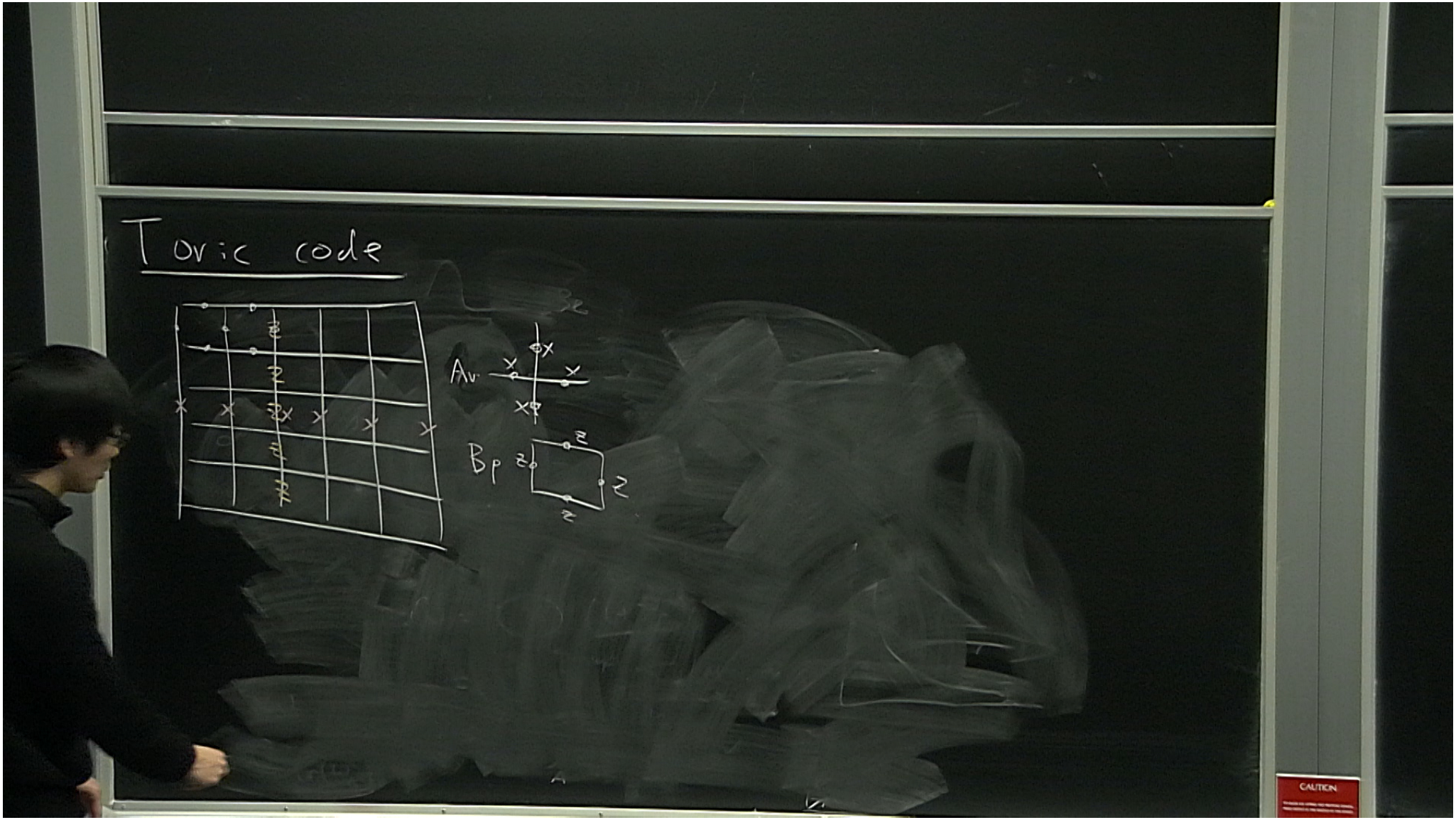
Title: IQC - Quantum Error Correction - Lecture 8

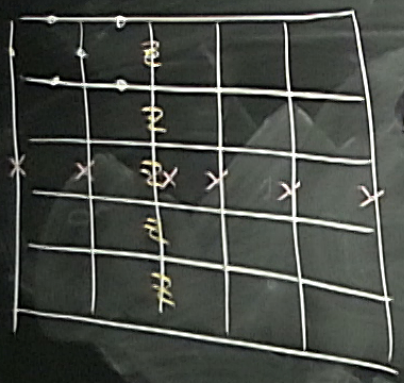
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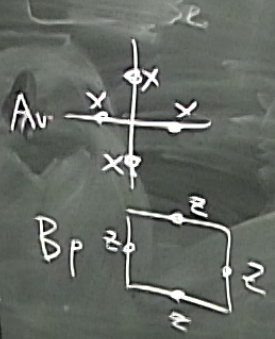
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Abstract:

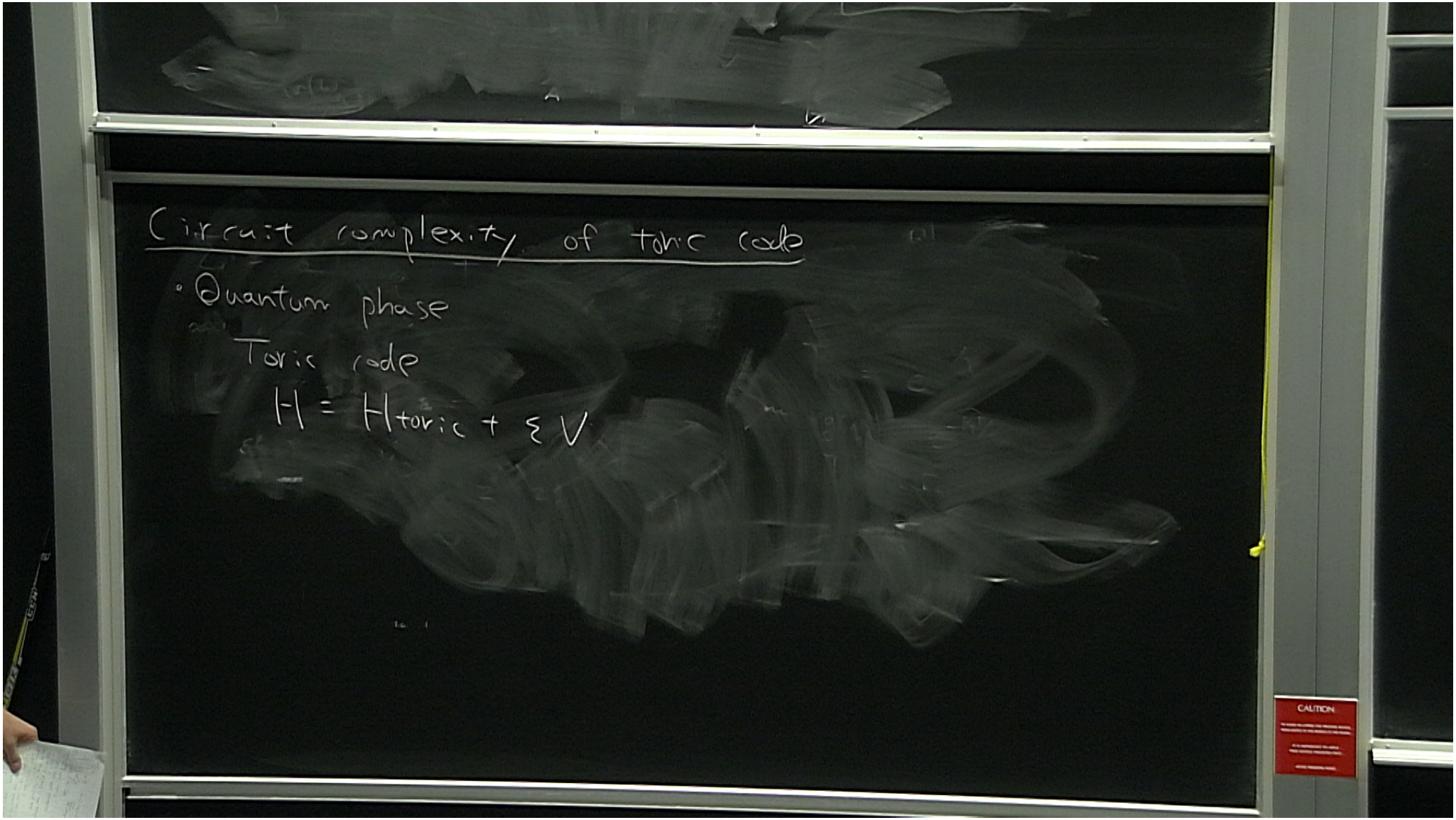

Toric code
Av.
Bp

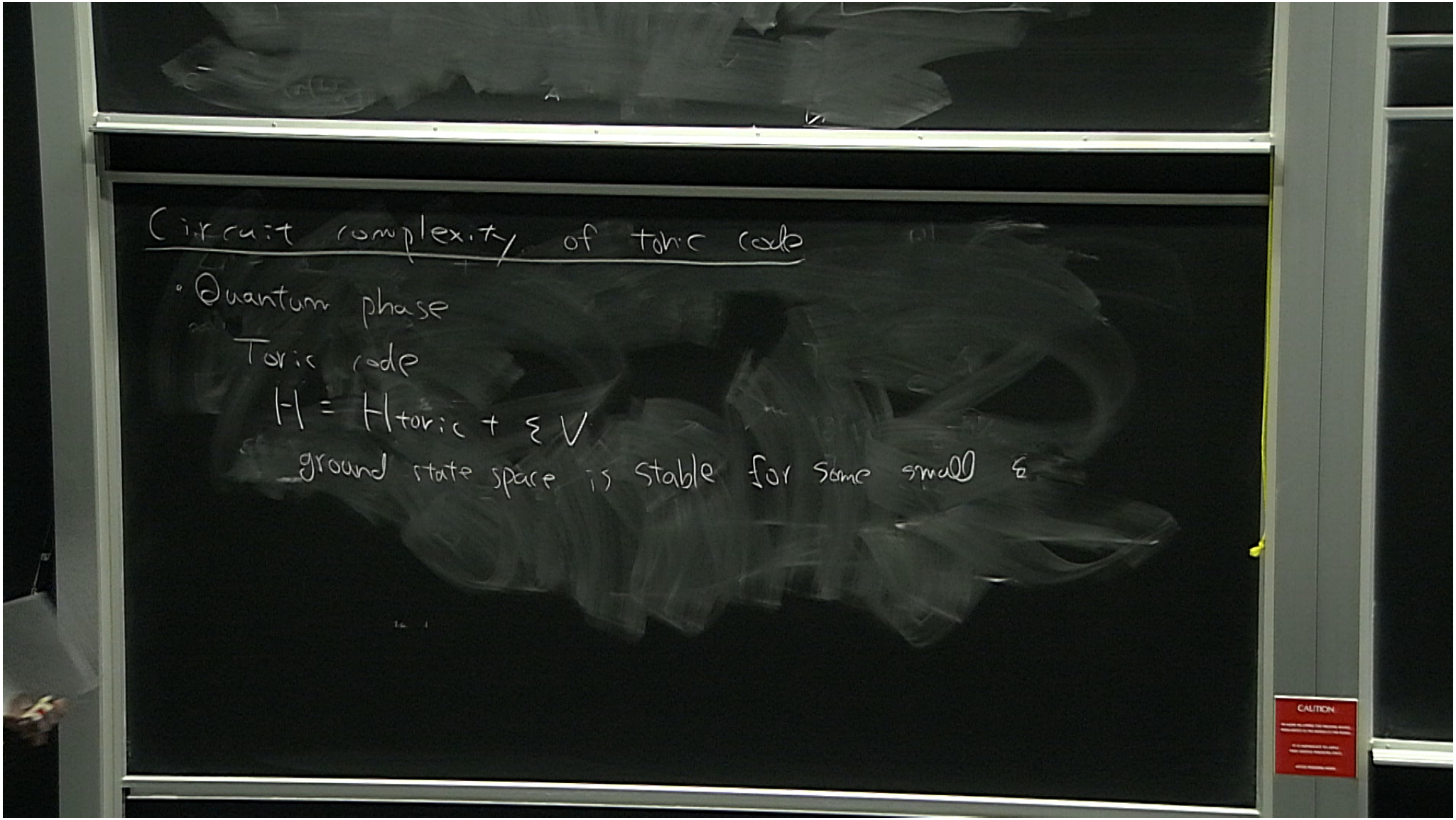
Toric code

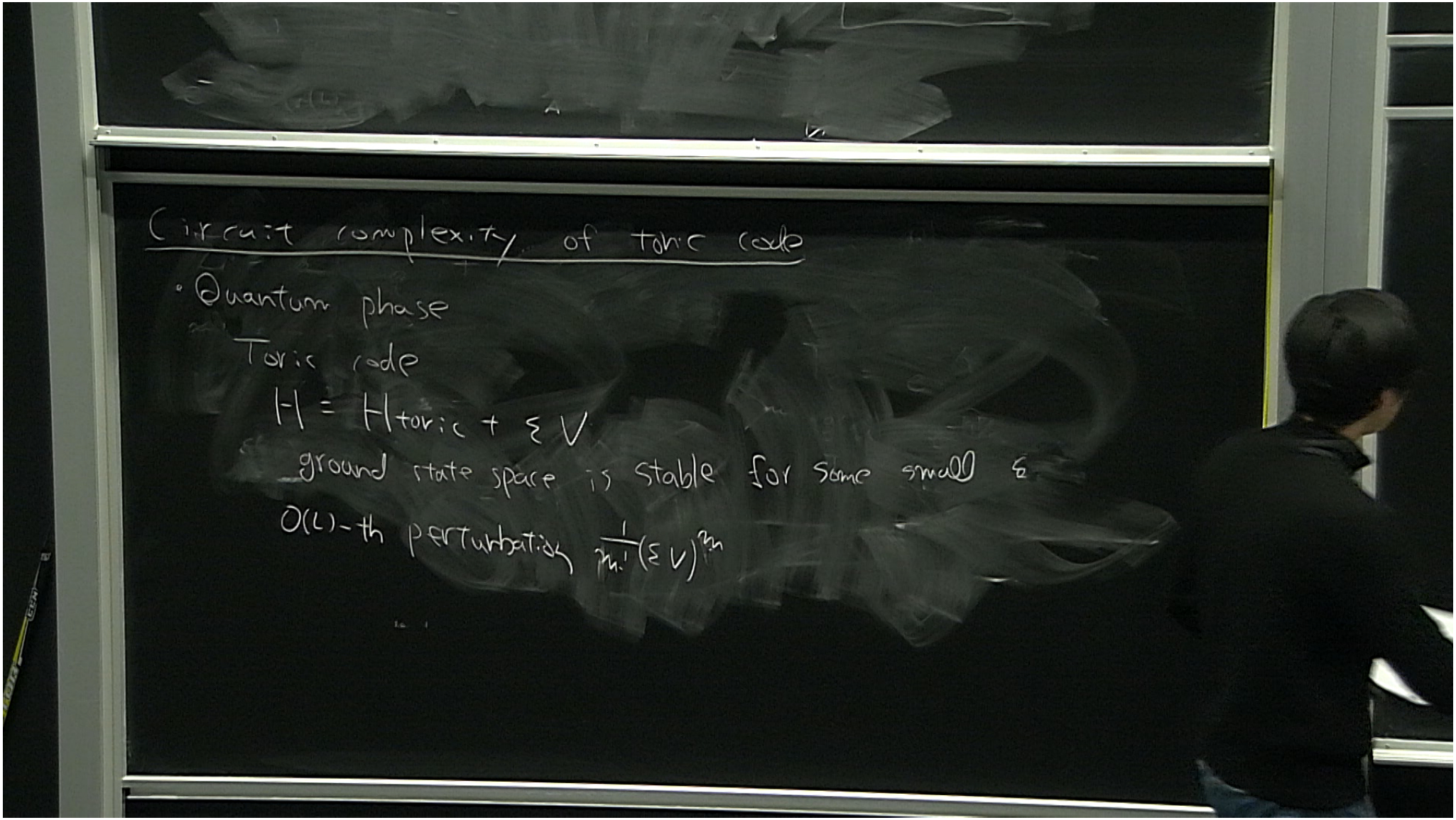


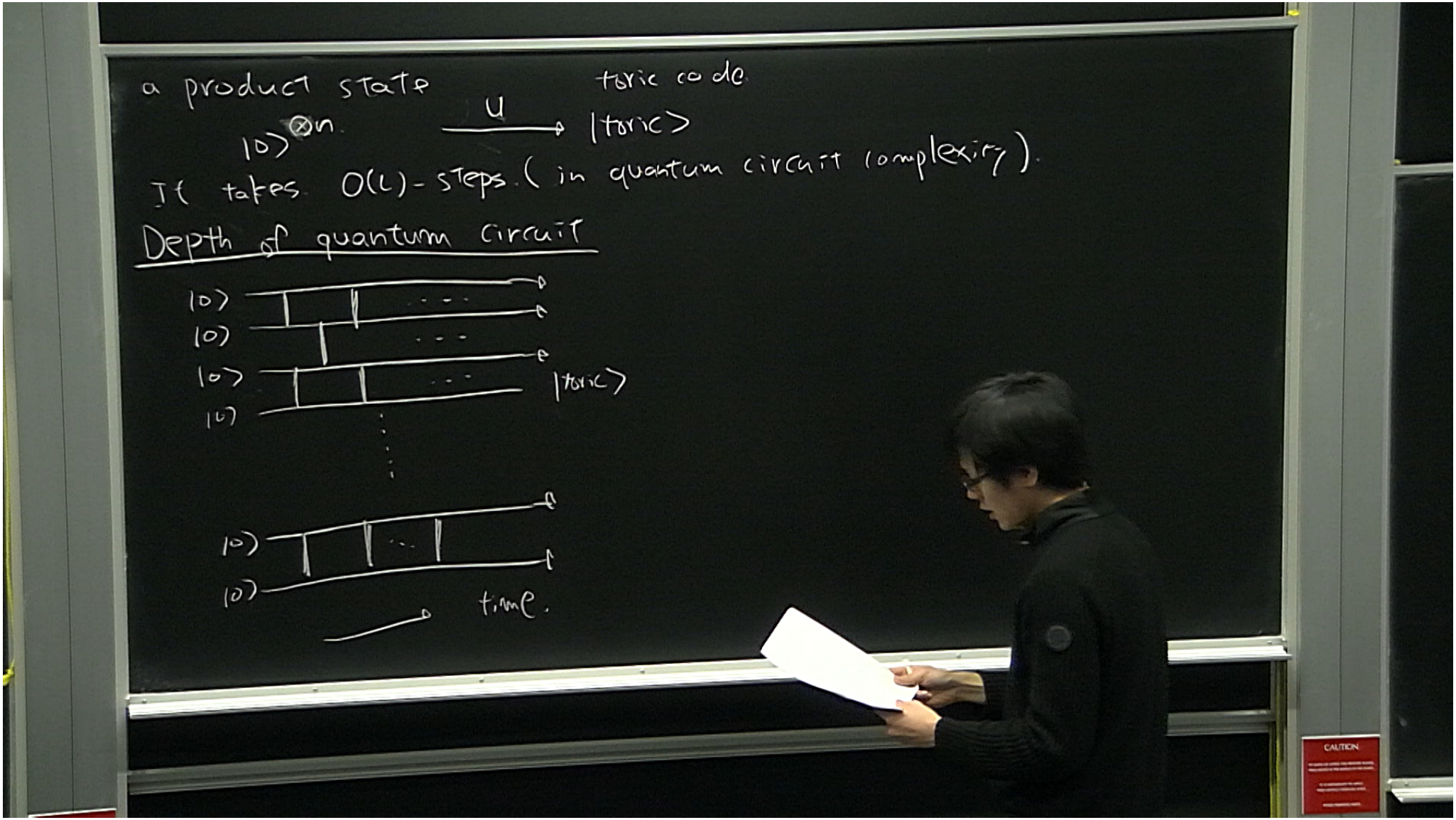


$|toric\rangle$: how complex ?


Circuit complexity of toric code
• Quantum phase
Toric code
H = H toric + ε V
CAUTION


Circuit complexity of toric code
• Quantum phase
Toric code
$H = H_{toric} + \varepsilon V$
ground state space is stable for some small ε
CAUTION

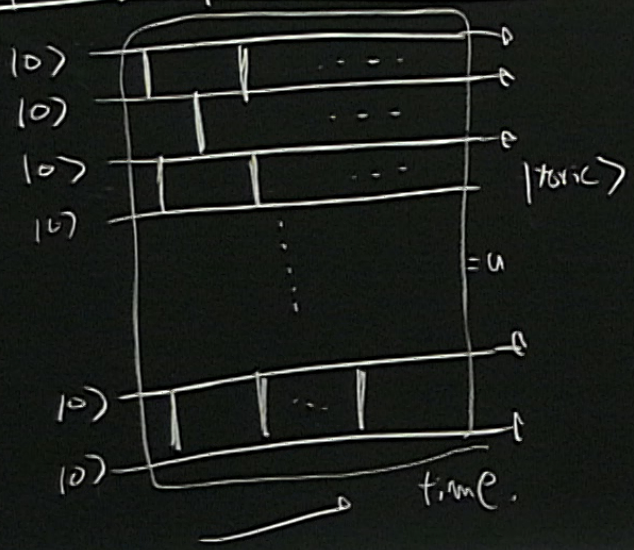

Circuit complexity of toric code
• Quantum phase
Toric code
$H = H_{toric} + \varepsilon V$
ground state space is stable for some small ε
$O(L)$-th perturbation $\frac{1}{n!}(\varepsilon V)^n$


a product state
toric code
$|0\rangle^{\otimes n}$
U
$|\text{toric}\rangle$
It takes O(L)-steps. (in quantum circuit complexity).
Depth of quantum circuit
$|0\rangle$
$|0\rangle$
$|0\rangle$
$|0\rangle$
$|\text{toric}\rangle$
$|0\rangle$
$|0\rangle$
time.

a product state $|0\rangle^{\otimes n}$ $\xrightarrow{U}$ toric code $|\text{toric}\rangle$

If takes $O(L)$-steps (in quantum circuit complexity).

Depth of quantum circuit



i). gates act on neighboring qubits

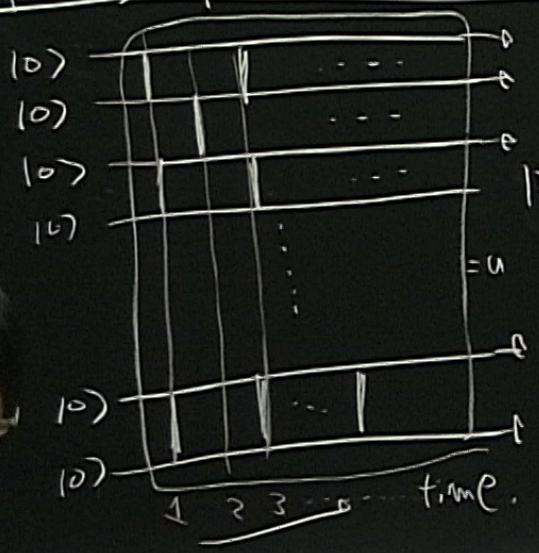
ii) $|\text{toric}\rangle = U|0\rangle^{\otimes n}$

- Circuit decomposition of U is not unique.

a product state $|0\rangle^{\otimes n}$ $\xrightarrow{U}$ toric code $|\text{toric}\rangle$

It takes $O(L)$-steps. (in quantum circuit complexity).

Depth of quantum circuit



i) gates act on neighboring qubits

ii) $|\text{toric}\rangle = U|0\rangle^{\otimes n}$

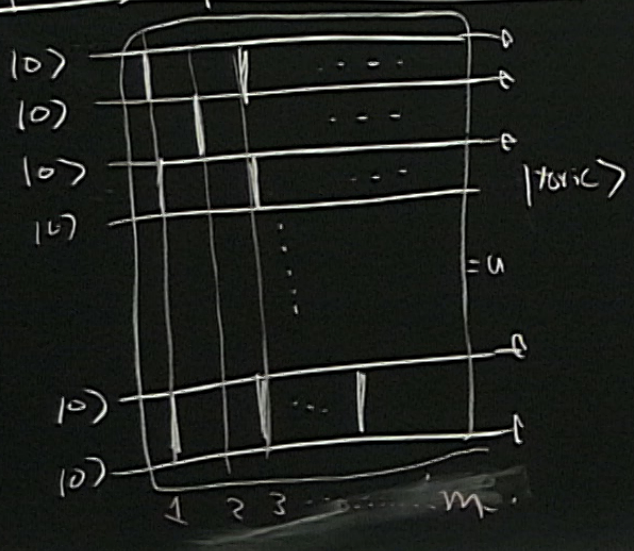
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- U s.t. $|\text{toric}\rangle = U|0\rangle^{\otimes n}$ is not unique.

We will consider the one with "minimal" steps.

a product state $|0\rangle^{\otimes n}$ $\xrightarrow{U}$ toric code $|\text{toric}\rangle$

It takes $O(L)$-steps. (in quantum circuit complexity)

Depth of quantum circuit



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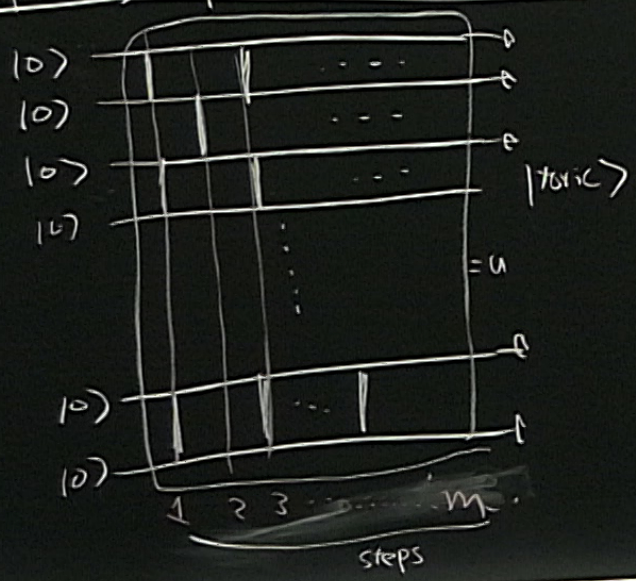
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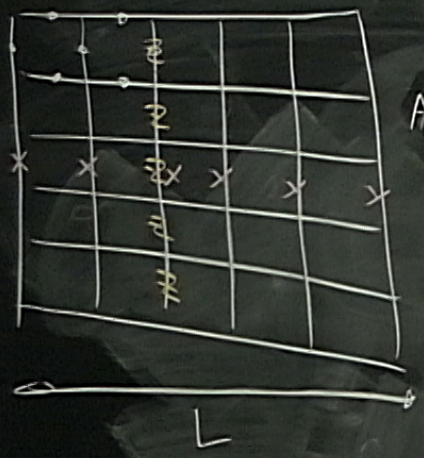
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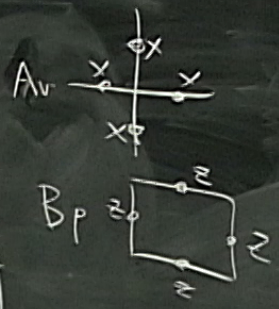
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↳ Depth of circuit.

