

Title: IQC - Quantum Error Correction - Lecture 9

Date: Mar 08, 2018 02:00 PM

URL: <http://pirsa.org/18030010>

Abstract:

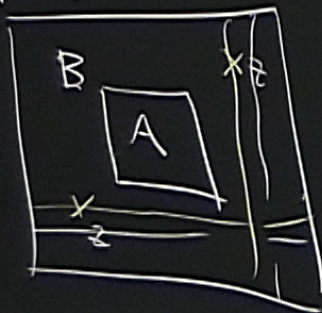
Lemma (duality)

Consider an arbitrary stab code with k logical qubits
 $A, B = A^\perp$: subsystems

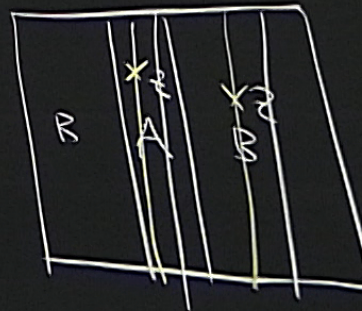
g_A : # of independent non-trivial logical ops

then $g_A + g_B = 2k$

eg) Toric code on a torus
($k=2$)



$$\begin{aligned}g_A &= 0 \\g_B &= 4 \\g_A + g_B &= 4\end{aligned}$$



$$\begin{aligned}g_A &= 2 \\g_B &= 2 \\g_A + g_B &= 4\end{aligned}$$

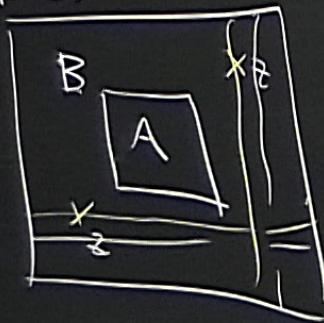
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Consider an arbitrary stab code with k logical qubits
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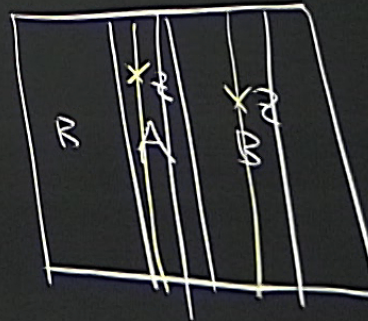
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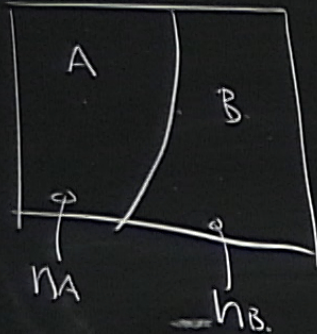


$$\begin{aligned} g_A &= 0 \\ g_B &= 4 \\ g_A + g_B &= 4 \end{aligned}$$



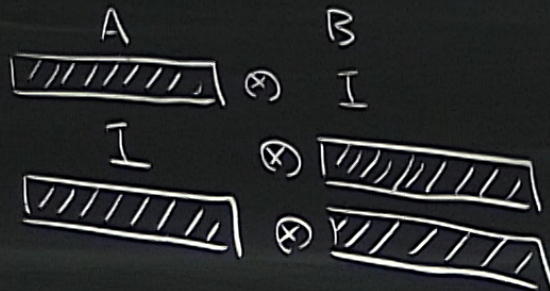
$$\begin{aligned} g_A &= 2 \\ g_B &= 2 \\ g_A + g_B &= 4 \end{aligned}$$

Proof



$$n_A + n_B = n$$

$n-k$ independent stabilizer generators



number of generators

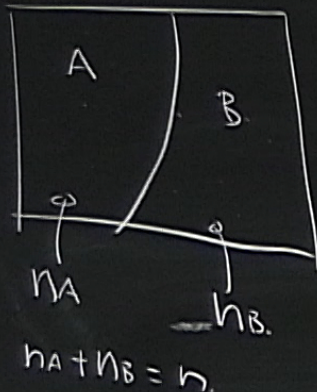
S_A

S_B

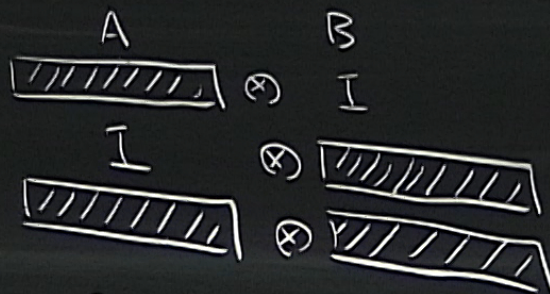
S_{AB}



Proof



$n-k$ independent stabilizer generators



number of generators

S_A

S_B

S_{AB}

$$S_A + S_B + S_{AB} = n - k$$

Logical op in A , $\mathcal{L}_A$.

- $[\mathcal{L}_A, S_j] = 0$

There are 2^{2n_A} Pauli ops on A (I, X, Y, Z)
 $n_A=1$
→ $S_A + S_{AB}$ constraints. → $2^{2n_A} / 2^{S_A + S_{AB}}$ Pauli op, commuting with S .

- $\mathcal{L}_A \approx \mathcal{L}_{A'}$ if $\mathcal{L}_{A'} = U \mathcal{L}_A U^\dagger$, $U \in S$.

For each $\mathcal{L}_A$, there are 2^{S_A} equivalent ops on A .

$$2^{n_A} - (S_A + S_{AB}) - S_A = 2^{g_A}$$

$$\rightarrow 2$$

$$g_A = 2n_A - 2S_A = S_{AB}.$$

Logical op in A . $\mathcal{L}_A$.

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(There are 2^{2n_A} Pauli ops on A (I, X, Y, Z) $n_A=1$
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$$g_A = 2n_A - 2S_A - S_{AB}.$$

$$g_B = 2n_B - 2S_B - S_{AB}.$$

Logical op in A. $\mathcal{L}_A$.

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$$g_A + g_B = 2(n_A + n_B) - 2(S_A + S_B + S_{AB}) = 2n - 2(n-k) = 2k$$

Logical op in A. $\mathcal{L}_A$.

