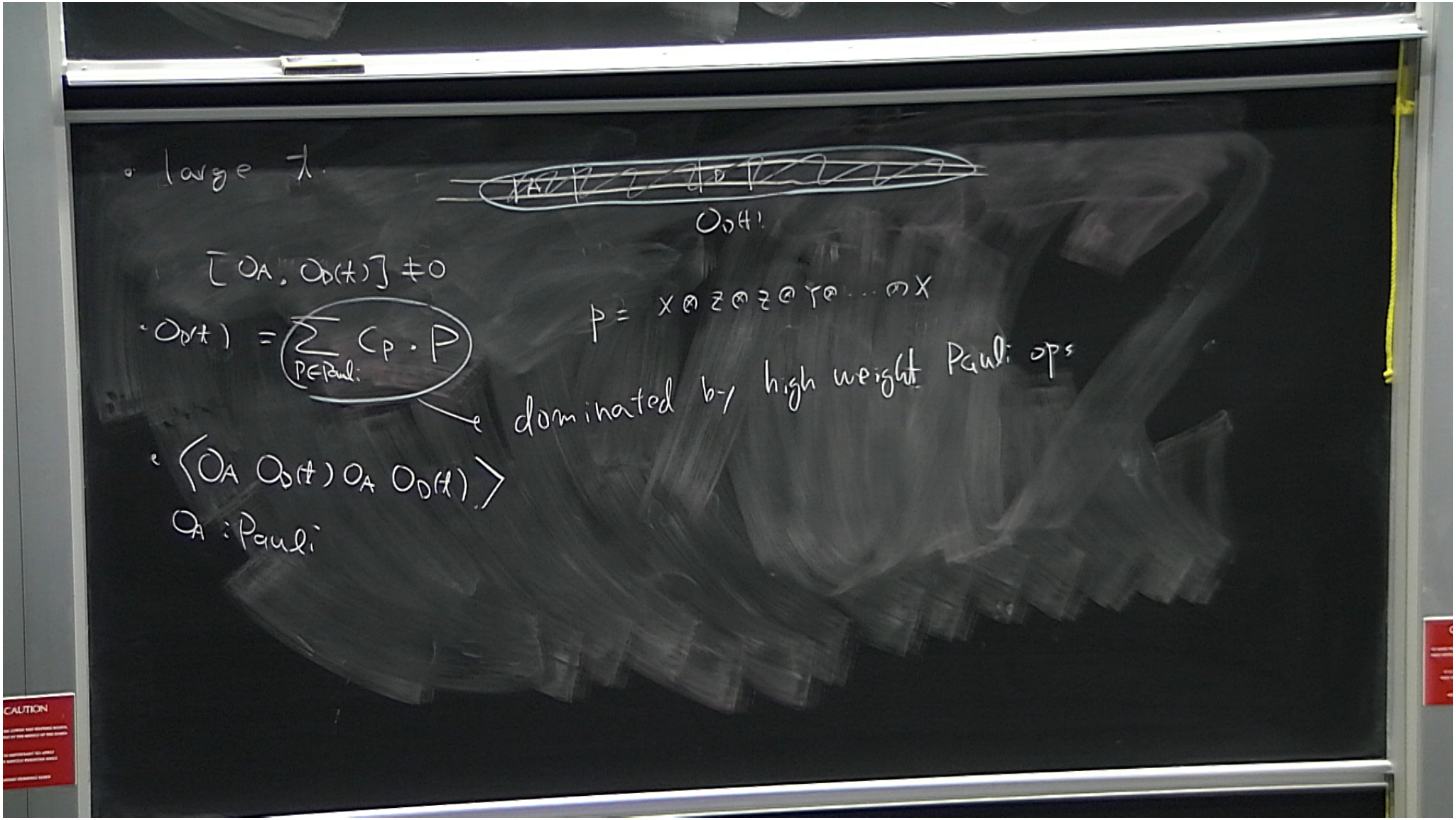


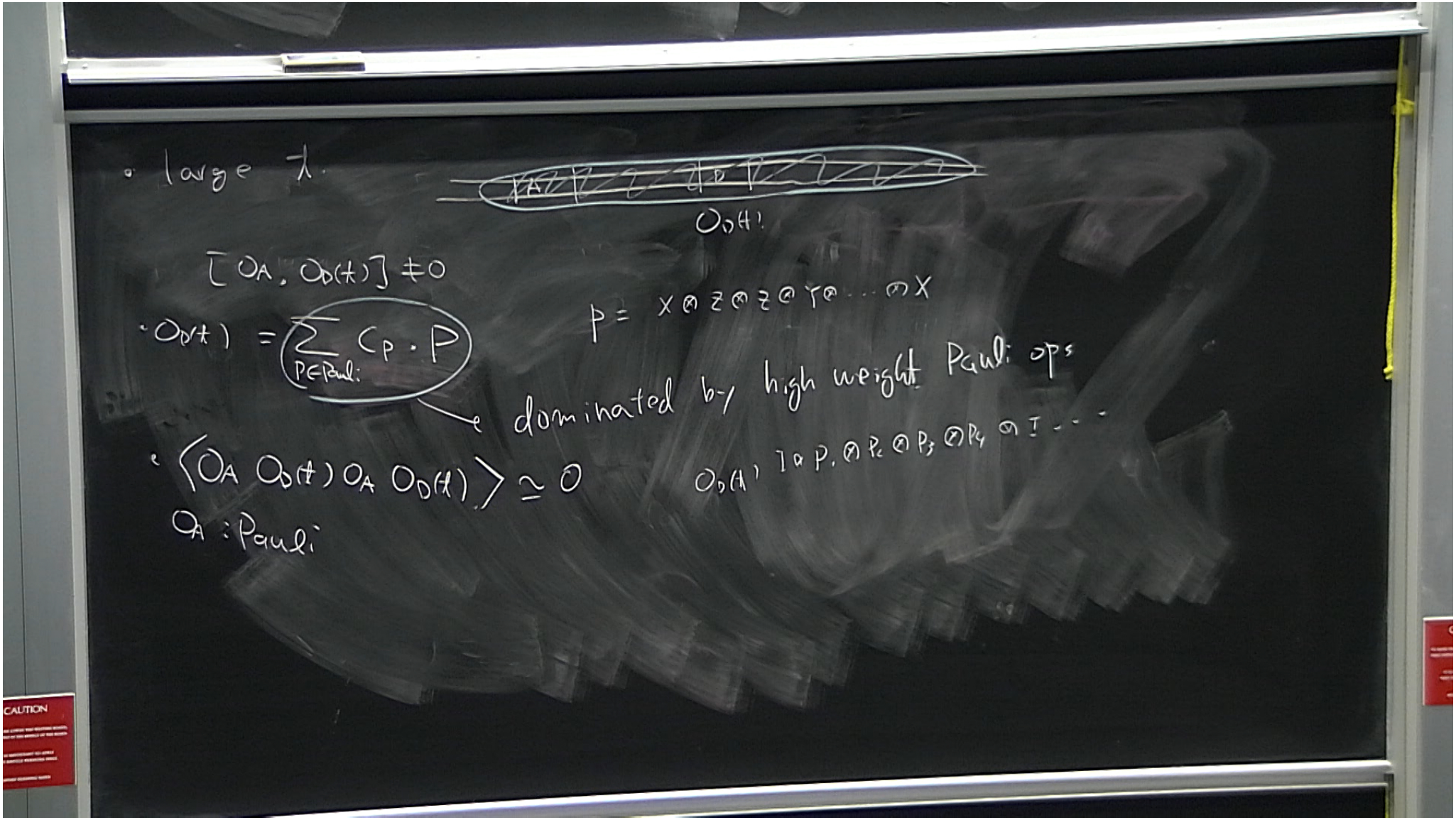
$$OTOC = \langle O_A\, O_D(t)\, O_A\, O_D(t) \rangle$$
$$= \langle O_A\, O_A\, O_D(t)\, O_D(t) \rangle$$