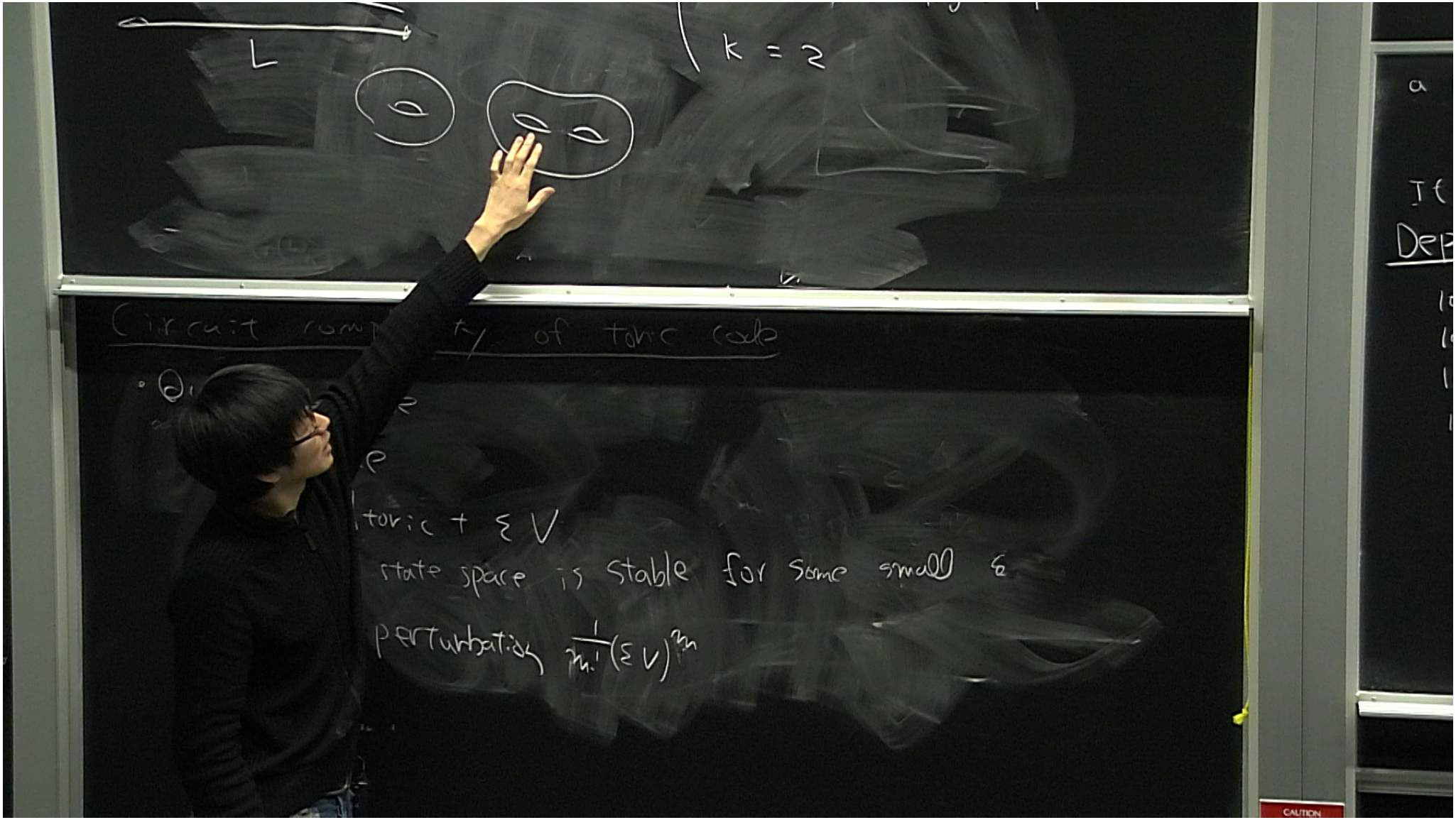
Toric code

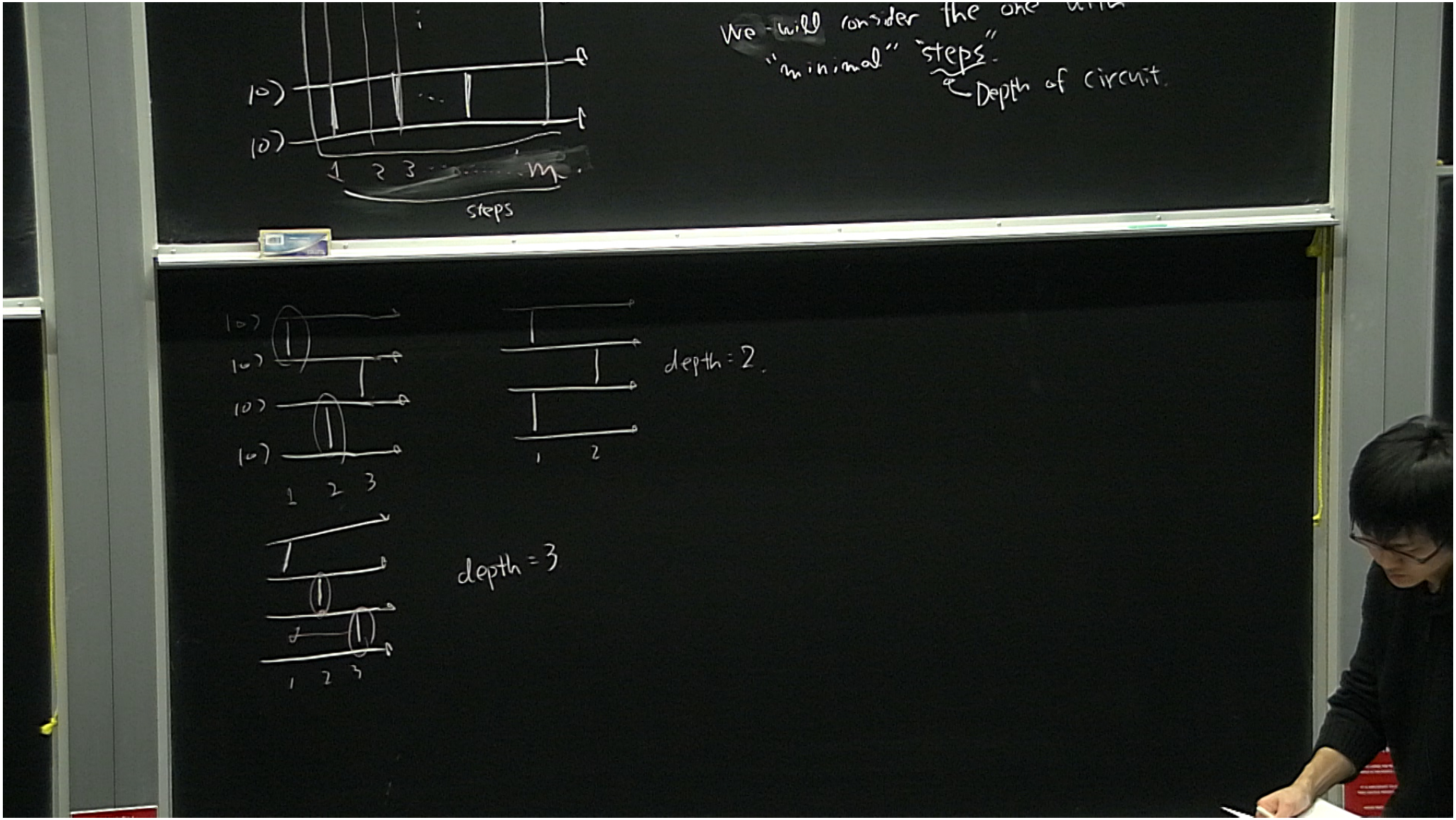


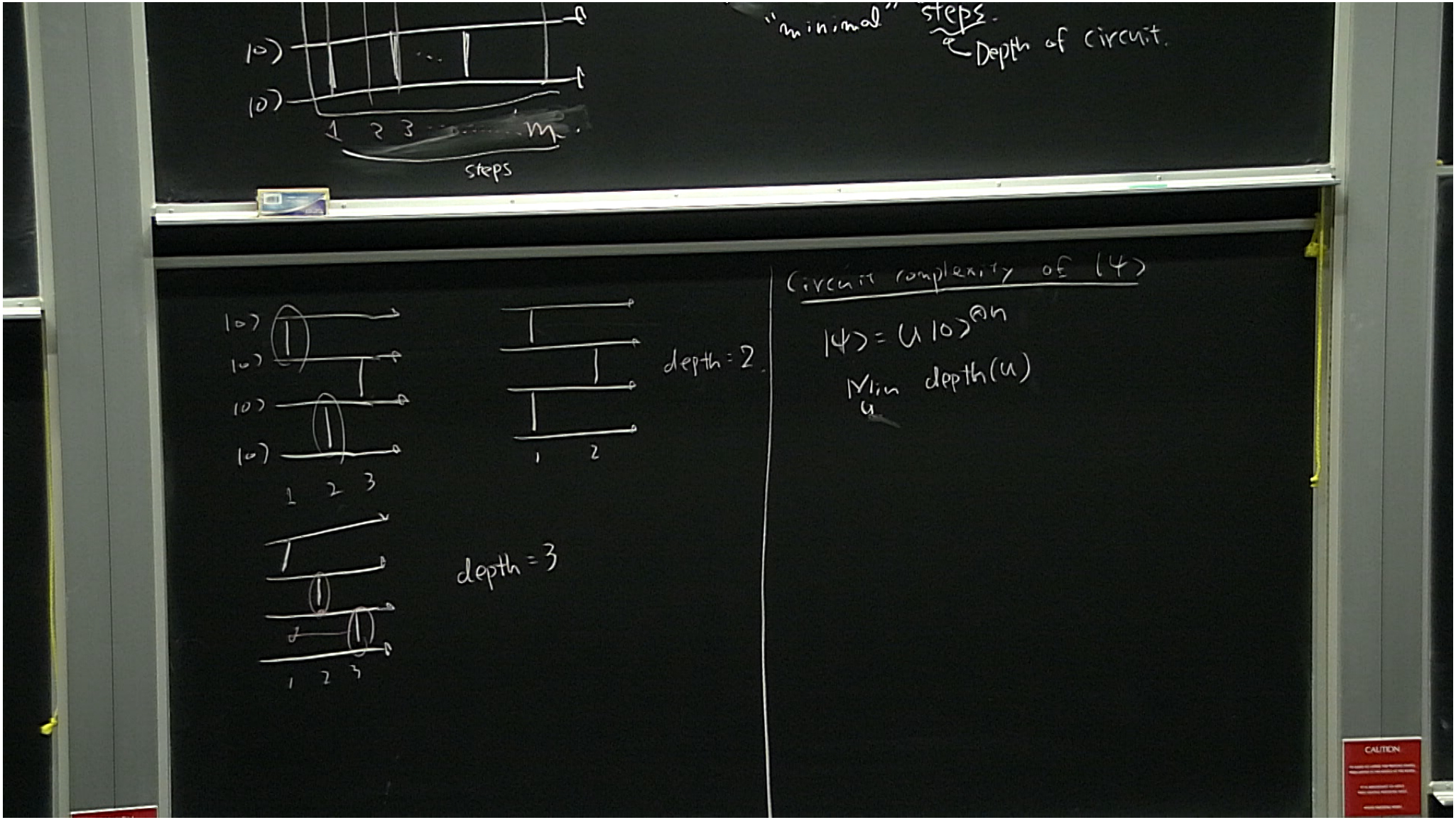


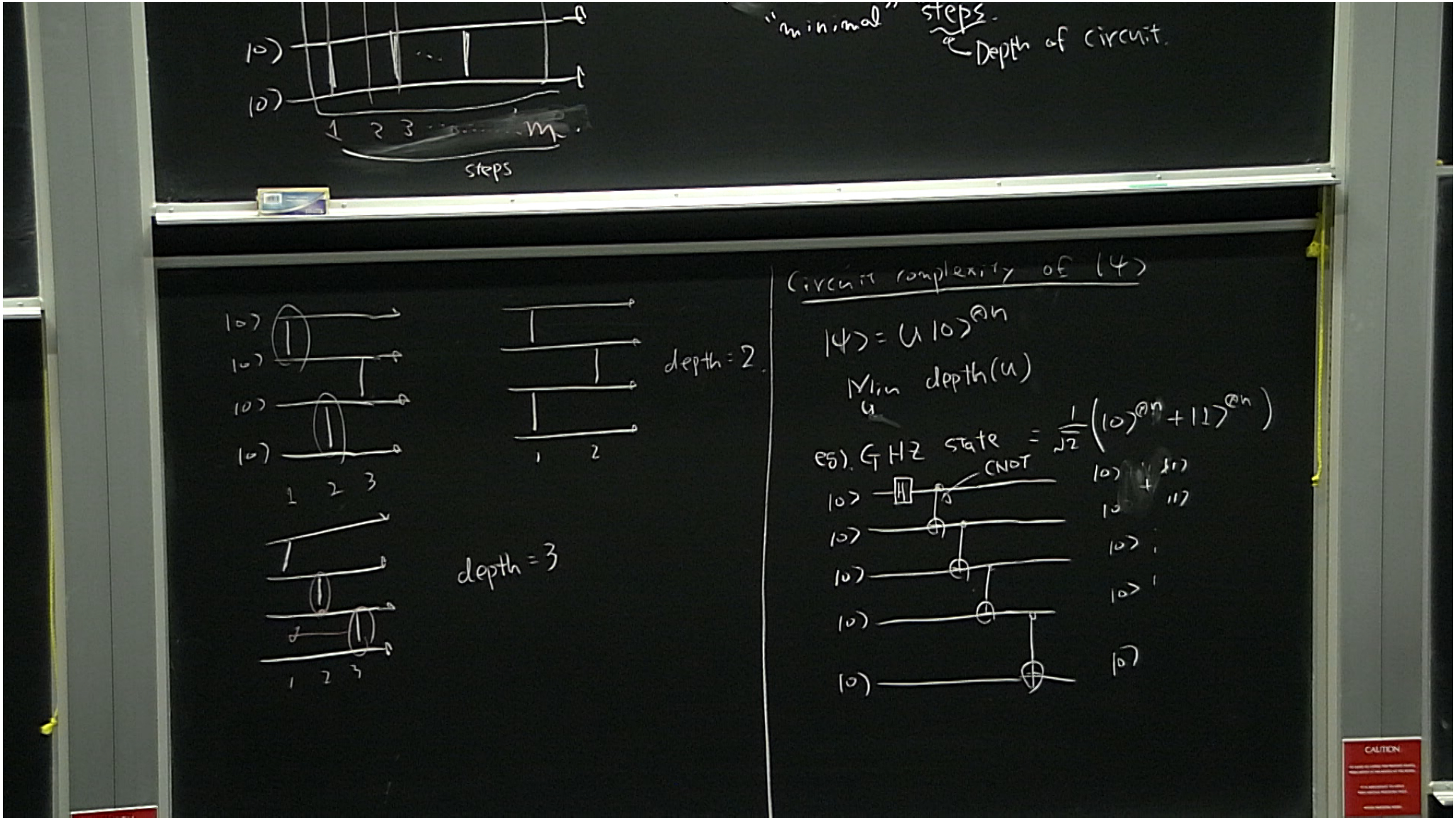
$|toric\rangle$: how complex ?

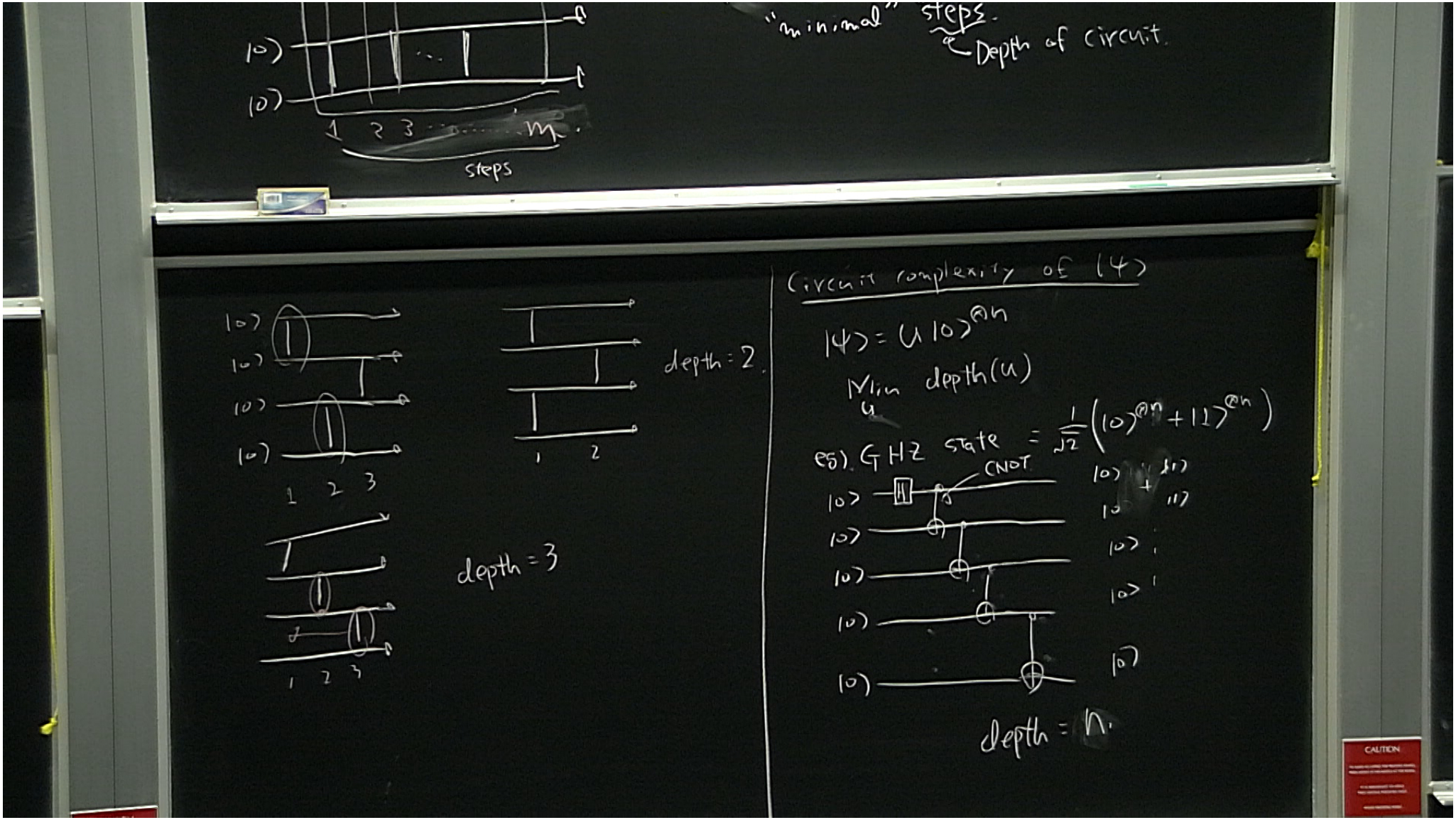
$$\begin{cases} N = 2L^2 \\ d = L \quad \text{minimal weight of logical op} \\ k = 2 \end{cases}$$


L
k = 2
Circuit
of toric code
toric + ε V
state space is stable for some small ε
perturbation


We will consider the one with
"minimal" "steps".
Depth of circuit.
|0⟩
|0⟩
1 2 3 ..... m
steps
|0⟩
|0⟩
|0⟩
|0⟩
1 2 3
depth = 2.
1 2
depth = 3
1 2 3


"minimal" steps.
Depth of circuit.
|0⟩
|0⟩
1 2 3 ... m.
steps
|0⟩
|0⟩
|0⟩
|0⟩
1 2 3
depth = 2
1 2
depth = 3
1 2 3
Circuit complexity of |ψ⟩
|ψ⟩ = U|0⟩^⊗n
Min depth(U)
U


|0⟩
|0⟩
1 2 3 ..... m.
steps
"minimal" steps.
Depth of circuit.
|0⟩
|0⟩
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Circuit complexity of |ψ⟩
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Min depth(U)
U
eg). GHZ state = 1/√2 (|0⟩^⊗n + |1⟩^⊗n)
CNOT
|0⟩
|0⟩
|0⟩
|0⟩
|0⟩
|0⟩ + |1⟩
|0⟩
|0⟩
|0⟩
|0⟩
CAUTION

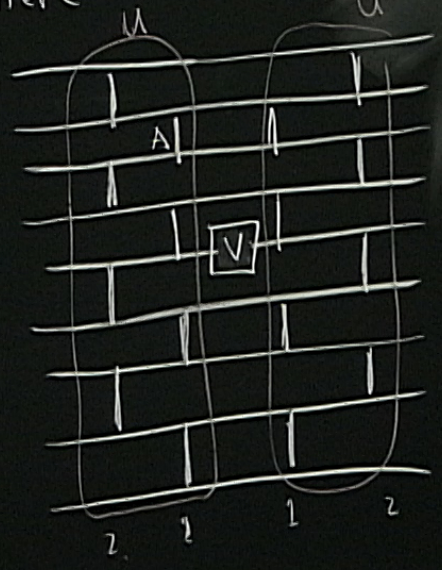

"minimal" steps.
Depth of circuit.
|0⟩
|0⟩
1 2 3 ... m.
steps
depth = 2.
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Circuit complexity of |ψ⟩
|ψ⟩ = U|0⟩^⊗n
Min depth(U)
U
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CNOT
depth = n.

Size of operators

Given 1-body operator V, how big $UVU^\dagger$ is ?

(Here U has depth m)

$UVU^\dagger$:



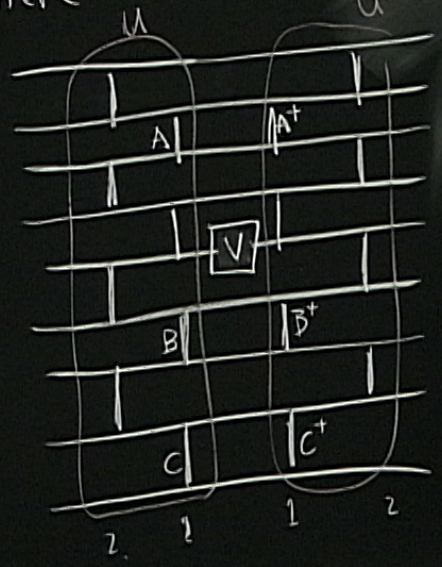
(m=2)

Size of operators

Given 1-body operator V, how big $UVU^\dagger$ is ?

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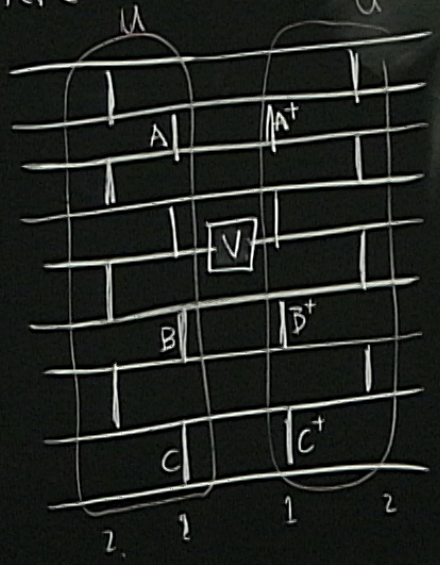
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Size of operators

Given 1-body operator V, how big $UVU^\dagger$ is ?

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$UVU^\dagger =$

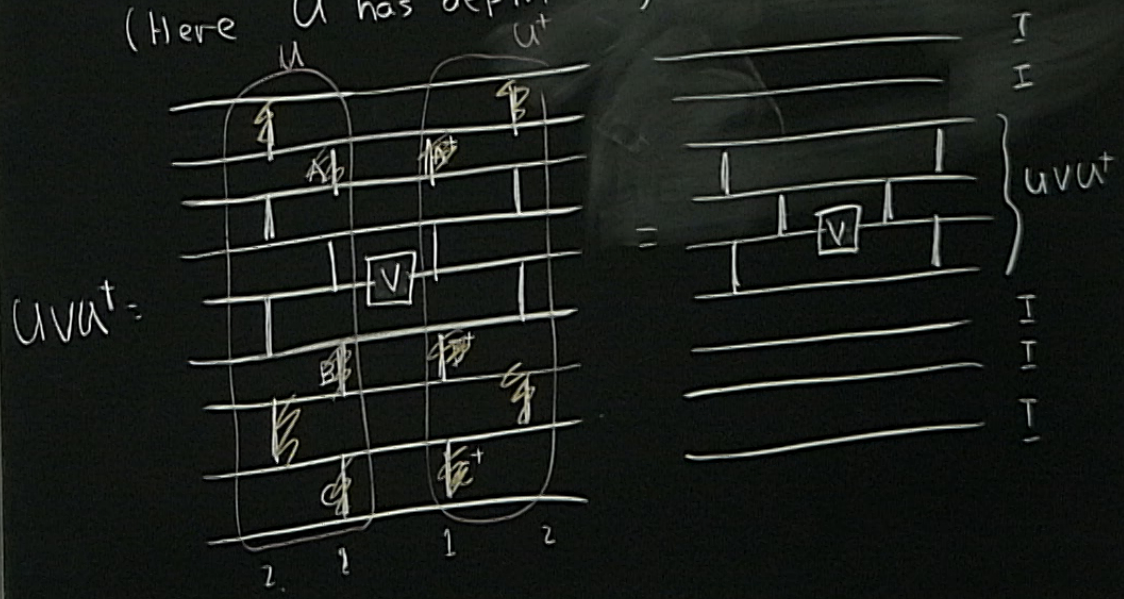


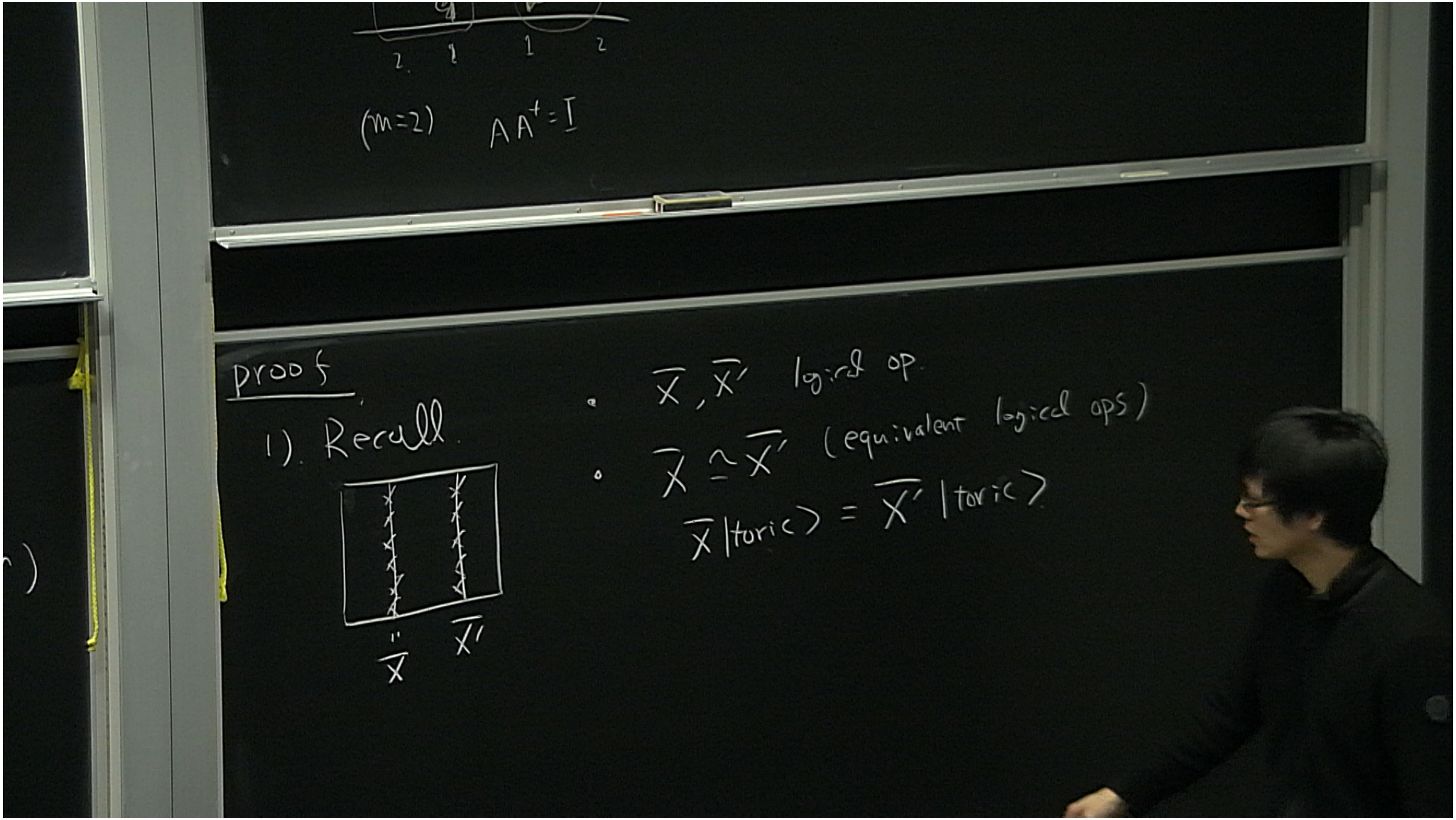
$(m=2) \quad AA^\dagger = I$

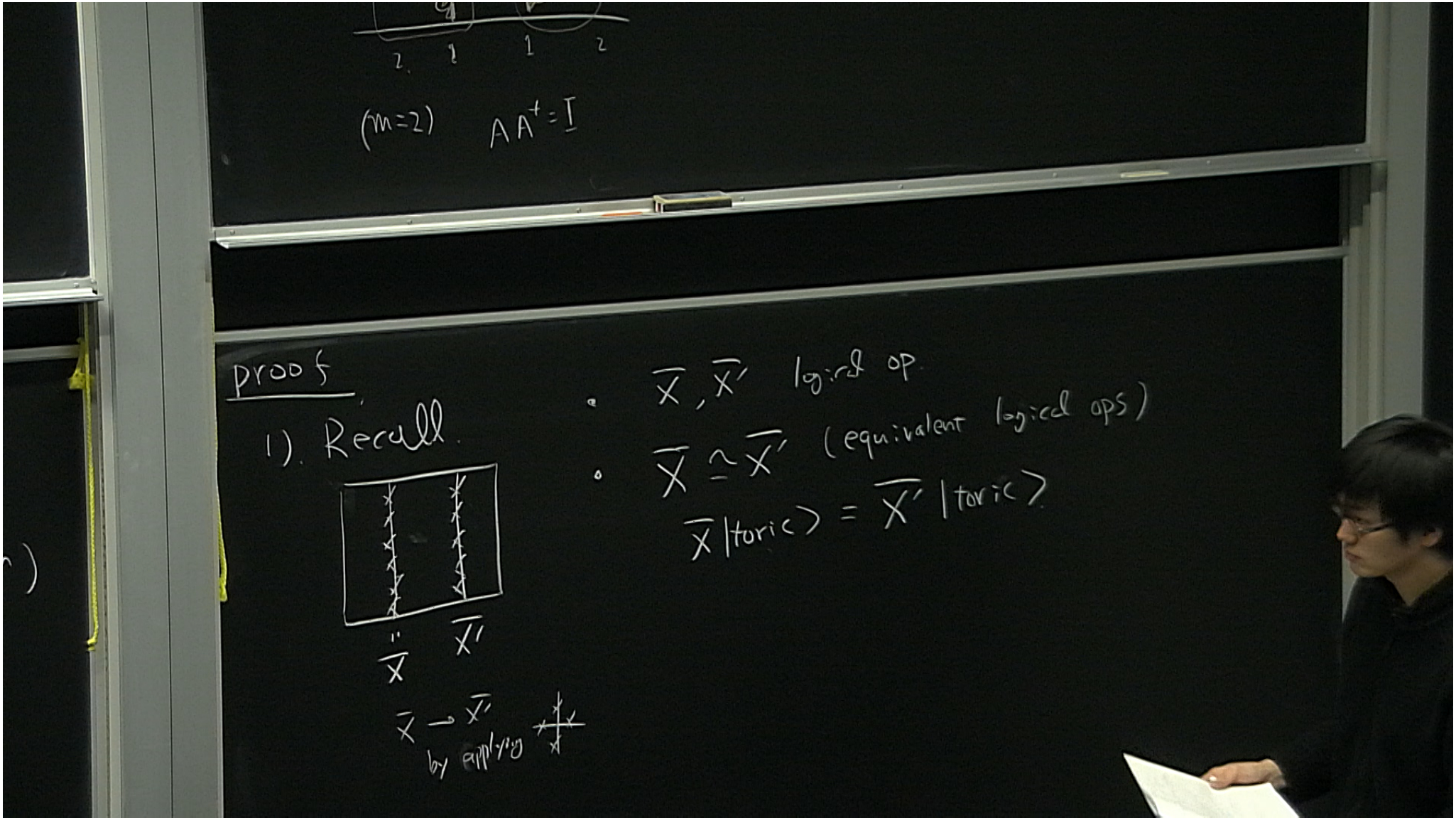
Size of operators

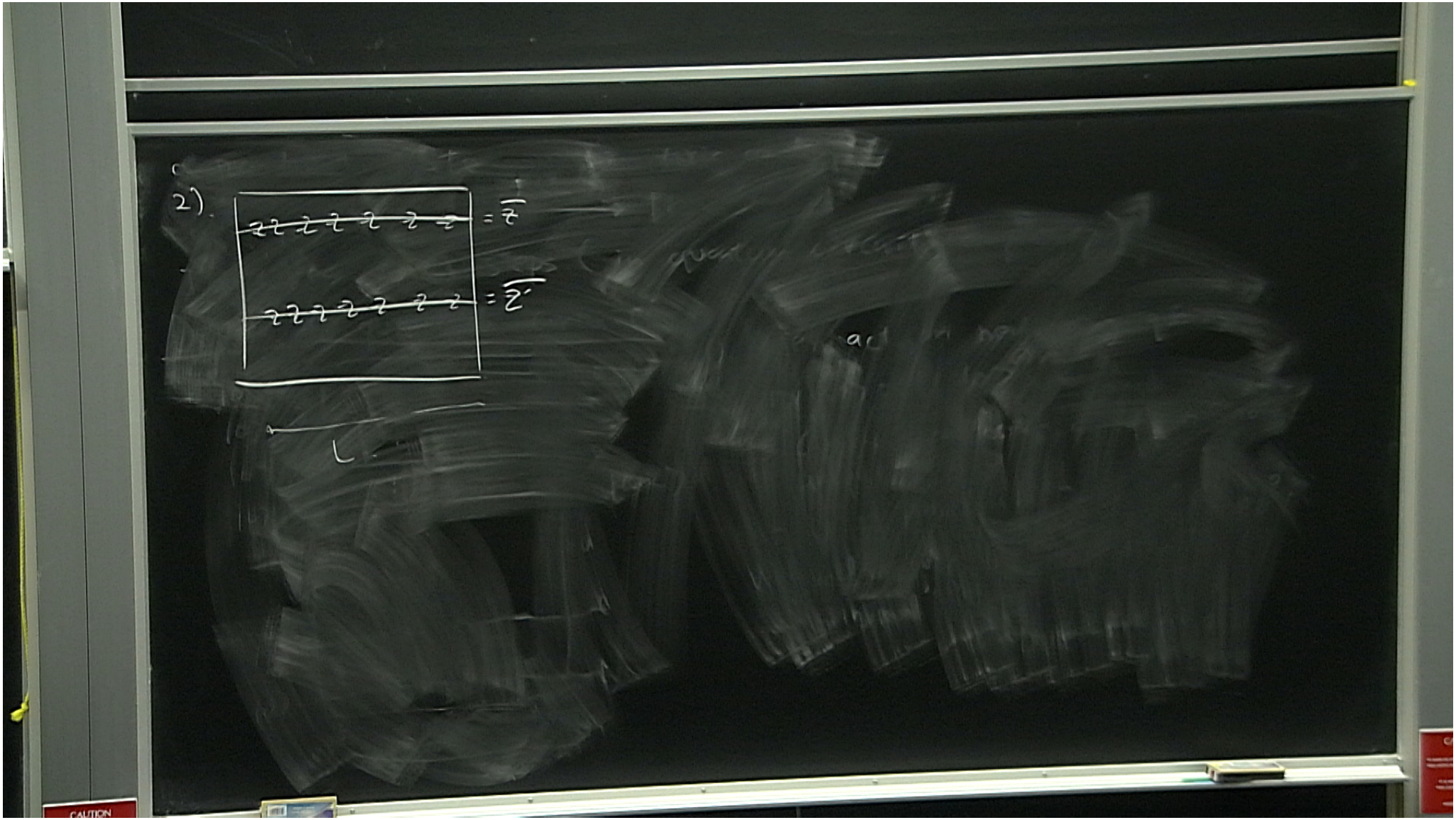
Given 1-body operator V, how big $UVU^\dagger$ is ?