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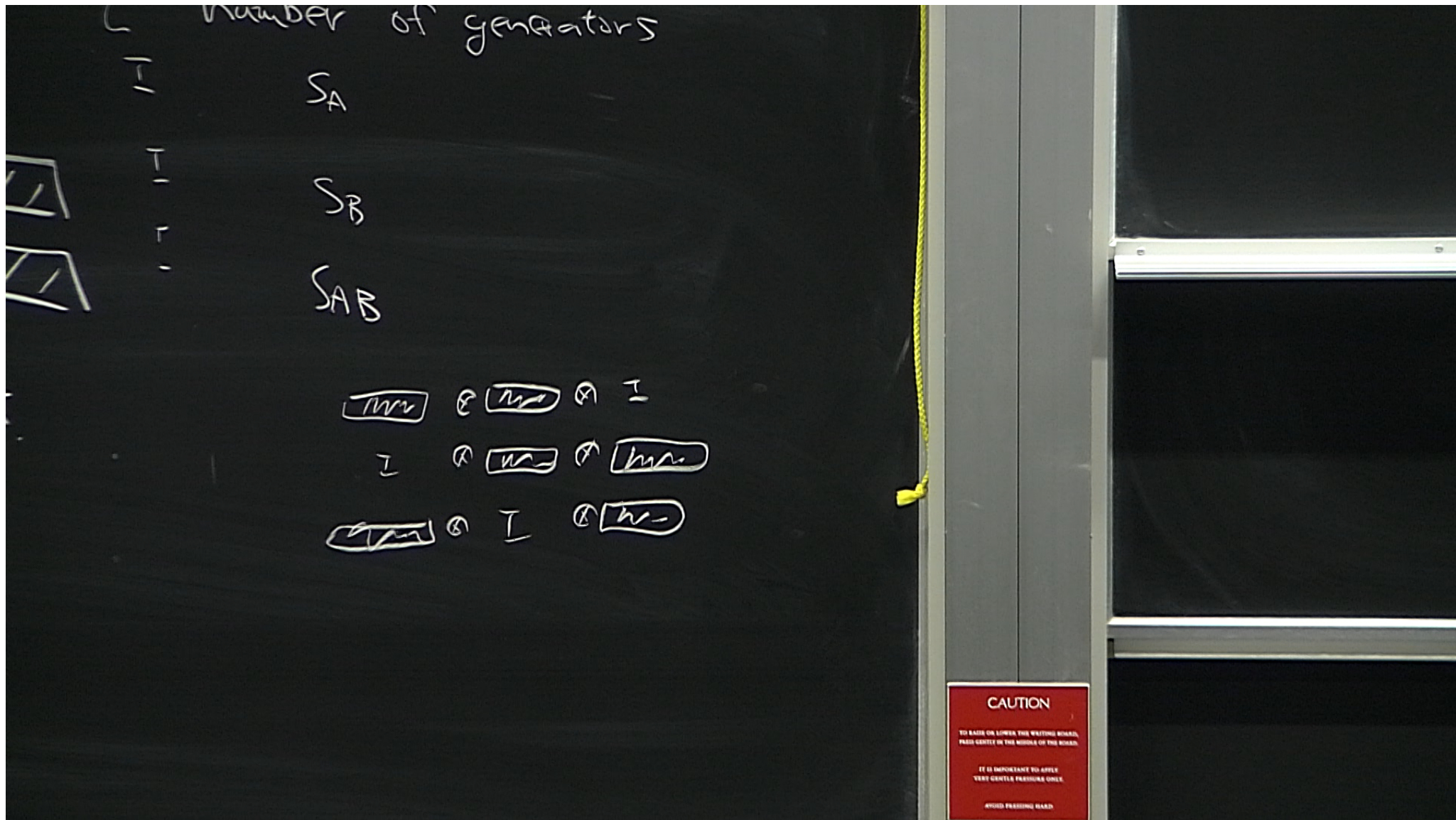
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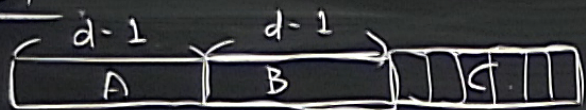
$$g_A + g_B = 2(n_A + n_B) - 2(S_A + S_B + S_{AB}) = 2n - 2(n-k) = 2k \quad \Rightarrow$$



Quantum Singleton bound

$$n - k \geq 2(d - 1)$$

proof



$$g_A = 0 \quad g_B = 0 \quad \nrightarrow \quad |C| \geq k$$

$$g_A = 0 \quad g_B = 0.$$

$$g_{BC} = 2k$$

suppose $|C| < k$.

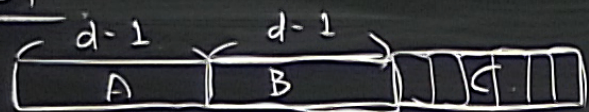
	C		B
1	q_1	0	r_1
2	q_2	0	r_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k$	q_{2k}	0	r_{2k}

independent.

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$$g_{B^c} = 2k$$

suppose $|C| < k$

	C	not ident. 1/	B
1	Q_1	0	r_1
2	Q_2	0	r_2
...	...	...	...
$2k$	Q_{2k}	0	r_{2k}
	I	0	r

independent.

$$Q_i \neq I$$

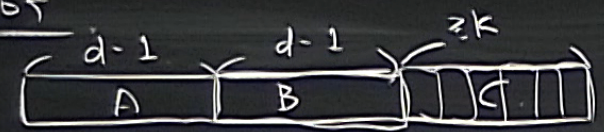
$Q_1, \dots, Q_{2k}$ Pauli

There are $2|C|$ indep Pauli op

Quantum Singleton bound

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independent

$$n \geq k + 2(d - 1)$$

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$$g_{B^c} = 2k$$

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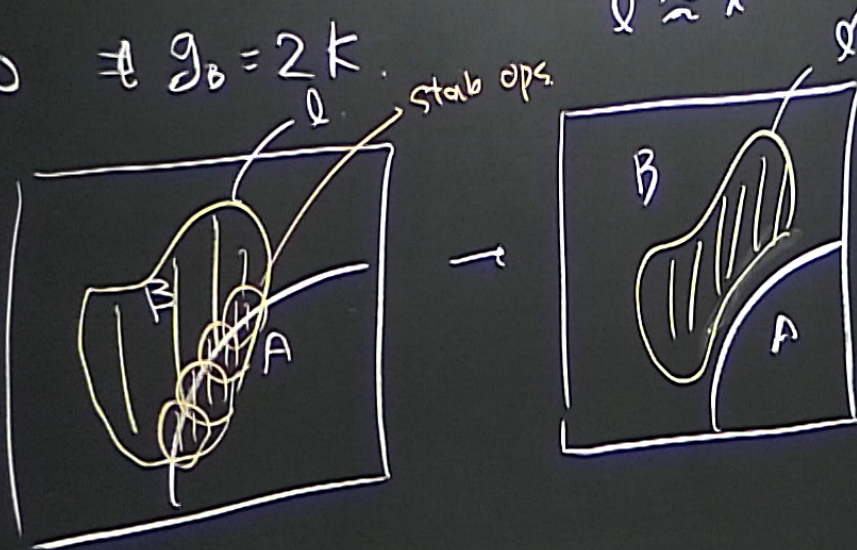
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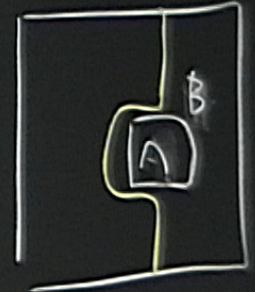
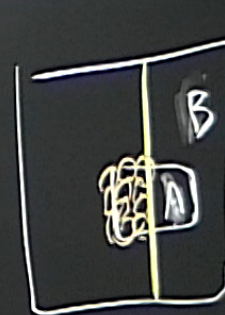
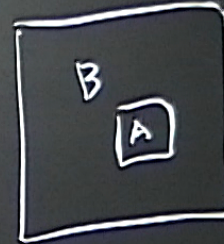
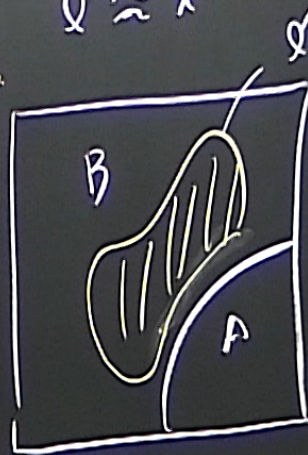
If there is no logical op on A, then, all logical ops can be supported on B.

proof

$$g_A = 0 \quad \& \quad g_B = 2k$$



$$Q' \approx Q.$$



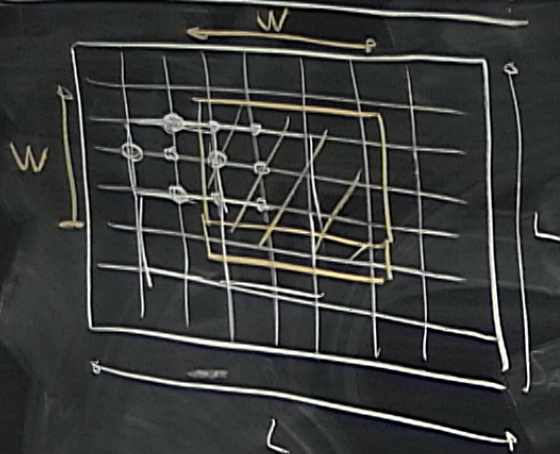
• Erasure error on A.

$$g_B = 2k.$$

Measurement of logical ops on B

$g_A = 0 \rightarrow$ Erasure error on A is recoverable.

