Title: IQC - Quantum Error Correction - Lecture 10

Date: Mar 15, 2018 02:00 PM

URL: http://pirsa.org/18030011

Abstract:

Clifford hierarchy (n qubits)

m-th level

C_0 : 0-th level $\quad \mathbb{1}, e^{i\theta}\mathbb{1}$

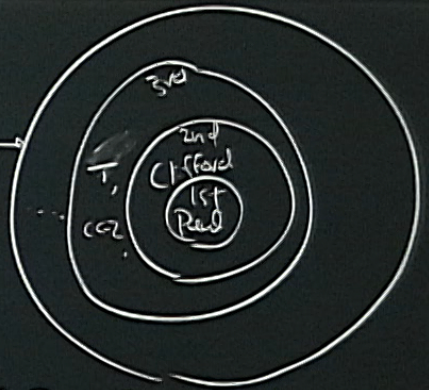
C_1 : 1-st level $\quad$ Paul: ops. X, Y, Z

C_2 : 2-nd level $\quad$ Clifford gate, CNOT, C-Z, S-gate

$U P U^\dagger = Q \quad P, Q \in$ Pauli

3rd level

4th level $\quad$ Non-Clifford gates $\quad$ T-gate, CCZ-gate



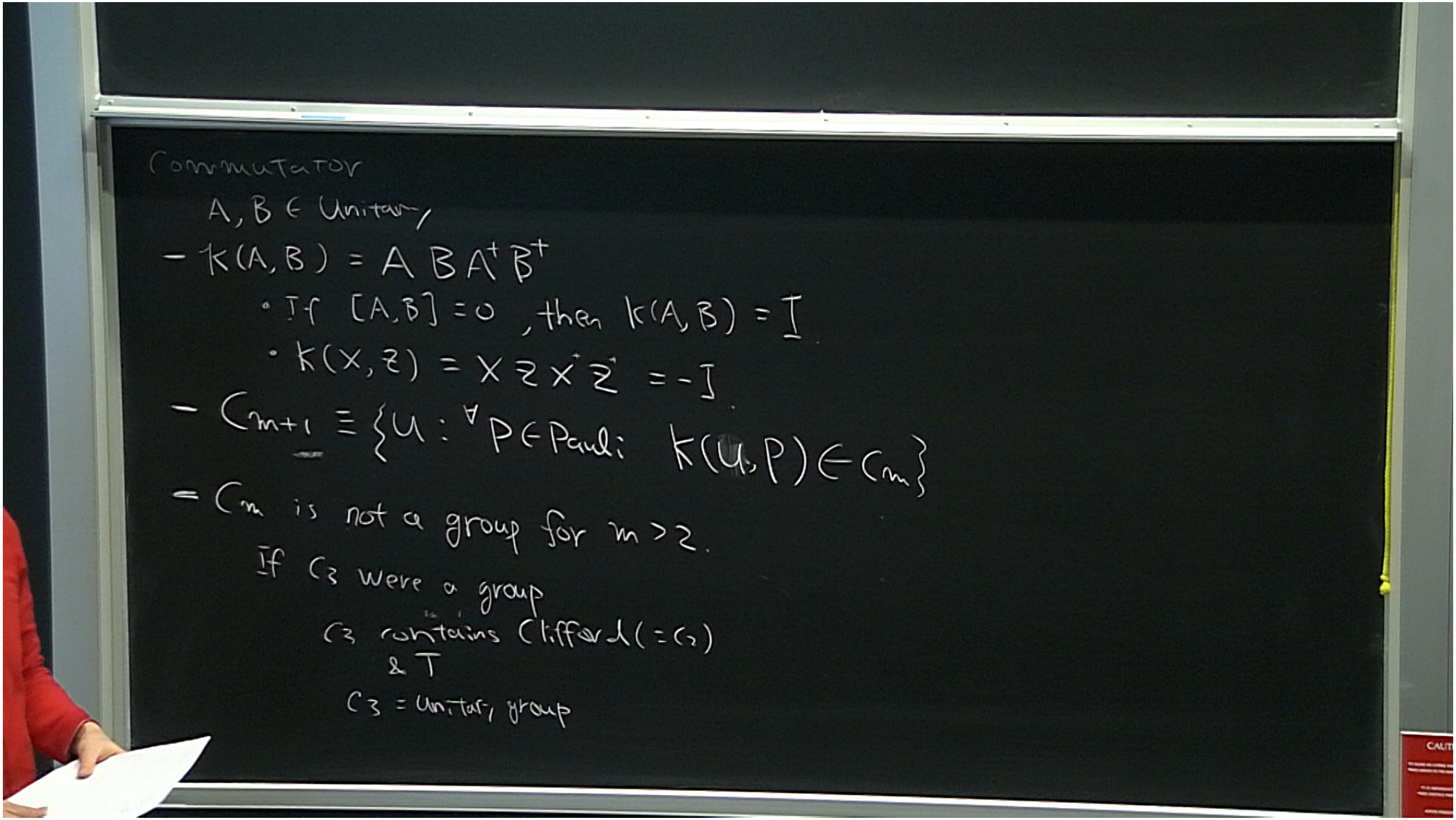
Commutator

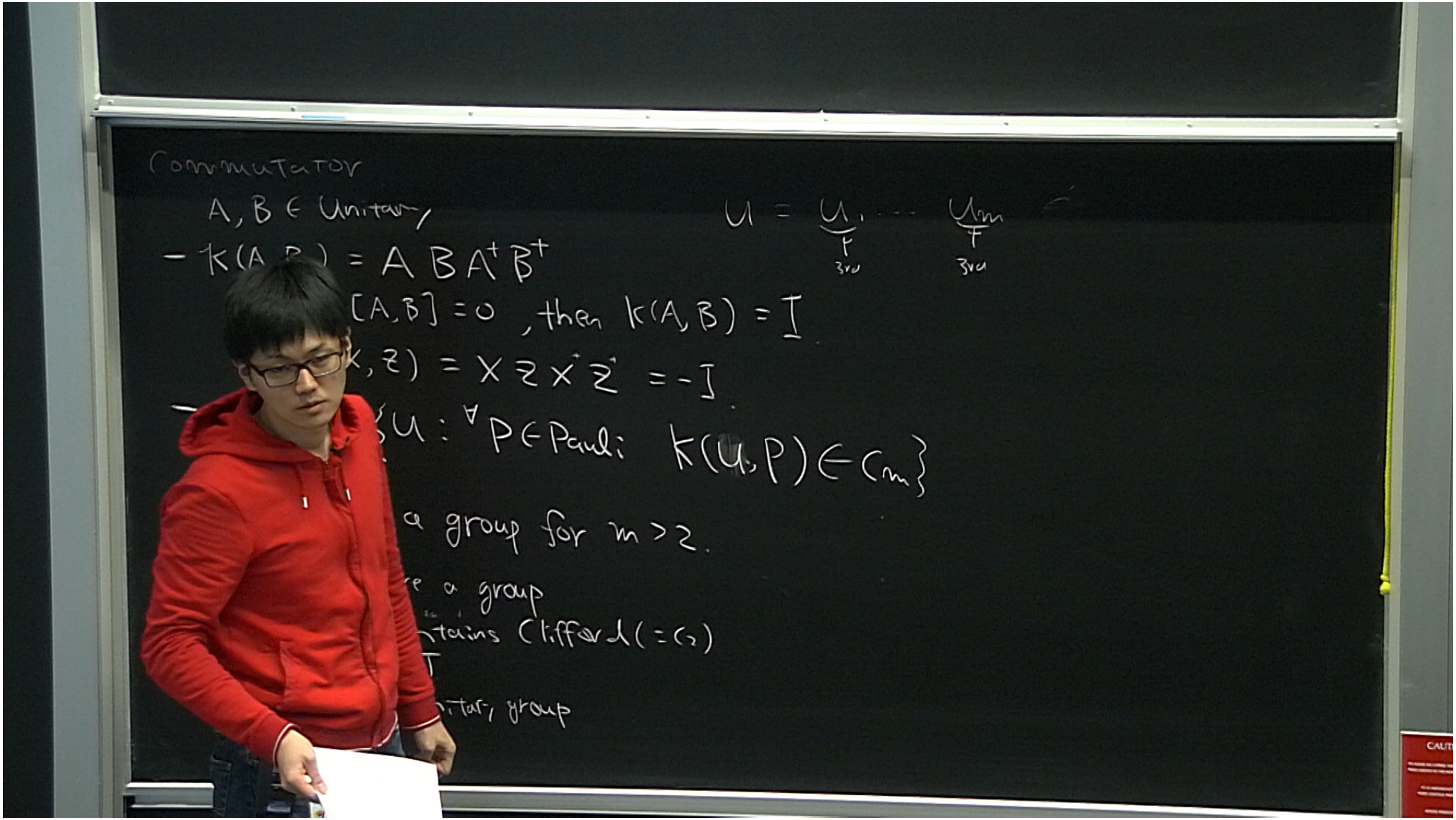
$A, B \in$ Unitary

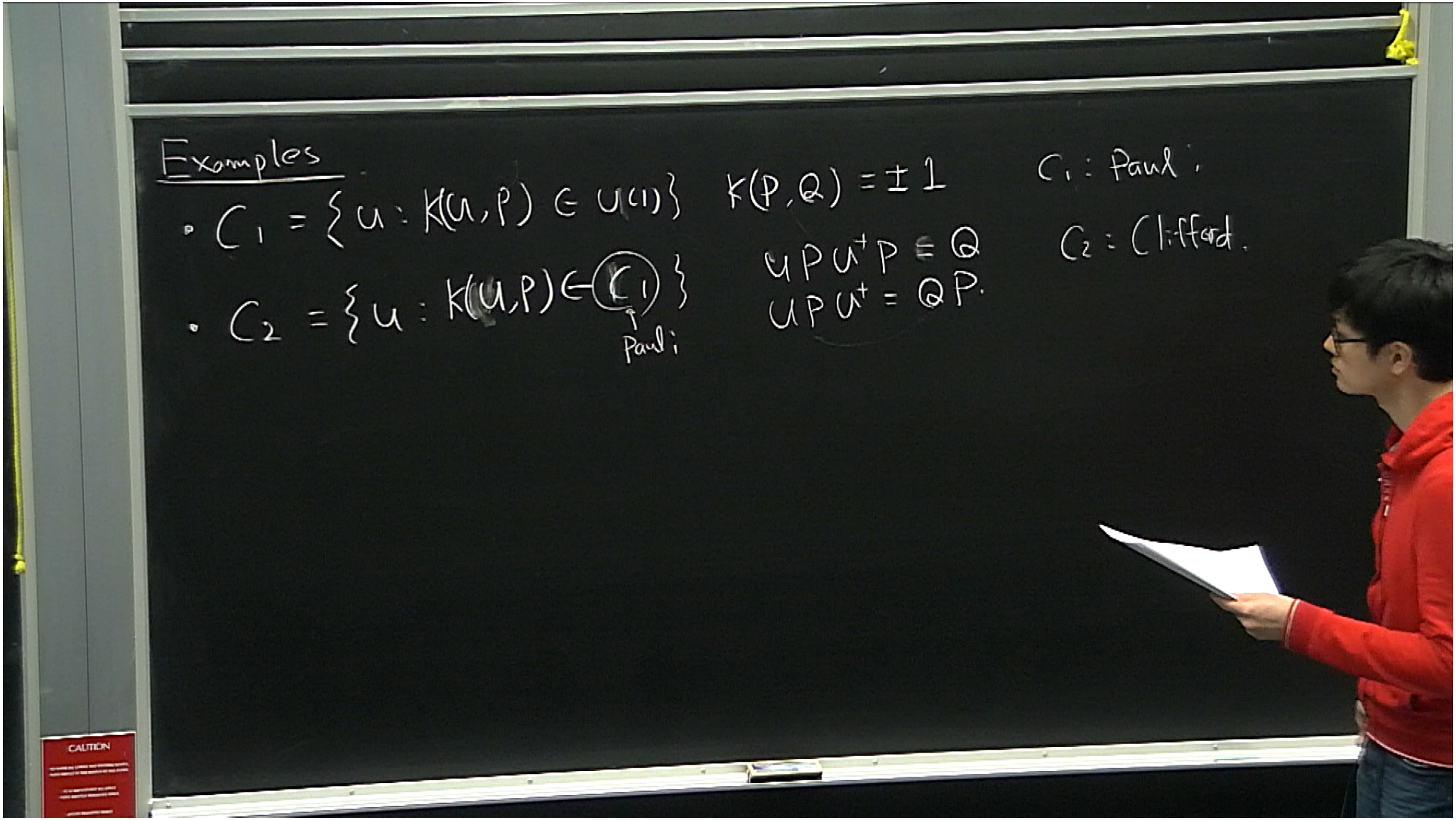
$K(A,B) = A B A^\dagger B^\dagger$

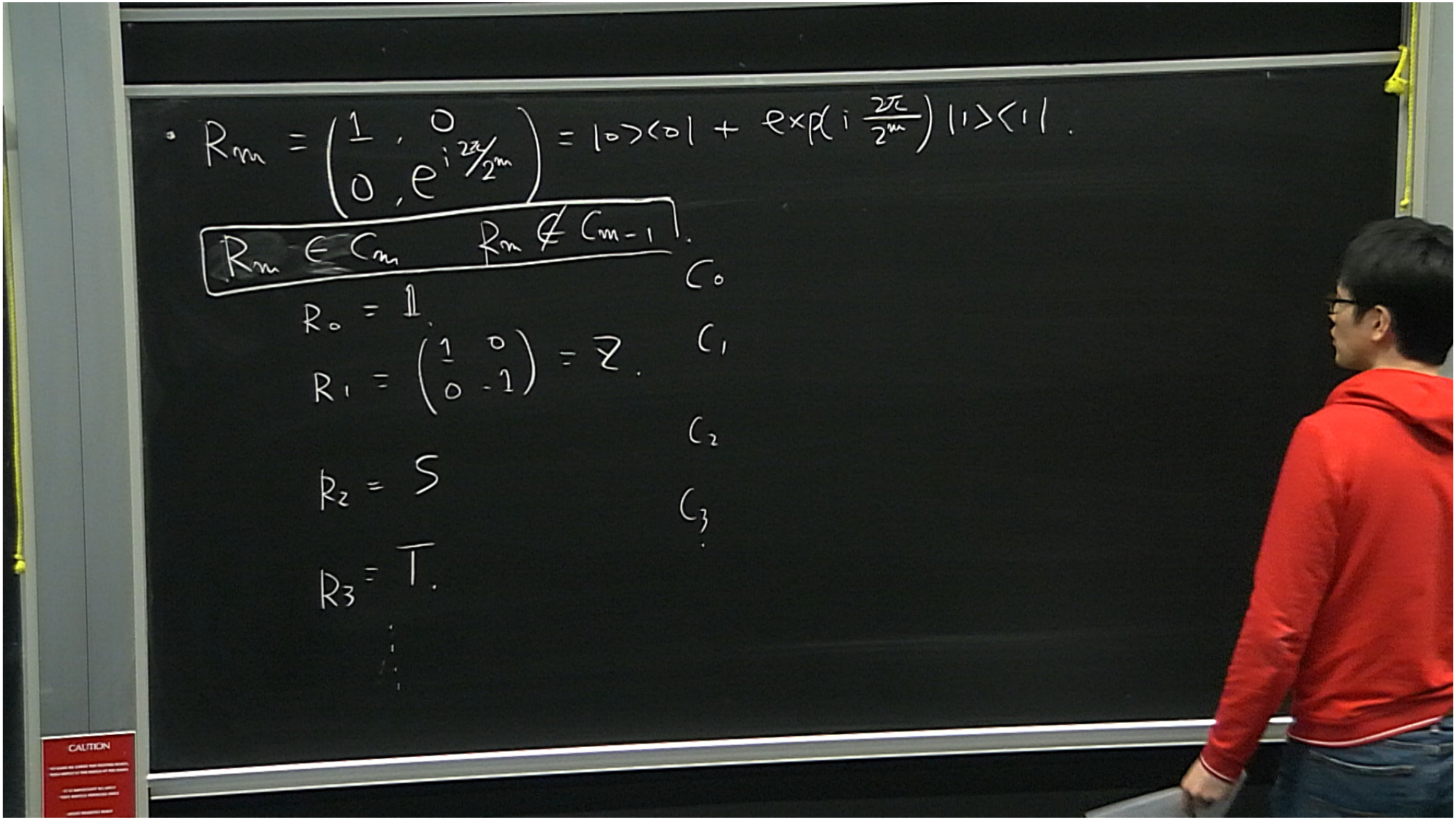
- If $[A,B] = 0$, then $K(A,B) = I$
- $K(X,Z) = X Z X^\dagger Z^\dagger = -I$