(Here U has depth m) → 2m-body (in 1D geometry)



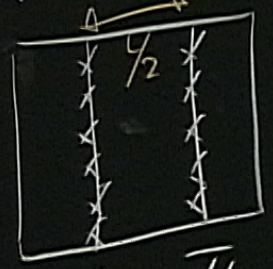

2 1 1 2
(m=2) $AA^t = I$
Proof
1). Recall
$\bar{X}$ $\bar{X}'$
• $\bar{X}, \bar{X}'$ logical op.
• $\bar{X} \simeq \bar{X}'$ (equivalent logical ops)
$\bar{X}|toric\rangle = \bar{X}'|toric\rangle$


2 1 1 2
(m=2) $AA^t = I$
Proof
1). Recall
$\bar{X}$ $\bar{X}'$
$\bar{X} \to \bar{X}'$
by applying
• $\bar{X}, \bar{X}'$ logical op.
• $\bar{X} \simeq \bar{X}'$ (equivalent logical ops)
$\bar{X}|toric\rangle = \bar{X}'|toric\rangle$


2)

Proof

1). Recall

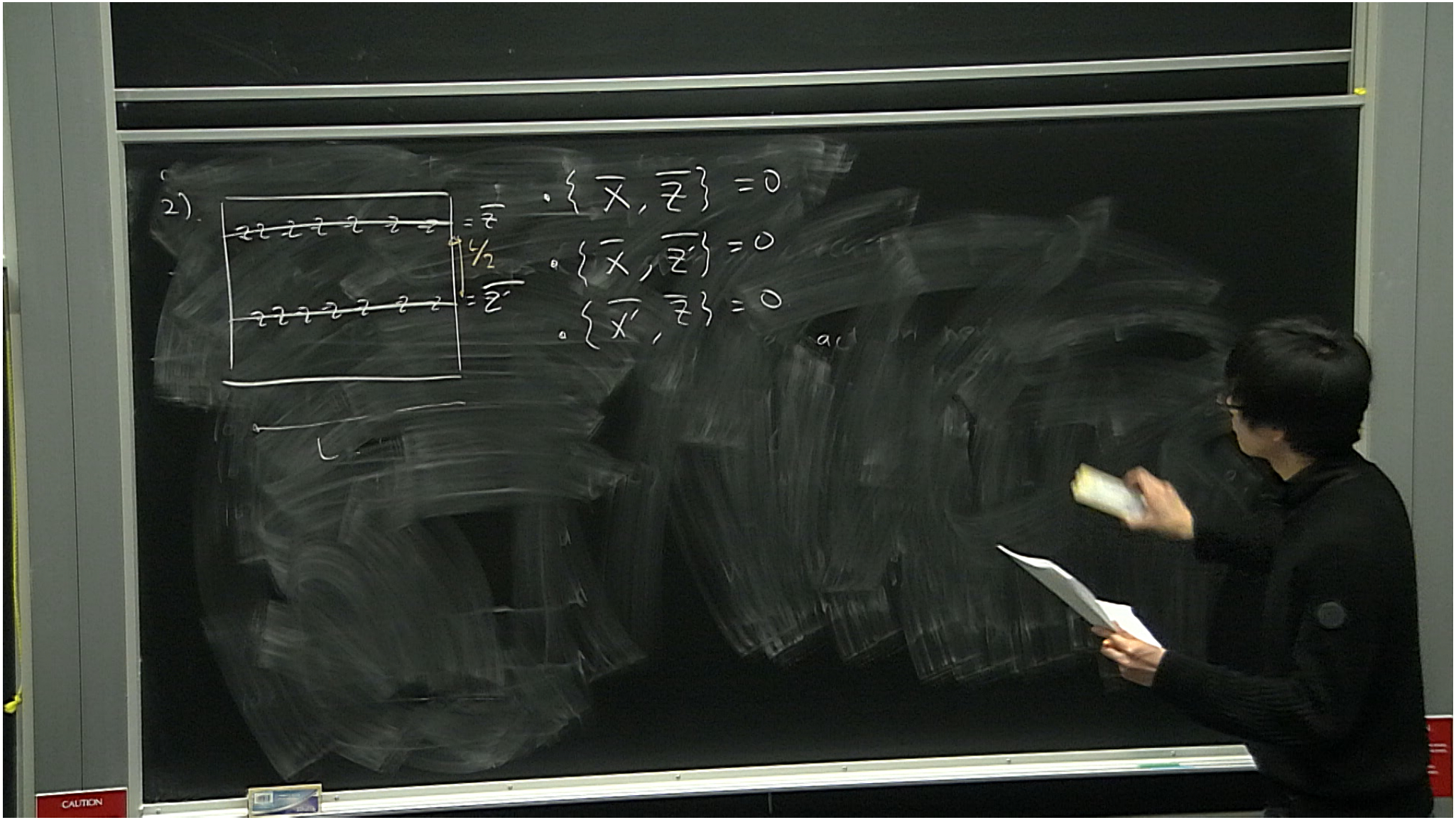


$\bar{X} \to \bar{X}'$ by applying

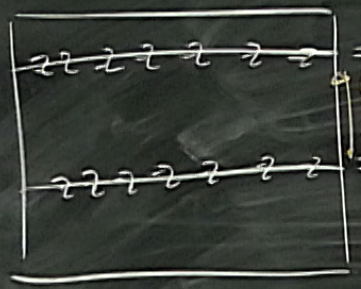
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$\bar{X}|toric\rangle = \bar{X}'|toric\rangle$

CAUTION


2)
$= \bar{z}$
$L/2$
$= \bar{z}'$
$\{\bar{x}, \bar{z}\} = 0$
$\{\bar{x}, \bar{z}'\} = 0$
$\{\bar{x}', \bar{z}\} = 0$
CAUTION

2)



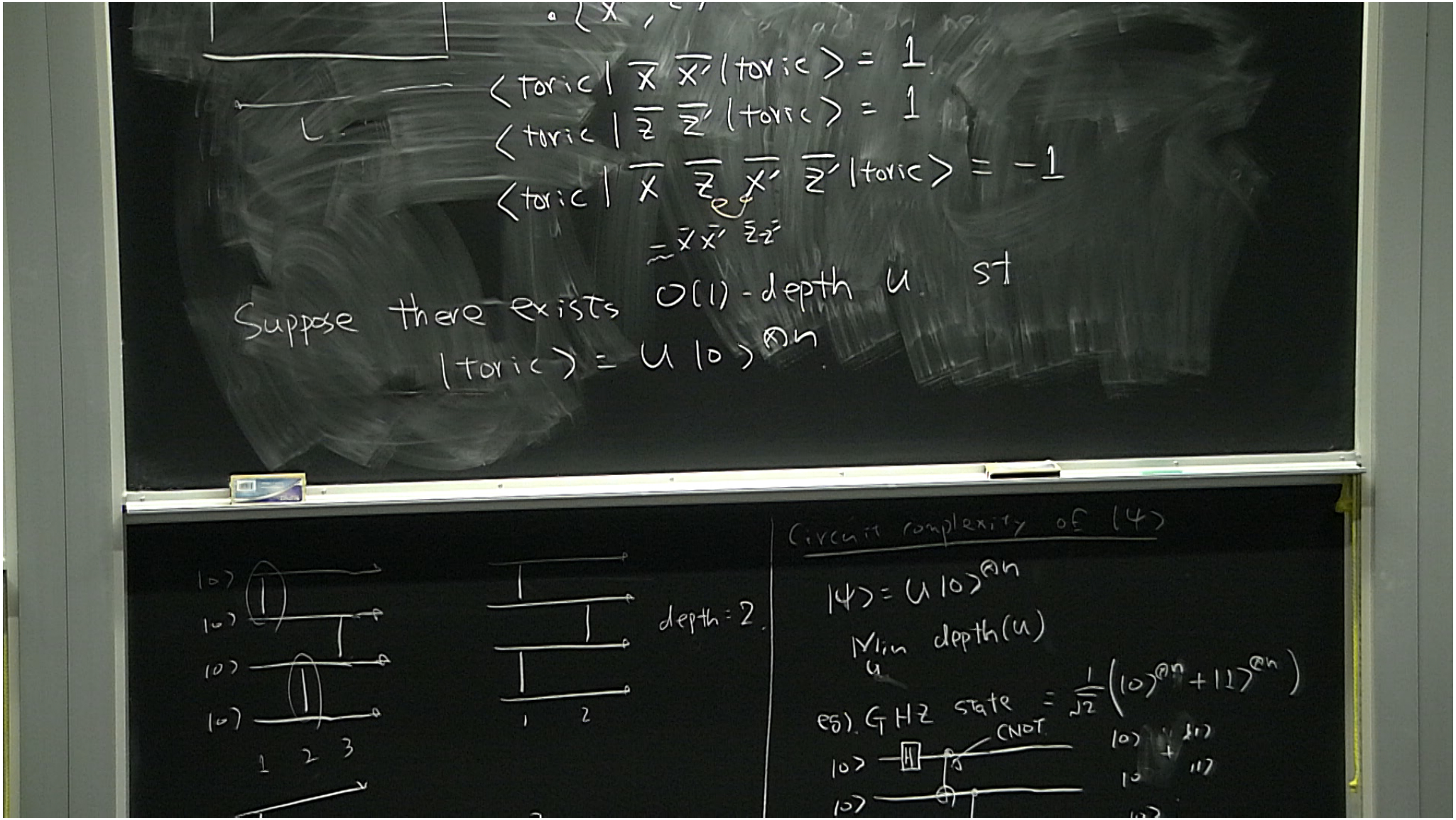
- $\{\bar{x}, \bar{z}\} = 0$
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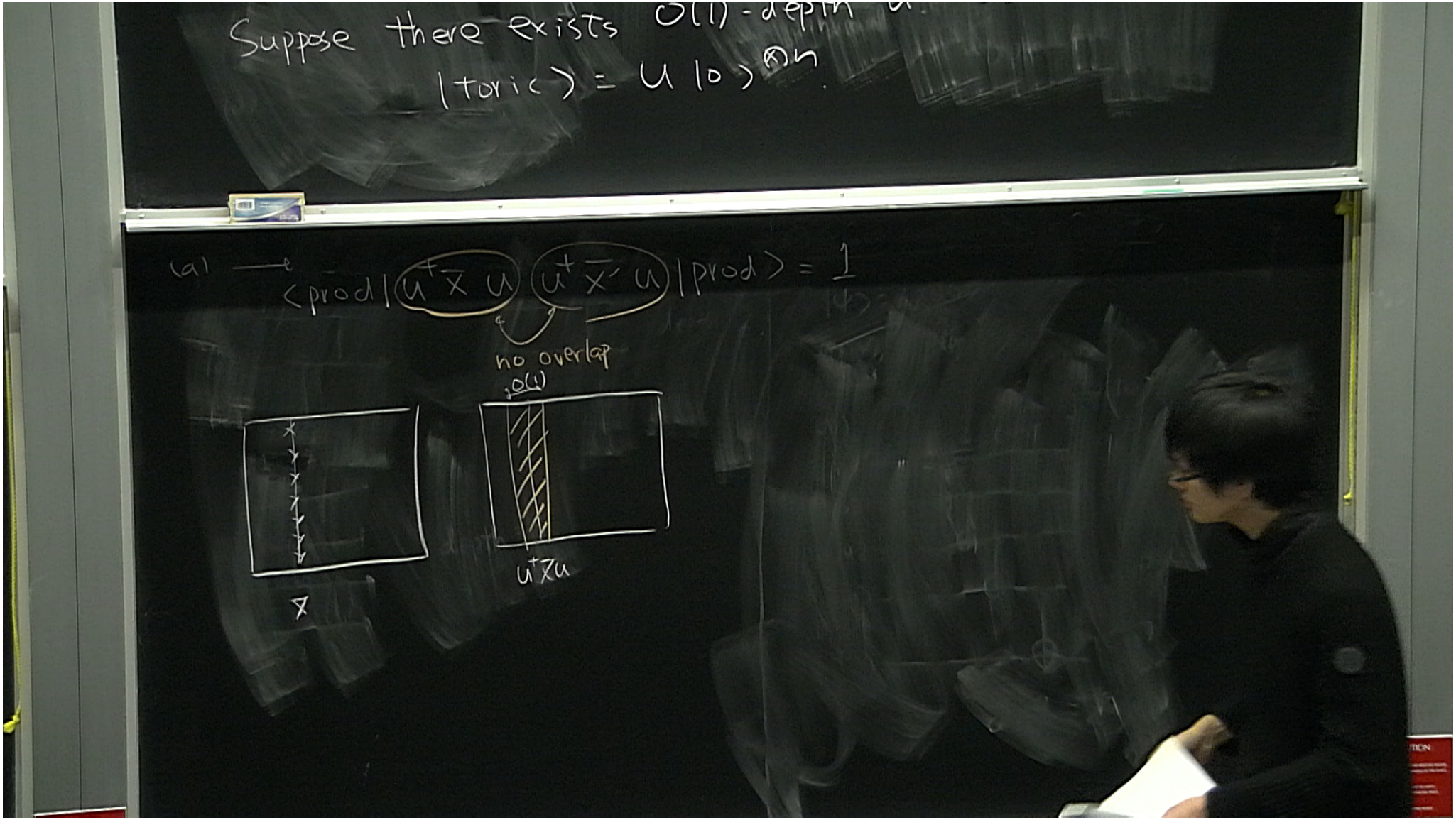
$\langle toric | \bar{x}\,\bar{x}' | toric \rangle = 1$

$\langle toric | \bar{z}\,\bar{z}' | toric \rangle = 1$

$\langle toric | \bar{x}\,\bar{z}\,\bar{x}'\,\bar{z}' | toric \rangle = -1$

$= \bar{x}\bar{x}'\,\bar{z}\bar{z}'$

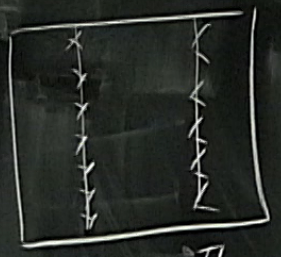

⟨toric| X̄ X̄' |toric⟩ = 1
⟨toric| Z̄ Z̄' |toric⟩ = 1
⟨toric| X̄ Z̄ X̄' Z̄' |toric⟩ = -1
≃ X̄X̄' Z̄Z̄'
Suppose there exists O(1)-depth U st
|toric⟩ = U |0⟩^⊗n
depth = 2
1 2 3
1 2
Circuit complexity of |ψ⟩
|ψ⟩ = U |0⟩^⊗n
Min depth(U)
U
eg). GHZ state = 1/√2 (|0⟩^⊗n + |1⟩^⊗n)
CNOT
H

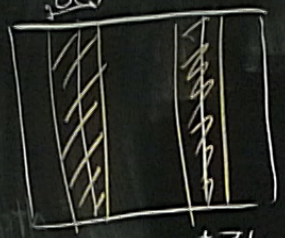

Suppose there exists O(1)-depth
|toric⟩ = U |0⟩^⊗n
(a) ⟨prod| u†x u u†z u |prod⟩ = 1
no overlap
O(1)
X
u†Zu

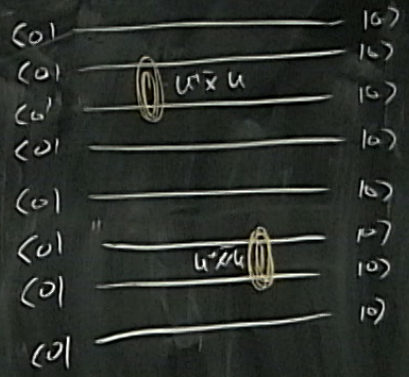
$|toric\rangle = U|0\rangle$

(a) $\longrightarrow$ $\langle prod | U^{+}_{\bar{x}} U \; U^{+}_{\bar{x}'} U | prod \rangle = 1$

no overlap

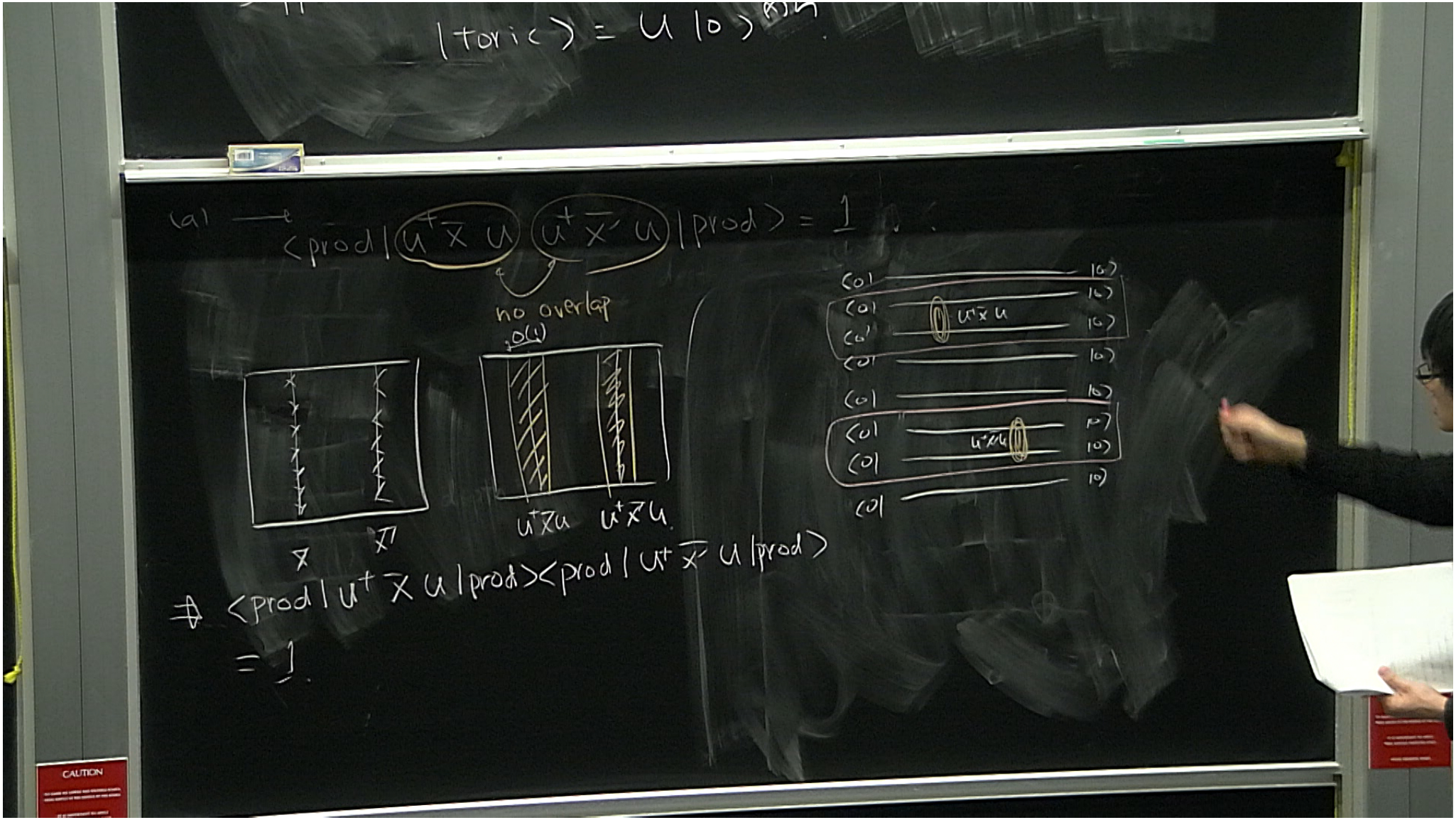


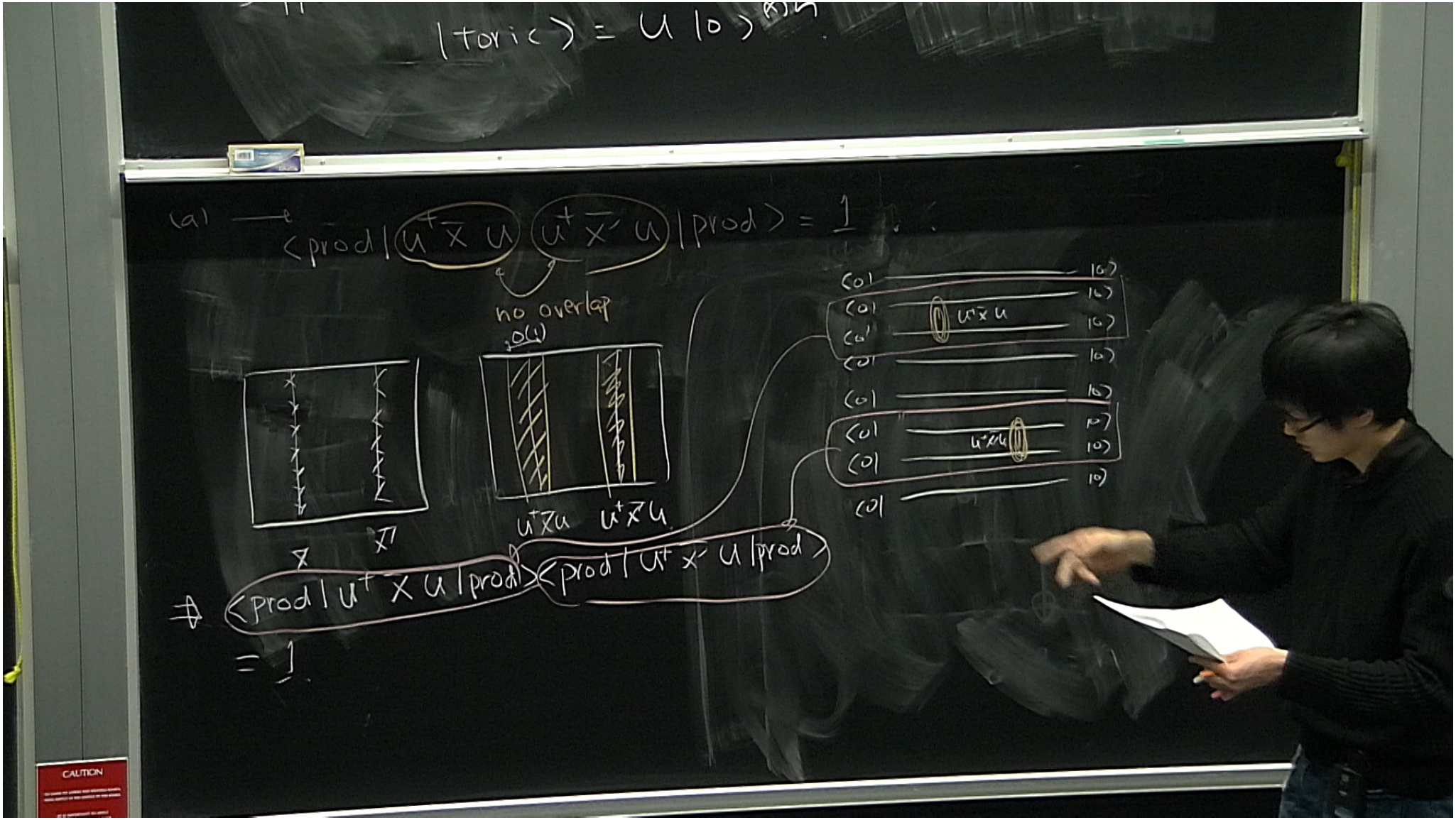


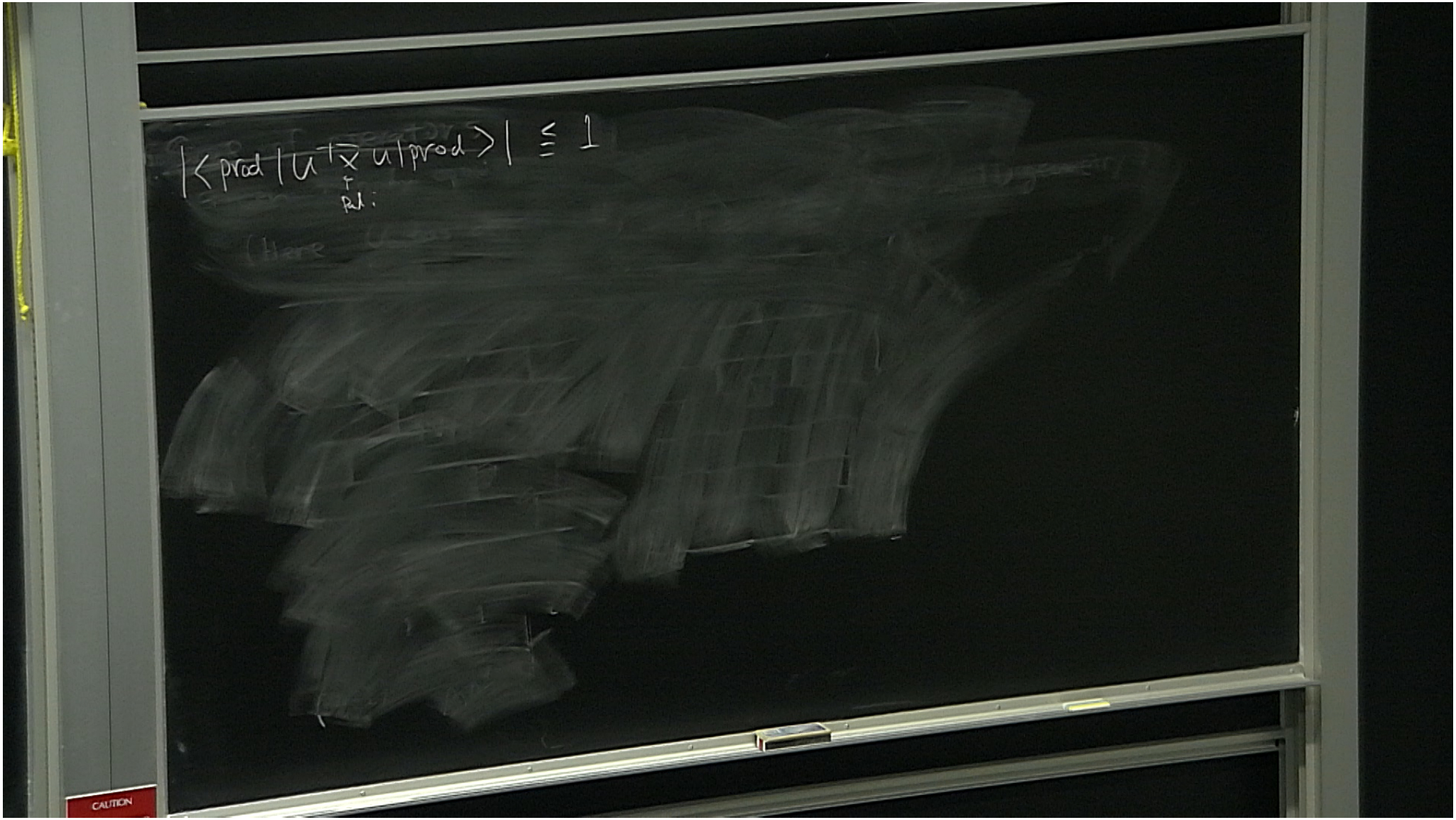


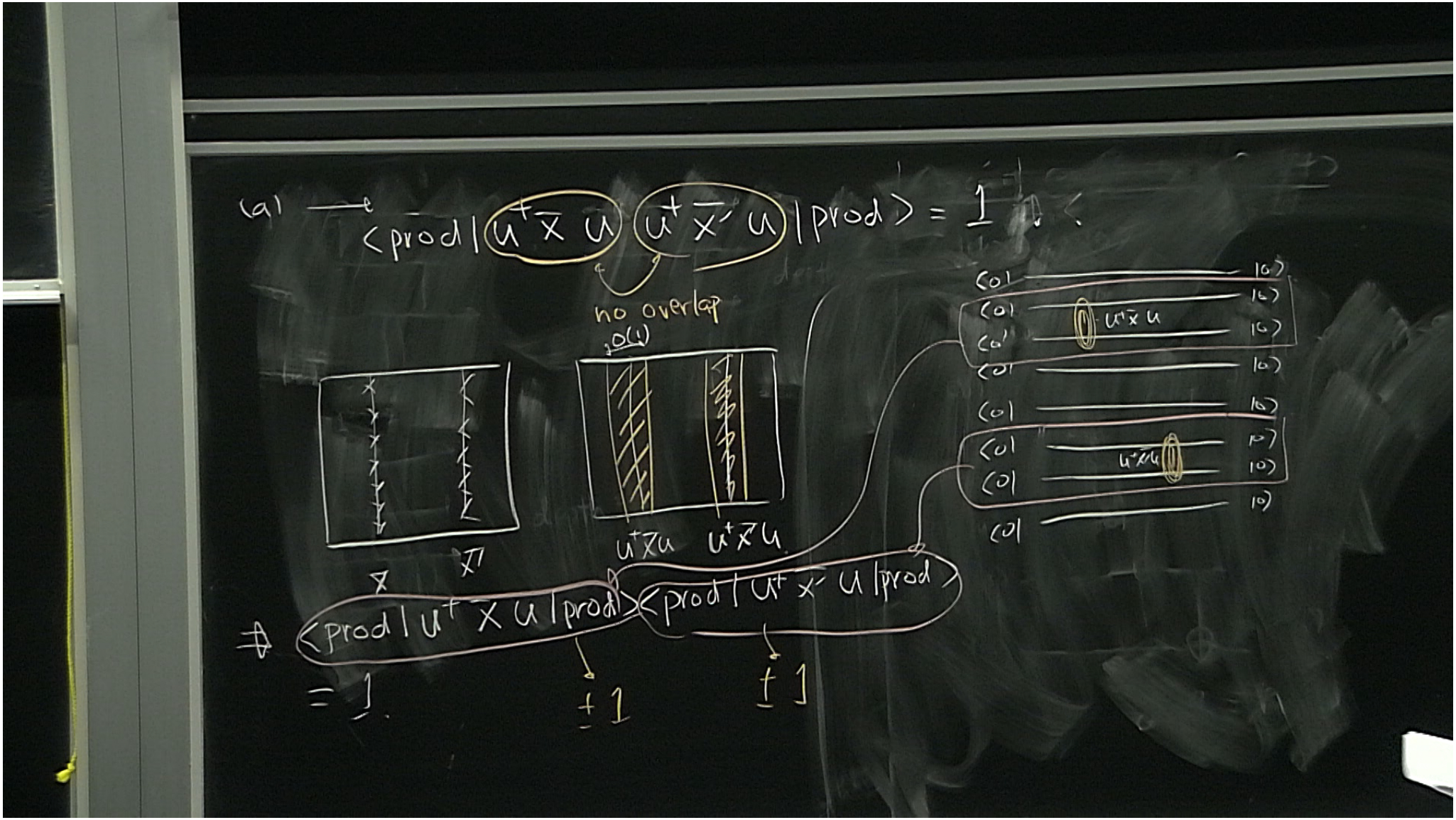
$\neq \langle prod | U^{+}_{\bar{x}} U | prod \rangle \langle prod | U^{+}_{\bar{x}'} U | prod \rangle$