if O_A, O_D are Pauli ops

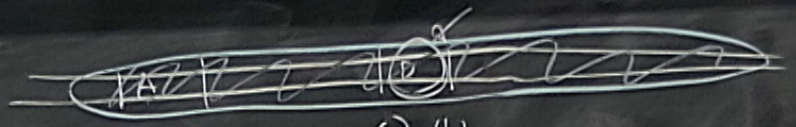
$$OTOC = 1$$


· large t.
$O_D(t)$
$[O_A, O_D(t)] \neq 0$
$O_D(t) = \sum_{P \in \text{Pauli}} c_P \cdot P$
dominated by high weight Pauli ops
$P = X \otimes Z \otimes Z \otimes Y \otimes \cdots \otimes X$
$\langle O_D(t)\, O_A\, O_D(t) \rangle$
O_A : Pauli


large t.
$O_B(t)$
$[O_A, O_B(t)] \neq 0$
$O_B(t) = \sum_{P \in \text{Pauli}} c_P \cdot P$
$P = X \otimes Z \otimes Z \otimes Y \otimes \cdots \otimes X$
dominated by high weight Pauli ops
$\langle O_A O_B(t) O_A O_B(t) \rangle$
O_A : Pauli


• large t.
$O_D(t)$
$[O_A, O_D(t)] \neq 0$
$O_D(t) = \sum_{P \in \text{Pauli}} c_P \cdot P$
dominated by high weight Pauli ops
$P = X \otimes Z \otimes Z \otimes Y \otimes \cdots \otimes X$
$O_D(t) \; I \otimes P_1 \otimes P_2 \otimes P_3 \otimes P_4 \otimes I \cdots$
$\langle O_A O_D(t) O_A O_D(t) \rangle \simeq 0$
O_A : Pauli

- large t.



$[O_A, O_D(t)] \neq 0$

- $O_D(t) = \sum_{P \in \text{Pauli}} c_P \cdot P$ ← dominated by high weight Pauli ops