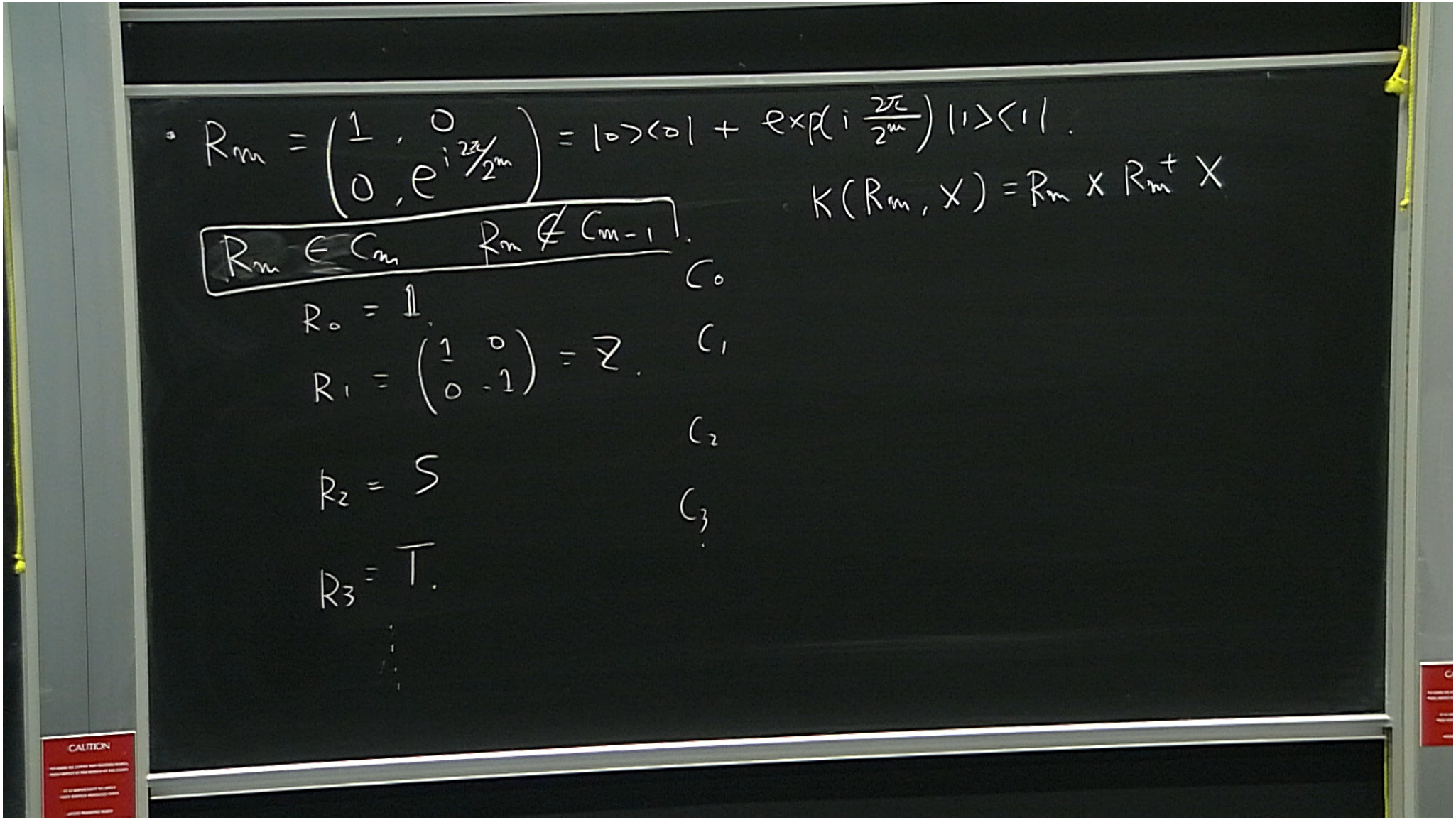
$C_{m+1} \equiv \{U : \forall P \in \text{Pauli}: K(U,P) \in C_m\}$

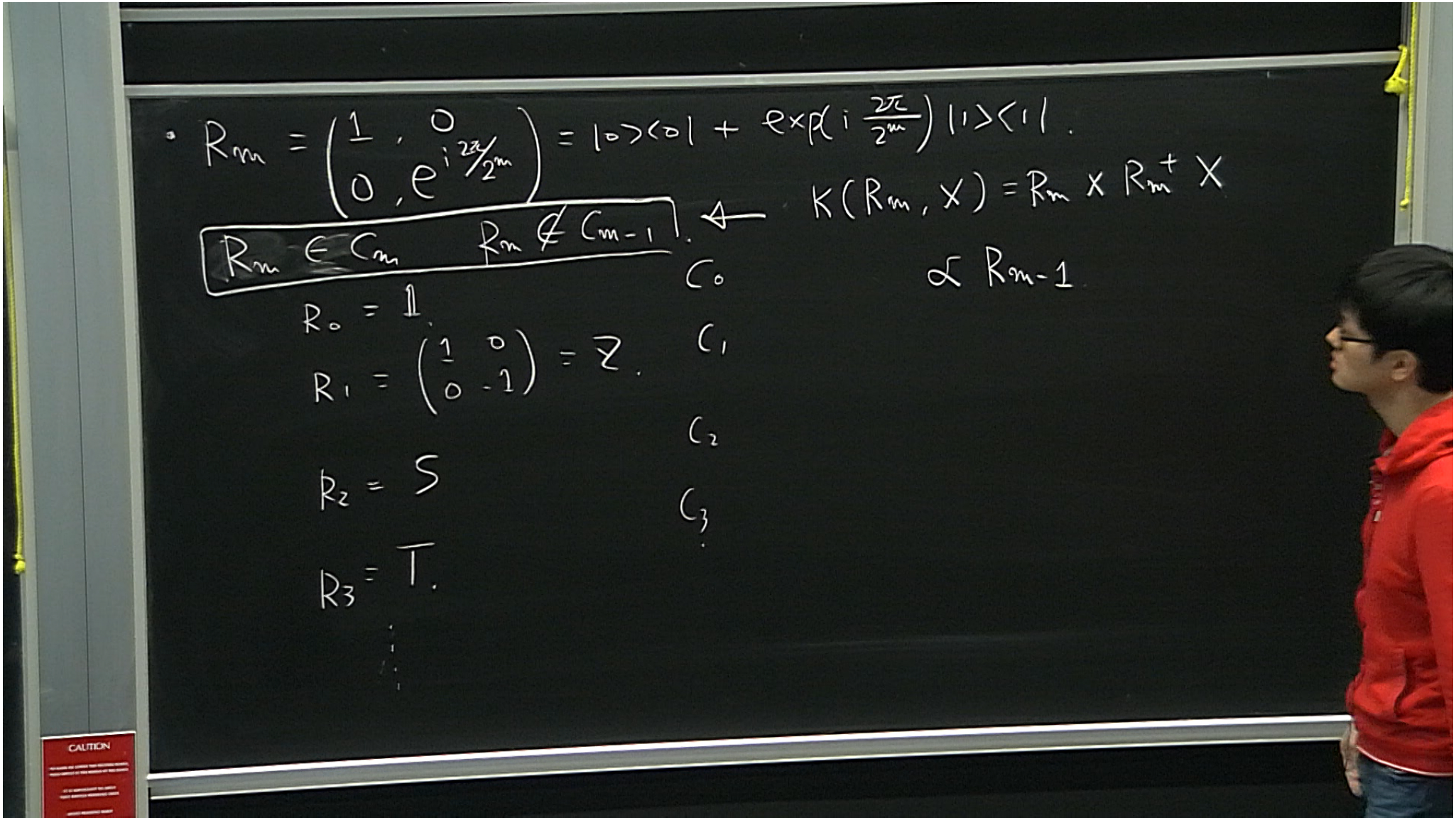

Commutator
A, B ∈ Unitary
- k(A,B) = A B A† B†
• If [A,B] = 0, then k(A,B) = I
• k(X,Z) = X Z X† Z† = -I
- C_{m+1} ≡ {U : ∀P ∈ Pauli: k(U,P) ∈ C_m}
= C_m is not a group for m > 2.
If C_3 were a group
C_3 contains Clifford (= C_2)
& T
C_3 = Unitary group


Commutator
A, B ∈ Unitary
$k(A,B) = ABA^\dagger B^\dagger$
$[A,B] = 0$, then $k(A,B) = I$
$k(X,Z) = XZX^\dagger Z^\dagger = -I$
$\{U : \forall P \in Pauli,\ k(U,P) \in C_m\}$
a group for $n > 2$.
a group
contains Clifford ($= C_2$)
T
unitary group
$U = U_1 \cdots U_m$
F
3rd
F
3rd


Examples
• $C_1 = \{U : K(U,P) \in U(1)\}$
• $C_2 = \{U : K(U,P) \in C_1\}$
Pauli
$K(P,Q) = \pm 1$
$UPU^\dagger P \equiv Q$
$UPU^\dagger = QP$.
C_1: Pauli.
C_2: Clifford.
CAUTION


$R_m = \begin{pmatrix} 1 & 0 \\ 0 & e^{i 2\pi/2^m} \end{pmatrix} = |0\rangle\langle 0| + \exp(i \frac{2\pi}{2^m}) |1\rangle\langle 1|$.
$R_m \in C_m \quad R_m \notin C_{m-1}$.
C_0
$R_0 = \mathbb{1}$.
C_1
$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$.
C_2
$R_2 = S$
C_3
$R_3 = T$.
CAUTION


$R_m = \begin{pmatrix} 1 & 0 \\ 0 & e^{i 2\pi/2^m} \end{pmatrix} = |0\rangle\langle 0| + \exp(i \frac{2\pi}{2^m}) |1\rangle\langle 1|$.
$K(R_m, X) = R_m X R_m^\dagger X$
$R_m \in C_m \quad R_m \notin C_{m-1}$.
$R_0 = \mathbb{1}$.
$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$.
$R_2 = S$
$R_3 = T$.
C_0
C_1
C_2
C_3
CAUTION

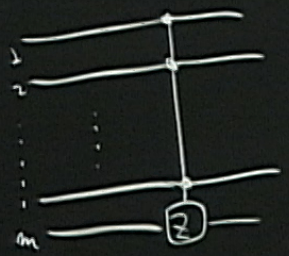

$R_m = \begin{pmatrix} 1 & 0 \\ 0 & e^{i 2\pi/2^m} \end{pmatrix} = |0\rangle\langle 0| + \exp(i \frac{2\pi}{2^m}) |1\rangle\langle 1|$.
$R_m \in C_m \quad R_m \notin C_{m-1}$.
$K(R_m, X) = R_m X R_m^\dagger X$
$\propto R_{m-1}$
$R_0 = \mathbb{1}$.
$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$.
$R_2 = S$
$R_3 = T$.
C_0
C_1
C_2
C_3
CAUTION

• Control-Z gate.

$C^{\otimes m-1}Z\,|i_1, i_2, \cdots, i_m\rangle = (-1)^{i_1 \cdots i_m}\,|i_1, \cdots, i_m\rangle$ ← product.

$i_1, \cdots, i_{m-1} = 1$. then apply Z on m-th qubit

"-1" only when $i_1 = i_2 = \cdots = i_m = 1$.



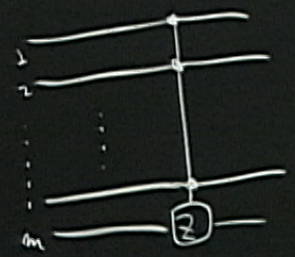
$C^{\otimes m-1}Z \in C_m$.

• (ontrol-Z gate (m-qubit)

$C^{\otimes m-1}Z\,|i_1, i_2, \cdots, i_m\rangle = (-1)^{i_1 \cdots i_m}\,|i_1, \cdots, i_m\rangle$ ← product.

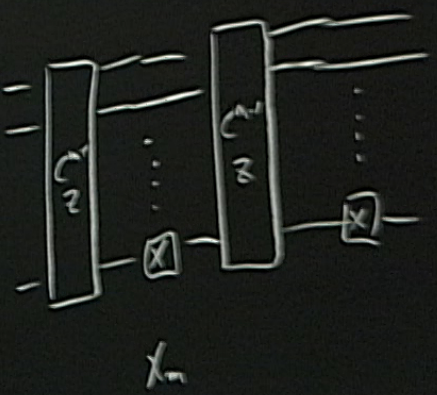
$i_1, \cdots, i_{m-1} = 1$. then apply Z on m-th qubit

"-1" only when $i_1 = i_2 = \cdots = i_m = 1$.

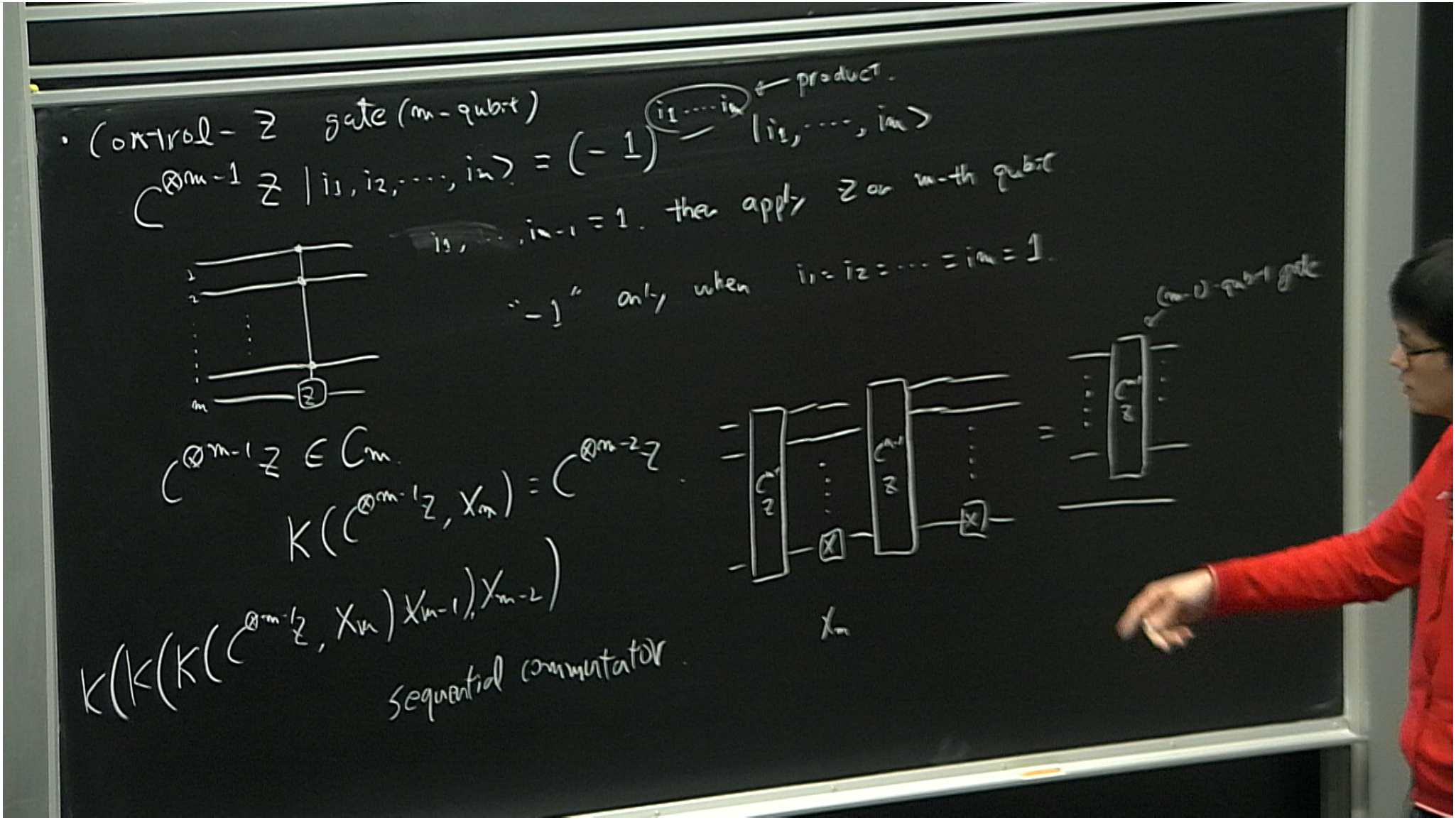


$C^{\otimes m-1}Z \in C_m$.

$K(C^{\otimes m-1}Z, X_m) = C^{\otimes m-2}Z$.



X_m


• (control-Z gate (m-qubit)
$C^{\otimes m-1}Z\,|i_1, i_2, \cdots, i_m\rangle = (-1)^{i_1 \cdots i_m}\,|i_1, \cdots, i_m\rangle$
product
$i_1, \cdots, i_{m-1} = 1$. then apply Z on m-th qubit
"-1" only when $i_1 = i_2 = \cdots = i_m = 1$
$C^{\otimes m-1}Z \in C_m$
$K(C^{\otimes m-1}Z, X_m) = C^{\otimes m-2}Z$
$K(K(K(C^{\otimes m-1}Z, X_m)X_{m-1}), X_{m-2})$
sequential commutator
$(m-1)$-qubit gate
X_m

- Reed-Muller code $[n, k, d]$

$[2^{m+1}-1, 1, 3]$ R_m gate $m=3$ 15-qubit code T gate

- Hypercube code $C^{(2^m-1)}Z$ gate

$[2^m, m, 2]$

- Above two codes can be obtained from m-dim color code with appropriate boundaries

- Reed-Muller code $[2^{m+1}-1, 1, 3]$ $[n, k, d]$ R_m gate $m=3$ 15-qubit code T gate

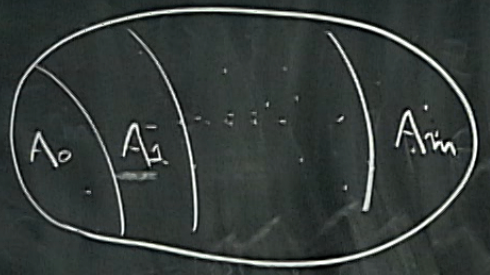
- Hypercube code $[2^m, m, 2]$ $C^{m-1}Z$ gate

- Above two codes can be obtained from m-dim color code with appropriate boundaries

Transversal gate

Lemma

Consider n-qubit stab code



m+1 subsystems.