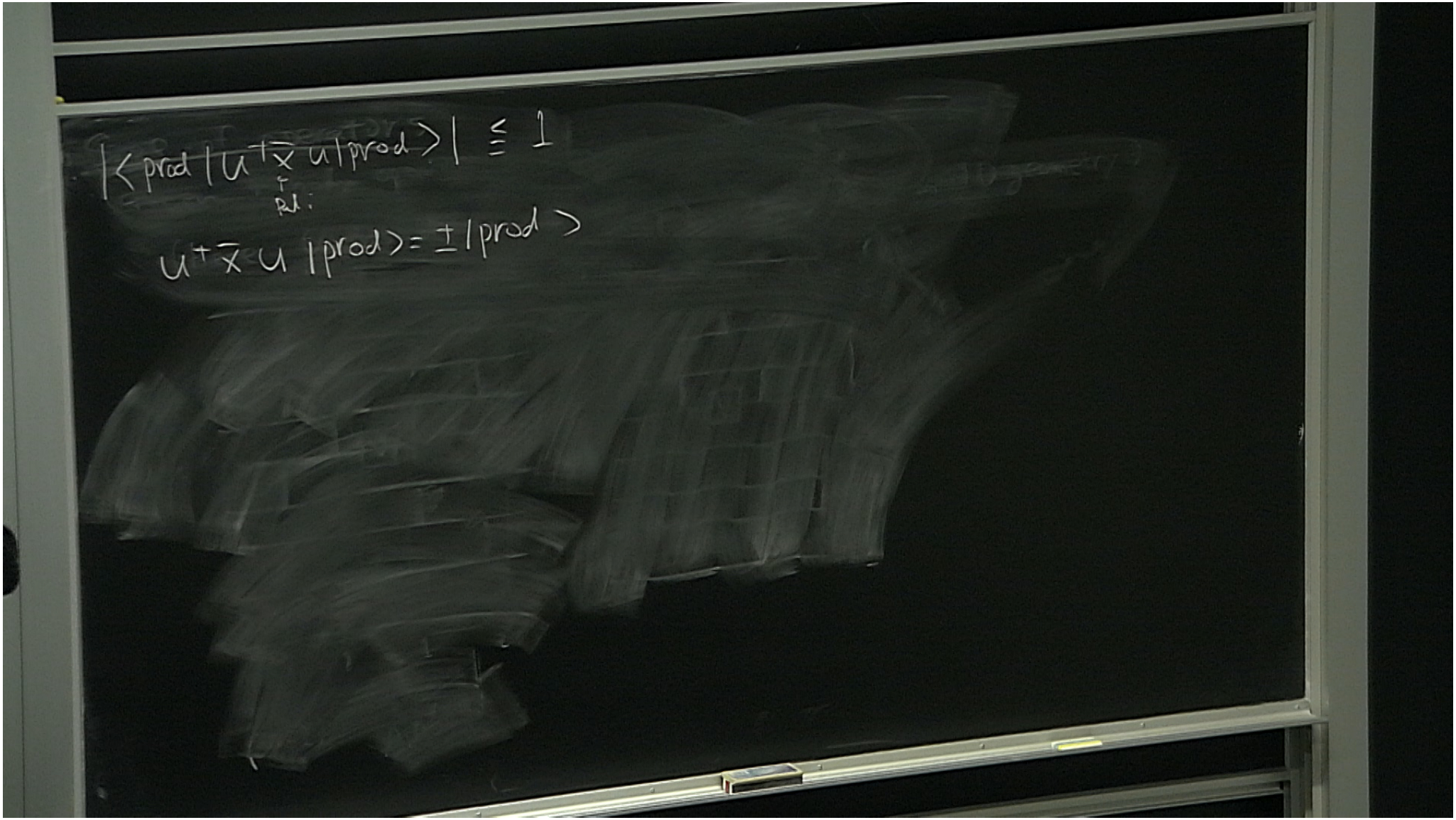
$= 1$

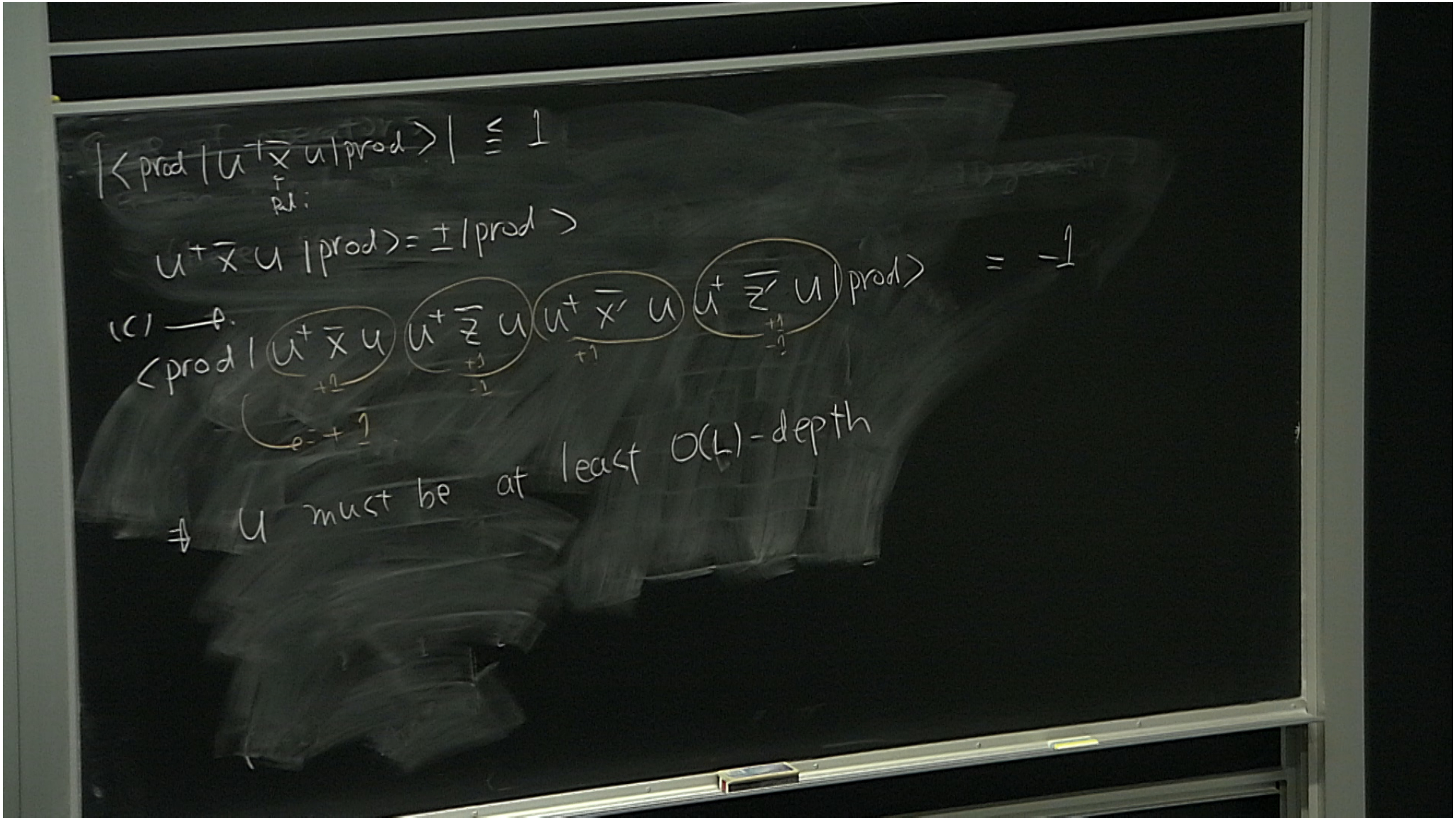

|toric⟩ = U |0⟩
(a)
⟨prod| u⁺ x̄ u u⁺ x̄ u |prod⟩ = 1
no overlap
x̄
x̄'
u⁺x̄u
u⁺x̄u
⟨0|
|0⟩
u⁺x̄u
u⁺x̄u
⟨prod| u⁺ x̄ u |prod⟩⟨prod| u⁺ x̄ u |prod⟩
= 1
CAUTION

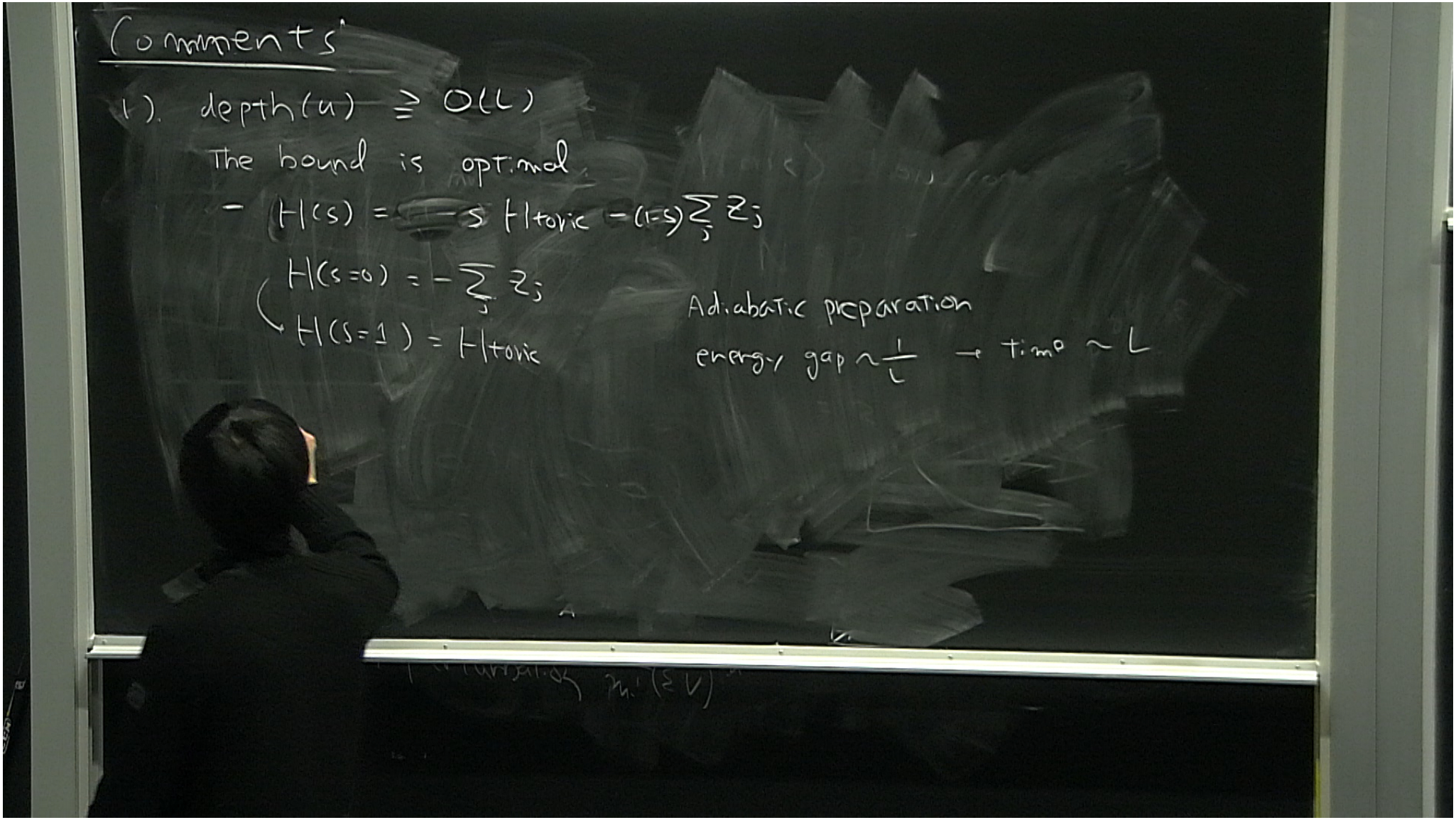

|toric⟩ = U |0⟩
(a)
⟨prod| U⁺ X U U⁺ X U |prod⟩ = 1
no overlap
O(L)
U⁺XU U⁺XU
X X
⟨prod| U⁺ X U |prod⟩ ⟨prod| U⁺ X U |prod⟩
= 1
⟨0| |0⟩
U⁺XU
U⁺XU
CAUTION


$|\langle prod | U^{\dagger} \sum_{i} U | prod \rangle| \leq 1$


(a)
⟨prod| U⁺ X̄ U U⁺ X̄' U |prod⟩ = 1
no overlap
O(1)
X
X'
U⁺XU
U⁺X'U
U⁺X̄U
⟨prod| U⁺ X̄ U |prod⟩ ⟨prod| U⁺ X̄' U |prod⟩
= 1
±1
±1


$|\langle prod | U^{\dagger} \bar{x} U | prod \rangle| \leq 1$
$U^{\dagger} \bar{x} U | prod \rangle = \pm | prod \rangle$


$|\langle prod | U^\dagger \bar{X} U | prod \rangle| \leq 1$
pf:
$U^\dagger \bar{X} U |prod\rangle = \pm |prod\rangle$
(c) ⟶ ρ.
$\langle prod | U^\dagger \bar{X} U \quad U^\dagger \bar{Z} U \quad U^\dagger \bar{X'} U \quad U^\dagger \bar{Z'} U | prod \rangle = -1$
+1 +1 +1 +1 -1 -1
$e = +1$
$\Rightarrow$ U must be at least O(L)-depth


Comments
1). depth(u) ≥ O(L)
The bound is optimal
- H(s) = s H_toric - (1-s) Σ_j Z_j
H(s=0) = - Σ_j Z_j
H(s=1) = H_toric
Adiabatic preparation
energy gap ~ 1/L → time ~ L

Comments

1). depth(u) $\geq O(L)$

The bound is optimal.

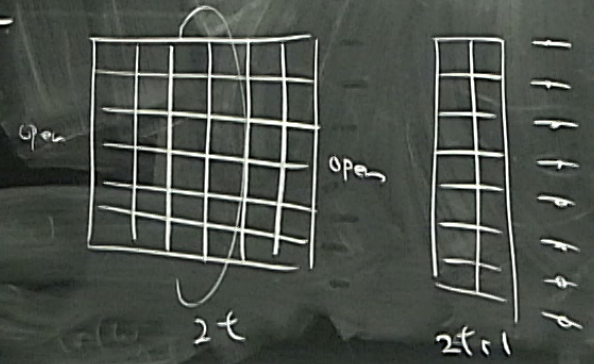
- $H(s) = -s\, H_{\text{toric}} - (1-s)\sum_j Z_j$

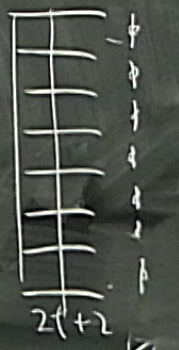
  $H(s=0) = -\sum_j Z_j$

  $H(s=1) = H_{\text{toric}}$

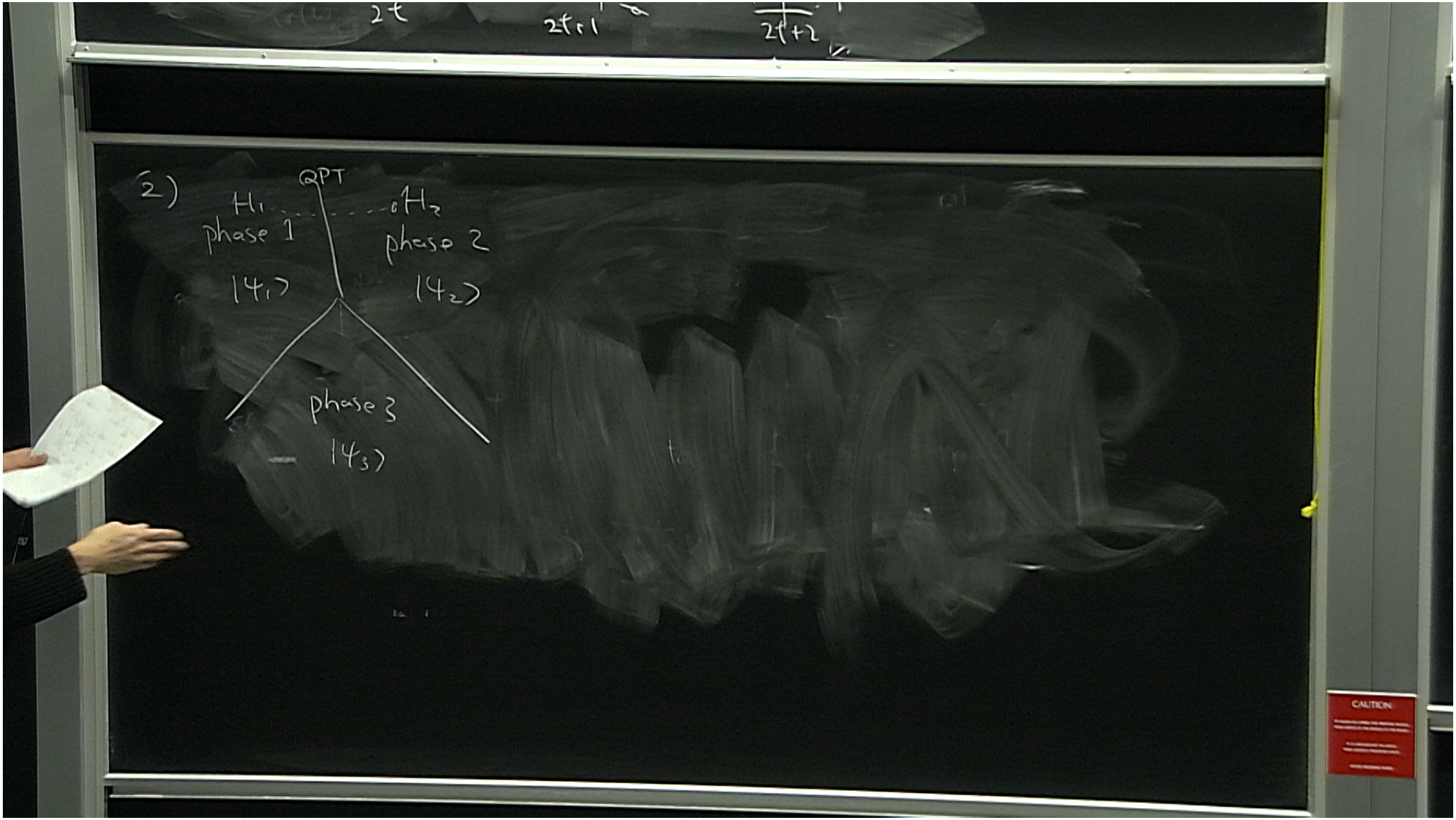
Adiabatic preparation

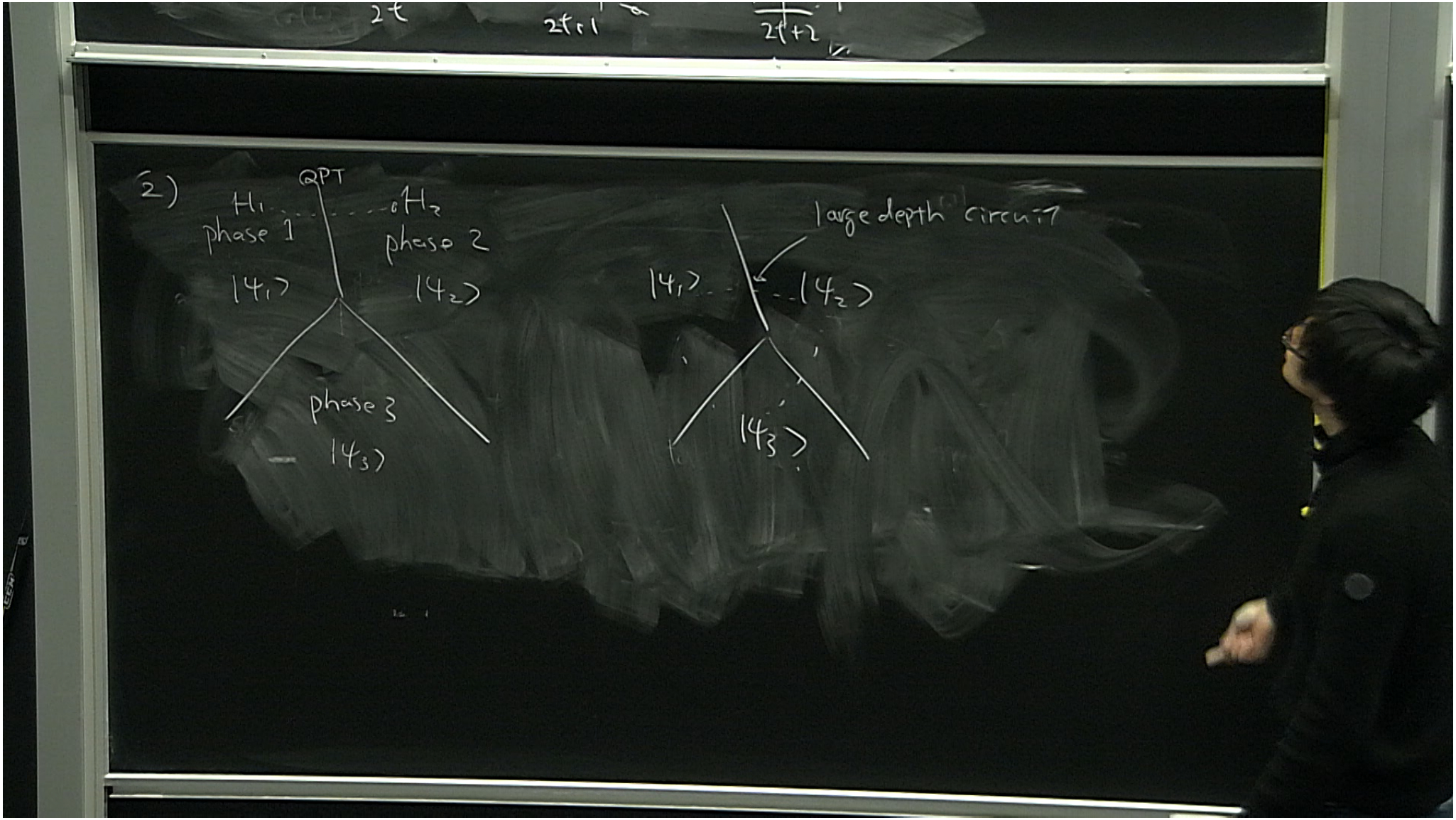
energy gap $\sim \frac{1}{L}$ → time $\sim L$

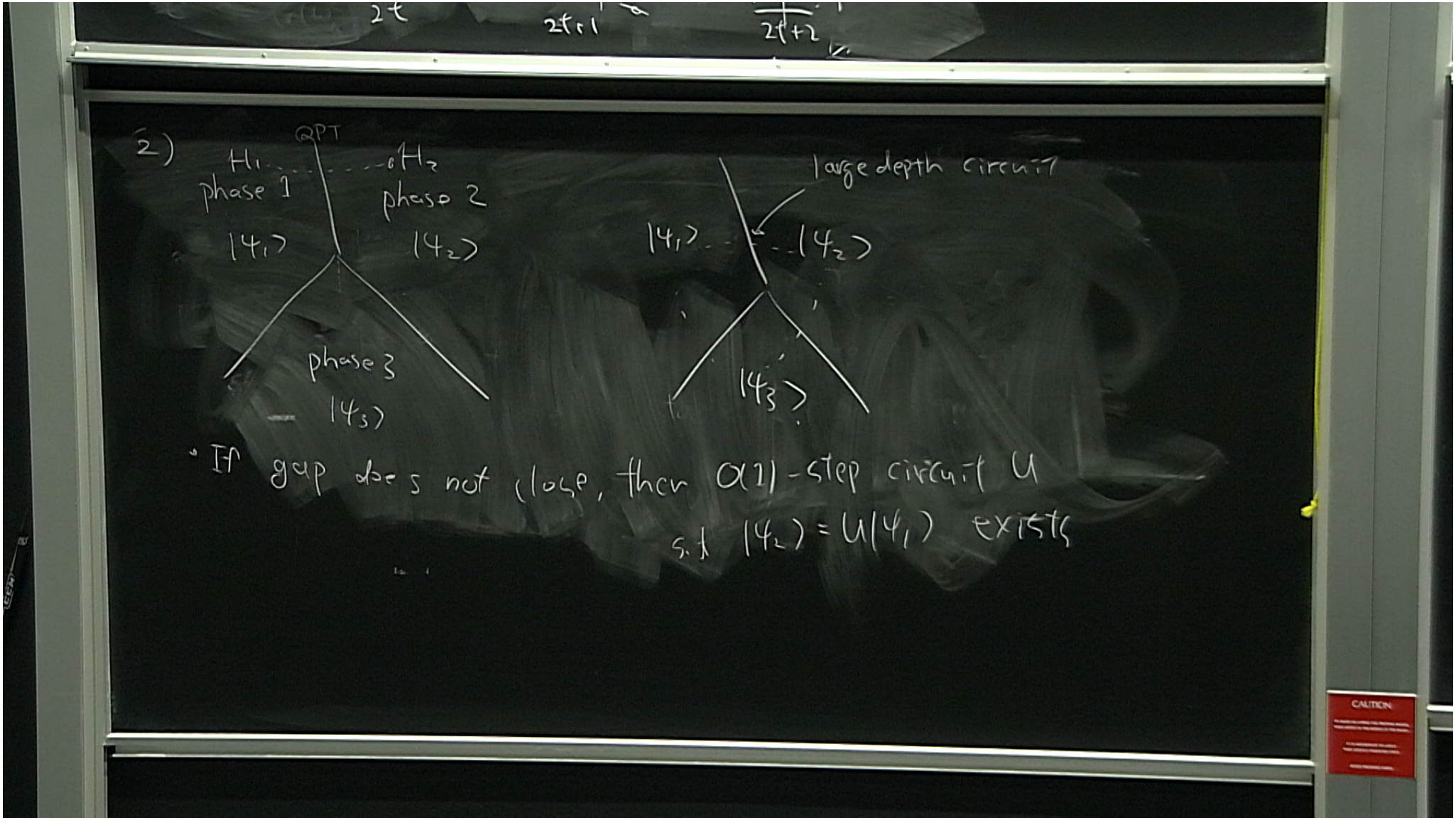




2)


2t
2t+1
2t+2
2)
QPT
H_1
phase 1
$|4_1\rangle$
H_2
phase 2
$|4_2\rangle$
phase 3
$|4_3\rangle$
CAUTION


2t
2t+1
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QPT
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large depth circuit
$|4_1\rangle$
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2t
2t+1
2t+2
2)
QPT
H_1
phase 1
$|\psi_1\rangle$
H_2
phase 2
$|\psi_2\rangle$
phase 3
$|\psi_3\rangle$
large depth circuit
$|\psi_1\rangle$
$|\psi_2\rangle$
$|\psi_3\rangle$
• If gap does not close, then $O(1)$-step circuit U
s.t. $|\psi_2\rangle = U|\psi_1\rangle$ exists

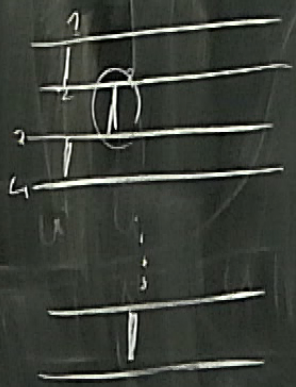
3) Generalizations

eg, U is "symmetric"