Def of load rules



• L is "big".

• S_j : geometrically, load can be supported on $w \times w$ box

$$w \sim \alpha(1)$$

$$|| = - \sum_j S_j$$

Code distance of local code

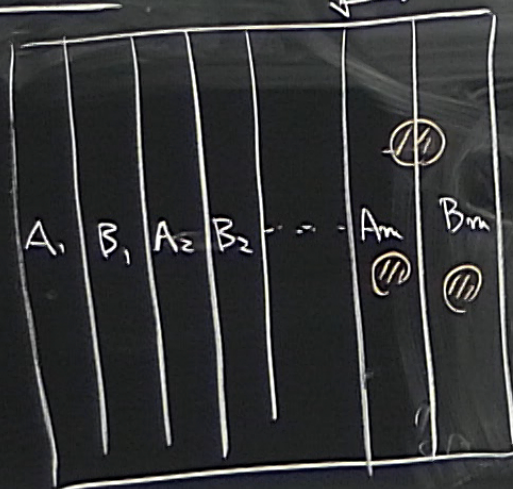
Consider a D -dimensional local stab code. Then

$$d \leq O(L^{D-1})$$

d : code distance.

$$n = L^D$$

proof ($D=2$)



$$A = \bigcup_j A_j$$

$$B = \bigcup_j B_j$$

$$g_A + g_B = 2k > 0$$

$$w \log g_A > 0$$

Code distance of local code

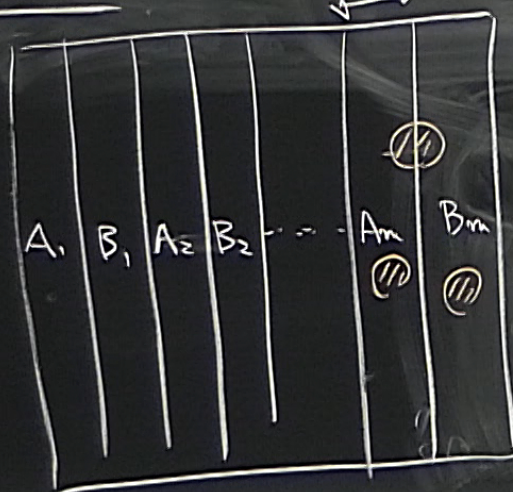
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logical op l

$$l = l_{A_1} \otimes l_{A_2} \dots \otimes l_{A_m}$$

Code distance of local code

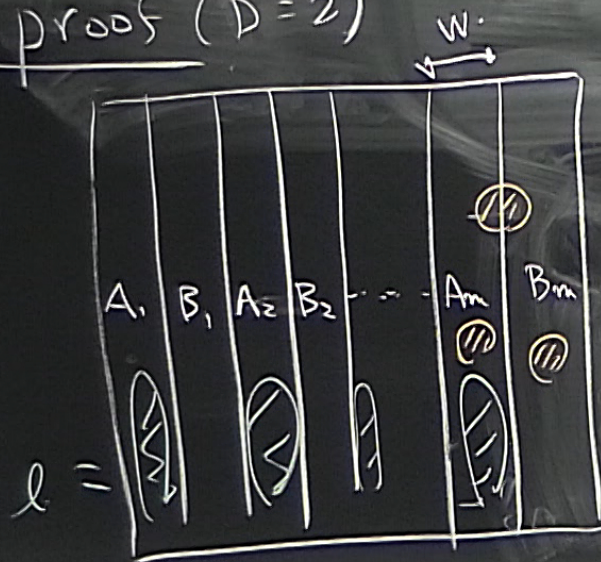
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logical $\exists p$ l

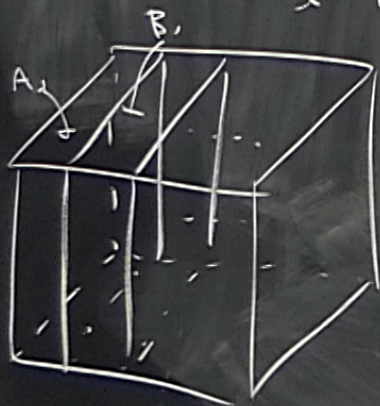
$$l = l_{A1} \otimes l_{A2} \dots \otimes l_{Am}$$

$$[Q, S_j] = 0$$

$$[Q_{A_1}, S_j] = 0 \quad [Q_{A_2}, S_j] = 0$$

$Q_{A_1}, Q_{A_2}, \dots, Q_{A_n}$ are logical ops.

Q_{A_1} is a non-trivial logical op supported on a region with WL qubits

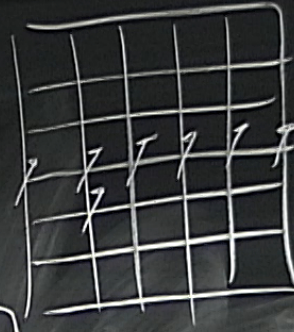
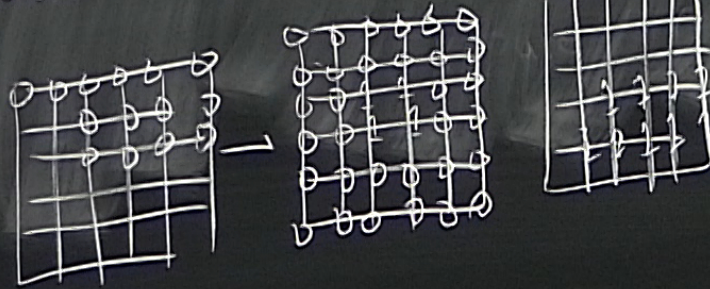


WL^{D-1} qubits

• Classical Ferromagnet (2D)

$$H = - \sum_{i,j} z_i z_j$$

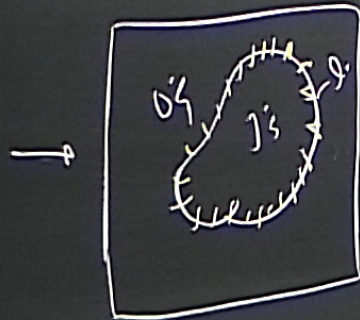
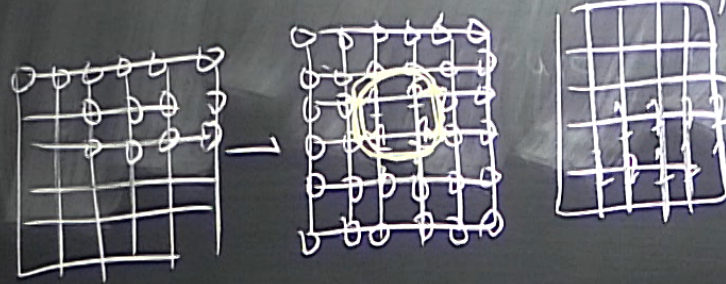
Stable memory at Finite temperature