ℓ : transversal op

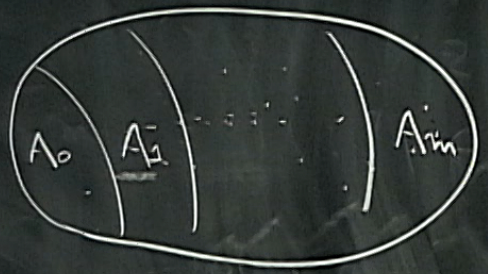
$A_0 \otimes A_1 \cdots \otimes A_m$

$\ell = \ell_0 \otimes \ell_1 \cdots \otimes \ell_m$

Transversal gate

Lemma

Consider n-qubit stab code



$m+1$ subsystems.

ℓ : transversal op

$A_0 \otimes A_1 \cdots \otimes A_m$

$\ell = \ell_0 \otimes \ell_1 \cdots \otimes \ell_m$

$\nexists\ \ell \in C_m$.

Assume

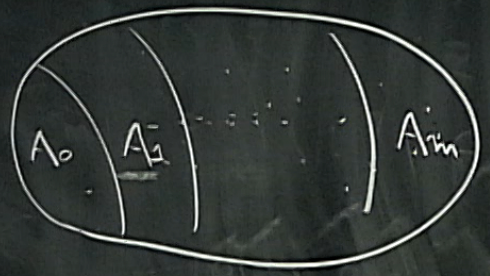
A_j does not support any logical op.

$g_{A_j} = 0$.

Transversal gate

Lemma

Consider n-qubit stab code



$m+1$ subsystems.

A_0 A_1 . . . . A_{m-1}

ℓ : transversal op

$A_0 \otimes A_1 \cdots \otimes A_m$

$\ell = \ell_0 \otimes \ell_1 \cdots \otimes \ell_m$

$\Rightarrow \ell \in C_m$.

Assume.

A_j does not support any logical op.

$g_{A_j} = 0$.

Transversal gate

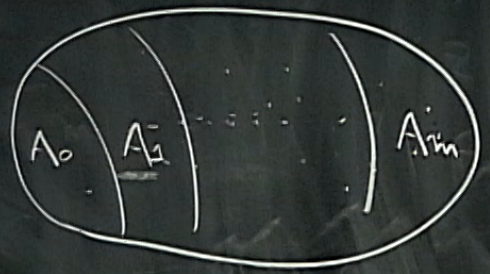
Lemma

Consider n-qubit stab code

ℓ : transversal op

$A_0 \otimes A_1 \cdots \otimes A_m$

$\ell = \ell_0 \otimes \ell_1 \cdots \otimes \ell_m$



$m+1$ subsystems.

$\nexists\ \ell \in C_m$.

A_0 A_1 $\cdots$ A_{n-1}

n subsystems

$g_{A_j} = 0$ $\quad \ell \in C_n$.

Assume.

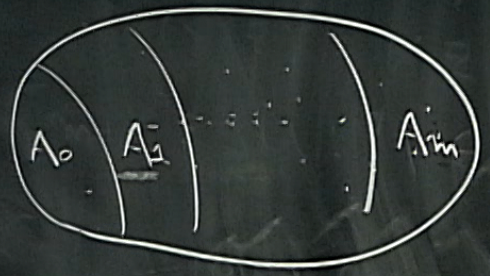
A_j does not support any logical op.

$g_{A_j} = 0$.

Transversal gate

Lemma

Consider n-qubit stab code



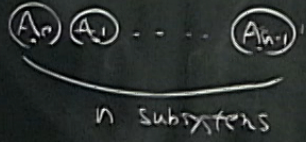
$n+1$ subsystems.

ℓ : transversal op

$A_0 \otimes A_1 \cdots \otimes A_m$

$\ell = \ell_0 \otimes \ell_1 \cdots \otimes \ell_m$

$\nexists\ \ell \in C_m$.



n subsystems

$g_{A_j} = 0 \quad \ell \in C_n$

Assume.

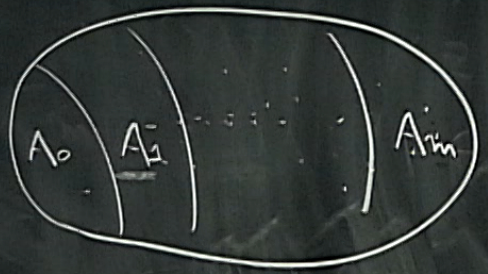
A_j does not support any logical op.

$g_{A_j} = 0$.

Transversal gate

Lemma

Consider n-qubit stab code



$m+1$ subsystems.

ℓ : transversal op

$A_0 \otimes A_1 \cdots \otimes A_m$

$\ell = \ell_0 \otimes \ell_1 \cdots \otimes \ell_m$

$\nexists\ \ell \in C_m$.

Assume.

A_j does not support any logical op.

$g_{A_j} = 0$.

$(A_0)(A_1)\cdots(A_{n-1})$

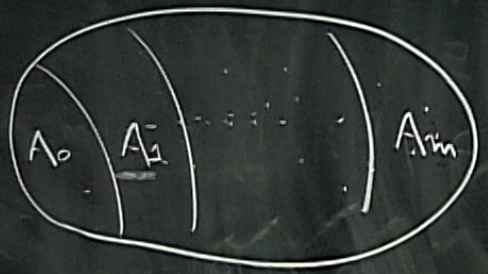
n subsystems

$g_{A_j} = 0 \quad \ell \in C_{n-1}$.

Transversal gate

Lemma

Consider n-qubit stab code



$n+1$ subsystems.

ℓ : transversal op

$A_0 \otimes A_1 \cdots \otimes A_m$

$\ell = \ell_0 \otimes \ell_1 \cdots \otimes \ell_m$

$\nexists\ \ell \in C_n$

Assume.

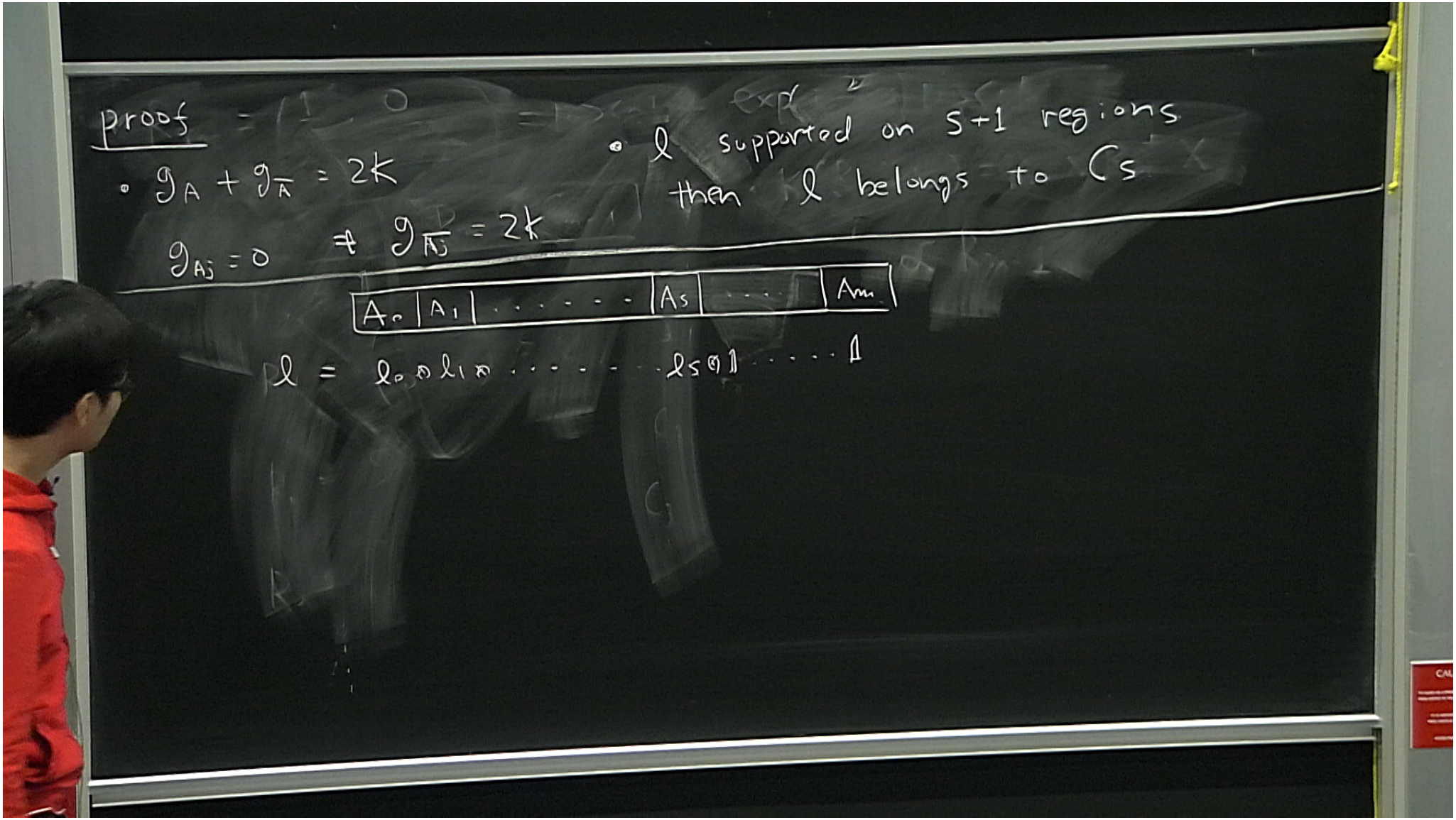
A_j does not support any logical op.

$g_{A_j} = 0.$

A_0 A_1 $\cdots$ A_{n-1}

n subsystems

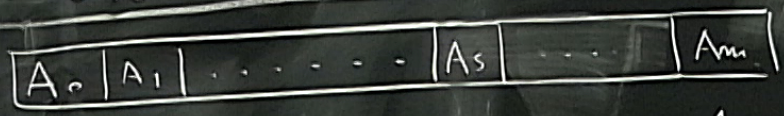
$g_{A_j} = 0 \Rightarrow \ell \in C_{n-1}$.


proof
$g_A + g_{\bar{A}} = 2k$
$g_{A_j} = 0 \neq g_{\bar{A}_j} = 2k$
• ℓ supported on $s+1$ regions then ℓ belongs to C_s
A_0 A_1 A_s A_m
$\ell = \ell_0 \ell_1 \ell_s$

proof

- $g_A + g_{\bar{A}} = 2k$
- $g_{A_j} = 0 \neq g_{\bar{A}_j} = 2k$

- ℓ supported on $s+1$ regions then ℓ belongs to C_s



$\ell = \ell_0 \otimes \ell_1 \otimes \cdots \otimes \ell_s \otimes \mathbb{1} \otimes \cdots \mathbb{1}$

Pauli logical op $\longrightarrow \bar{P} = \mathbb{1} \otimes P_1 \otimes \cdots \otimes P_s \otimes \cdots P_m$

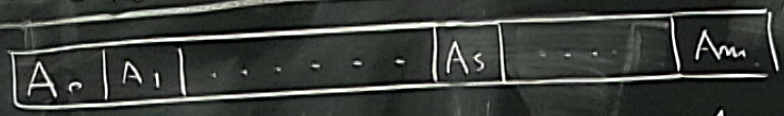
$k(\ell, \bar{P}) = \mathbb{1} \otimes \mathbb{1} \otimes 0 \cdots \otimes \mathbb{1} \otimes \mathbb{1} \cdots \mathbb{1}$

proof

- $g_A + g_{\bar{A}} = 2k$

$g_{A_j} = 0 \neq g_{\bar{A}_j} = 2k$

- ℓ supported on $s+1$ regions then ℓ belongs to C_s



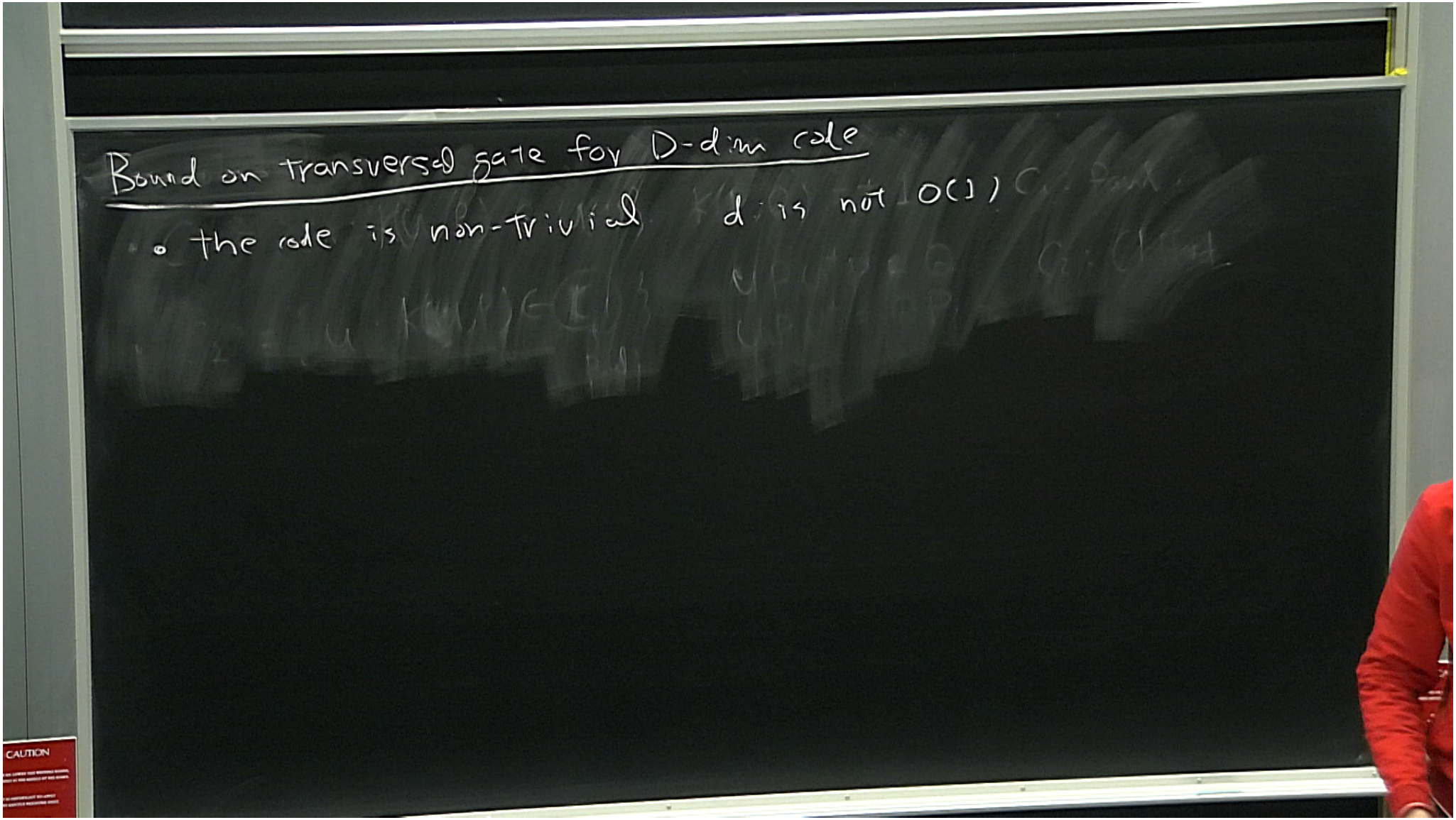
$\ell = \ell_0 \otimes \ell_1 \otimes \cdots \ell_s \otimes 1 \cdots 1$

Pauli logical op $\longrightarrow \bar{P} = 1 \otimes P_1 \otimes \cdots P_s \otimes \cdots P_m$

$K(\ell, \bar{P}) = 1 \otimes \cdots \otimes 1 \cdots 1$

$K(\ell_1, P_1)$ $K(\ell_2, P_2)$ — s regions.

$K(\ell, \bar{P}) \in C_{s-1} \Rightarrow \ell \in C_s$.


Bound on transversal gate for D-dim code
• the code is non-trivial d is not O(1)
CAUTION

Bound on transversal gate for D-dim code

- the code is non-Trivial d is not $O(1)$
- Transversal gate $\in C_D$.