$S = \bigotimes_j X_j$

n qubits

$[U, S] = 0$



$[U_{12}, S] = 0$

$[U_{23}, S] = 0$

$[U_{34}, S] = 0$

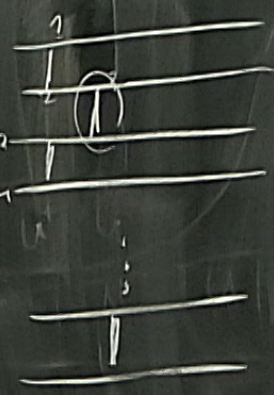
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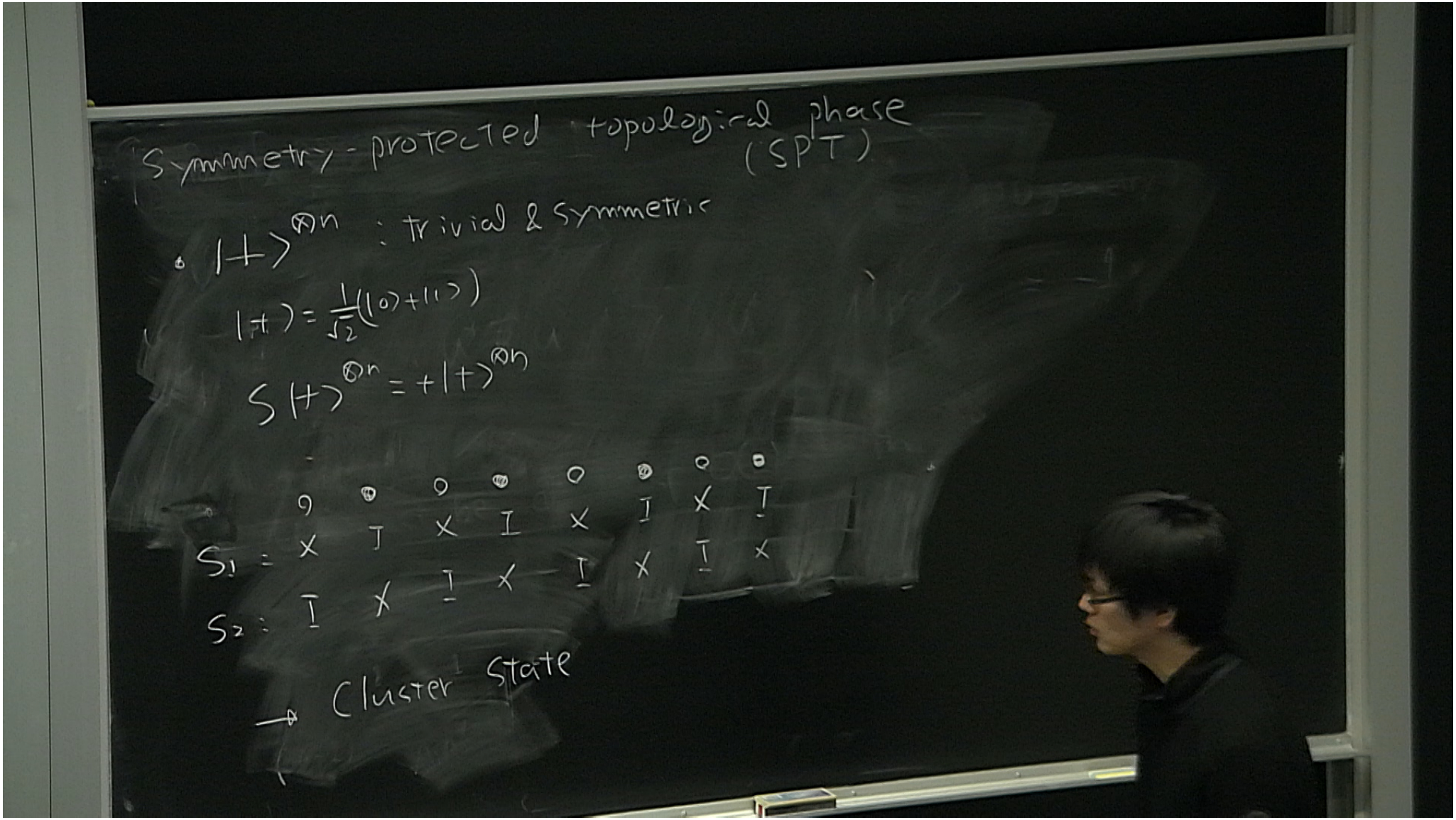


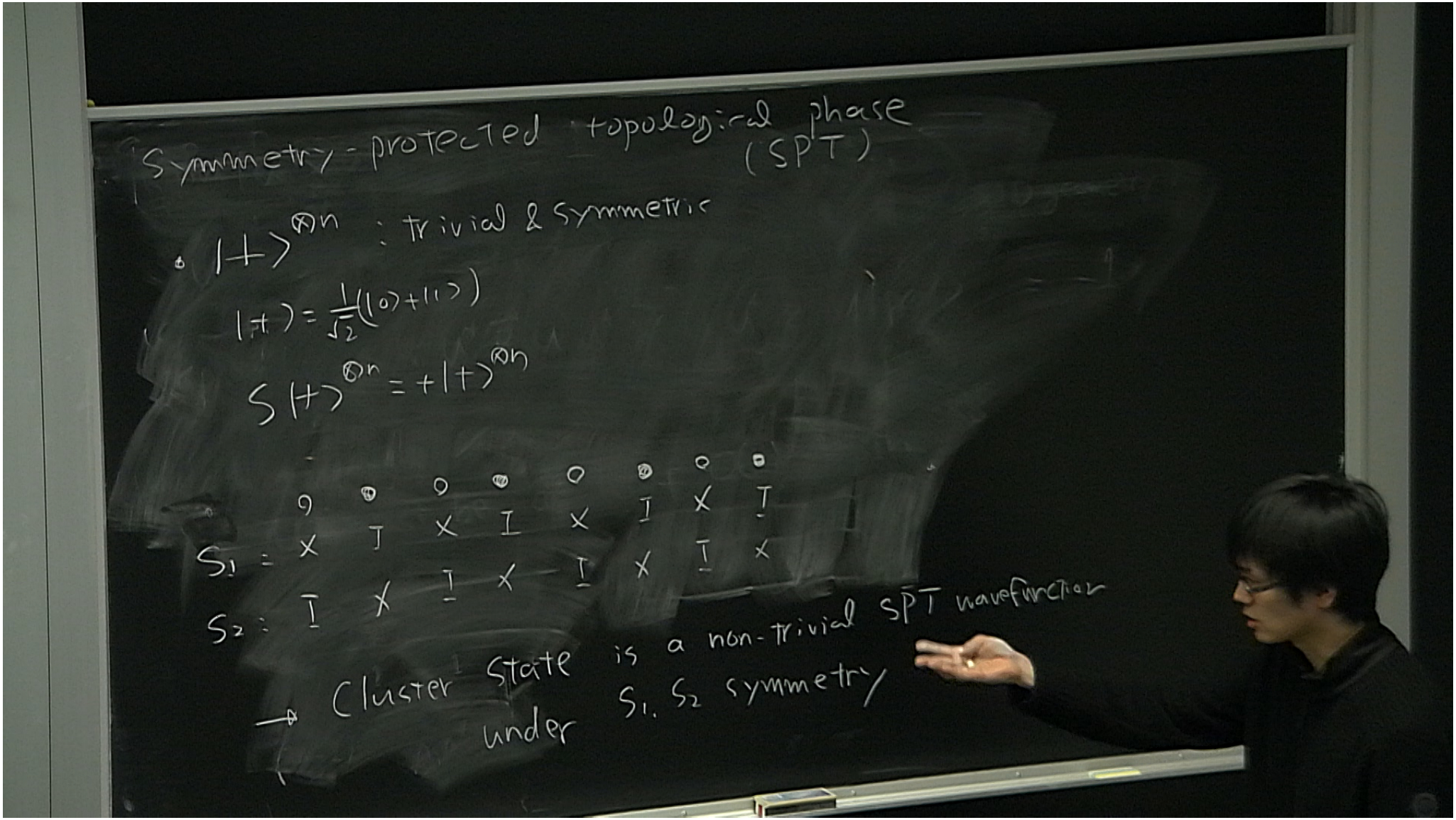
$[U_{12}, S] = 0$

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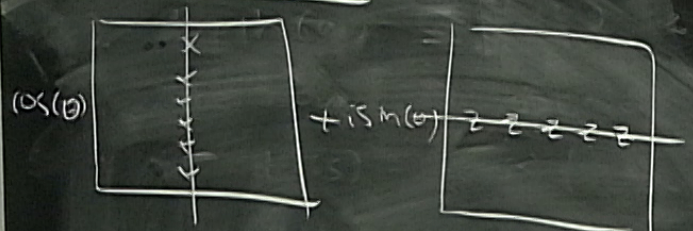
$[U_{34}, S] = 0$

$|4,7\rangle$ $|\phi_i\rangle$


Symmetry-protected topological phase (SPT)
• $|+\rangle^{\otimes n}$: trivial & symmetric
$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
$S|+\rangle^{\otimes n} = +|+\rangle^{\otimes n}$
S_1 : X I X I X I X I
S_2 : I X I X I X I X
→ Cluster State


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S_1 : X I X I X I X I
S_2 : I X I X I X I X
→ Cluster state is a non-trivial SPT wavefunction under S_1, S_2 symmetry

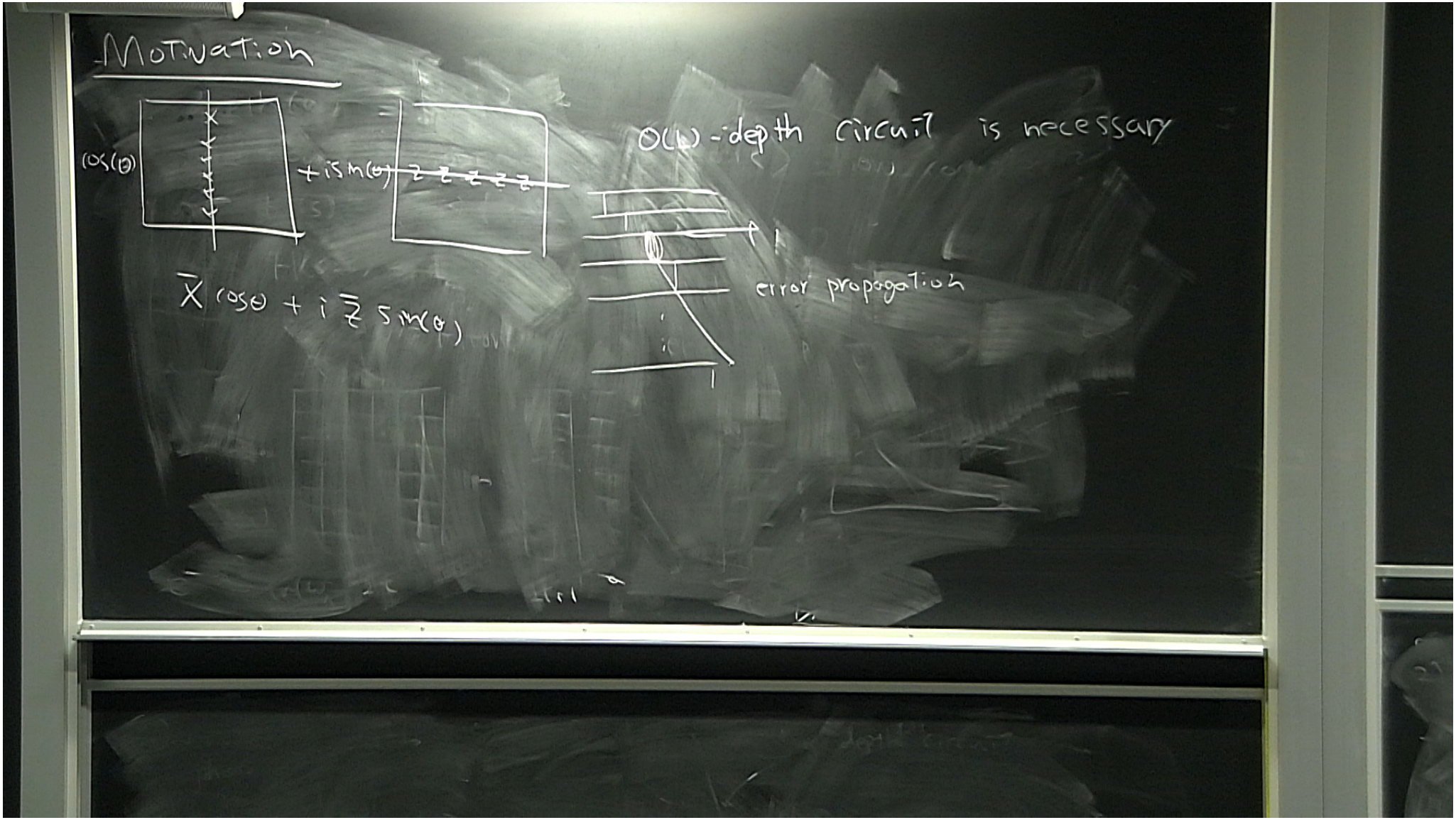
Motivation

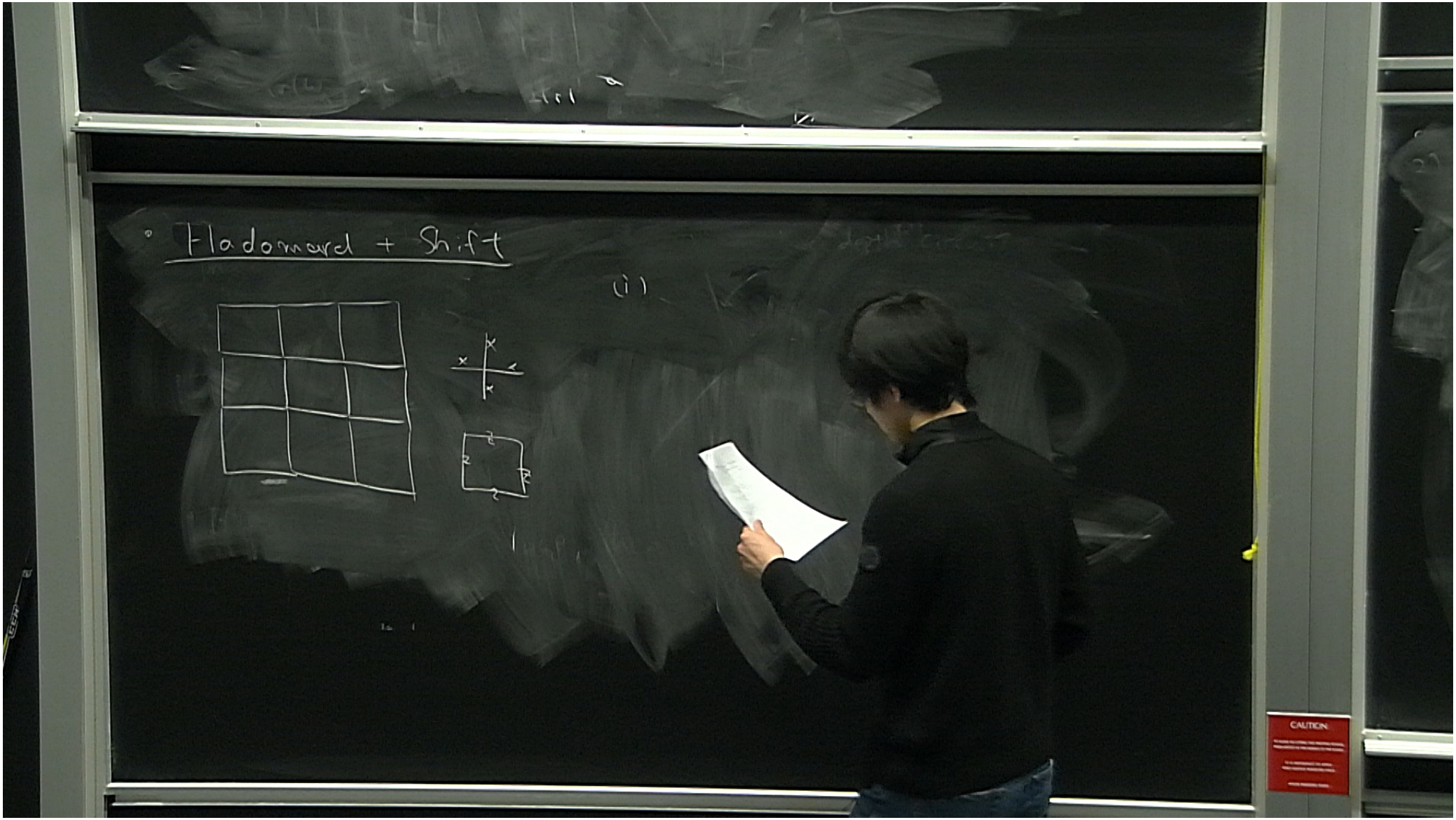


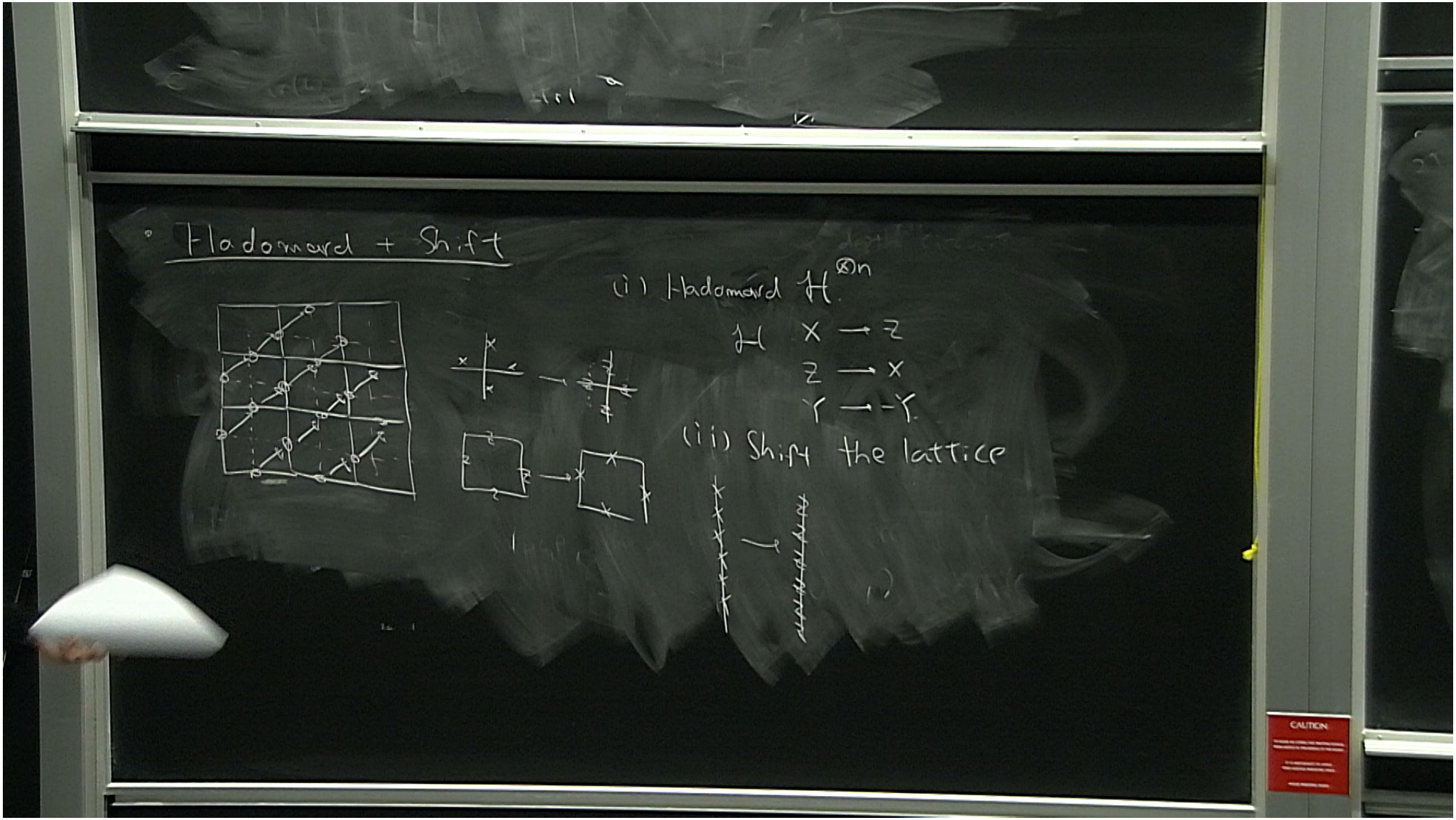
$\bar{X}\cos\theta + i\bar{Z}\sin\theta$