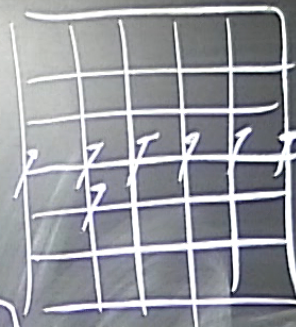
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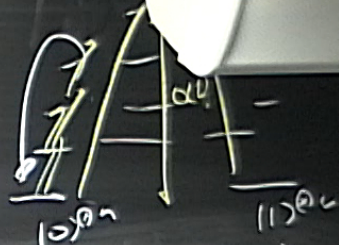
excitation energy
 $\approx$ length



or "Energy barrier" $\approx O(L)$

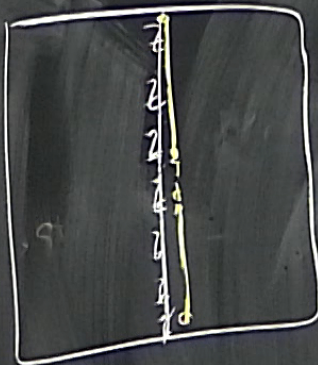
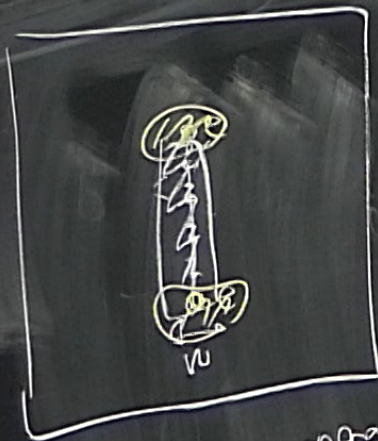
memory Time

$$\exp\left(\frac{\Delta_{\text{barrier}}}{T}\right)$$



"Self-correcting" quantum memory

- 2D toric code is not a self-correcting memory.



memory time $\sim \alpha(1)$

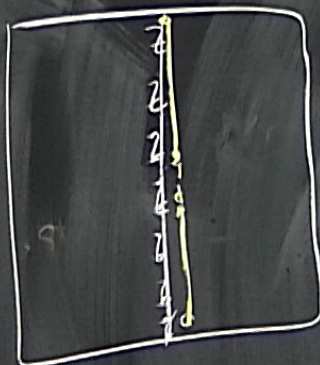
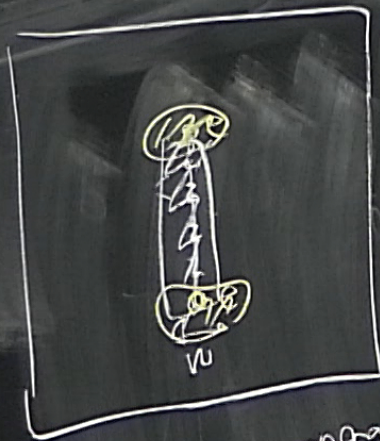
$$|\psi_0\rangle \xrightarrow[\text{QEC}]{\text{thermal noise}} |\psi\rangle$$

energy barrier $\sim \alpha(1)$.

- 4D toric code is self-correcting

"Self-correcting" quantum memory

- 2D toric code is not a self-correcting memory.



memory time $\sim \mathcal{O}(1)$

$$|4_0\rangle \xrightarrow[\text{QEC}]{\text{threshold}} \text{X}$$

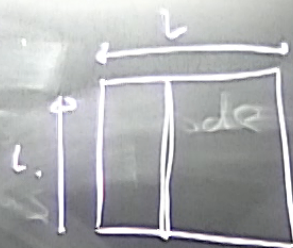
gs

energy barrier $\sim \mathcal{O}(1)$.

- 4D toric code is self-correcting
- 3D ??

Consider 2-D local rule

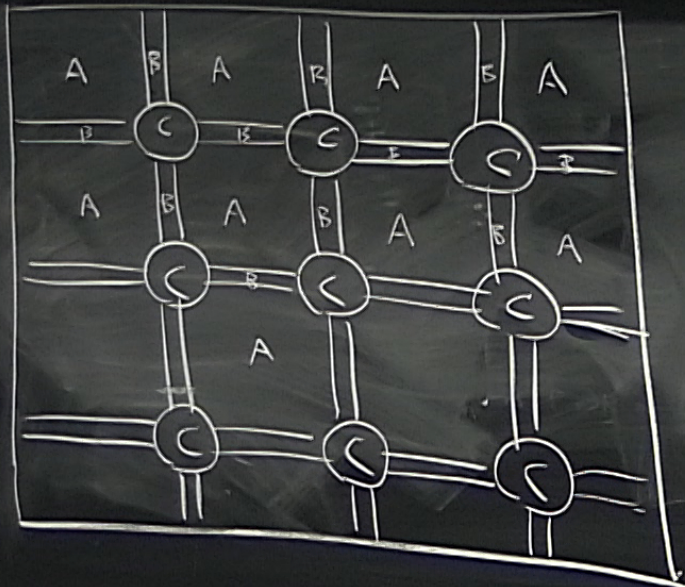
$$k d^2 \leq O(L^2)$$

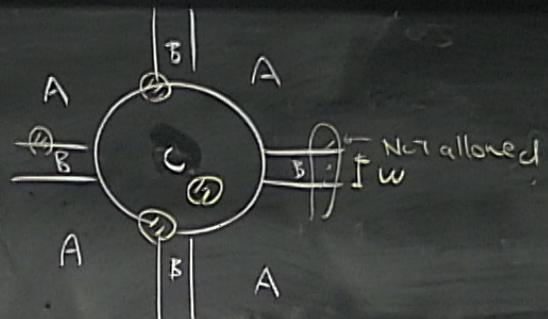
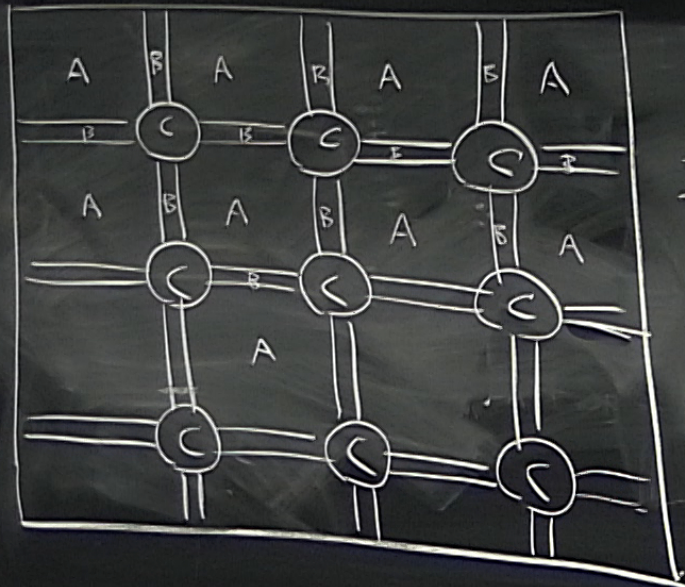