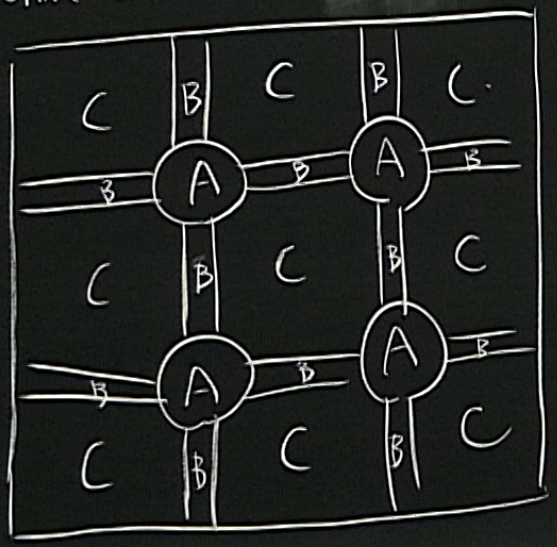
Bound on transversal gate for D-dim code

Theorem

- The code is non-Trivial. d is not $O(1)$
- Transversal gate $\in C_D$.
  - D-dim color code $R_D \in C_D$.

proof

($D=2$)



Bound on transversal gate for D-dim code

Theorem

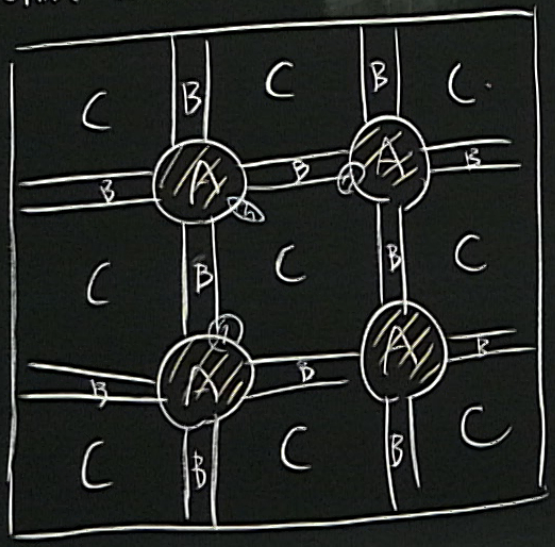
- The code is non-Trivial. d is not $O(1)$
- Transversal gate $\in C_D$.

– D-dim color code $R_D \in C_D$.

Region A $\quad \partial_A = 0$

proof

$(D=2)$



Bound on transversal gate for D-dim code

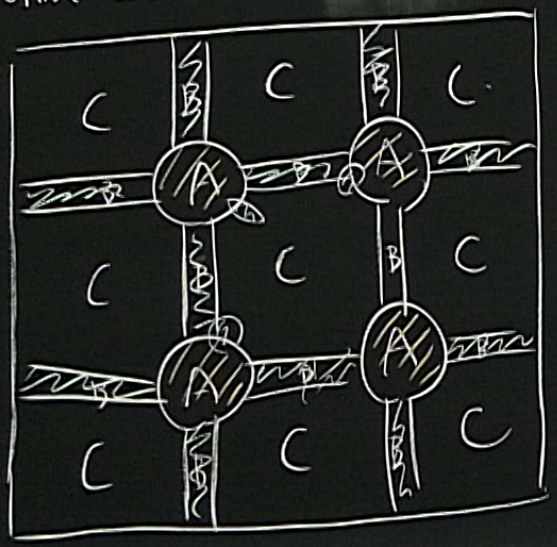
Theorem

- The code is non-Trivial d is not $O(1)$
- Transversal gate $\in C_D$.

– D-dim color code $R_D \in C_D$.

proof

($D=2$)



Region A $g_A = 0$

Region B $g_B = 0$.

$g_C = 0$.

$\Rightarrow l \in C_2$

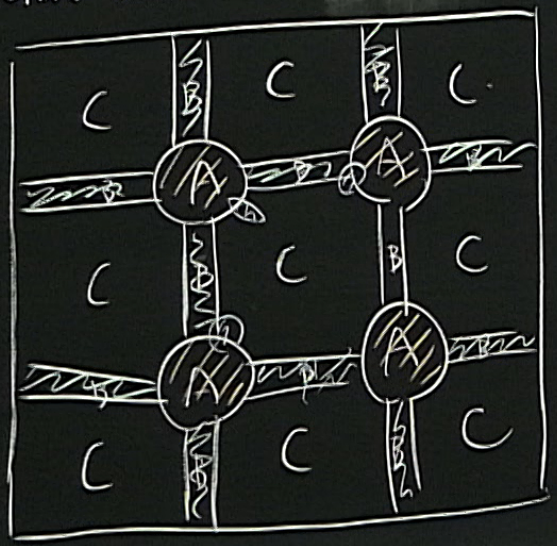
3 subsystems

Bound on transversal gate for D-dim code

Theorem
- The code is non-Trivial (d is not $O(1)$)
- Transversal gate $\in C_D$.

– D-dim color code $R_D \in C_D$.

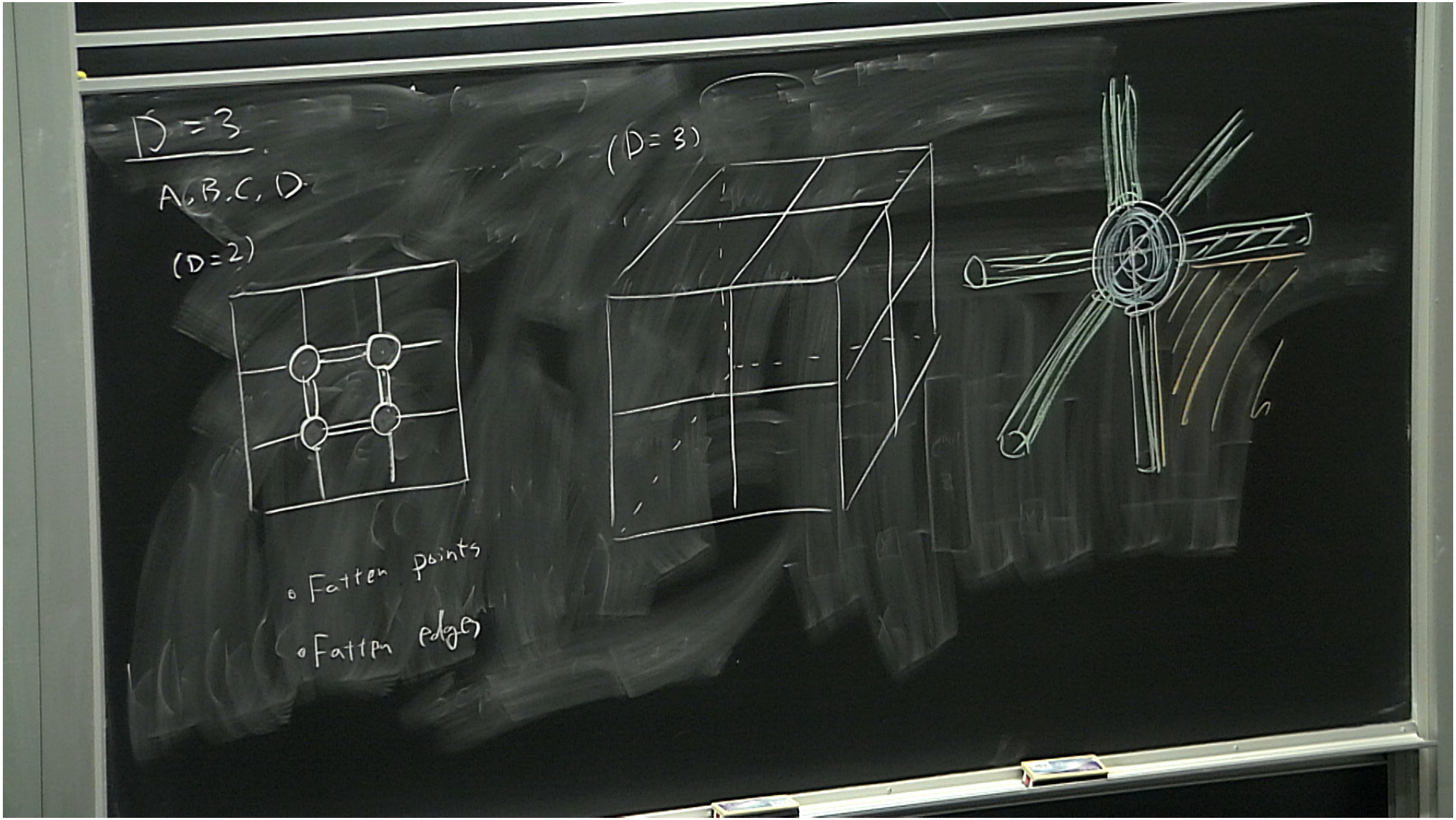
proof
($D=2$)

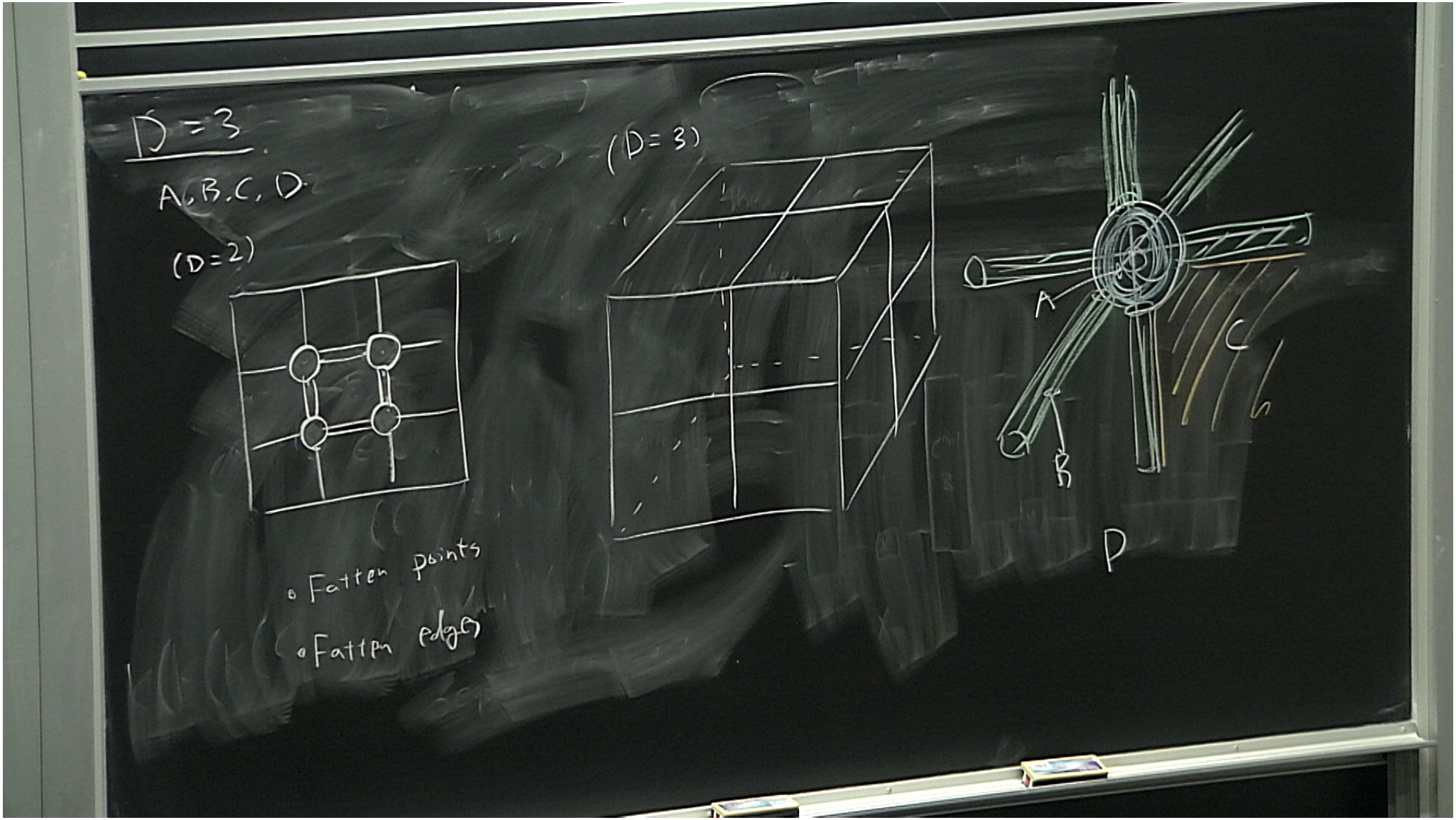


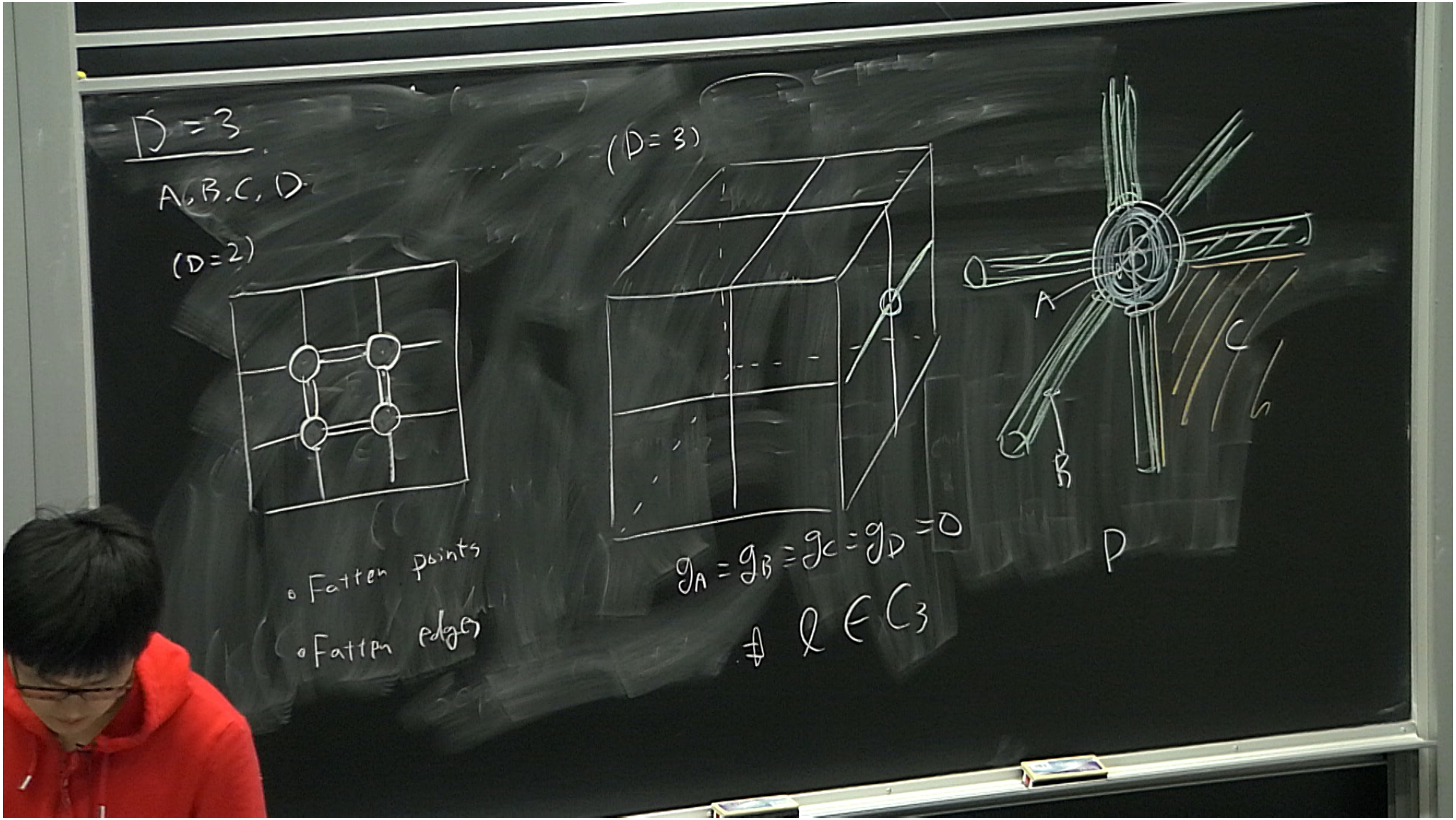
Region A $g_A = 0$
Region B $g_B = 0$
Region C $g_C = 0$
} 3 subsystems