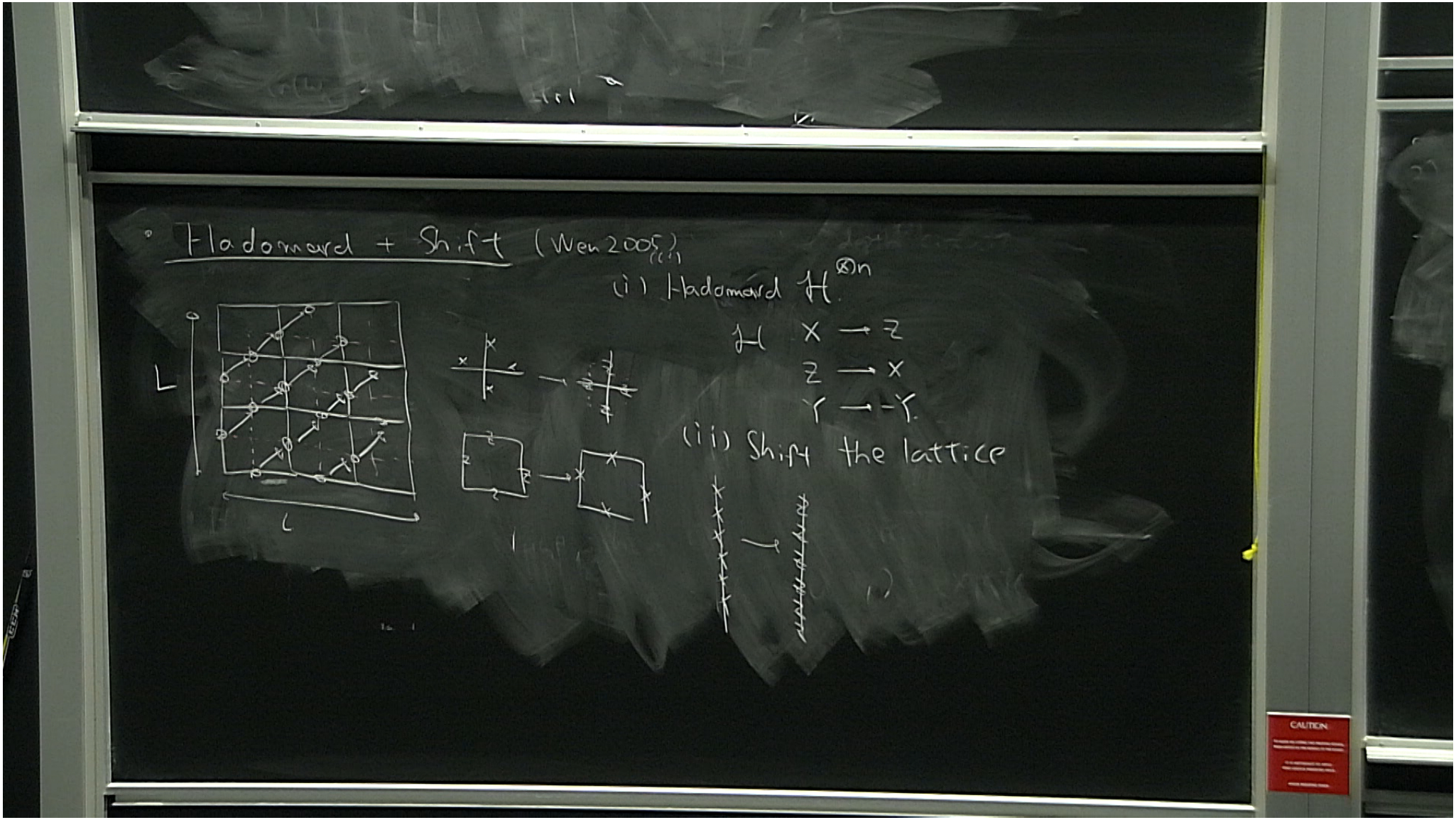


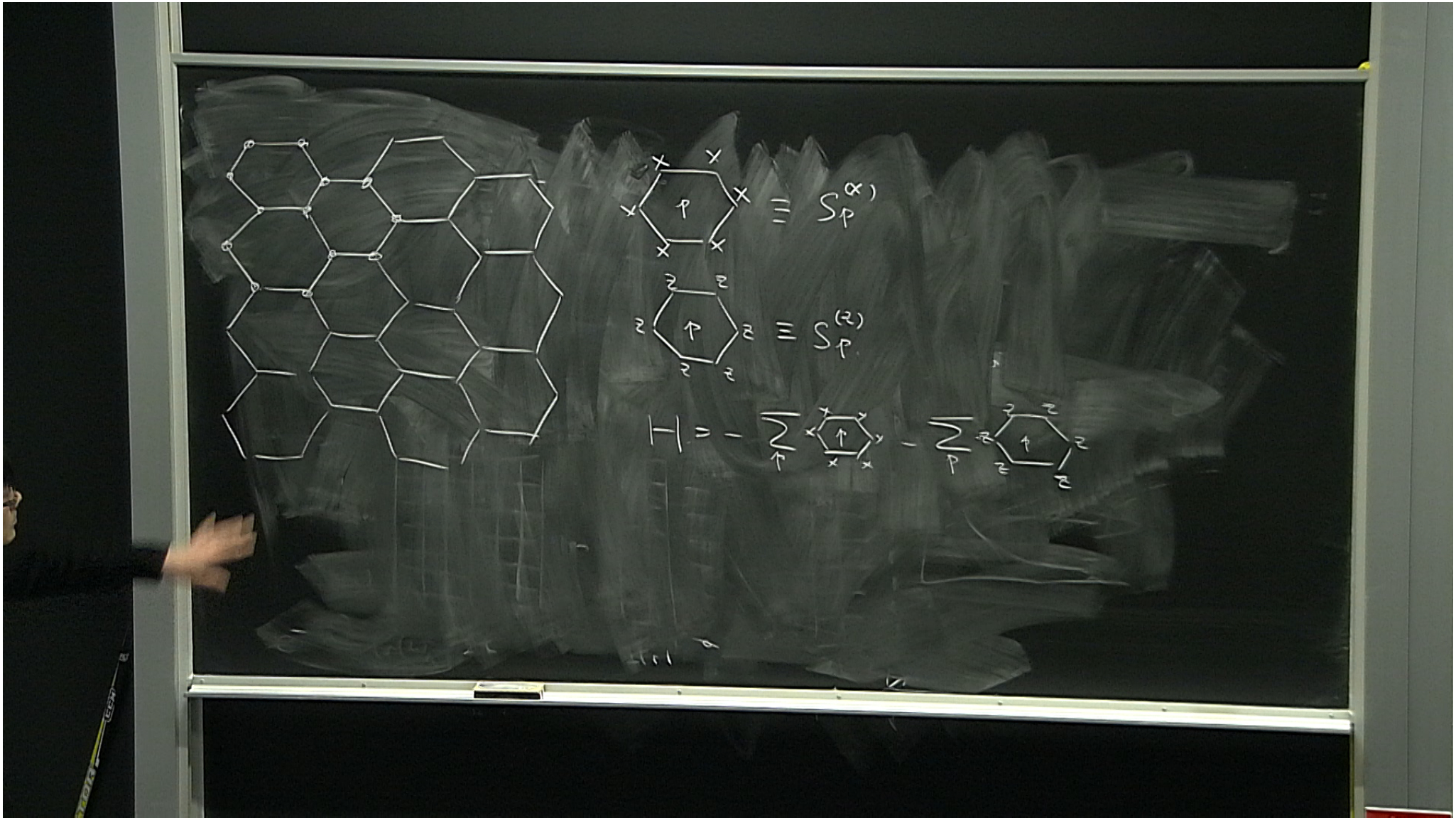
O(L)-depth circuit is necessary

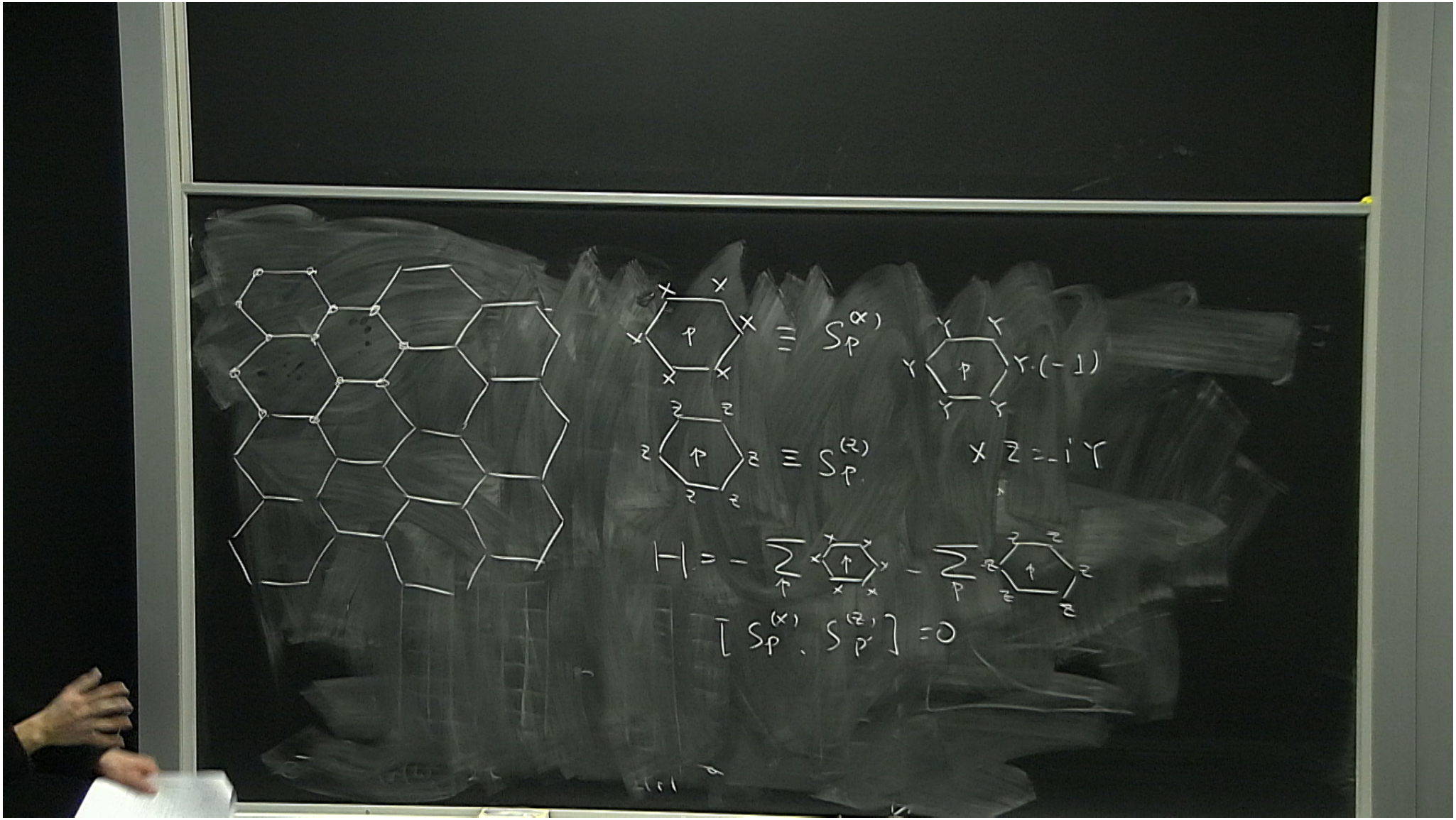

Motivation
cos(θ)
+ i sin(θ)
O(1)-depth circuit is necessary
error propagation

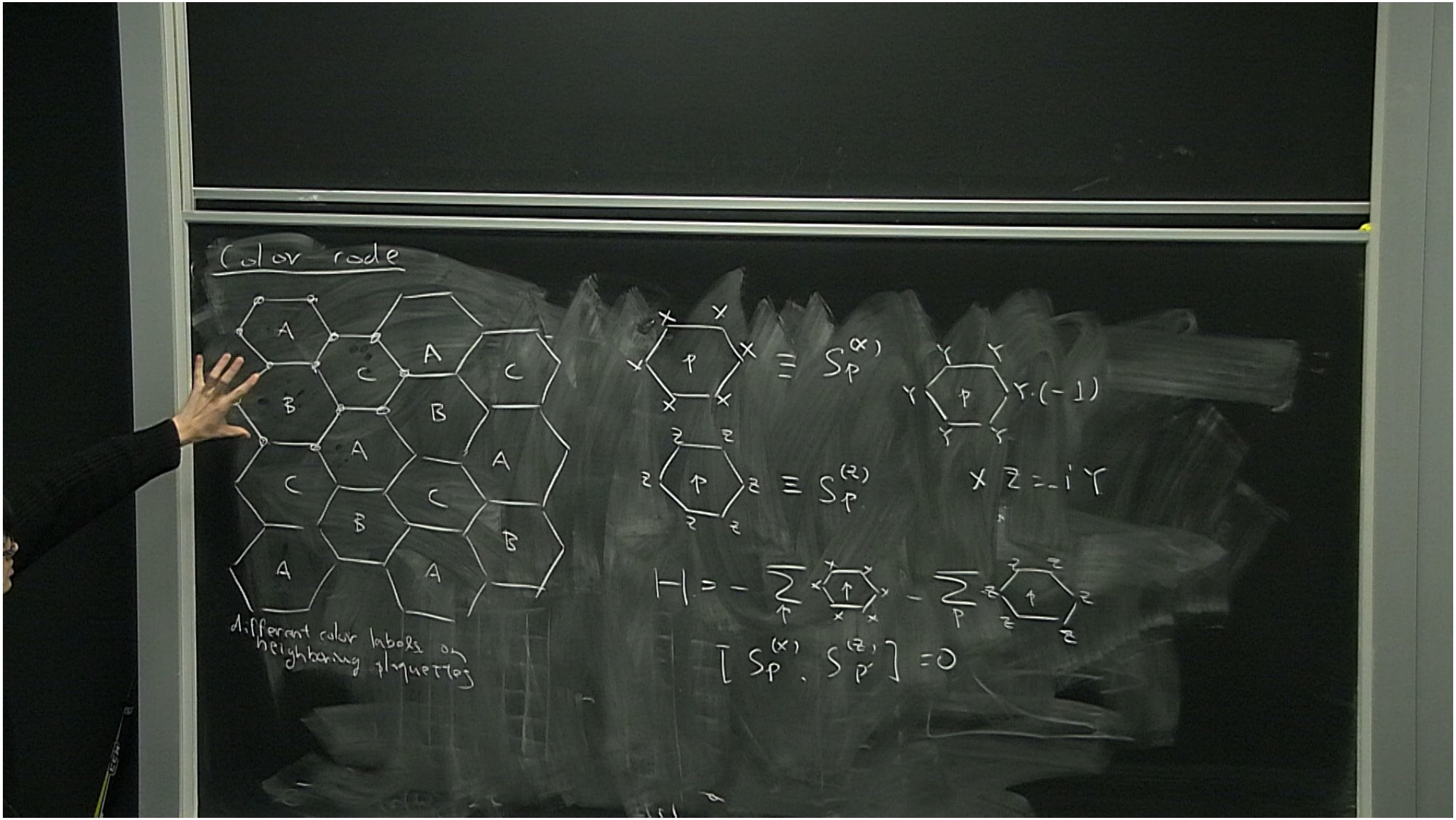

Hadamard + Shift
(i)
CAUTION

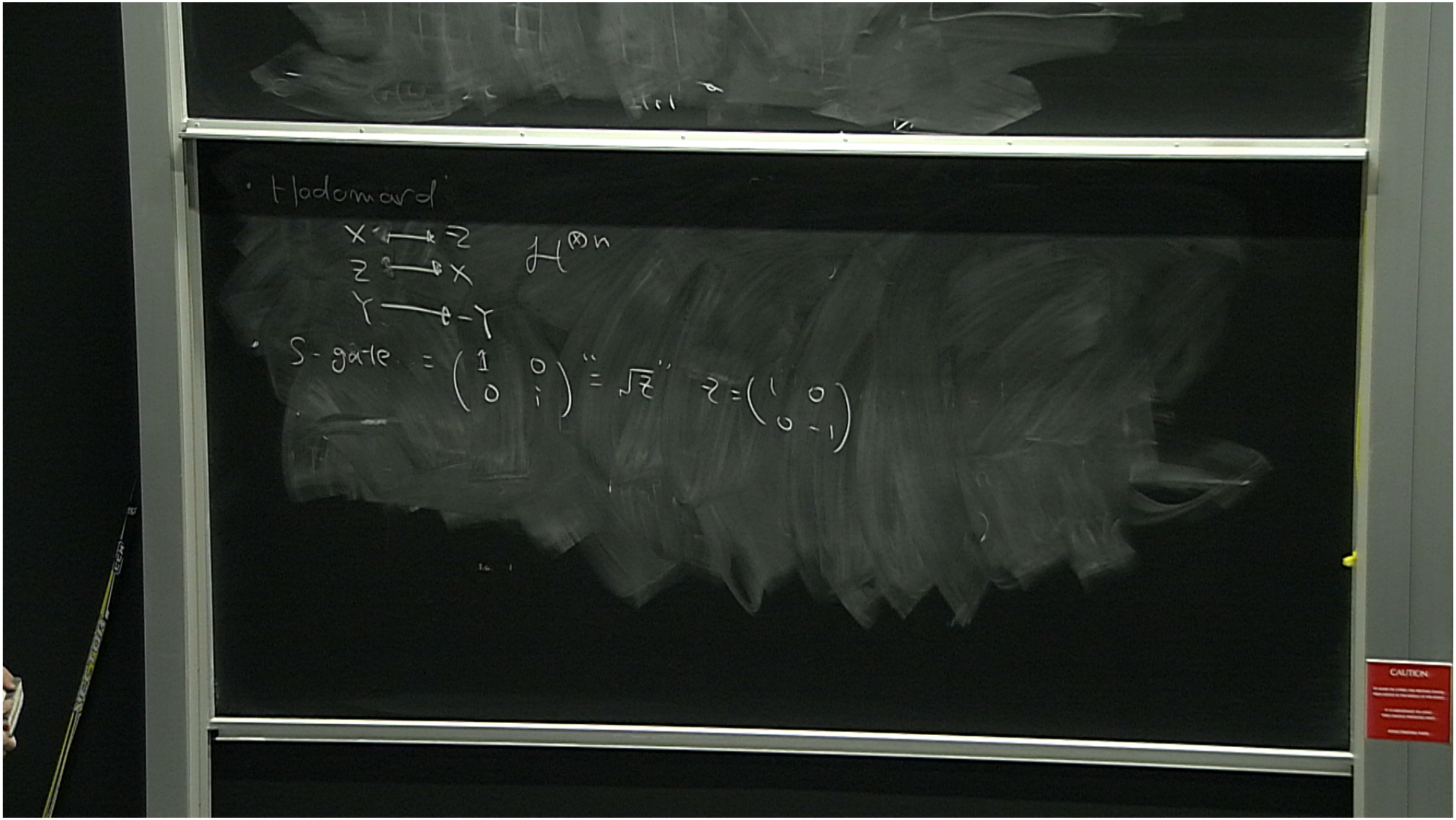

Hadamard + Shift
(i) Hadamard $H^{\otimes n}$
H $X \to Z$
$Z \to X$
$Y \to -Y$
(ii) Shift the lattice
CAUTION

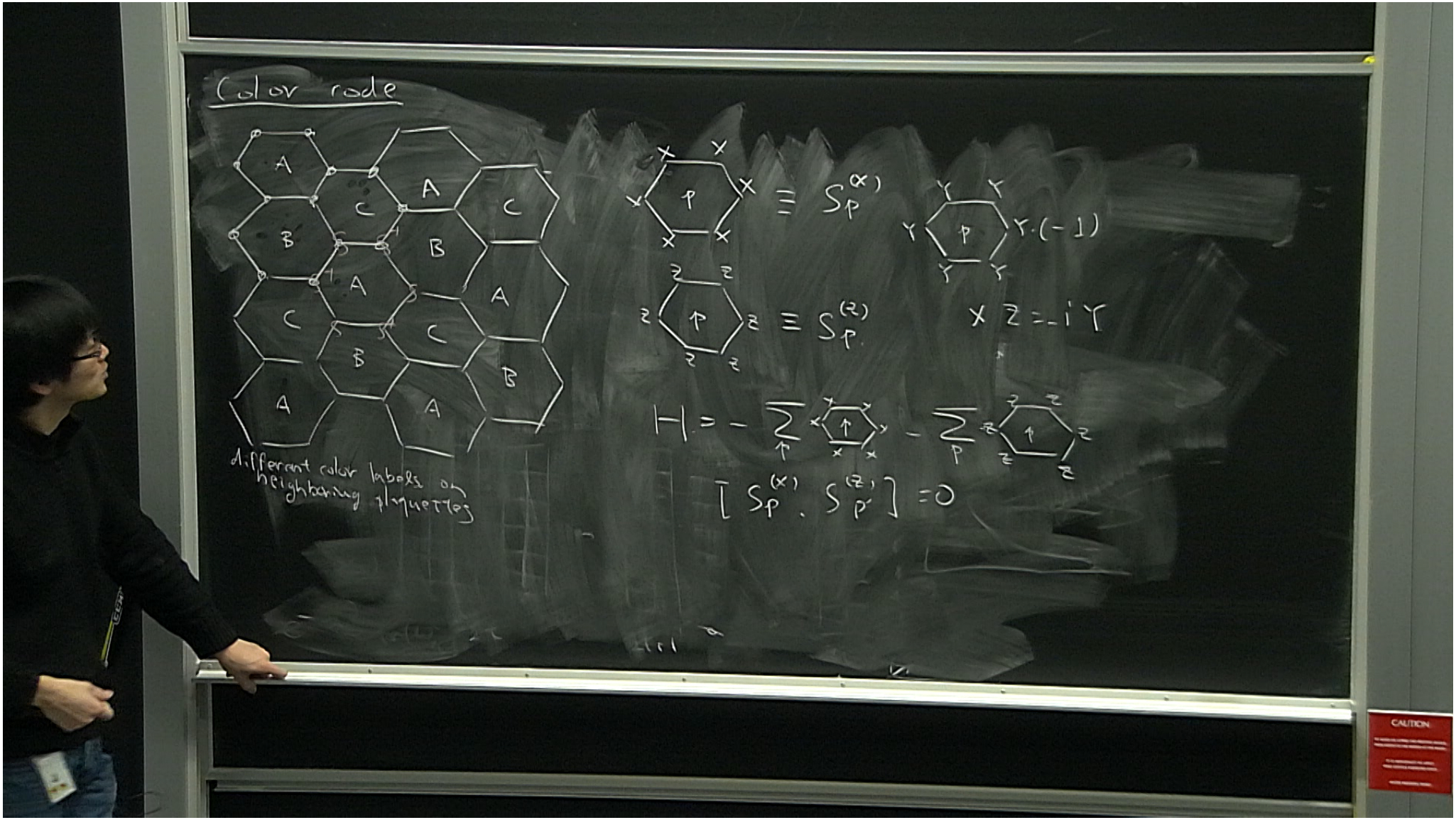

Hadamard + Shift (Wen 2005)
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H $X \to Z$
$Z \to X$
$Y \to -Y$
(ii) Shift the lattice
L
L


$\equiv S_p^{(x)}$
$\equiv S_p^{(z)}$
$H = -$


$\equiv S_p^{(x)}$
$\equiv S_p^{(z)}$
$X Z = -iY$
$H = -\sum_p$
$[S_p^{(x)}, S_{p'}^{(z)}] = 0$


Color code
A
B
C
different color labels on neighboring plaquettes
$S_p^{(x)}$
$S_p^{(z)}$
$XZ = -iY$
$H = -\sum_p S_p^{(x)} - \sum_p S_p^{(z)}$
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• Hadamard
X ⟷ Z
Z ⟷ X
Y ⟷ -Y
$H^{\otimes n}$
• S-gate $= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ "$= \sqrt{Z}$" $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$


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$X \longrightarrow Z$

$Z \longrightarrow X$

$Y \longrightarrow -Y$

$H^{\otimes n}$

S-gate $= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \sqrt{Z}$ $\quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad S^{\otimes n}$

$X \longrightarrow Y$

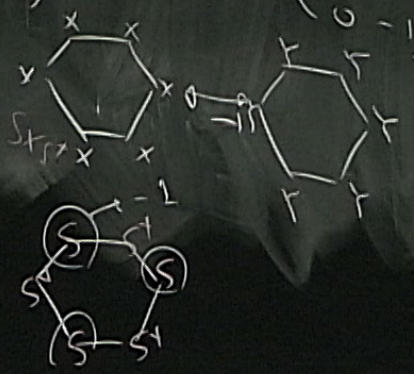
$Y \longrightarrow -X$

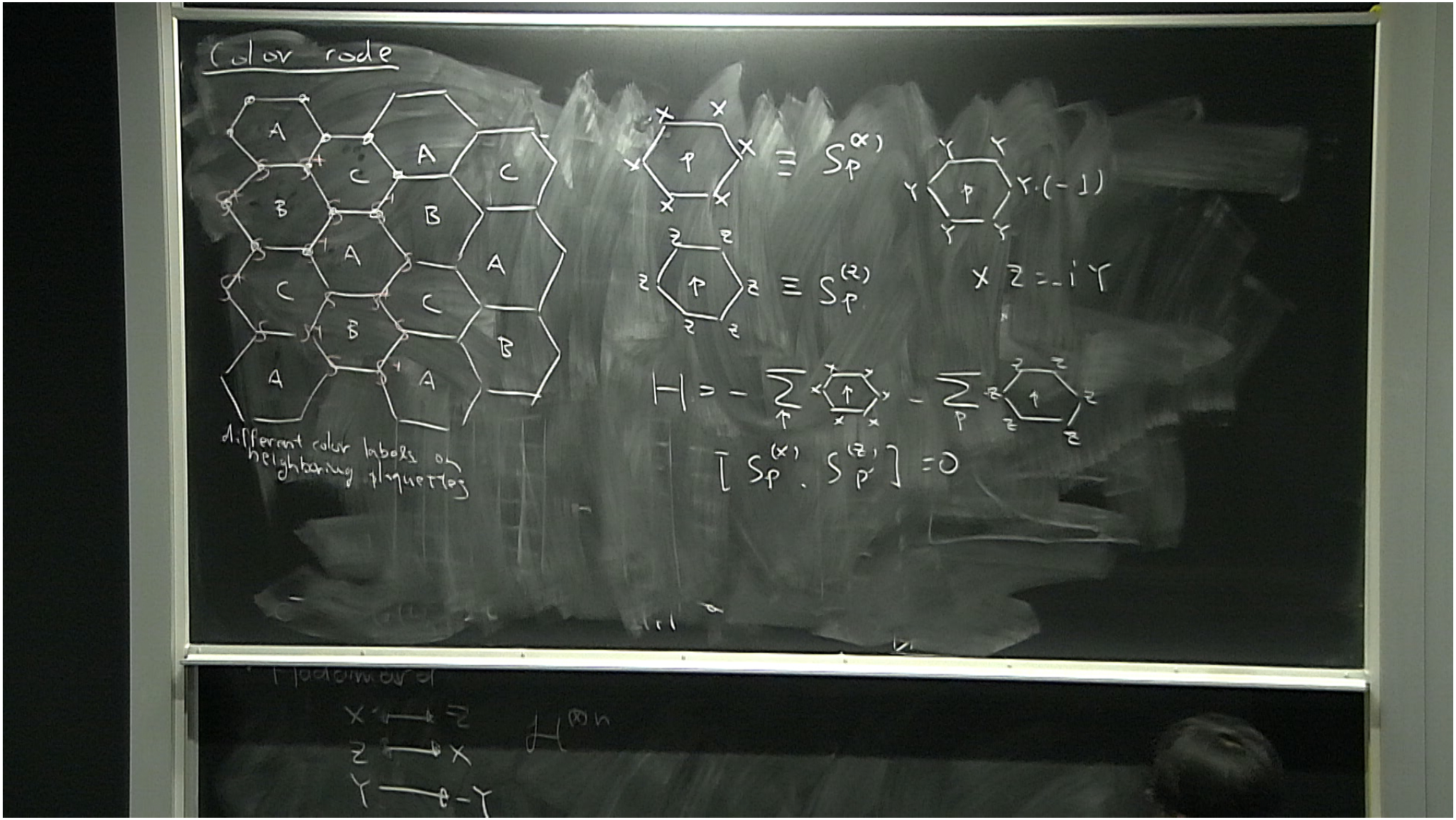
$Z \longrightarrow Z$

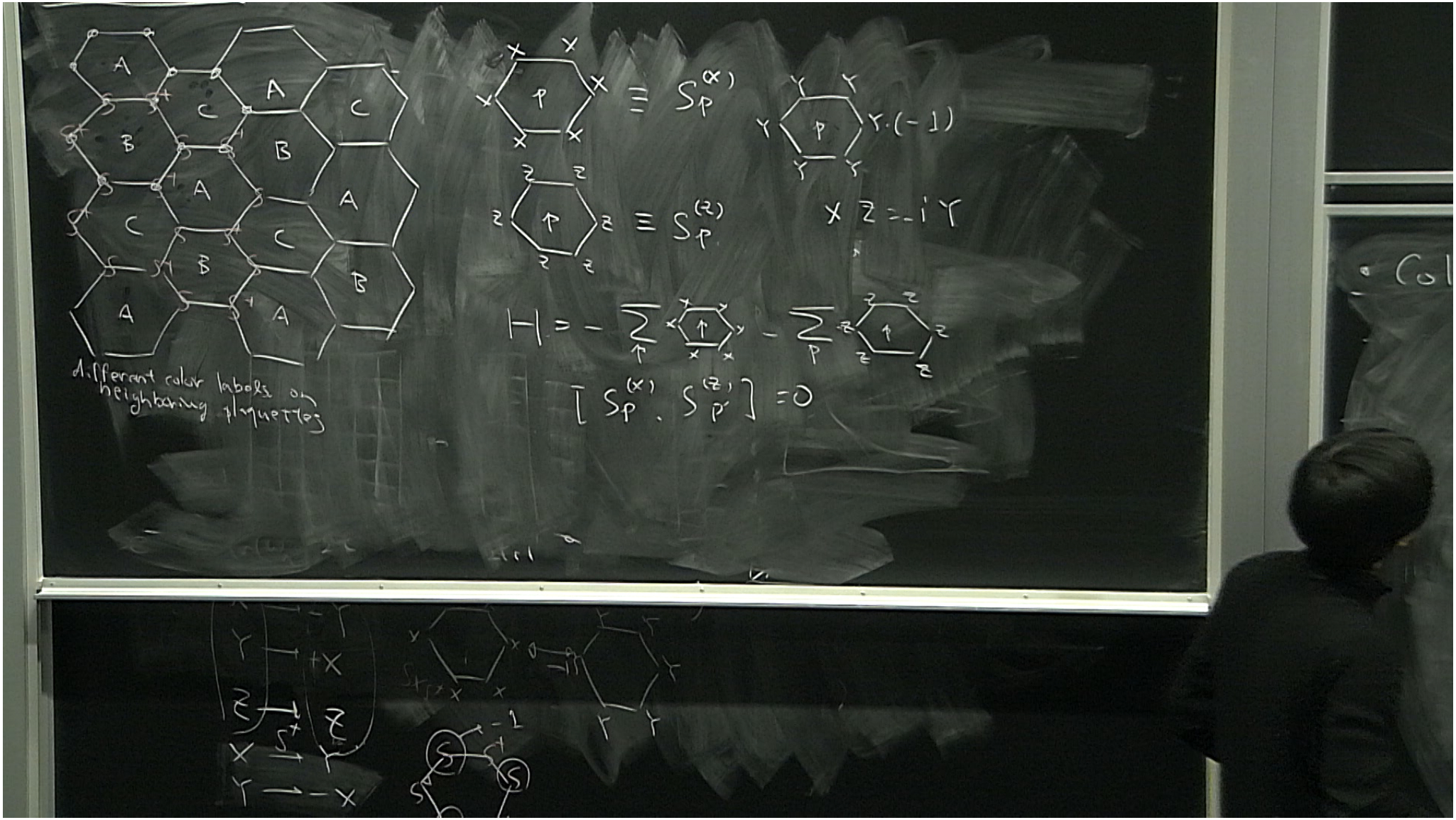
$X \longrightarrow Y$

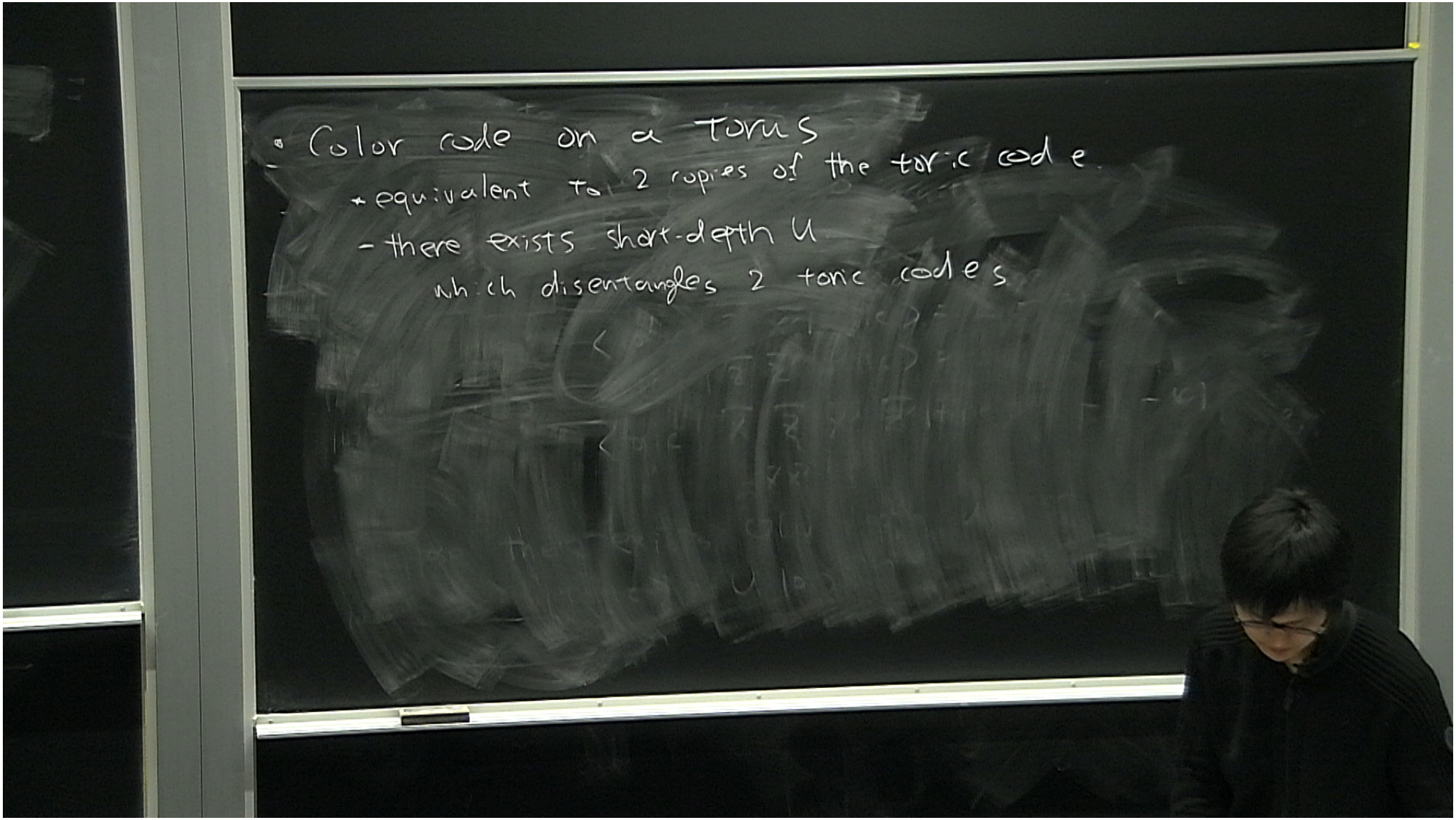
$Y \longrightarrow -X$

$Z \longrightarrow Z$

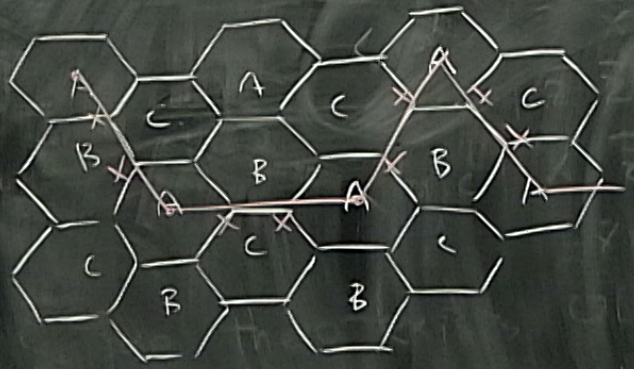



Color code
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different color labels on neighboring plaquettes
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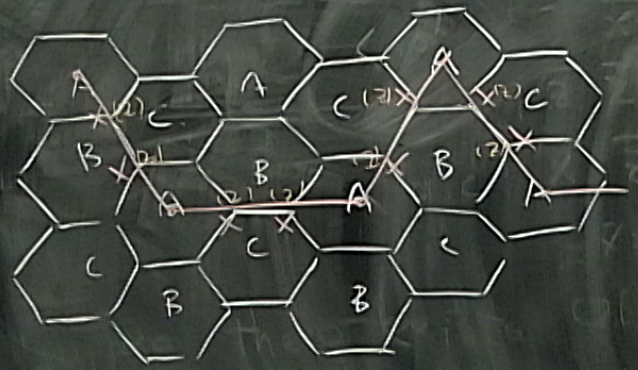

A
B
C
different color labels on neighboring plaquettes
$\equiv S_p^{(x)}$
$\equiv S_p^{(z)}$
$\times (-1)$
$xz = -iy$
$H = -\sum_p - \sum_p$
$[S_p^{(x)}, S_{p'}^{(z)}] = 0$


• Color code on a torus
* equivalent to 2 copies of the toric code.
- there exists short-depth U
which disentangles 2 toric codes

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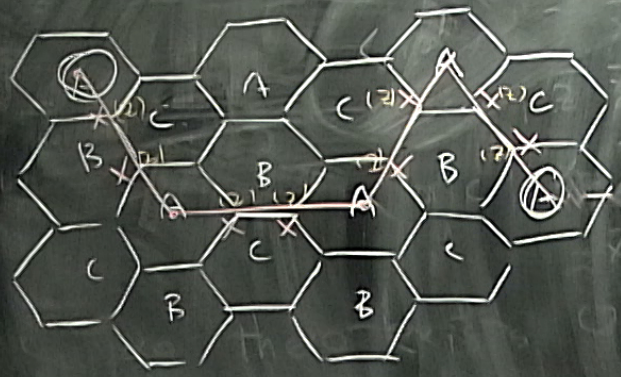


X-string on A, B, C.
Z-string on A, B, C.

* equivalent to 2 copies of the toric code.