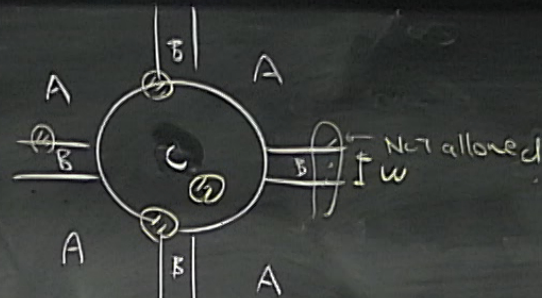
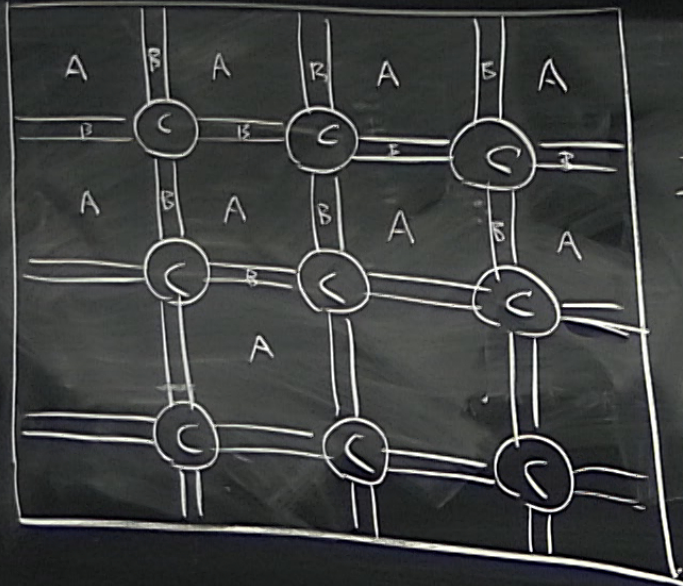
eg toric code on a torus

$$k=2, d=L$$

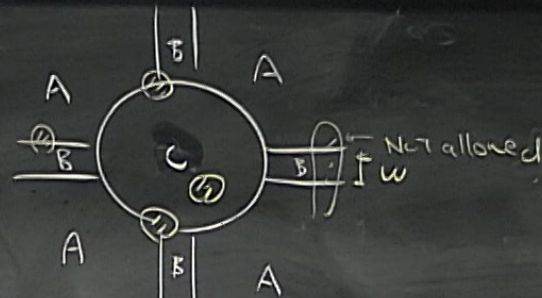
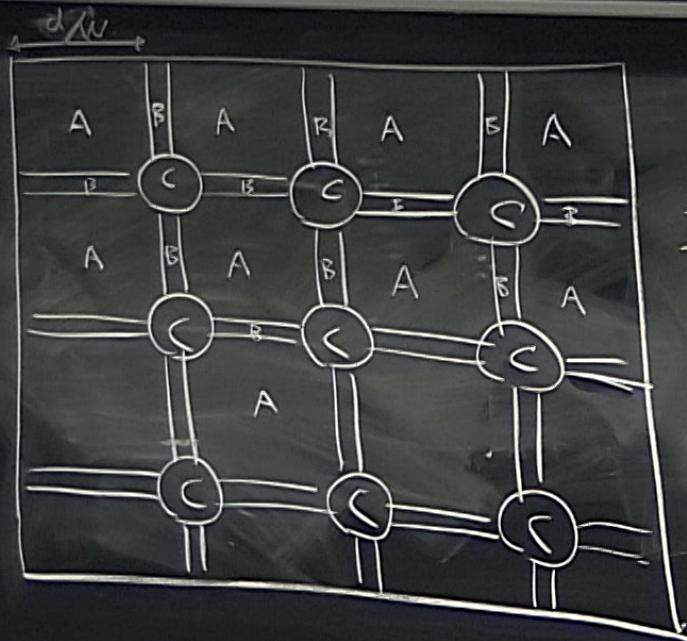
$$\text{LHS} = L^2$$



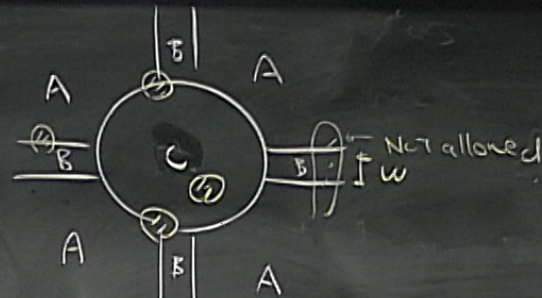
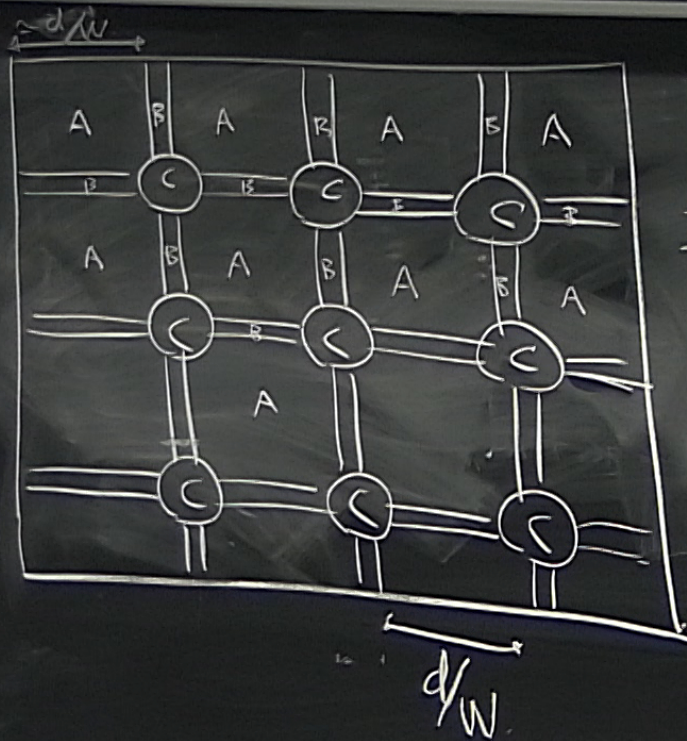




No stab generator act on 2 regions with the same label.



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No stab generator act on 2 regions with the same label.

$$\begin{aligned}
 (a) \quad g_c &= 0 & |c| < d \\
 g_B &= 0 & |B| < d \\
 \underline{g_A} &= 0 & |A| \sim d^2 \quad \leftarrow \text{will be proven}
 \end{aligned}$$