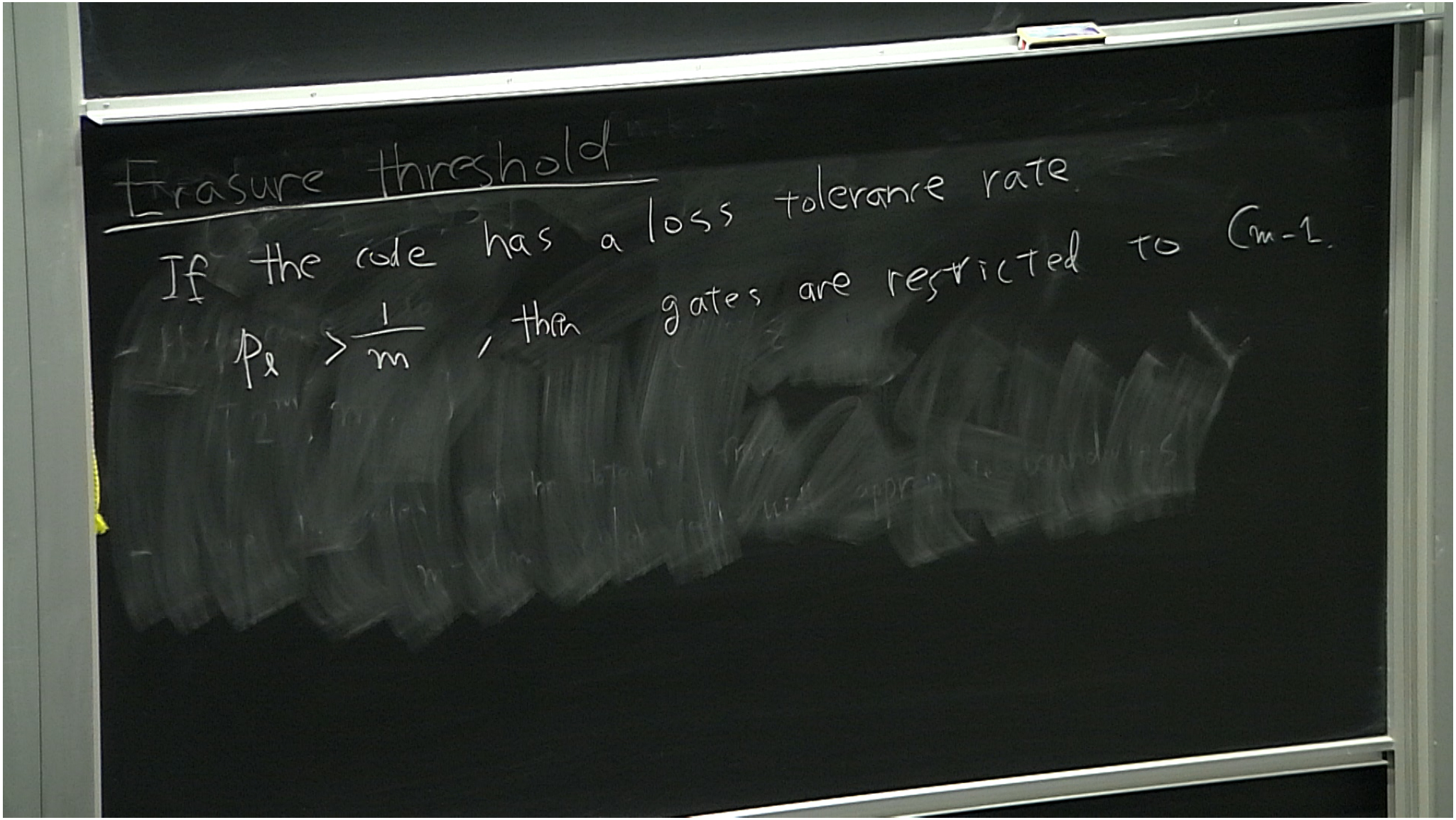
$\nexists\ l \in C_2$

$g_A + g_{\bar{A}} = 2k$


D=3
A, B, C, D
(D=2)
(D=3)
• Fatten points
• Fatten edges


D=3
A, B, C, D
(D=2)
(D=3)
A
C
B
D
Fatten points
Fatten edges

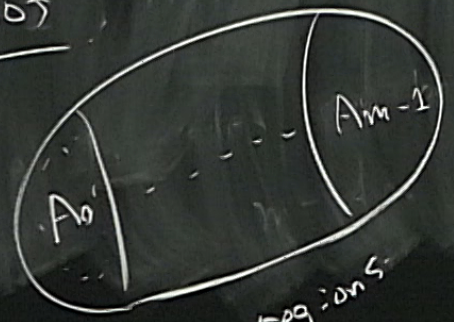

D=3
A, B, C, D
(D=2)
(D=3)
A
C
B
D
• Fatten points
• Fatten edges
$g_A = g_B = g_C = g_D = 0$
$\nexists\ \ell \in C_3$


Erasure threshold
If the code has a loss tolerance rate
$p_\ell > \frac{1}{m}$, then gates are restricted to C_{m-1}.

Erasure threshold

If the code has a loss tolerance rate $p_l > \frac{1}{m}$, then gates are restricted to C_{m-1}.

proof:



qubits on A_0 are chosen randomly with prob $\frac{1}{m}$

$g_{A_0} = 0$

$g_{A_j} = 0$

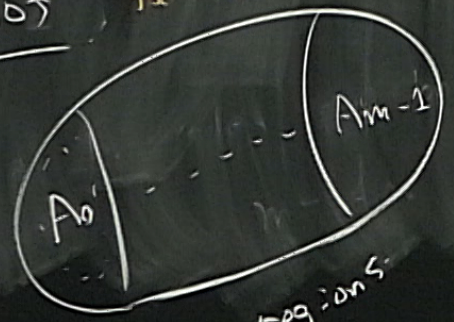
(If $g_{A_0} > 0$, then, we lose QI by losing qubits on A_0.

$\Rightarrow g \in C_{m-1}$

Erasure threshold

If the code has a loss tolerance rate $p_\ell > \frac{1}{m}$, then gates are restricted to C_{m-1}.

proof: $p_\ell = 0.5$



qubits on A_0 are chosen randomly with prob $\frac{1}{m}$

$g_{A_0} = 0$

$g_{A_j} = 0$

(If $g_{A_0} > 0$, then, we lose QI by losing qubits on A_0

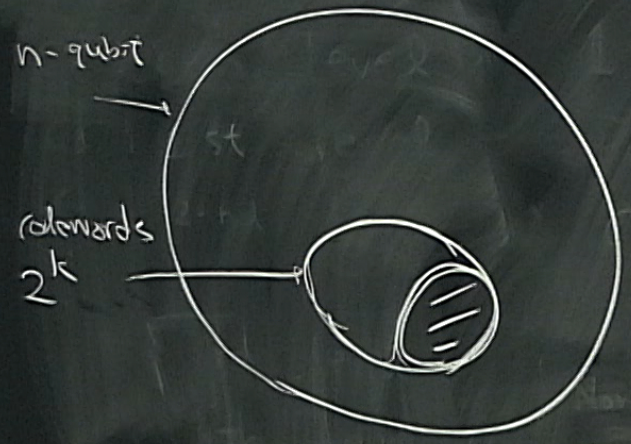
$\Rightarrow g \in C_{m-1}$

Subsystem code

- Stabilizer code $S = \langle S_1, S_2, \cdots S_{n-k} \rangle$

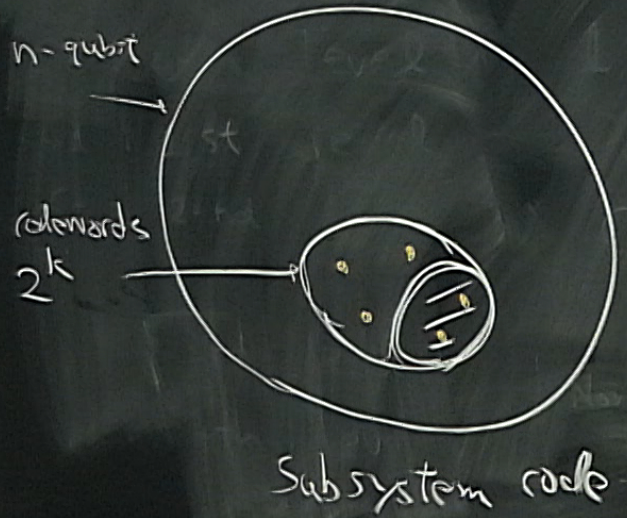
k : large

we use a part of codeword space.



Subsystem code

- Stabilizer code $S = \langle S_1, S_2, \cdots S_{n-k} \rangle$



k : large

we use a part of codeword space.

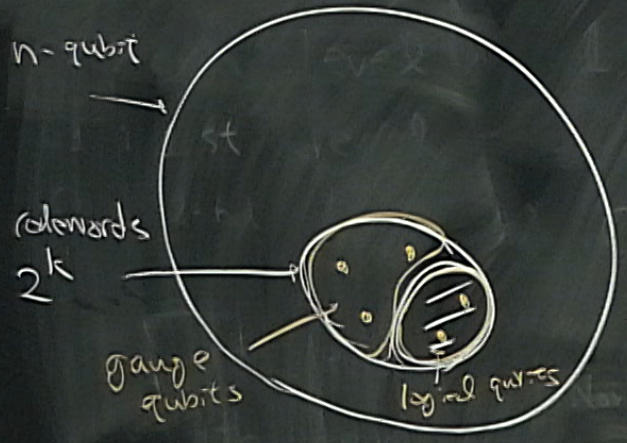
$$\mathcal{H} = \bigoplus_{(S_1 \cdots S_{n-k})} \mathcal{H}_{(S_1 \cdots S_{n-k})}$$

$$\mathcal{H}_{code} = \mathcal{H}_{(+1, \cdots +1)}$$

$$\mathcal{H}_{code} = \mathcal{H}_{gauge} \otimes \mathcal{H}_{logical}$$

Subsystem code

- Stabilizer code $S = \langle S_1, S_2, \cdots S_{n-k} \rangle$



Subsystem code

k : large

we use a part of codeword space.

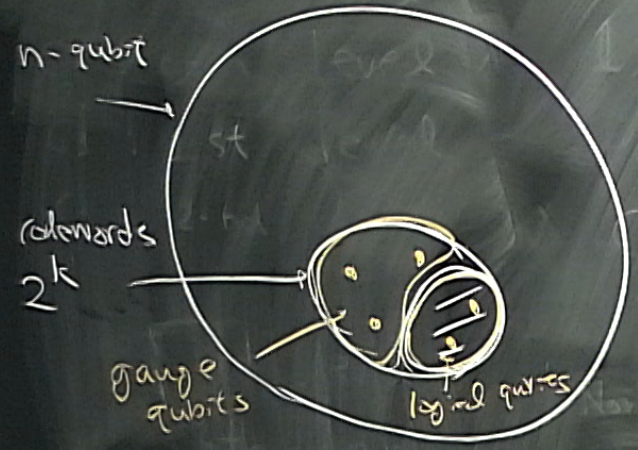
$$\mathcal{H} = \bigoplus_{(s_1 \cdots s_{n-k})} \mathcal{H}_{(s_1 \cdots s_{n-k})}$$

$$\mathcal{H}_{code} = \mathcal{H}_{(+1, \cdots +1)}$$

$$\mathcal{H}_{code} = \mathcal{H}_{gauge} \otimes \mathcal{H}_{logical}$$

Subsystem code

- Stabilizer code $S = \langle S_1, S_2, \cdots S_{n-k} \rangle$



Subsystem code

k : large

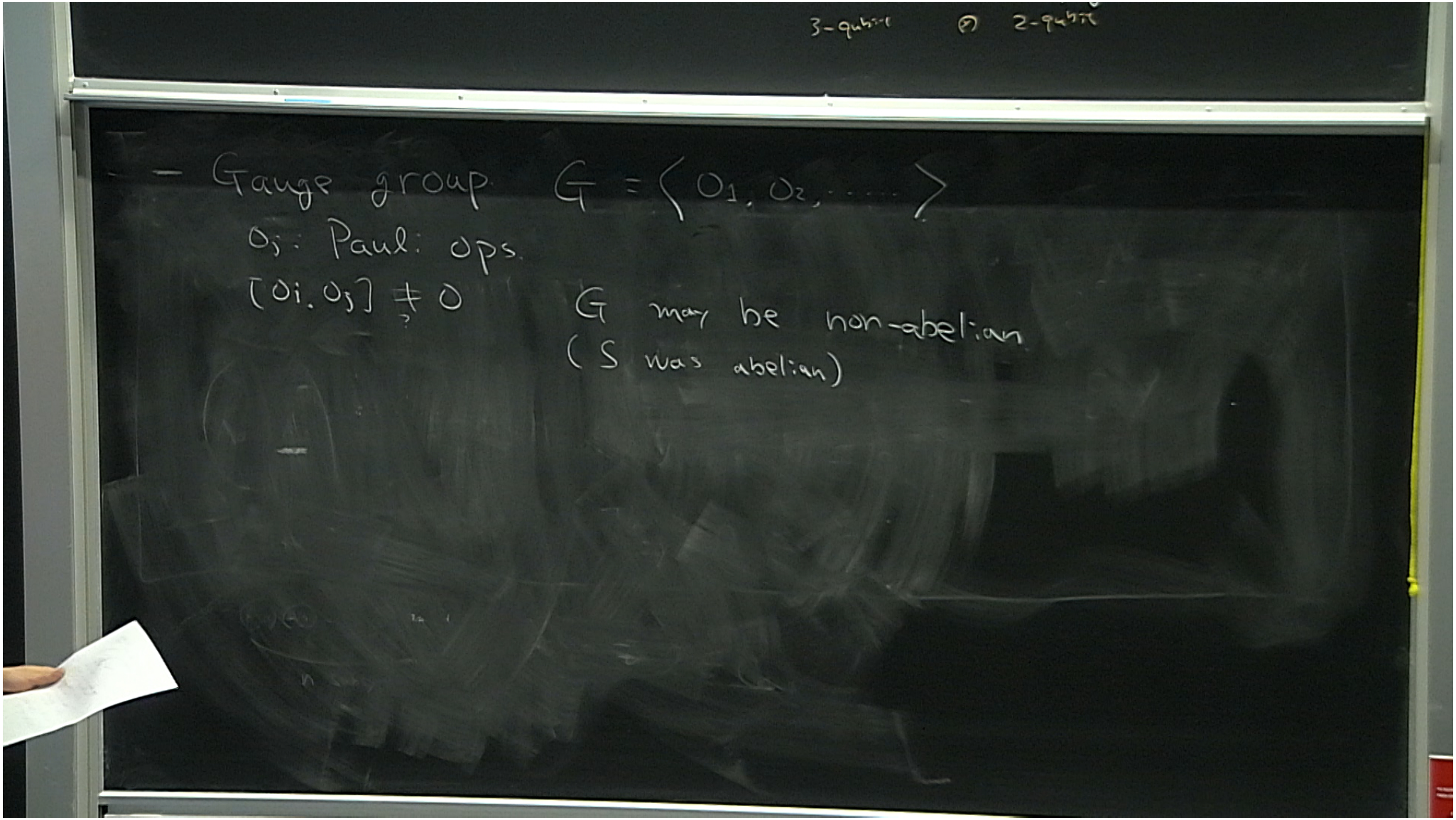
we use a part of codeword space.

$$\mathcal{H} = \bigoplus_{(S_1 \cdots S_{N-k})} \mathcal{H}_{(S_1 \cdots S_{n-k})}$$

$$\mathcal{H}_{code} = \mathcal{H}_{(+1, \cdots +1)}$$

$$\mathcal{H}_{code} = \mathcal{H}_{gauge} \otimes \mathcal{H}_{logical}$$

3-qubit $\otimes$ 2-qubit


3-qubit
2-qubit
- Gauge group $G = \langle O_1, O_2, \ldots \rangle$
O_j: Pauli ops.
$[O_i, O_j] \neq 0$
G may be non-abelian
(S was abelian)