- there exists short-depth U

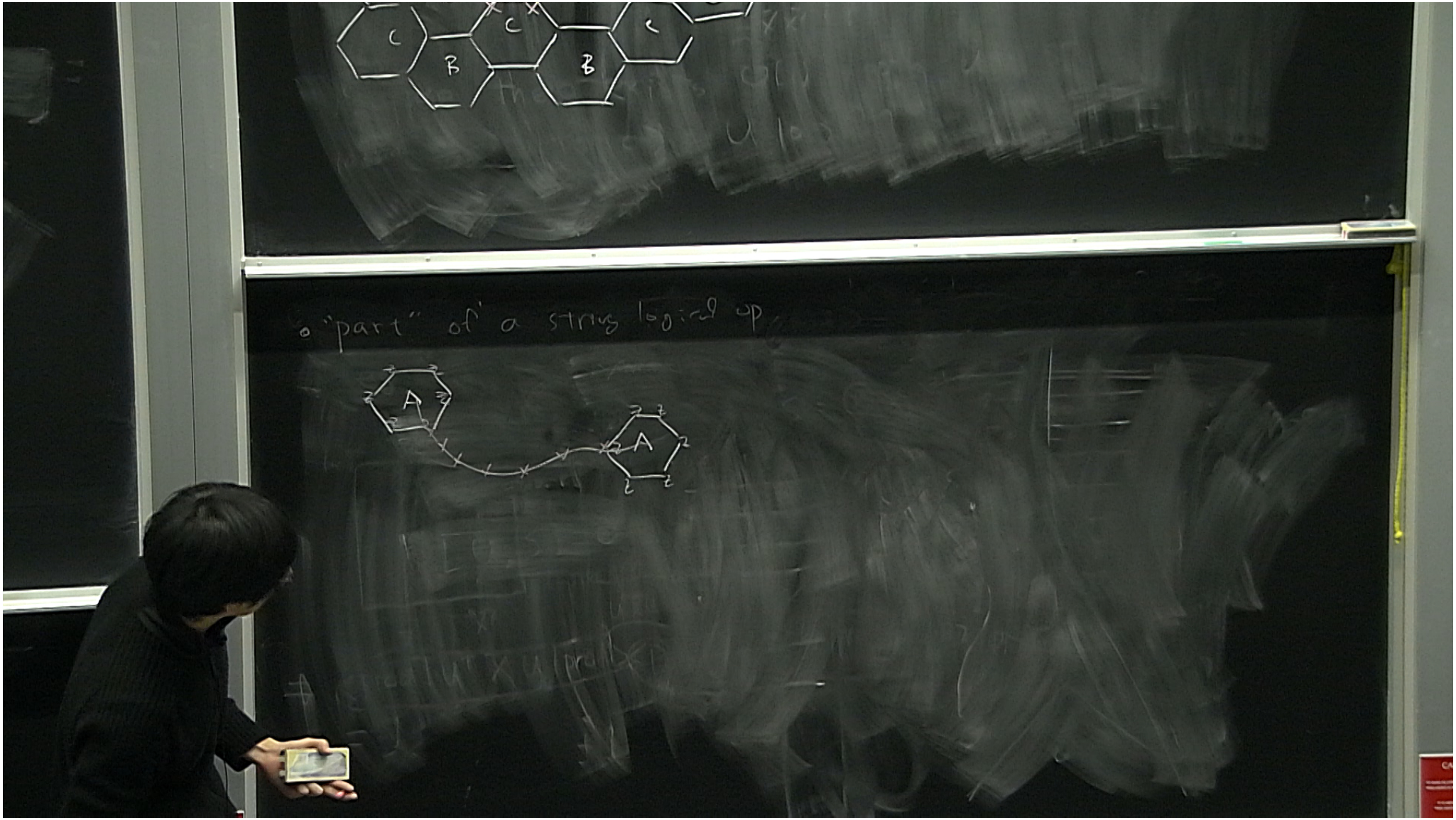
which disentangles 2 toric codes

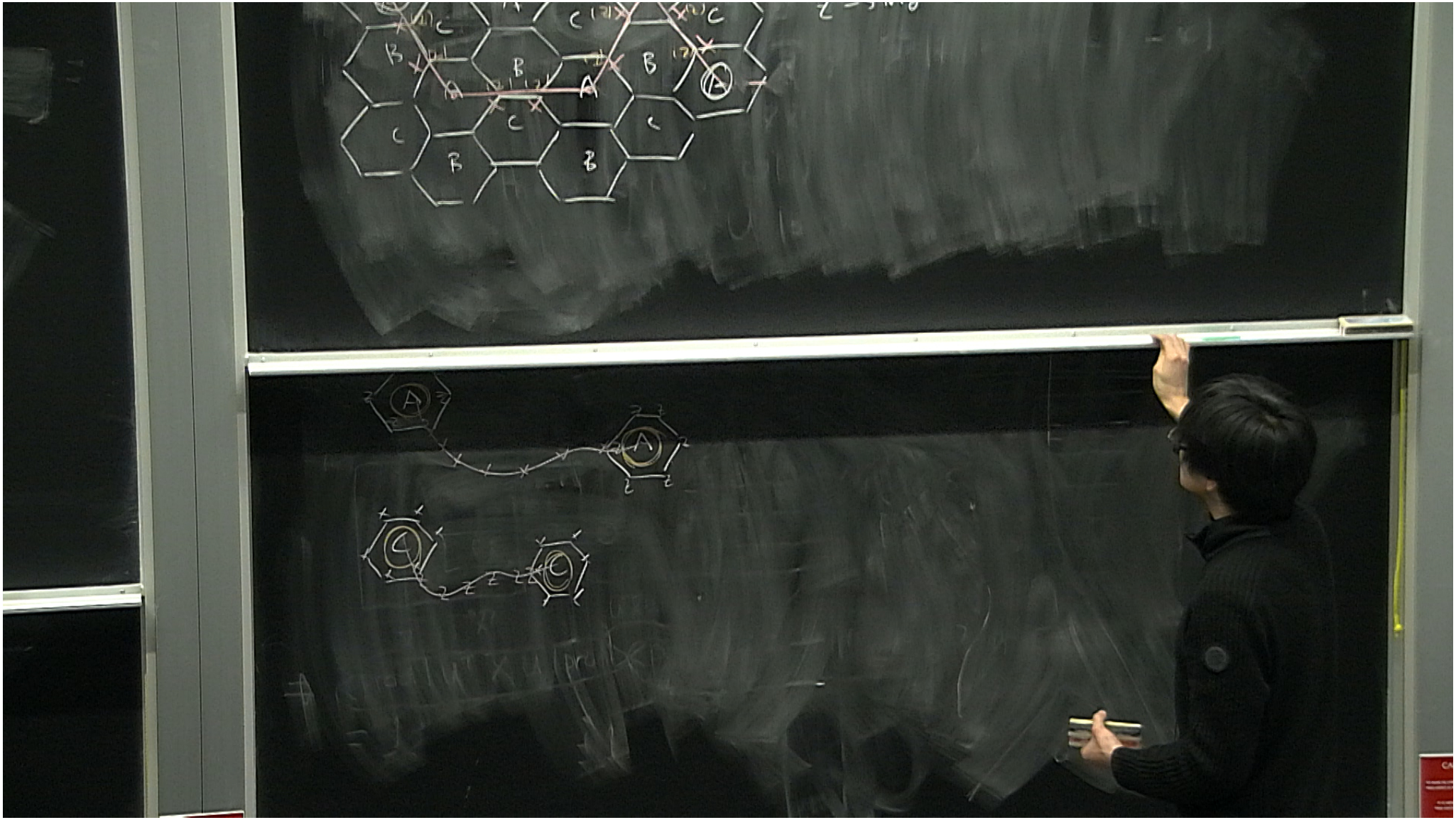


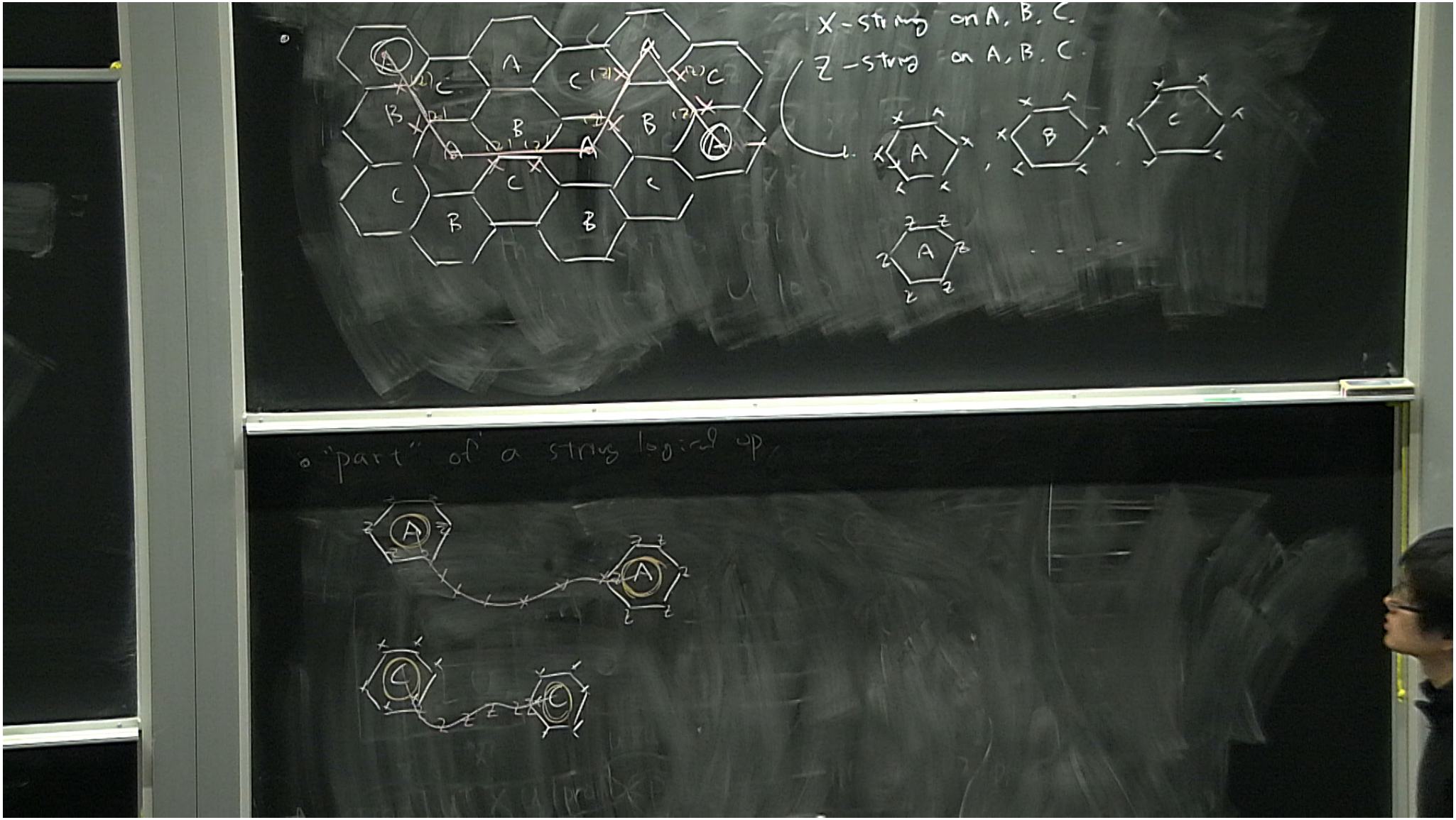
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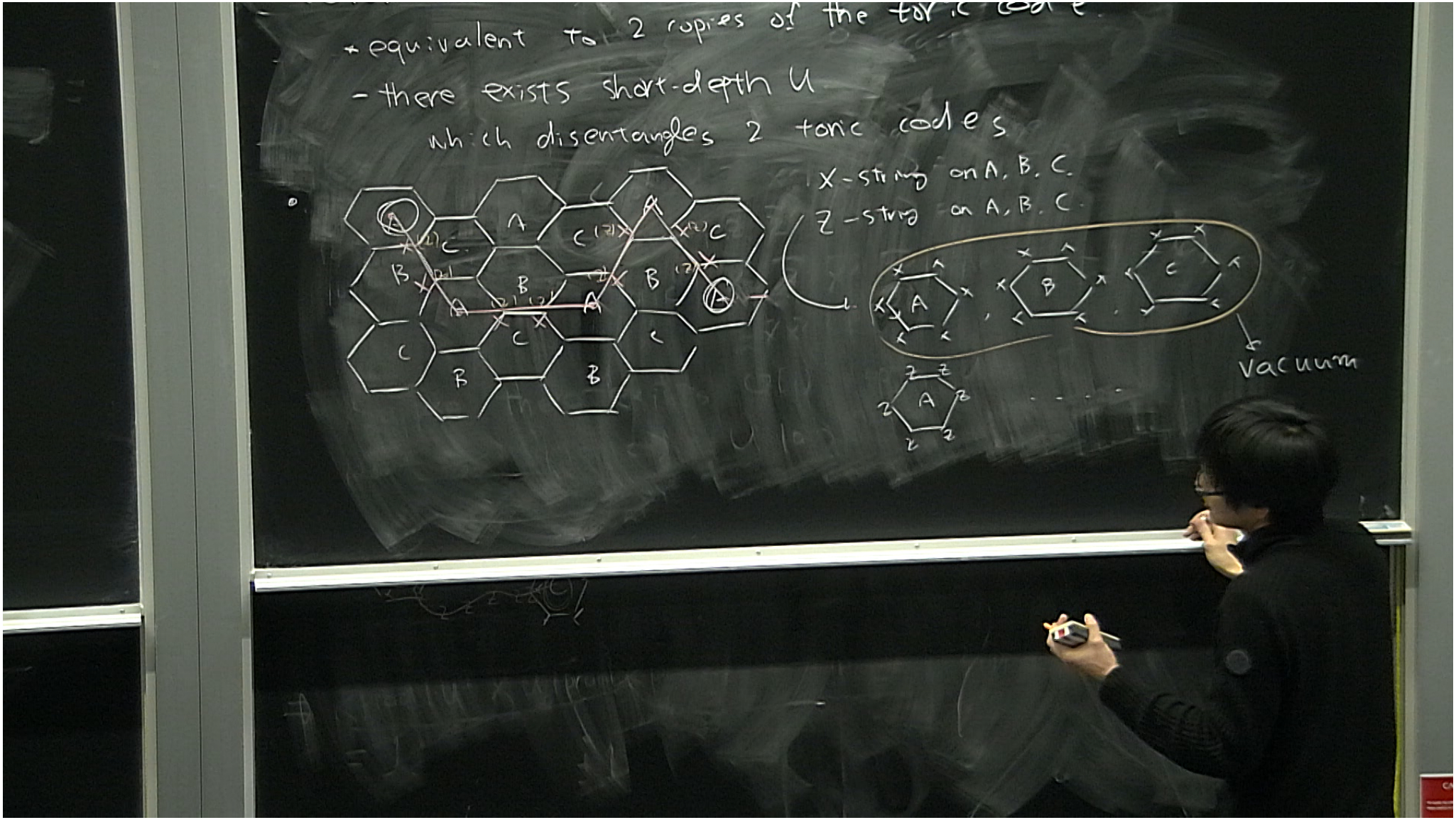
Z-string on A, B, C.

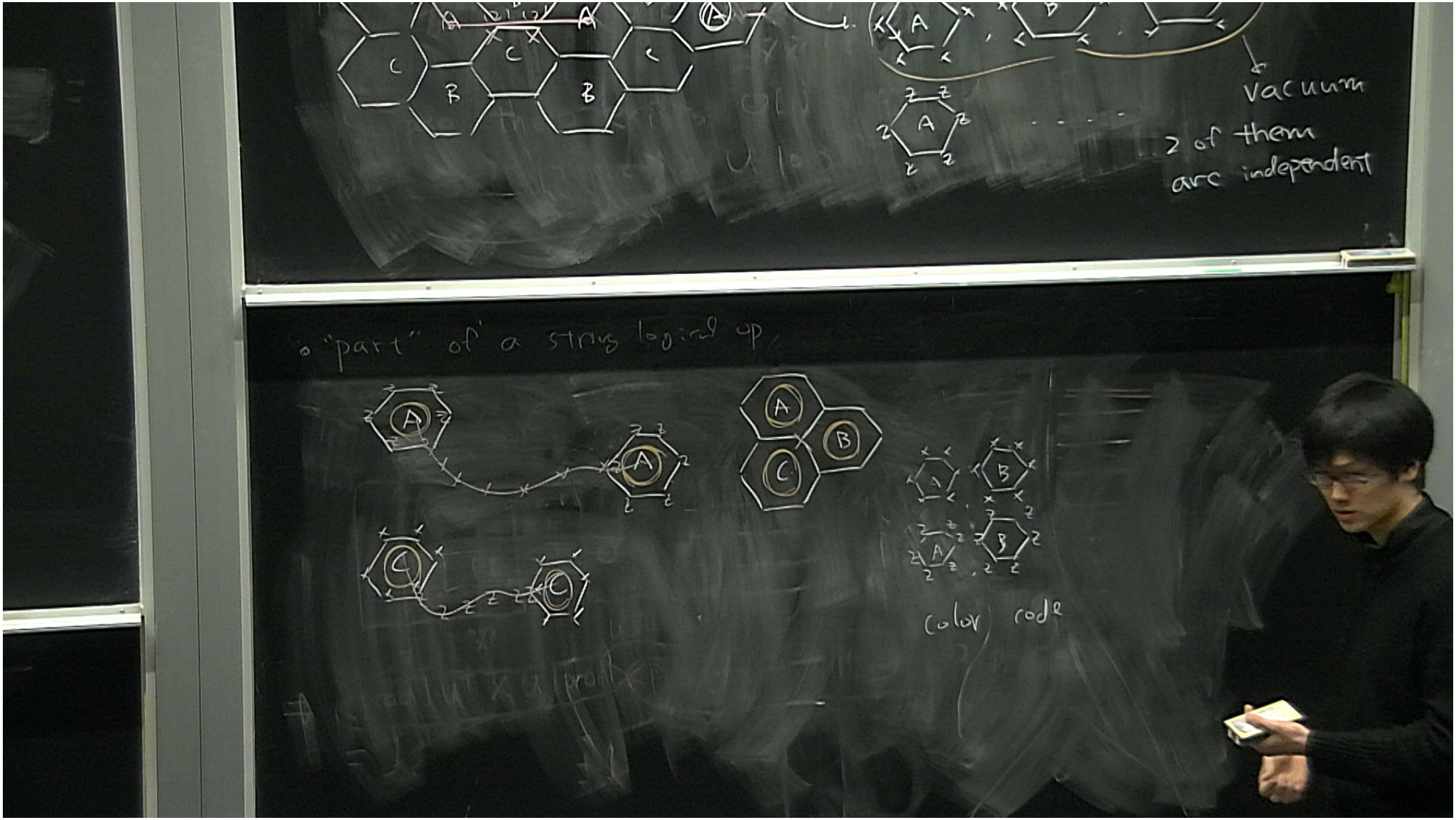
* "part" of a string logical op

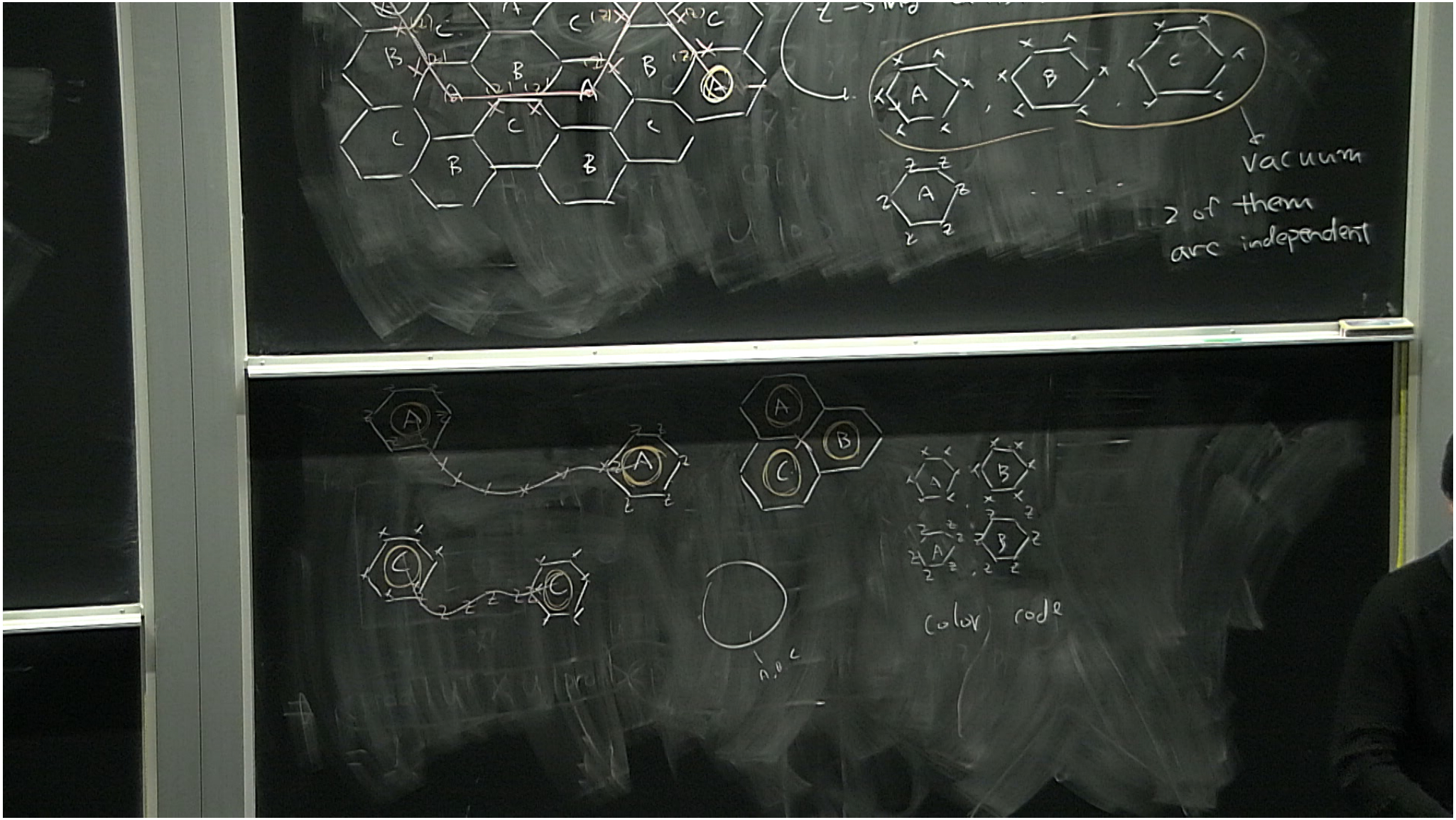

C
C
C
B
B
"part" of a string logical op.
A
A

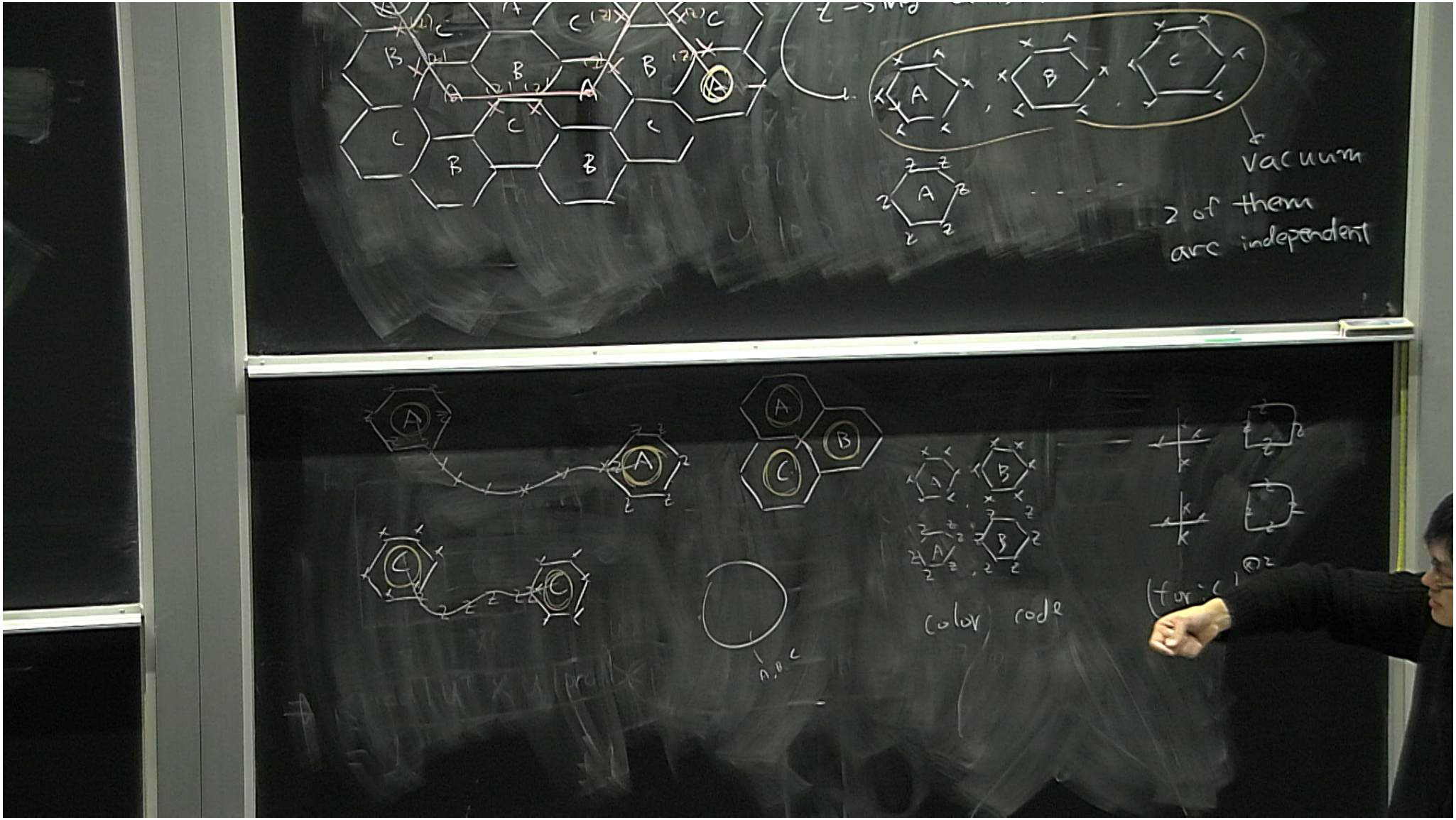

A
B
C
A
B
C
A
A
A
C
C

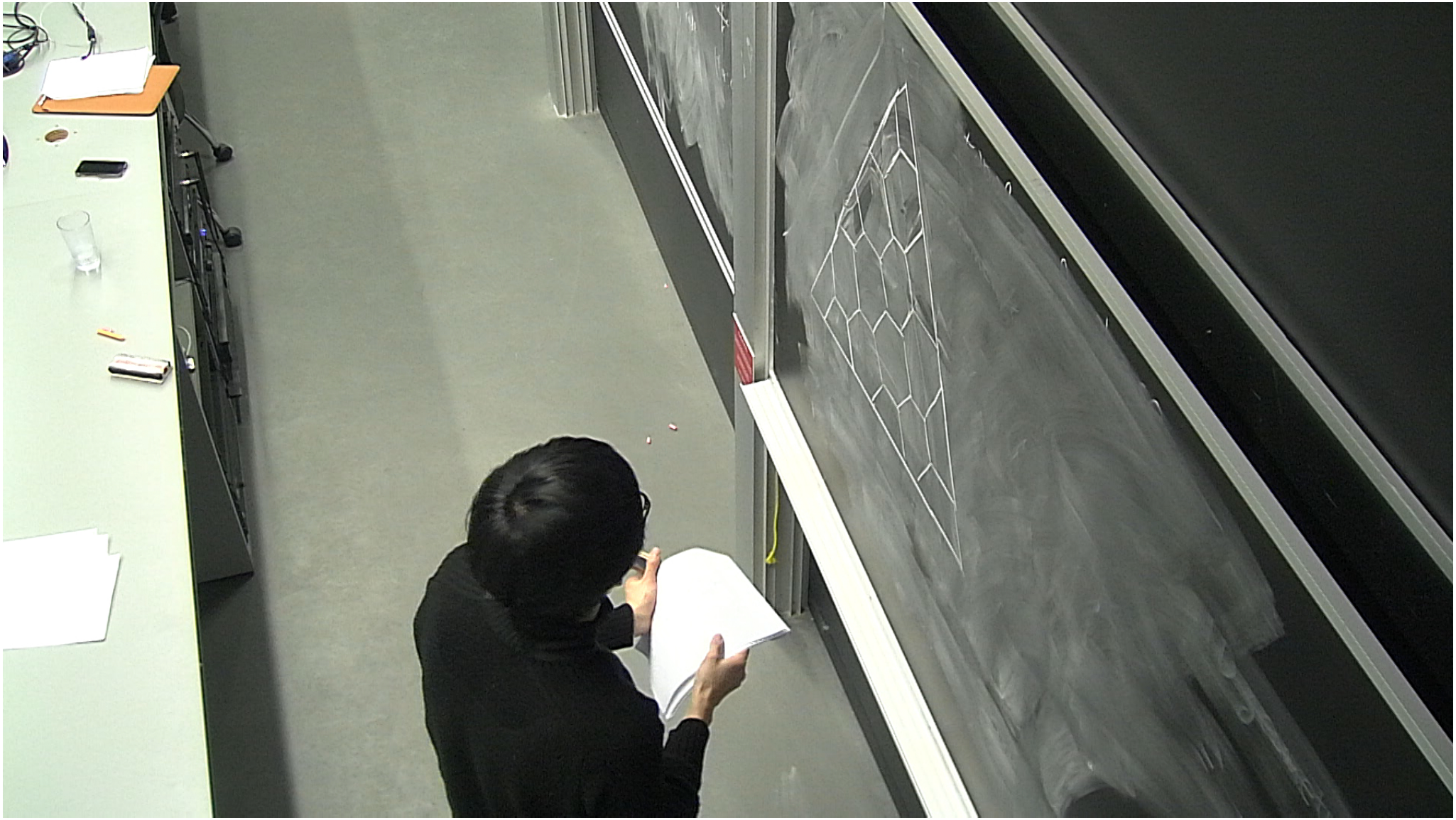

X-string on A, B, C.
Z-string on A, B, C.
A
B
C
"part" of a string looped up.