$$(b) \quad g_A = 0 \quad g_B = 0 \quad g_{Bc} = 2K \quad (\leftarrow g_n = 0)$$

$\Rightarrow$ From the Singlet bound proof

$$|c| \geq K$$

$$\left(\frac{L}{d}\right)^2 \geq K$$

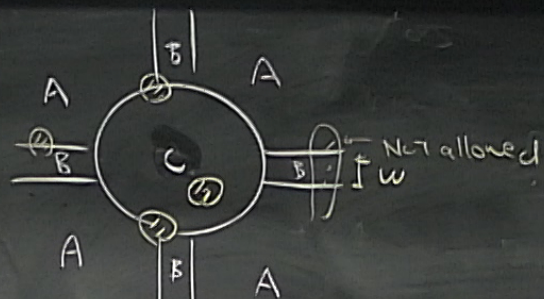
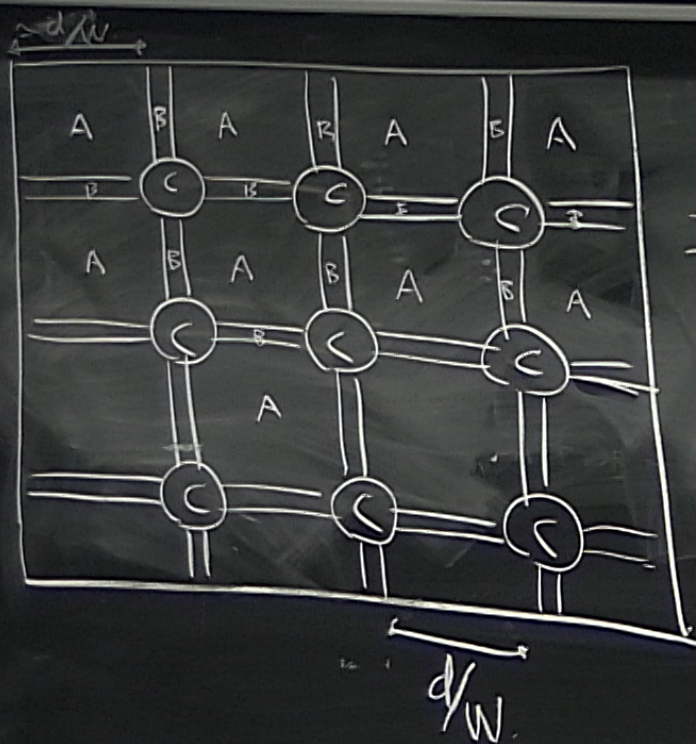
$$(c) : \sim \left(\frac{L}{d}\right)^2$$

$$K d^2 \geq O(L^2)$$

$$|c|$$

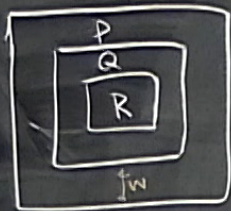
CAUTION

CAUTION



No ^{local} stab generator act on 2 regions with the same label.

Lemma (holographic principle)



$\overline{PQR}$

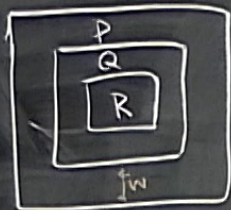
- Assume there is a logical op on PQR, l
 - Assume $g_a = 0 \quad g_R = 0$
- then l can be supported on P

CAUTION

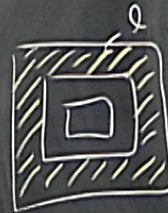
USE ONLY WITH PROPER HANDLING
INSTRUCTIONS IN THE MANUAL
DO NOT TOUCH THE SURFACE
OF THE BOARD WITH YOUR HANDS

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PQR



CAUTION

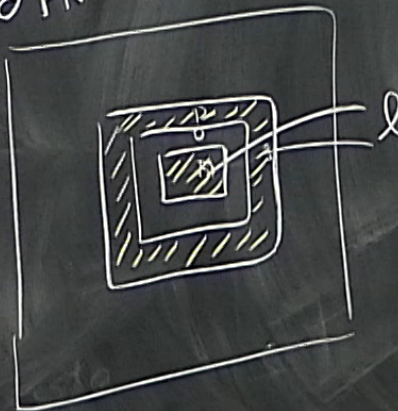
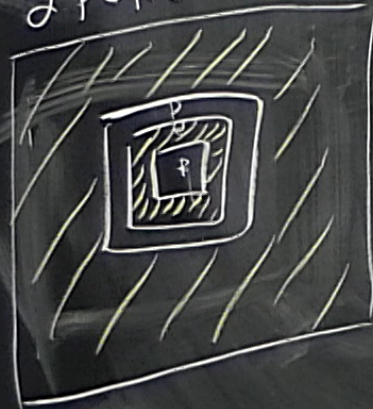
CAUTION

proof

$$g_{PQR} = 1, \quad g_Q = g_R = 0$$

$$g_{\overline{PQR}} = 2k - 1$$

$$g_{\overline{PQRQ}} = 2k - 1 \quad \Rightarrow \quad g_{PR} = 1$$

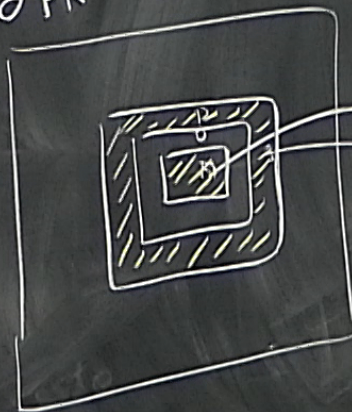
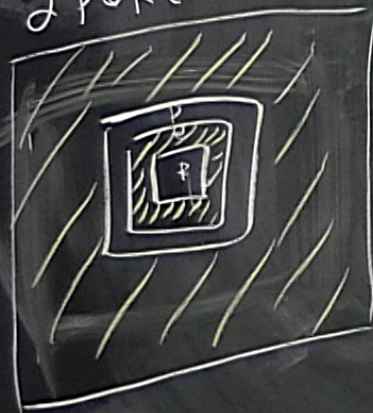


proof

$$g_{PQR} = 1, \quad g_Q = g_R = 0$$

$$g_{\overline{PQR}} = 2k - 1$$

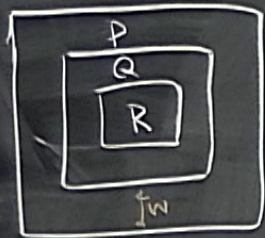
$$g_{\overline{PQR}Q} = 2k - 1 \Rightarrow g_{PR} = 1$$



$$g_R = 0$$

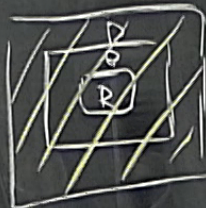
$$\Rightarrow g_P = 1$$

Lemma (holographic principle)

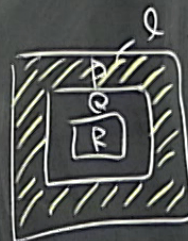


$\overline{PQR}$

- Assume there is a logical op on PQR, l
 - Assume $g_Q = 0$ $g_R = 0$
- then l can be supported on P



$g_{PQR} = 1$

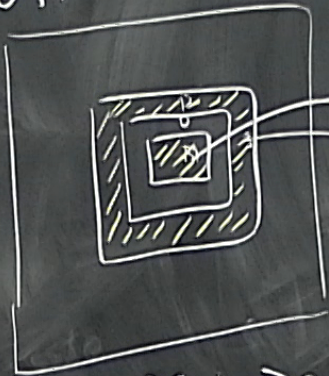


$g_P = 1$

$$g_{PQR} = 1, g_{Q} = 0$$

$$g_{\overline{PQR}} = 2k - 1$$

$$g_{\overline{PQR}} = 2k - 1 \Rightarrow g_{PR} = 1$$



$$g_R = 0$$

$$\Rightarrow g_P = 1$$

$$g_{\overline{PQR}} = 2k - 1 \Rightarrow g_P = 1$$

$$g_{\overline{P}} = 0?$$

• Quantum code ($T=0$)

$|4, \rangle, |4, \rangle$ have high complexity

"error-correction"

• Self-correcting memory ($T>0$)

" ρ_β " Gibbs ensemble has high complexity

• Quantum code ($T=0$)