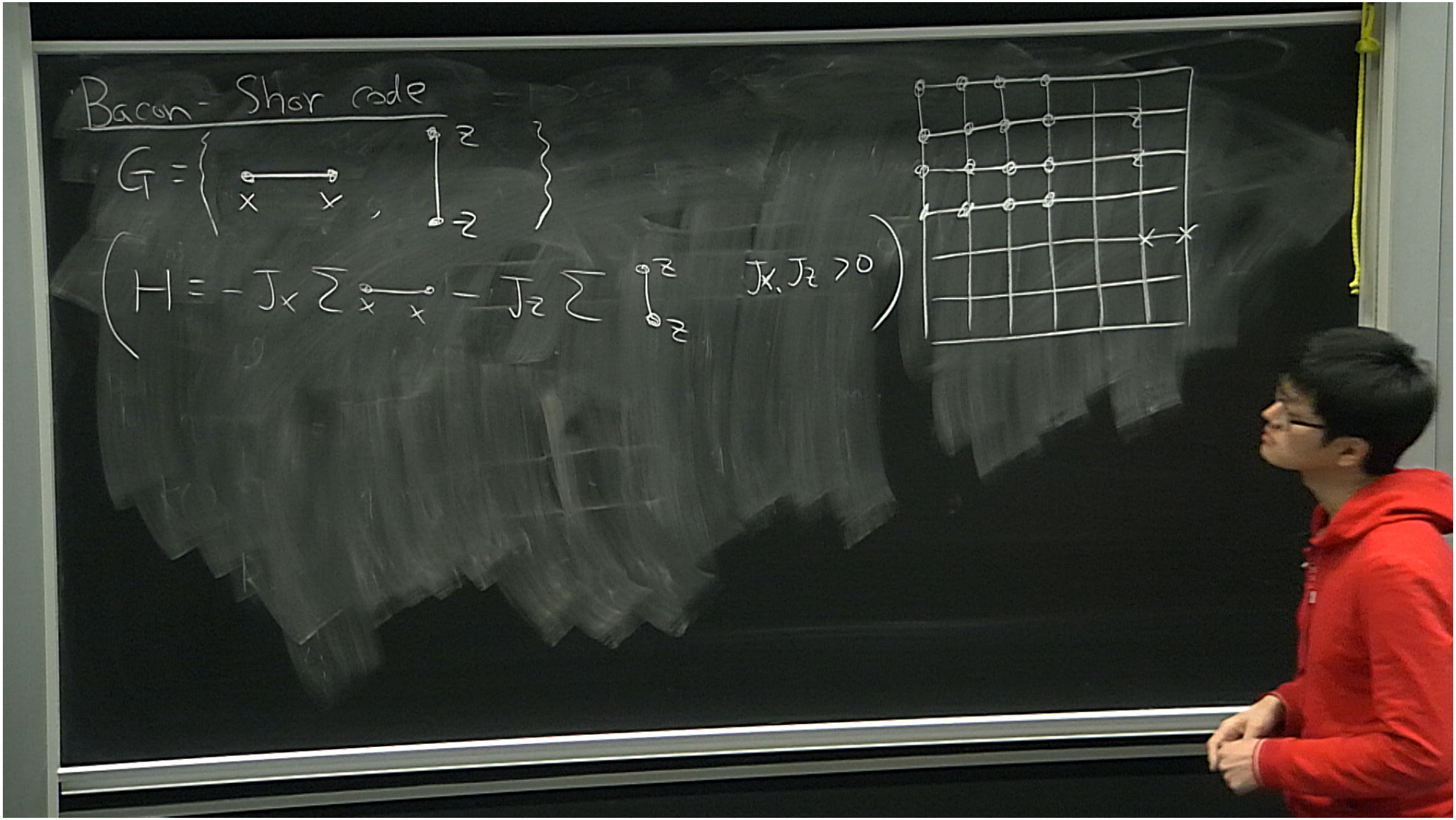
3-qubit ⊗ 2-qubit

- Gauge group $G = \langle O_1, O_2, \ldots \rangle$
  O_j: Pauli ops.
  $[O_i, O_j] \neq 0$
  G may be non-abelian
  (S was abelian)
- "Stabilizer group" $S \subseteq G$ $S = \langle S_1, \ldots, S_{n-m} \rangle$
  - center of G.
  $\begin{cases} [S_i, S_j] = 0 \\ [S_j, G] = 0 \end{cases}$

3-qubit ⊗ 2-qubit

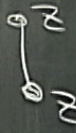
- Gauge group $G = \langle O_1, O_2, \ldots \rangle$
  O_j: Pauli ops.
  $[O_i, O_j] \neq 0$
  G may be non-abelian
  (S was abelian)
- "Stabilizer group" $S \subseteq G$ $S = \langle S_1, \ldots, S_{n-m} \rangle$
  = center of G.
  $[S_i, S_j] = 0$.
  $[S_j, G] = 0$
- Codeword space

$$P_S = \{|\psi\rangle : S_j|\psi\rangle = |\psi\rangle\}$$

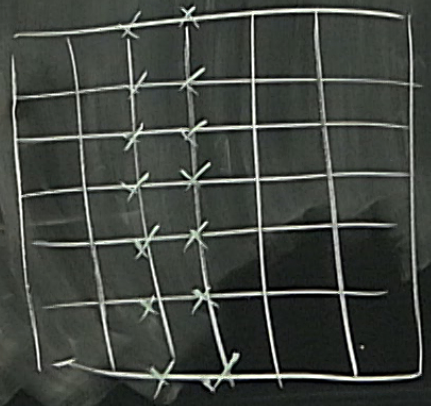

Bacon - Shor code
G = { X X , Z Z }
(H = - J_x Σ X X - J_z Σ Z Z   J_x, J_z > 0)

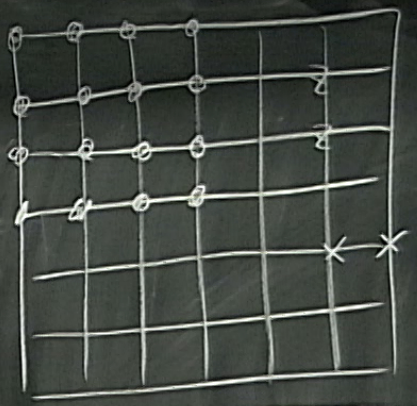
Bacon-Shor code

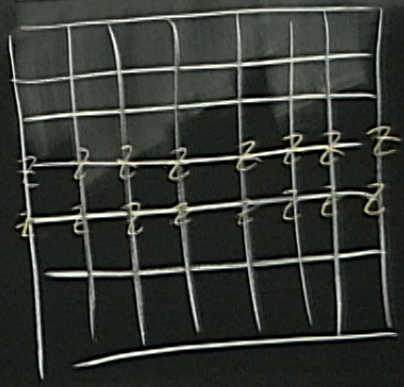
$G = \{$  $\}$

$\left(H = -J_x \sum$  $- J_z \sum$  $\quad J_x, J_z > 0 \right)$

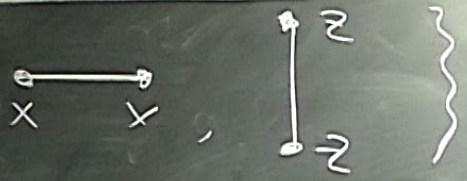
- Stabilizer group

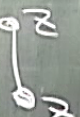




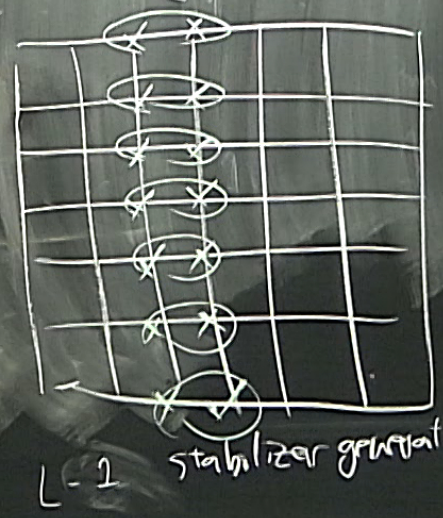


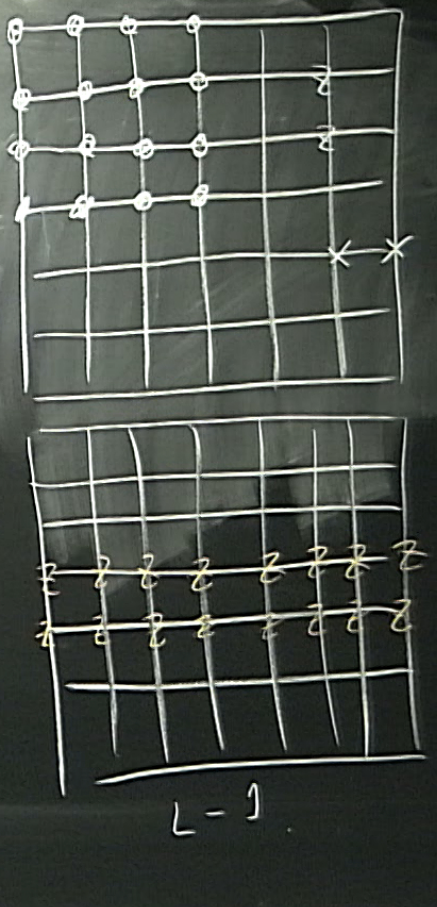
Bacon-Shor code

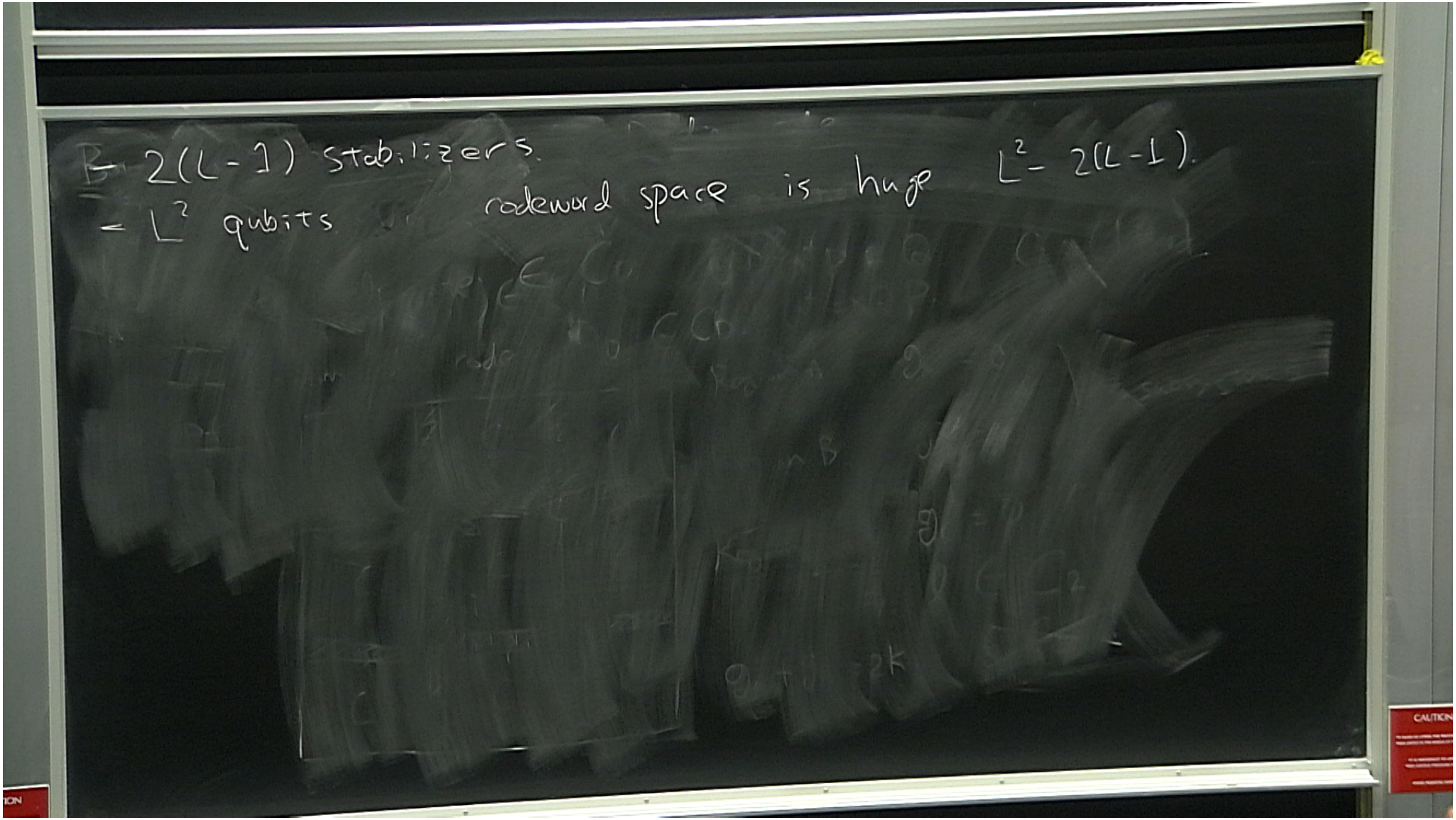
$G = \{$  $\}$

$\left(H = -J_x \sum$  $- J_z \sum$  $\quad J_x, J_z > 0 \right)$

- Stabilizer group
  - ✓ commute with G
  - ⊗ inside G






2(L-1) stabilizers.
L^2 qubits
codeword space is huge $L^2 - 2(L-1)$.

$2(L-1)$ stabilizers.

L^2 qubits codeword space is huge $L^2 - 2(L-1)$.

Hamiltonian Pground state satisfies $S_j = +1$ or not

but for CSS subsystem code, (terms are X only or Z only)

then $-H$. is a matrix with positive entries

$2(L-1)$ stabilizers

L^2 qubits codeword space is huge $L^2 - 2(L-1)$.

Hamiltonian Pground state satisfies $S_j = +1$ or not

but for CSS subsystem code. (terms are X only or Z only)

then $-H$. is a matrix with positive entries

Perron-Frobenius theorem

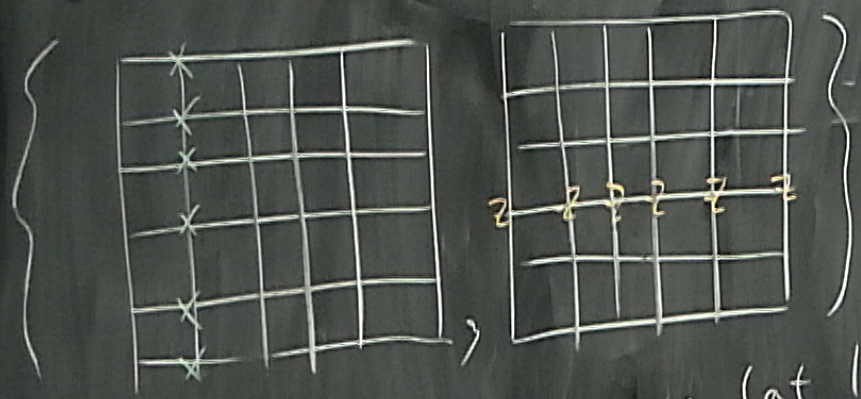
gs of H $|gs\rangle = \sum_{i_1 \ldots i_n} (\text{positive}) |i_1 \ldots i_n\rangle$

$\therefore$ Pground $\subseteq$ Ps.

Bare & dressed logical ops

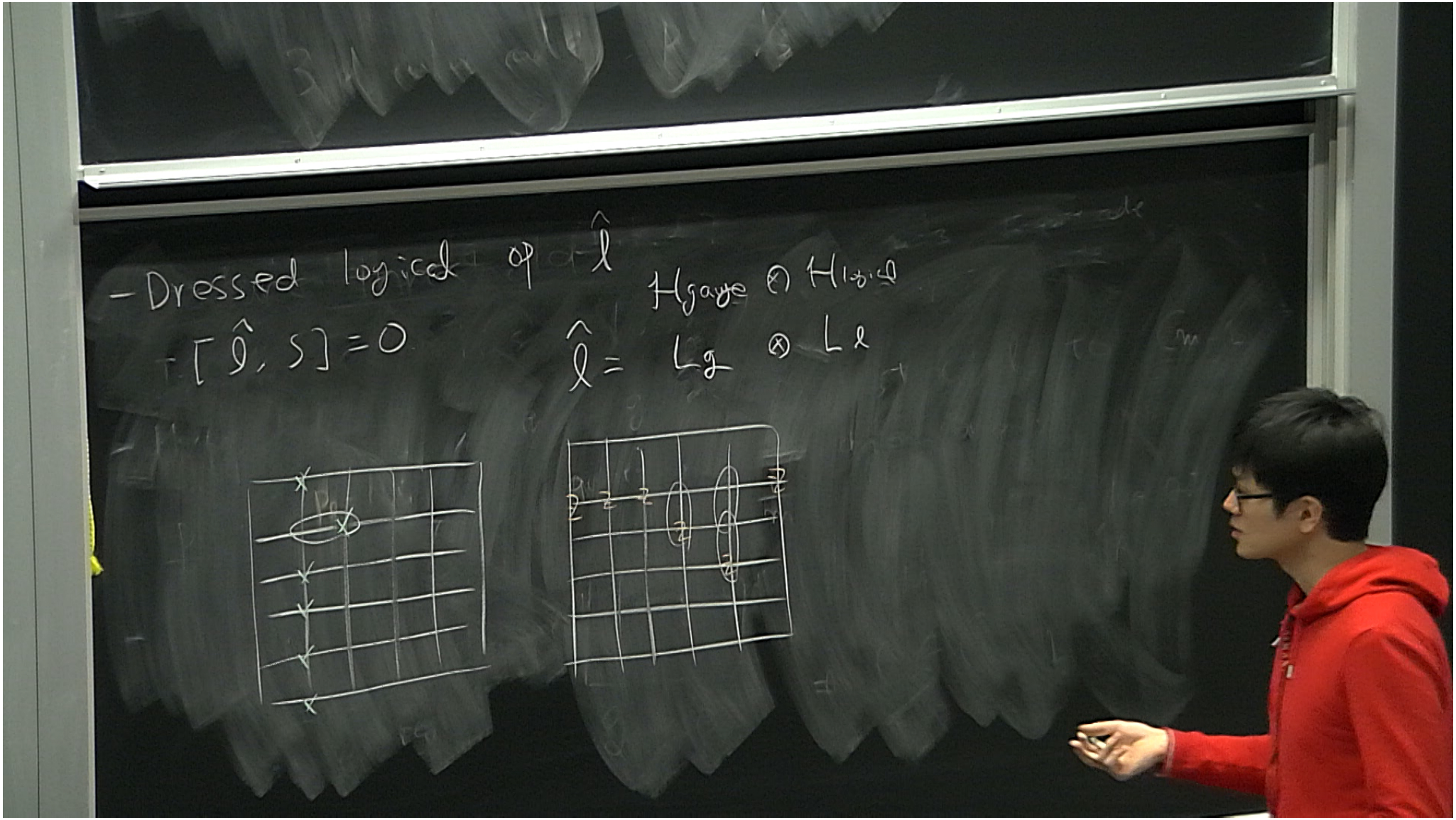
- Bare logical op $\bar{l}$ $\quad \mathcal{H}_{gauge} \otimes \mathcal{H}_{logical}$

$[\bar{l}, G] = 0 \quad \bar{l} = 1 \otimes L_{\bar{l}}$



$= 0.$

$\Rightarrow$ Hamiltonian is (at least) 2-fold degenerate.