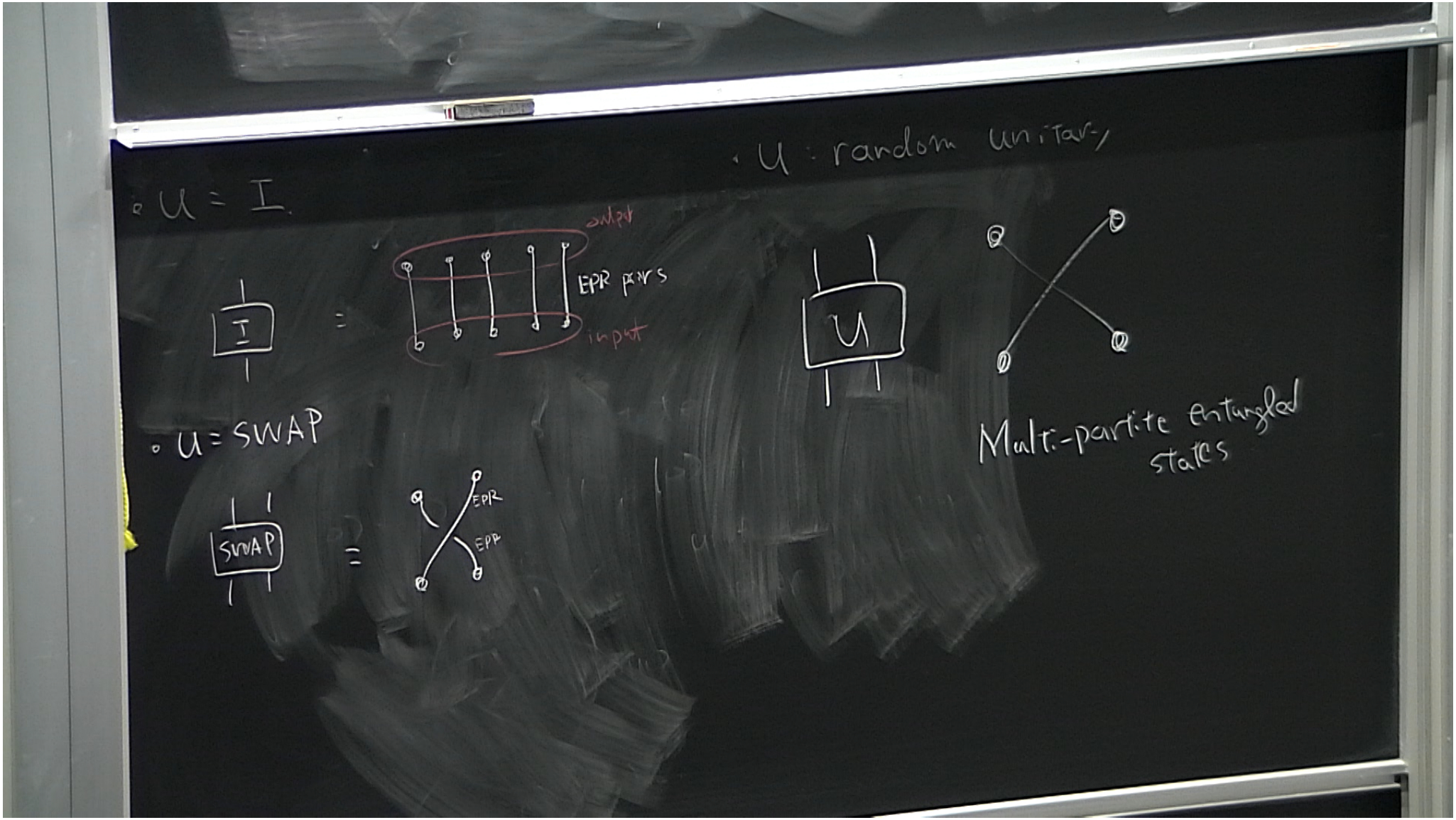
$P = X \otimes Z \otimes Z \otimes Y \otimes \cdots \otimes X$