$|4, \rangle, |4, \rangle$ have high complexity

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• Self-correcting memory ($T>0$)

" ρ_β " Gibbs ensemble has high complexity

→ Can ρ_β be prepared from classical ensemble
by short-depth circuit?

eg. $H = - \sum_j z_i z_j$

Claim

2D toric code is not thermally stable. $\sim \log(L)$ depth

circuit to prepare $\rho_\beta = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}$

Claim

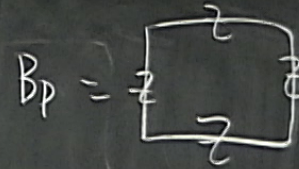
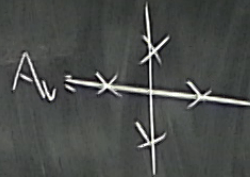
2D toric code is not thermally stable. $\sim \log(L)$ depth

circuit to prepare $\rho_B = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}$

$$\beta = \frac{1}{T}$$

Simplification

$$H_{\text{toric}} = - \sum_{v=1}^{L^2-1} A_v - \sum_{p=1}^{L^2-1} B_p$$



$$\prod_v A_v = I$$

$$\prod_p B_p = I$$

Imperfect Hamiltonian

$$H = - \sum_{j=1}^{n-k} S_j$$

$$H(\vec{m}) = - \sum_{j=1}^{n-k} m_j S_j \quad (m_j = 0, 1)$$

Imperfect Hamiltonian

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• Maximally mixed ensemble in the gs space: $\rho(\vec{m})$

$$\rho(\vec{m})$$

Imperfect Hamiltonian

$$H = - \sum_{j=1}^{n-k} S_j$$

$$Q_j = \frac{I + S_j}{2}$$

$$H(\vec{m}) = - \sum_{j=1}^{n-k} m_j S_j \quad (m_j = 0, 1)$$

• Maximally mixed ensemble in the gs space: $\rho(\vec{m})$

$$\rho(\vec{m}) = \frac{1}{2^k} \prod_{j=1}^{n-k} \left(\frac{I + S_j}{2} \right)^{m_j} \left(\frac{I}{2} \right)^{(1-m_j)}$$

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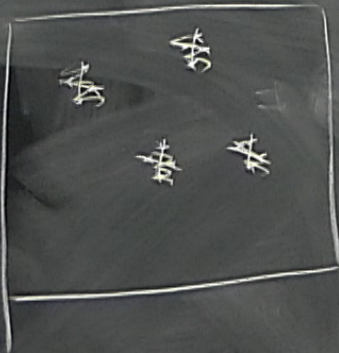
$$\text{tr}(\rho(\vec{m})) = 1$$

$$\text{tr}(I) = 2^n$$

Disentangling Imperfect Hamiltonian

$H(\vec{m})$ with prob p $m_j = 0$

($1-p$) $m_j = 1$
Claim: "most" of $H(\vec{m})$ are trivial.



Disentangling Imperfect Hamiltonian

$$H(\vec{m})$$

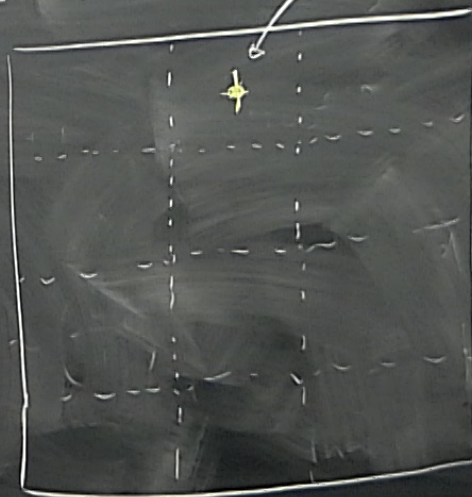
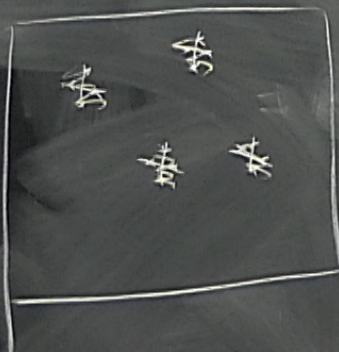
with prob p $m_j = 0$

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Claim: "most" of $H(\vec{m})$ are trivial.

$$p = \frac{2}{e^{2\beta} + 1}$$

v vertices



Disentangling Imperfect Hamiltonian

$$H(\vec{m})$$

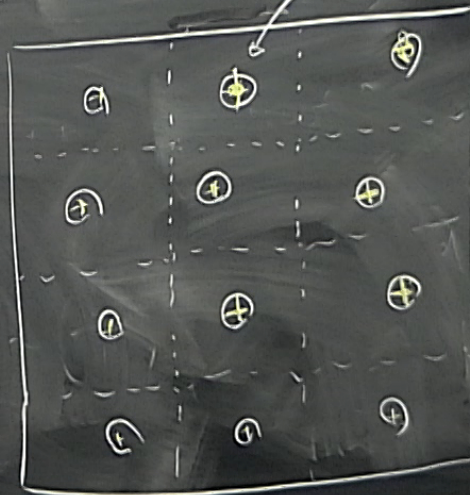
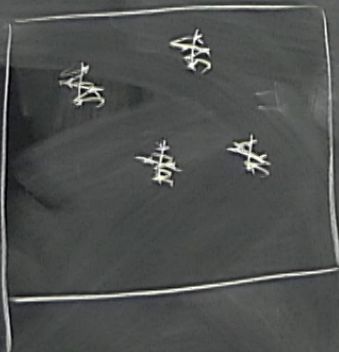
with prob p $m_j = 0$

$(1-p)$ $m_j = 1$

Claim: "most" of $H(\vec{m})$ are trivial.

$$p = \frac{2}{e^{2b} + 1}$$

v vertices



Disentangling Imperfect Hamiltonian

• $H(\vec{m})$

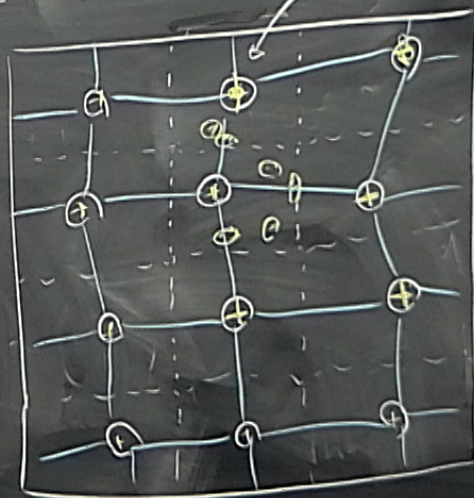
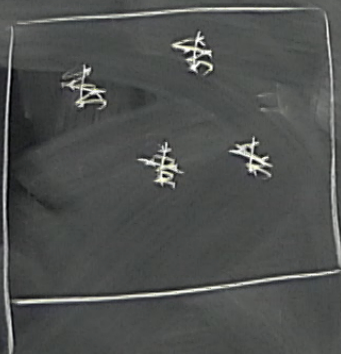
with prob p $m_j = 0$