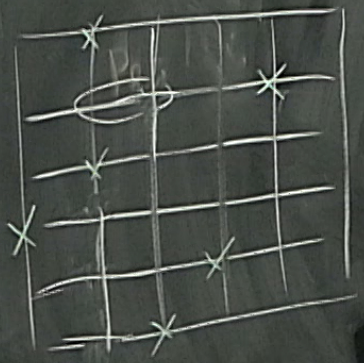

- Dressed logical op $\hat{l}$
$[\hat{l}, S] = 0$
$H_{gauge} \otimes H_{logical}$
$\hat{l} = L_g \otimes L_l$

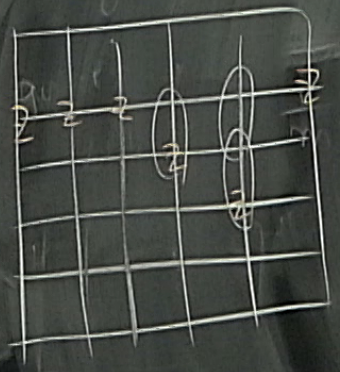
- Dressed logical $\hat{l}$

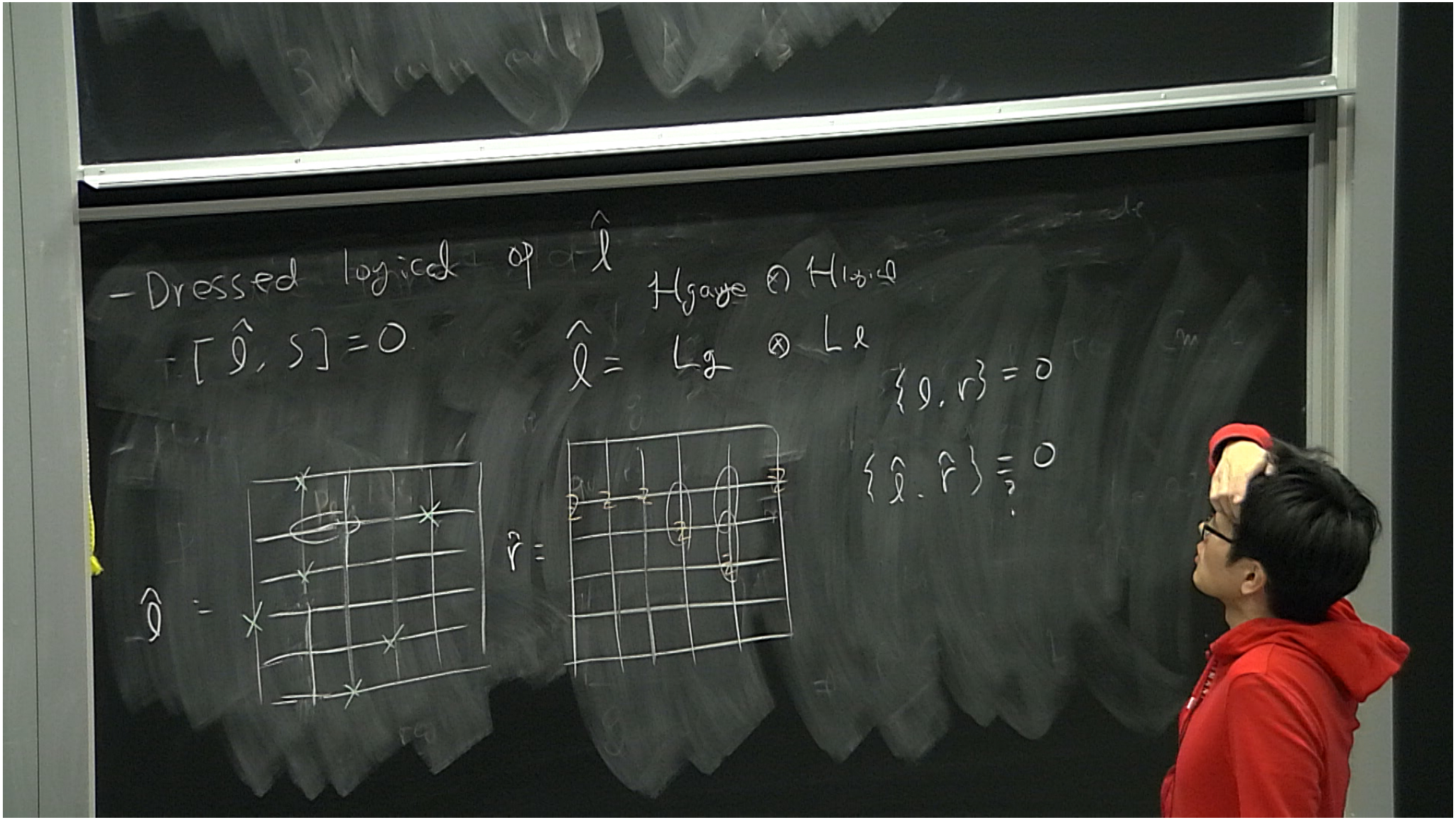
$\hat{l} = L_g \otimes L_e$

$H_{gauge} \otimes H_{logical}$

$-[\hat{l}, S] = 0$






- Dressed logical op $\hat{l}$
$[\hat{g}, S] = 0$
$\hat{l} = L_g \otimes L_\ell$
$\{g, r\} = 0$
$\{\hat{l}, \hat{r}\} = 0$
$\hat{g} =$
$\hat{r} =$

- Dressed logical qubit $\hat{l}$

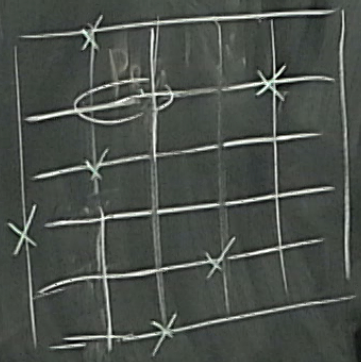
$\mathcal{H}_{gauge} \otimes \mathcal{H}_{logical}$