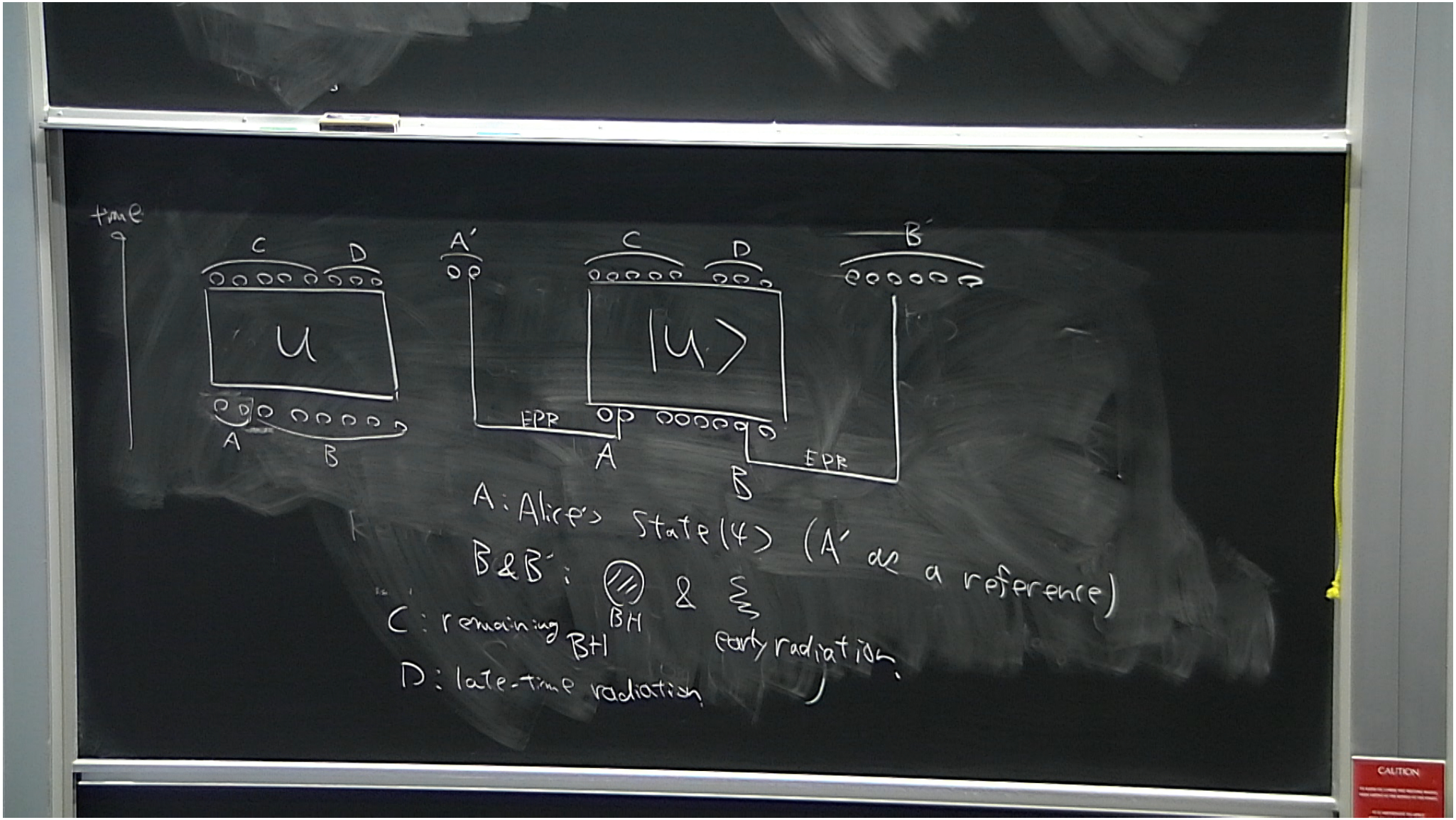
$O_D(t) \propto P_1 \otimes P_2 \otimes P_3 \otimes P_4 \otimes I \cdots$

- $\langle O_A O_D(t) O_A O_D(t) \rangle \simeq 0$

O_A : Pauli

"Scrambling" = decay of OTOC.


U = I
output
EPR pairs
input
U = SWAP
SWAP
EPR
EPR
U : random unitary
U
Multi-partite entangled states


time
C
D
U
A
B
A'
C
D
|U>
EPR
A
B
B'
EPR
A: Alice's state |U> (A' as a reference)
B&B': BH & EM
C: remaining BH
early radiation
D: late-time radiation