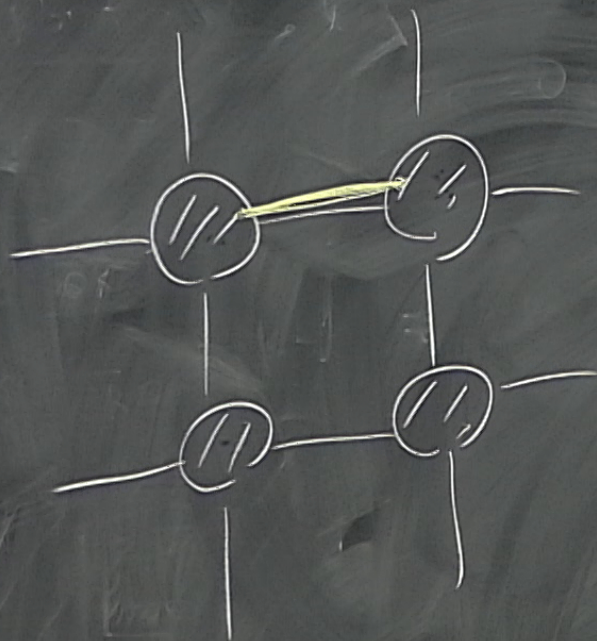
$(1-p)$ $m_j = 1$

Claim: "most" of $H(\vec{m})$ are trivial

$$p = \frac{2}{e^{2\beta} + 1}$$

• No stab generator act on 3 regions

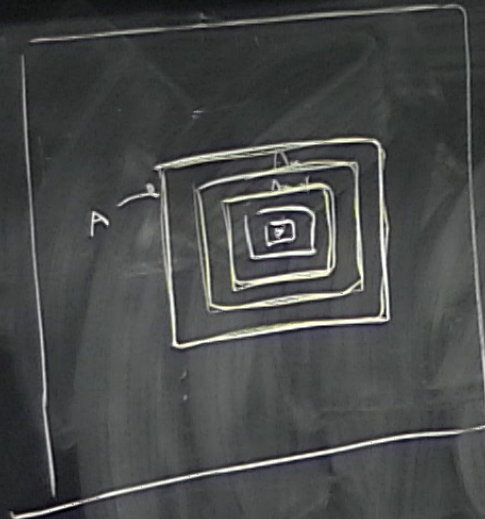




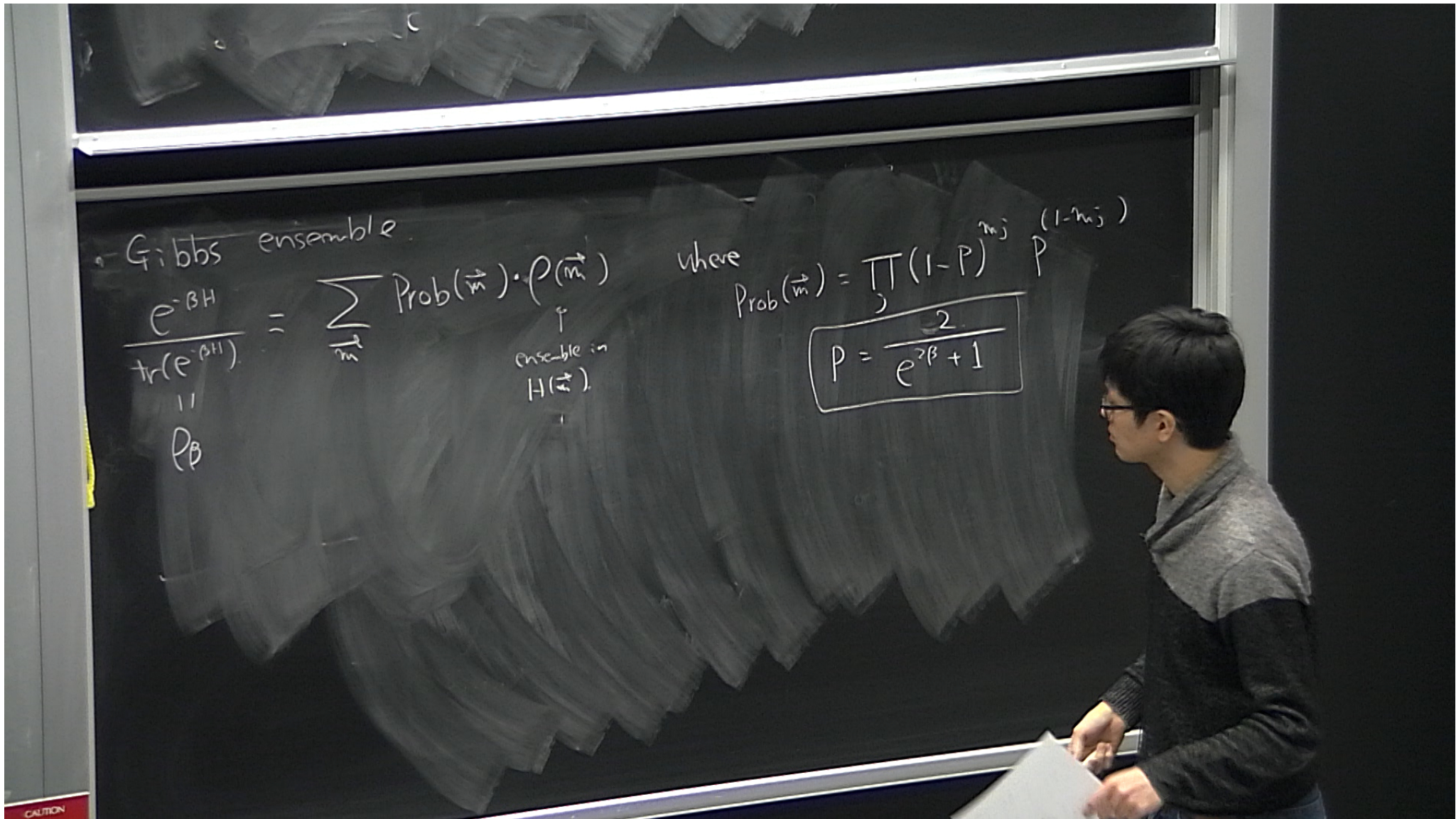
Stabilizers are 2-local

(a) Hamiltonian, 2-local, stabilizers
 $\Rightarrow \exists$ local unitary to bring $H(\vec{m})$
 into classical form.

(b) Encoding circuit for the toric code



$$|A_m| \leq d$$



Gibbs ensemble

$$\frac{e^{-\beta H}}{\text{tr}(e^{-\beta H})} = \sum_{\vec{m}} \text{Prob}(\vec{m}) \cdot \underset{\substack{\uparrow \\ \text{ensemble in} \\ H(\vec{m})}}{\rho(\vec{m})}$$

ρ_B

where

$$\text{Prob}(\vec{m}) = \prod_j (1-p)^{m_j} p^{(1-m_j)}$$

$$p = \frac{2}{e^{2\beta} + 1}$$

Gibbs ensemble

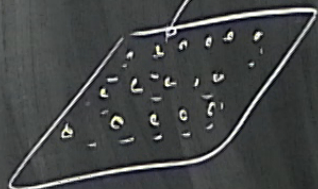
$$\frac{e^{-\beta H}}{\text{tr}(e^{-\beta H})} = \sum_{\vec{m}} \text{Prob}(\vec{m}) \cdot \rho(\vec{m})$$

ensemble in $H(\vec{m})$.

where

$$\text{Prob}(\vec{m}) = \prod_j (1-p)^{m_j} p^{(1-m_j)}$$

$$p = \frac{2}{e^{2\beta} + 1}$$



$$(S_j^z)^2 = I$$

$$e^{\beta S_j^z}$$

$$P_\beta$$