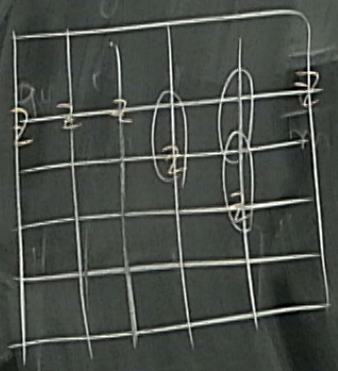
$[\hat{g}, S] = 0$

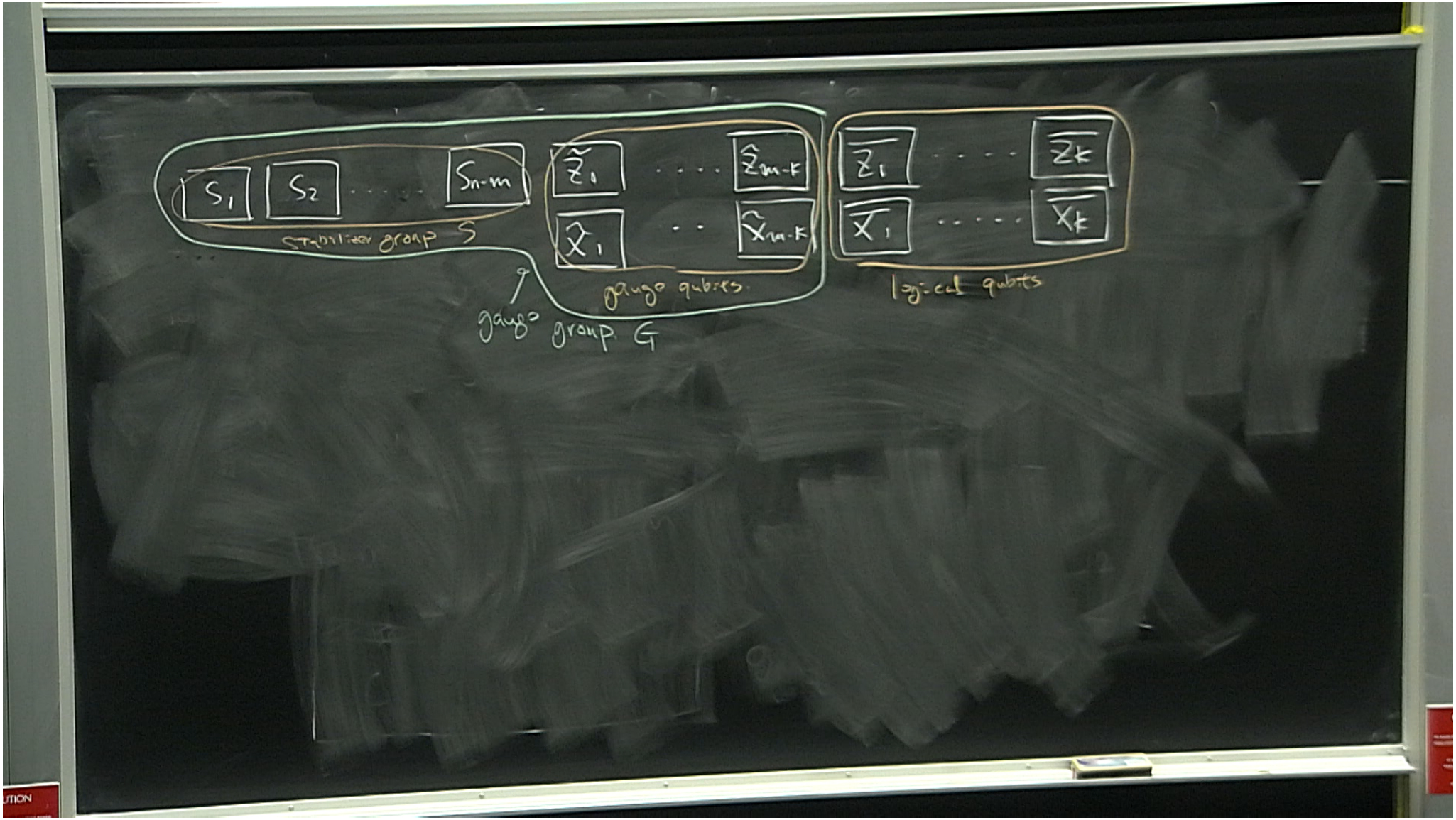
$\hat{l} = L_g \otimes L_l$

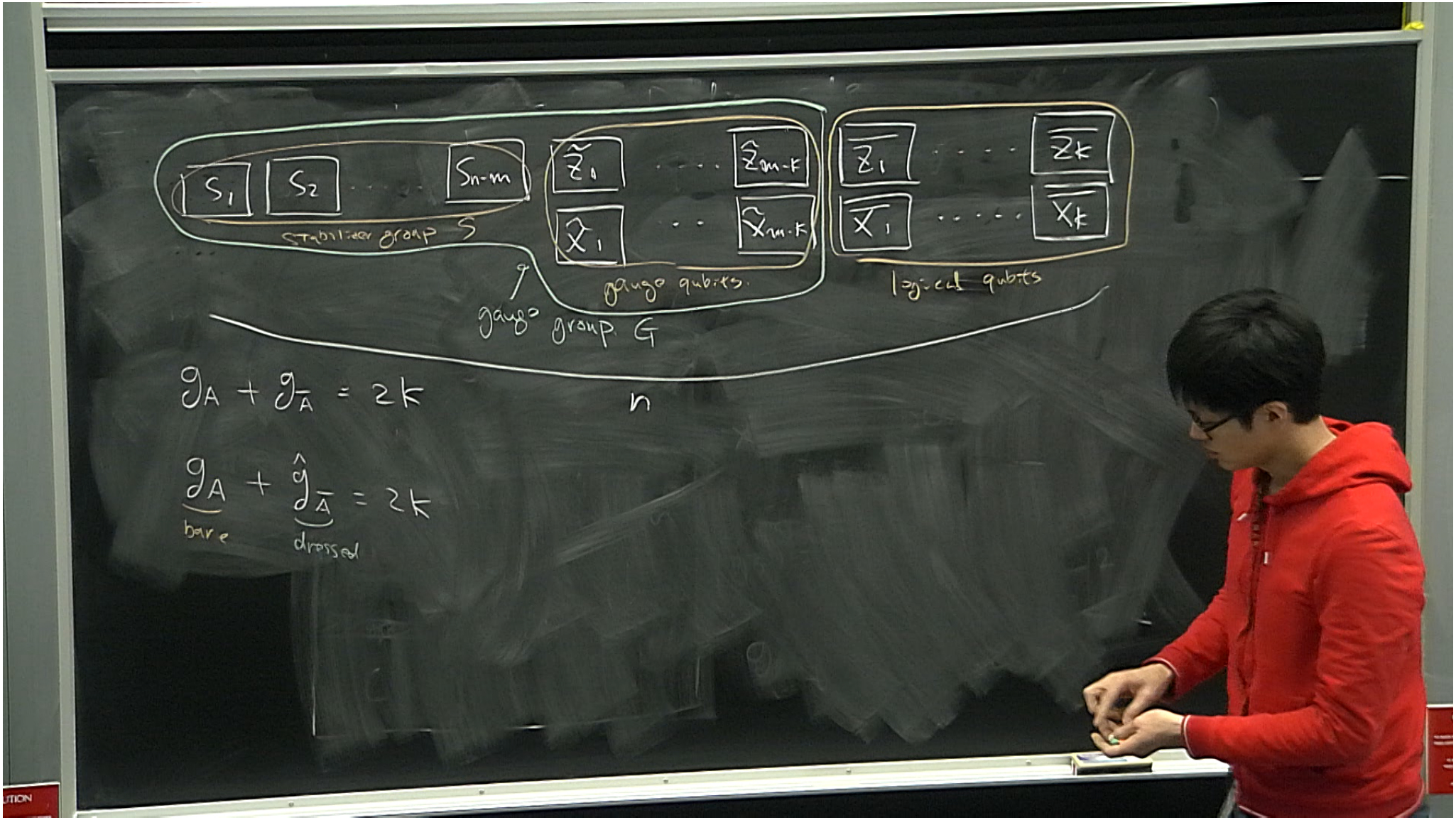
$\{\hat{g}, \hat{r}\} = 0$

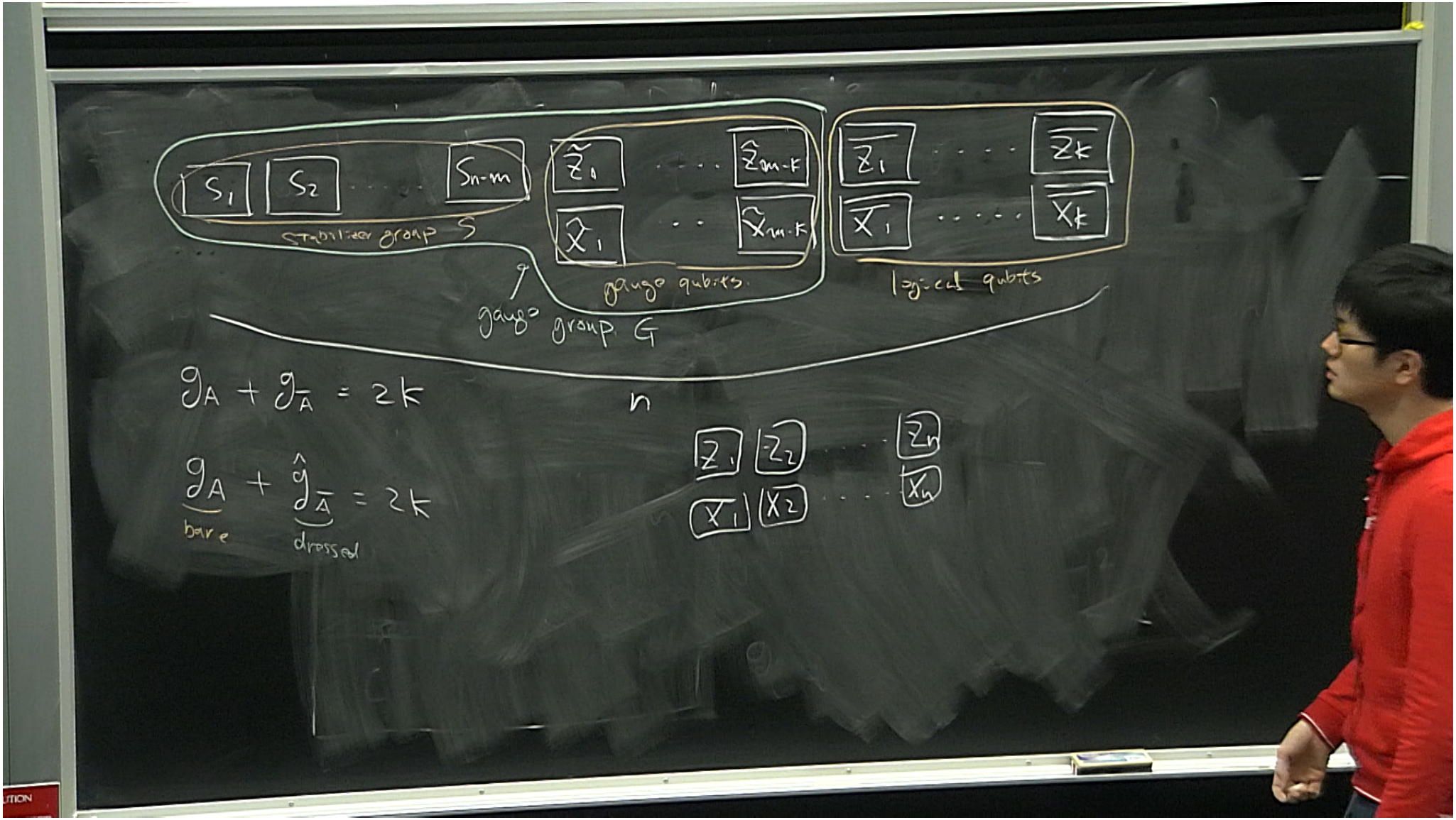
$\{\hat{l}, \hat{r}\} = 0$

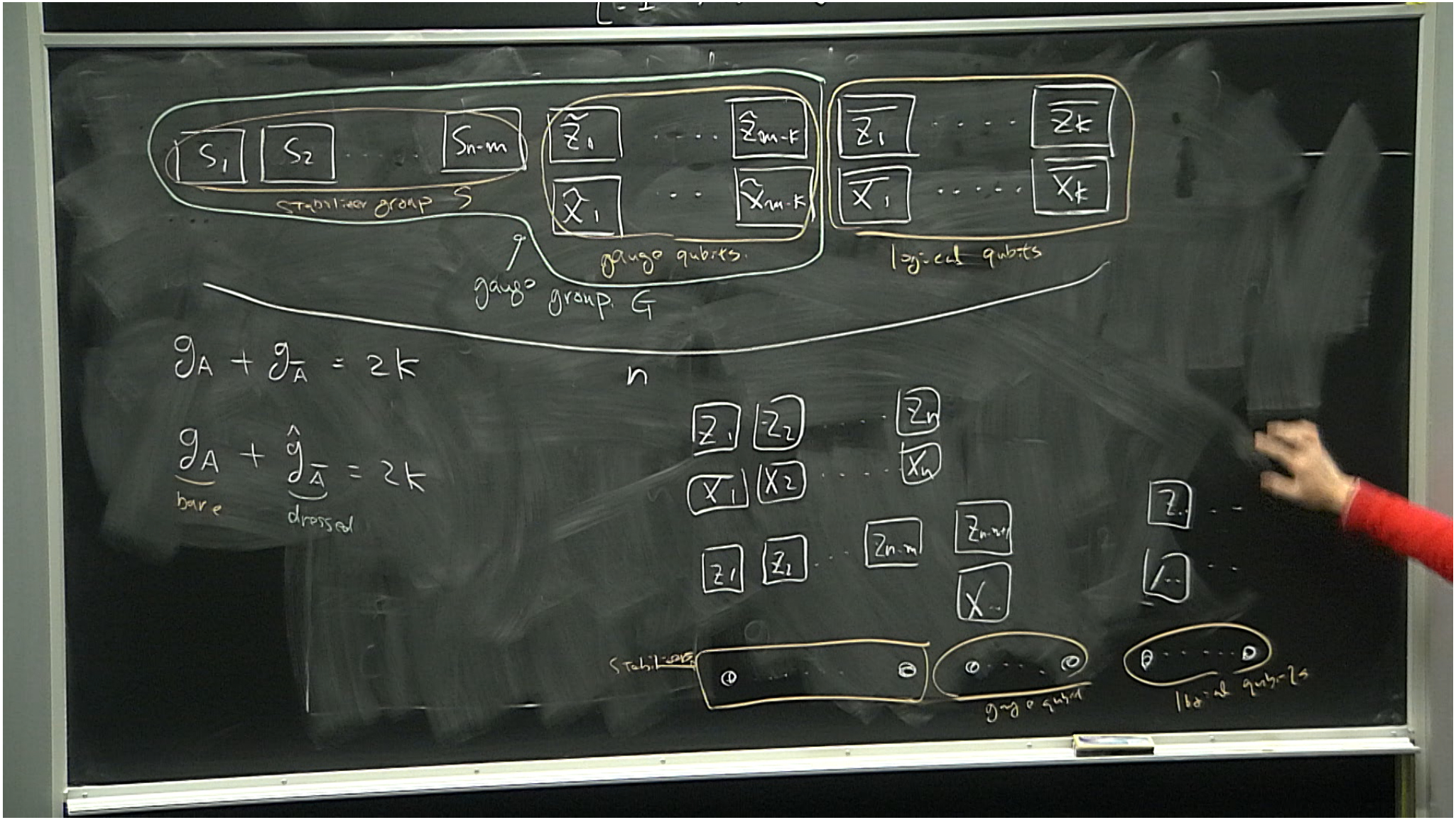
$\hat{g} =$ 

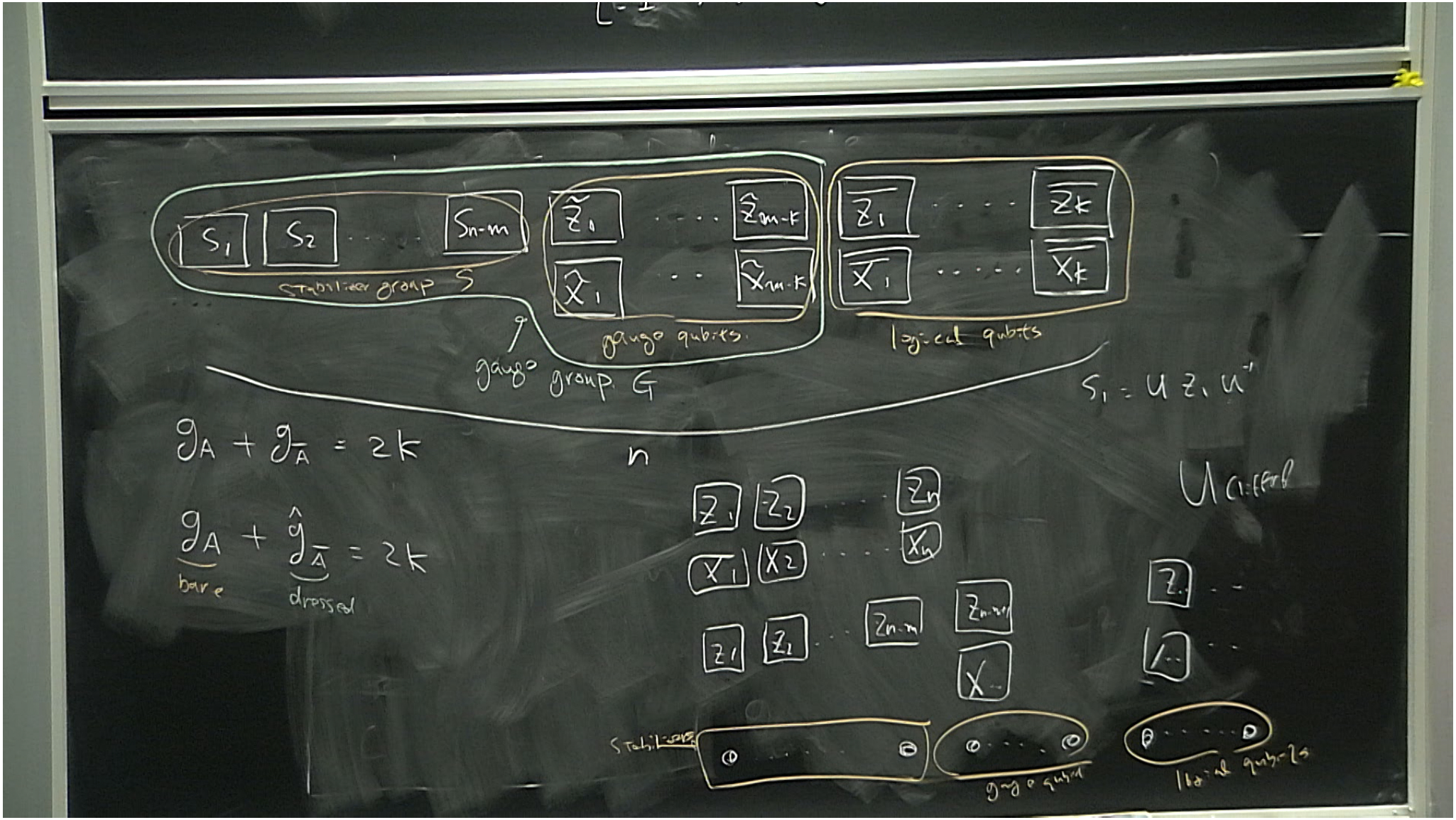
$\hat{r} =$ 

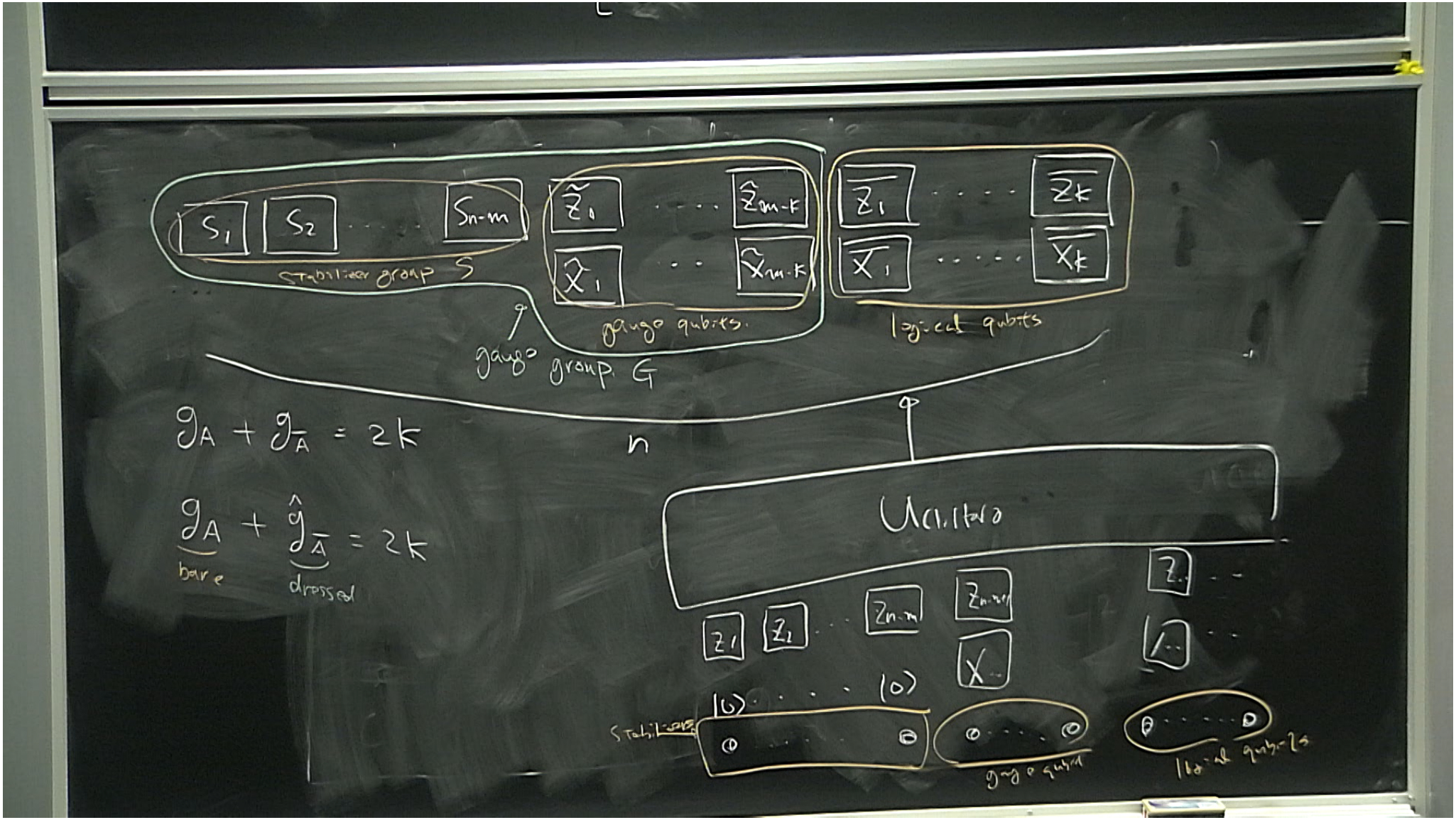

S_1
S_2
S_{n-m}
stabilizer group S
$\tilde{Z}_1$
$\hat{Z}_{m-k}$
$\tilde{X}_1$
$\hat{X}_{m-k}$
gauge qubits
gauge group G
$\bar{Z}_1$
$\bar{Z}_k$
$\bar{X}_1$
$\bar{X}_k$
logical qubits


S_1 S_2 S_{n-m}
stabilizer group S
$\tilde{Z}_1$ $\hat{Z}_{m-k}$
$\tilde{X}_1$ $\hat{X}_{m-k}$
gauge qubits
$\bar{Z}_1$ $\bar{Z}_k$
$\bar{X}_1$ $\bar{X}_k$
logical qubits
gauge group G
n
$g_A + g_{\bar{A}} = 2k$
$g_A + \hat{g}_{\bar{A}} = 2k$
bare
dressed


S_1 S_2 S_{n-m}
stabilizer group S
$\tilde{Z}_1$ $\tilde{Z}_{m-k}$
$\tilde{X}_1$ $\tilde{X}_{m-k}$
gauge qubits
$\bar{Z}_1$ $\bar{Z}_k$
$\bar{X}_1$ $\bar{X}_k$
logical qubits
gauge group G
$g_A + g_{\bar{A}} = 2k$
n
Z_1 Z_2 Z_n
X_1 X_2 X_n
$g_A + \hat{g}_{\bar{A}} = 2k$
bare
dressed


S_1 S_2 S_{n-m}
stabilizer group S
$\tilde{Z}_1$ $\hat{Z}_{m-k}$
$\tilde{X}_1$ $\hat{X}_{m-k}$
gauge qubits
$\bar{Z}_1$ $\bar{Z}_k$
$\bar{X}_1$ $\bar{X}_k$
logical qubits
gauge group G
n
$g_A + g_{\bar{A}} = 2k$
$g_A + \hat{g}_{\bar{A}} = 2k$
bare
dressed
Z_1 Z_2 Z_n
X_1 X_2 X_n
Z_1 Z_1 Z_{n-m}
Z_{n-m}
X
stabilizer
gauge qubits
logical qubits


S_1 S_2 S_{n-m}
Stabilizer group S
$\tilde{Z}_1$ $\hat{Z}_{m-k}$
$\tilde{X}_1$ $\hat{X}_{m-k}$
gauge qubits
$\bar{Z}_1$ $\bar{Z}_k$
$\bar{X}_1$ $\bar{X}_k$
logical qubits
gauge group G
n
$S_1 = U Z_1 U^\dagger$
$g_A + g_{\bar{A}} = 2k$
$\underbrace{g_A}_{\text{bare}} + \underbrace{\hat{g}_{\bar{A}}}_{\text{dressed}} = 2k$
Z_1 Z_2 Z_n
X_1 X_2 X_n
Z_1 Z_1 Z_{n-m}
Z_{n-m}
X
U Clifford
Stabilizers
gauge qubits
logical qubits

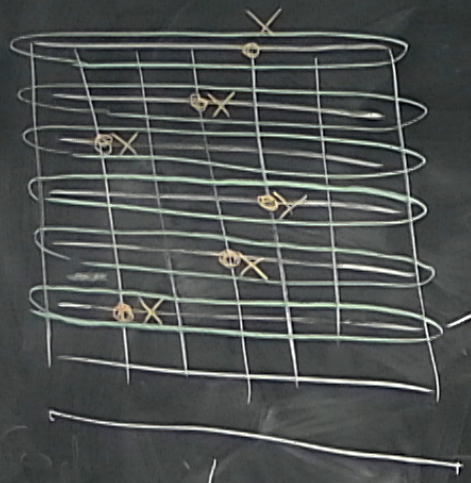

S_1 S_2 S_{n-m}
stabilizer group S
$\tilde{Z}_1$ $\hat{Z}_{m-k}$
$\tilde{X}_1$ $\hat{X}_{m-k}$
gauge qubits
gauge group G
$\overline{Z}_1$ $\overline{Z}_k$
$\overline{X}_1$ $\overline{X}_k$
logical qubits
$g_A + g_{\bar{A}} = 2k$
n
$\underline{g_A} + \underline{\hat{g}_{\bar{A}}} = 2k$
bare
dressed
Unitary
Z_1 Z_1 Z_{n-m} Z_{n-m} X
$|0\rangle$ $|0\rangle$
stabilizers
gauge qubits
logical qubits

3-qubit ⊗ 2-qubit

Bacon-Shor code

Erasure threshold is $p_L \to 0$ as $N \to \infty$

p : loss probability $< \frac{1}{2}$



pL qubits missing