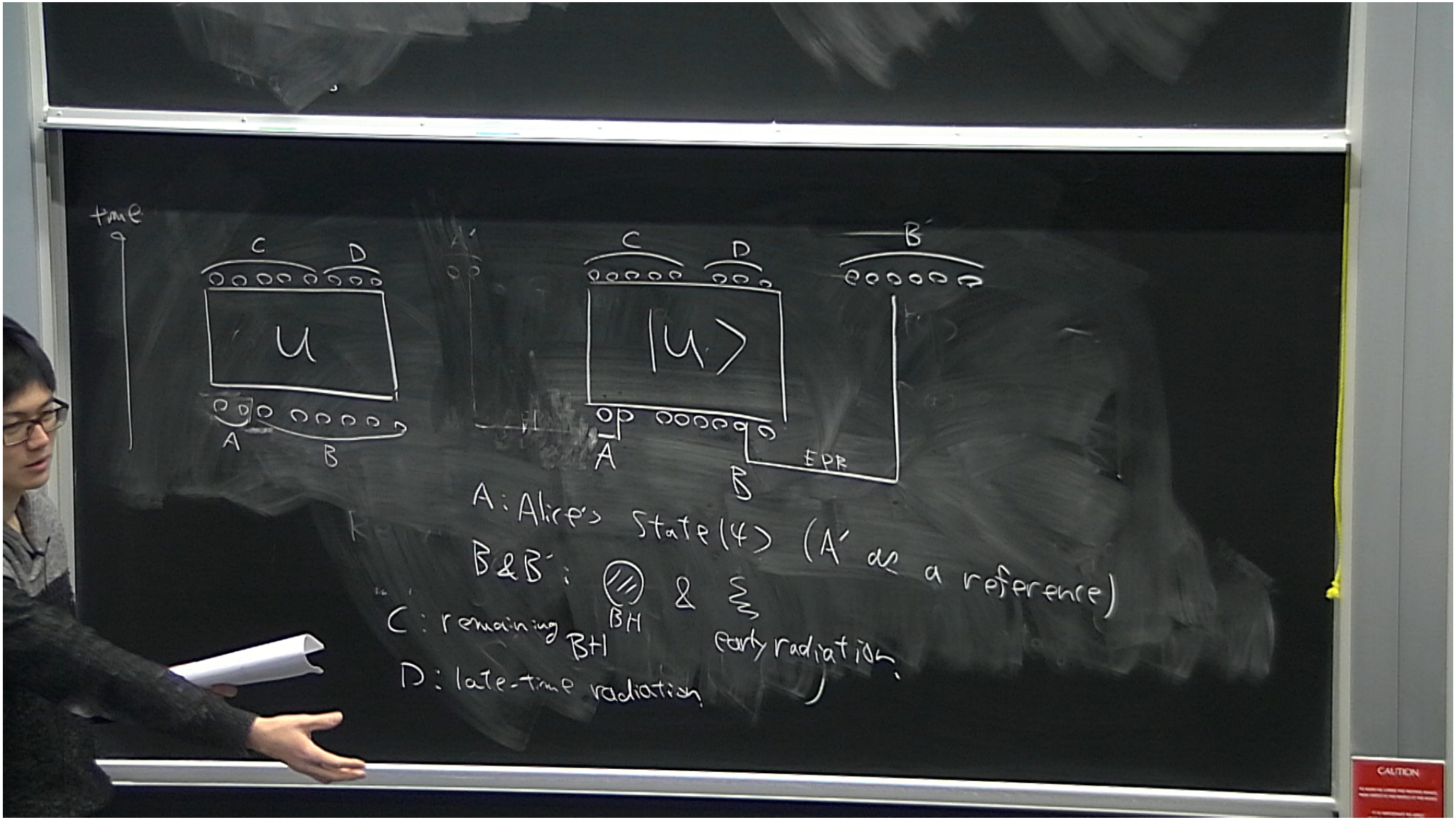
D: n_D qubits $\quad 4^{n_D}$ Pauli ops.

$$\langle \mathrm{OTOC} \rangle_{\mathrm{ave}} = \frac{1}{4^{n_A} 4^{n_D}} \sum_{O_A \in \mathrm{Pauli}} \sum_{O_X \in \mathrm{Pauli}} \langle O_A(0) O_D(t) O_A(0) O_D(t) \rangle$$

Theorem

$$-\log_2 \langle \mathrm{OTOC} \rangle_{\mathrm{ave}} = I^{(2)}(A', B'D)$$

$$I^{(2)}(A', B'D) = S^{(2)}_{A'} + S^{(2)}_{B'D} - S^{(2)}_{A'B'D}$$ (Rényi-2 mutual info)


time
C
D
U
A
B
A'
C
D
|U>
A
B
B'
EPR
A: Alice's state |U> (A' as a reference)
B&B': BH & EM
C: remaining BH early radiation.
D: late-time radiation
CAUTION