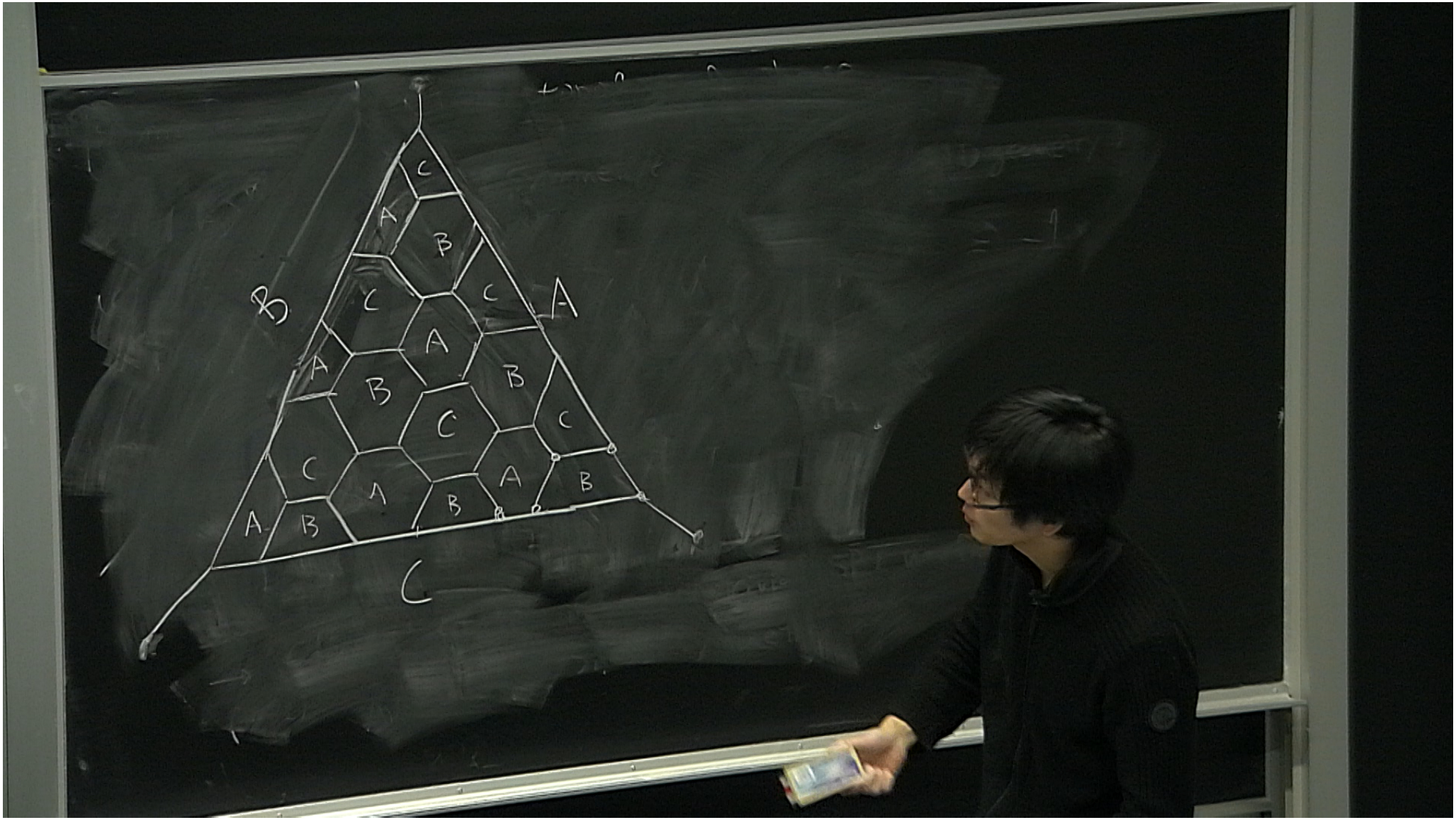

* equivalent to 2 copies of the toric code
- there exists short-depth U
which disentangles 2 toric codes
X-string on A, B, C
Z-string on A, B, C
vacuum

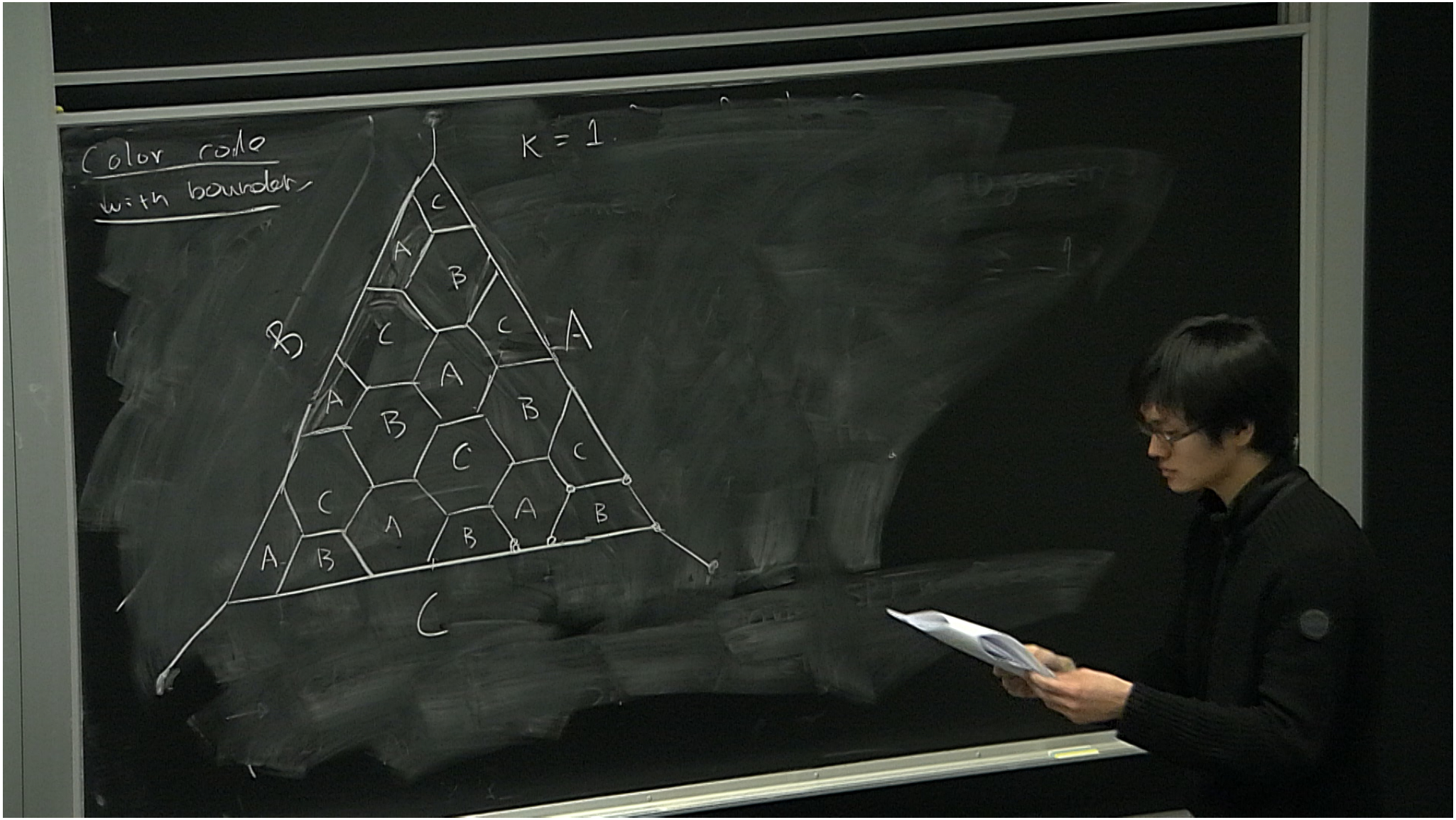

Vacuum
2 of them
are independent
"part" of a string logical op
color code

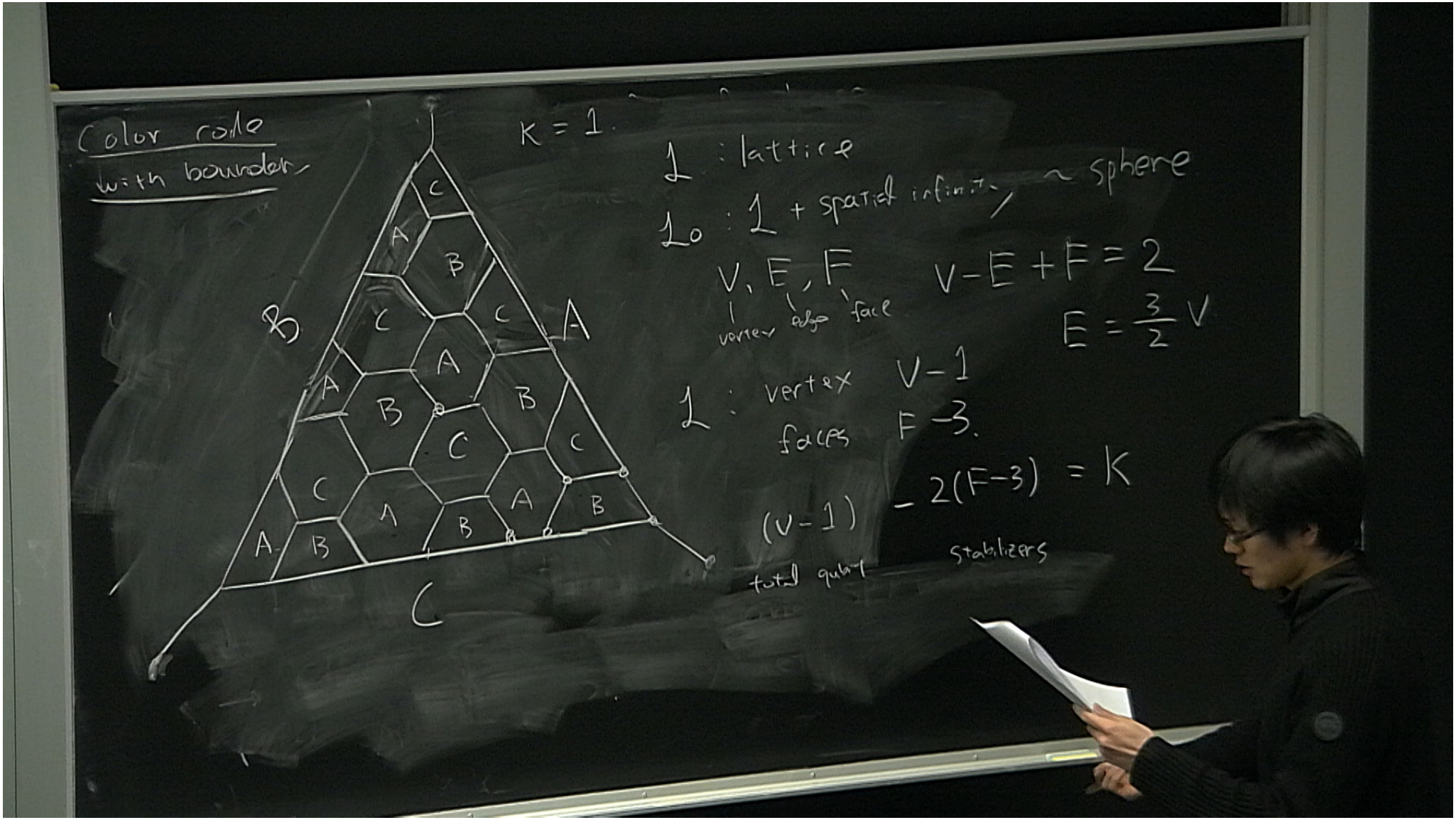

vacuum
2 of them
are independent
color code

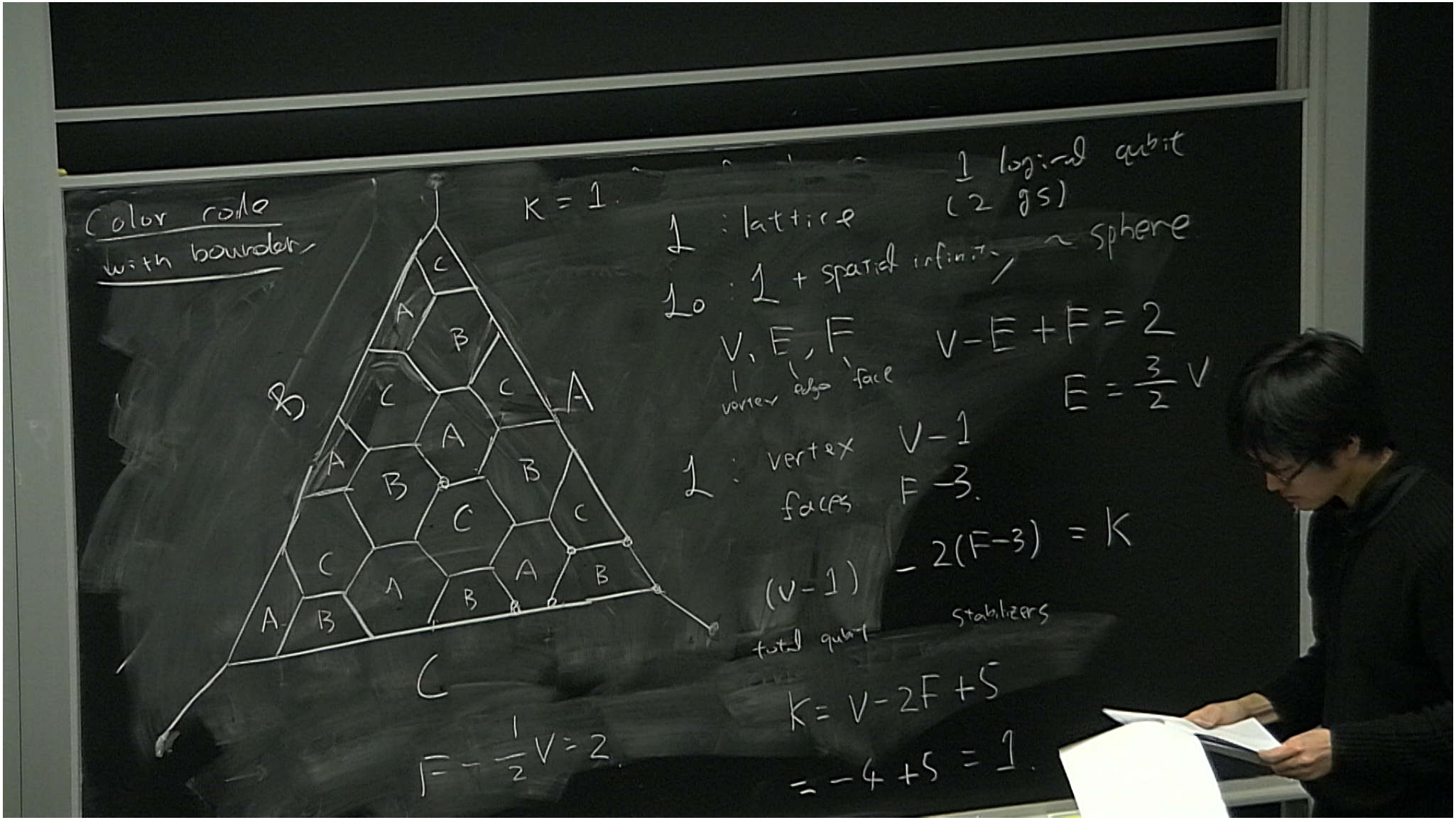

vacuum
2 of them
are independent
color code




B
A
C

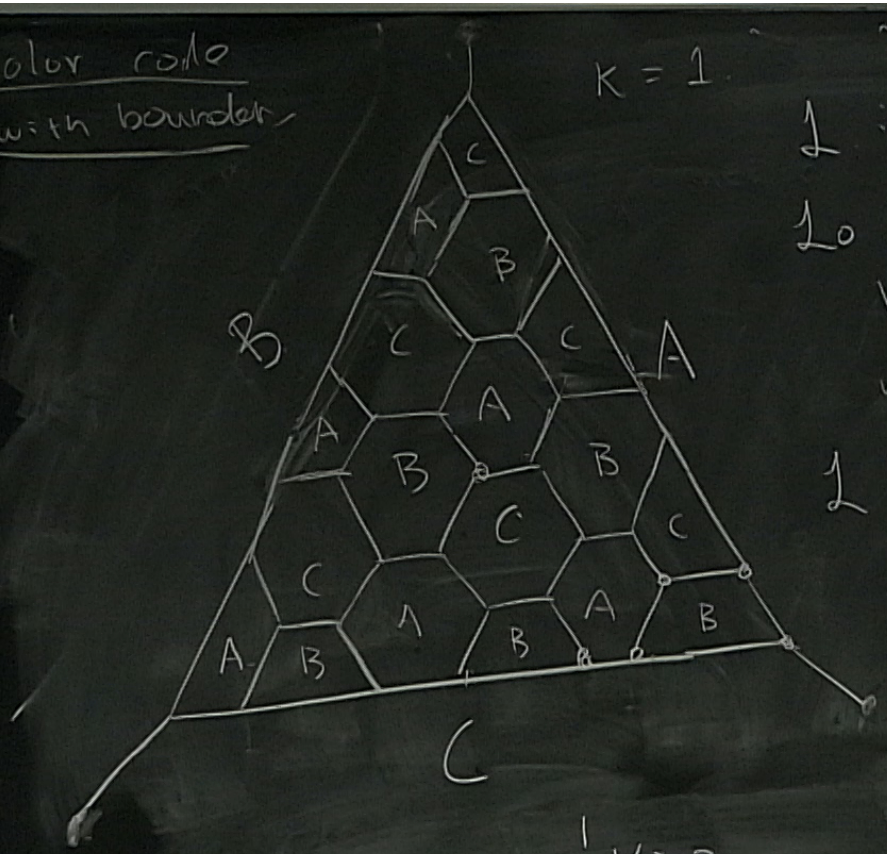

Color code
with boundary
K = 1
B
A
C


Color code
with boundary
K = 1
1 : lattice
1o : 1 + spatial infinity ~ sphere
V, E, F
vertex edge face
V - E + F = 2
E = 3/2 V
1 : vertex V - 1
faces F - 3.
(V - 1) - 2(F - 3) = K
total qubit
stabilizers
A
B
C


Color code
with boundary
K = 1.
1 : lattice
1 logical qubit
(2 gs)
1o : 1 + spatial infinity ~ sphere
V, E, F
vertex edge face
V - E + F = 2
E = 3/2 V
1 : vertex V - 1
faces F - 3.
(V - 1) - 2(F - 3) = K
total qubit
stabilizers
K = V - 2F + 5
= -4 + 5 = 1.
F - 1/2 V = 2.
A
B
C

Color code

with boundary



$K = 1$

1 : lattice (2 gs) ~ sphere

1o : 1 + spatial infinity

V, E, F (vertex, edge, face) $V - E + F = 2$

$E = \frac{3}{2} V$

1 : vertex $V - 1$

faces $F - 3$.

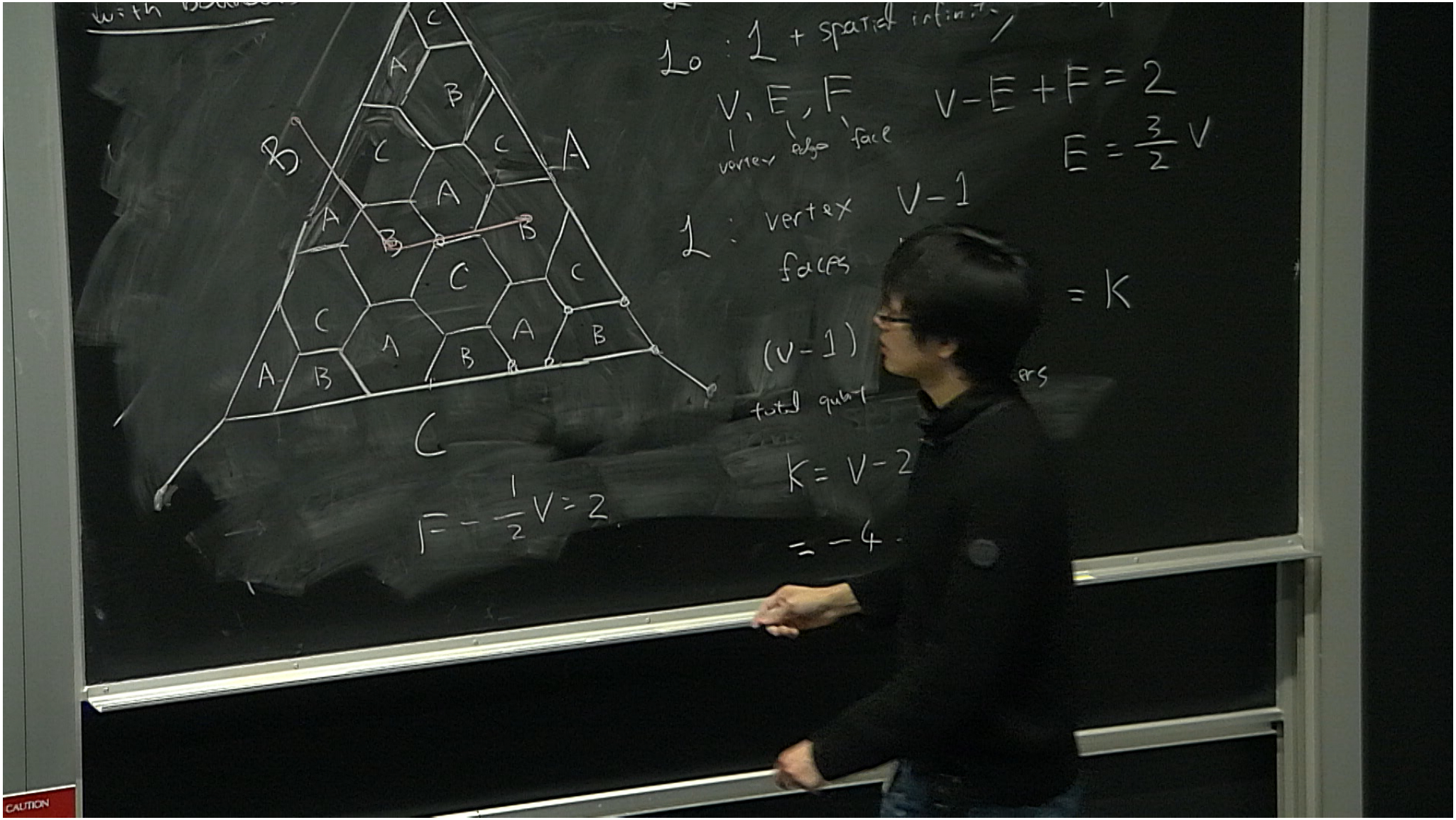
$(V-1) - 2(F-3) = K$

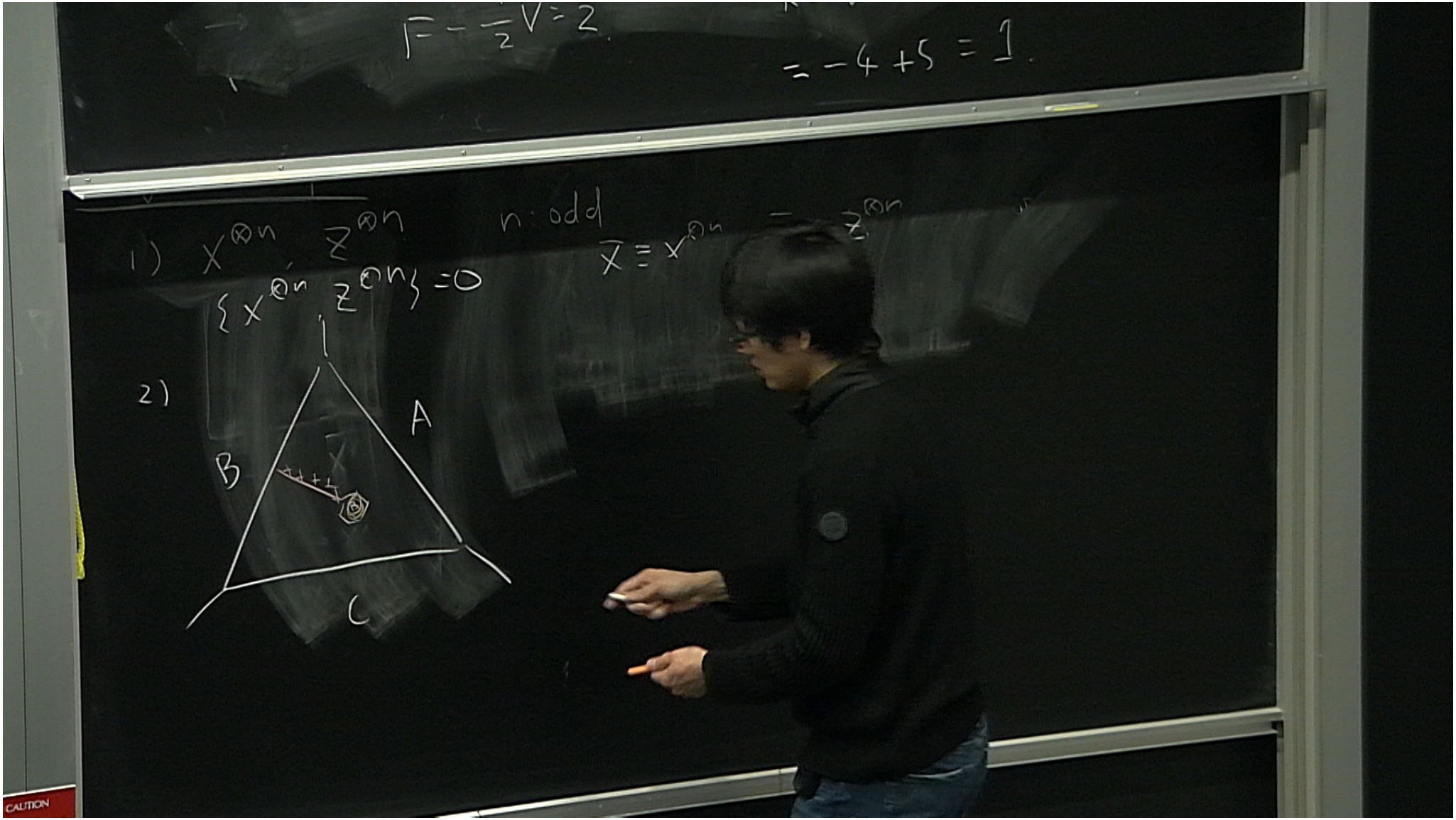
(total qubits) (stabilizers)

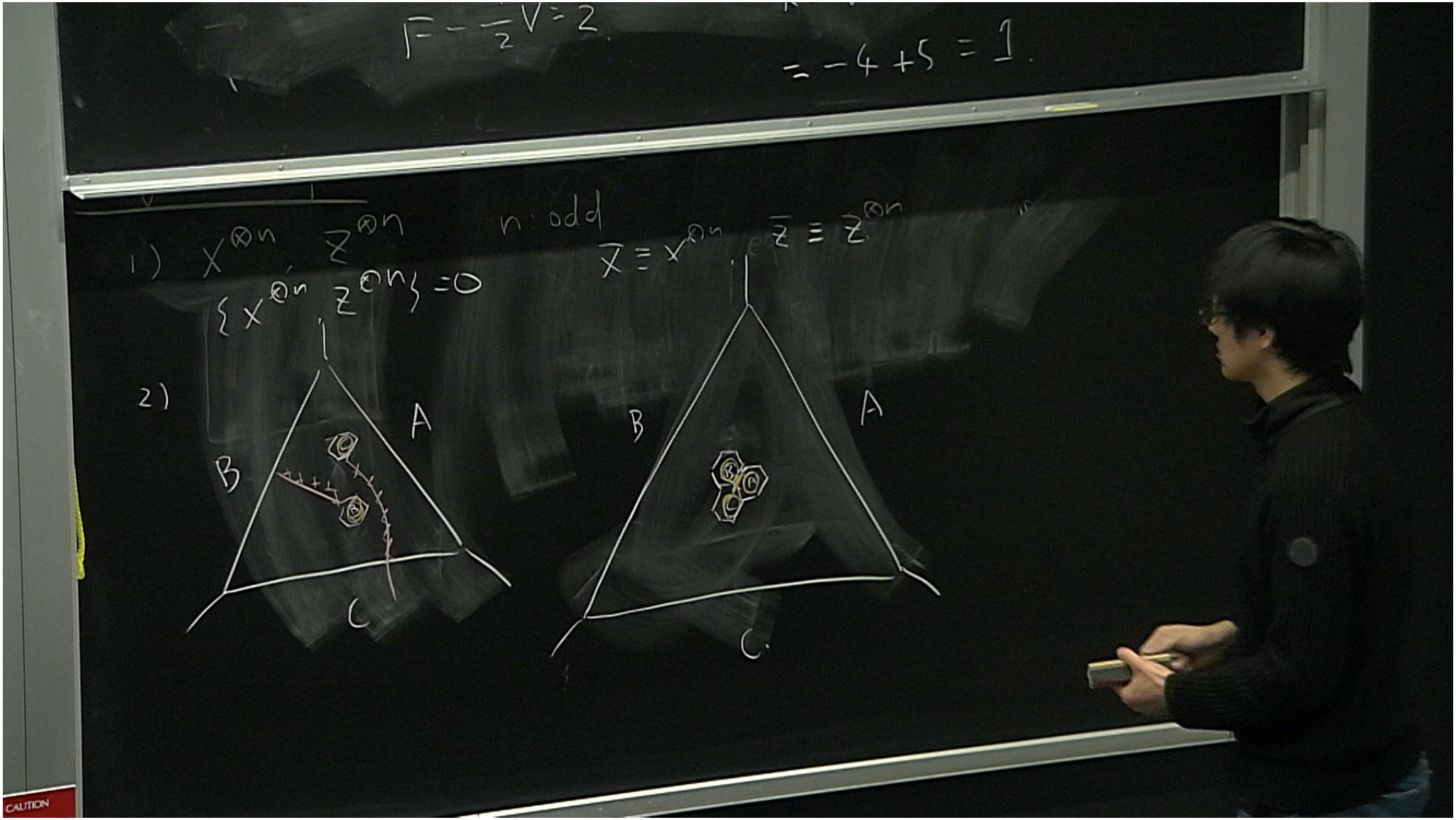
$K = V - 2F + 5$

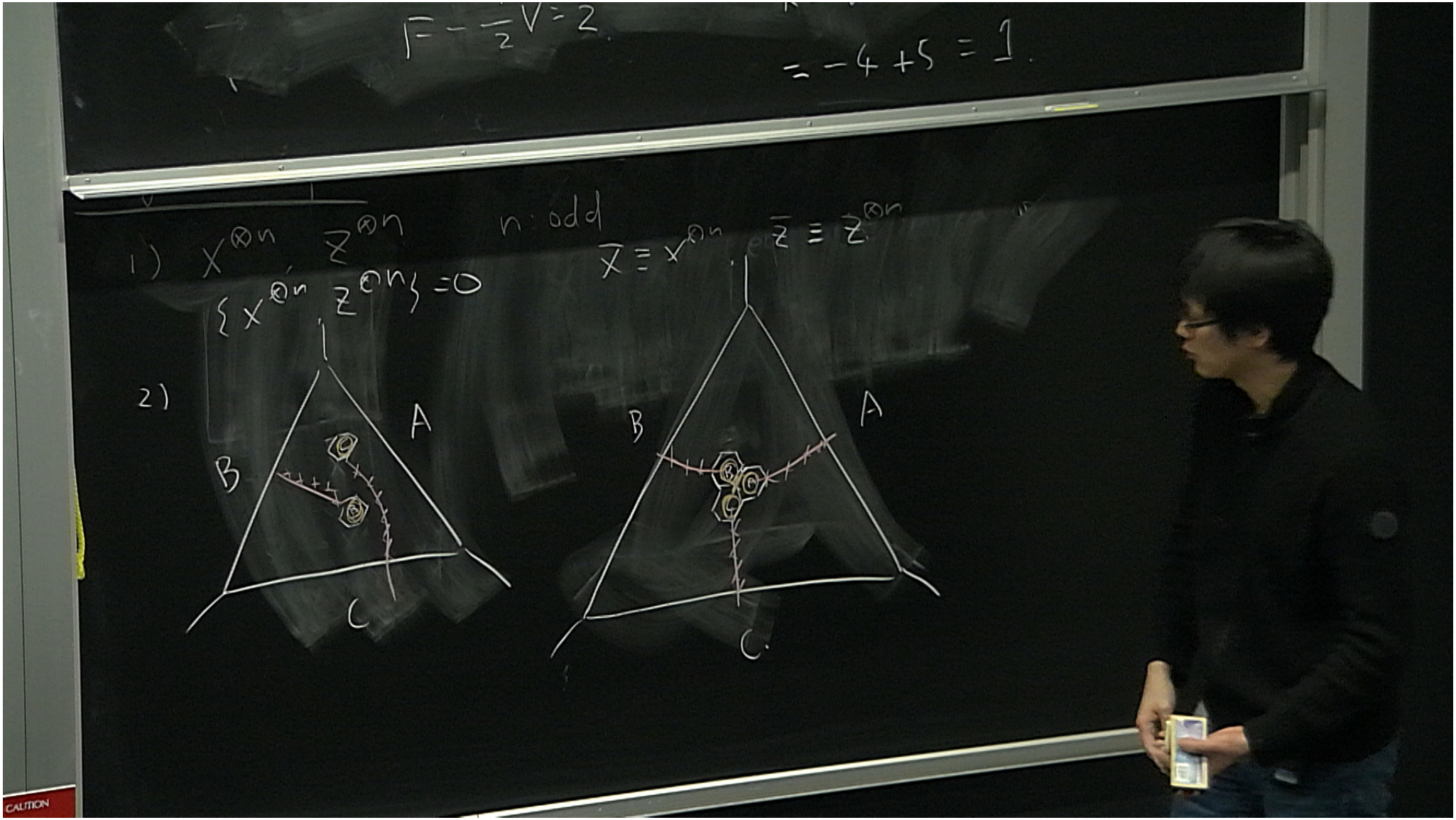
$= -4 + 5 = 1$.

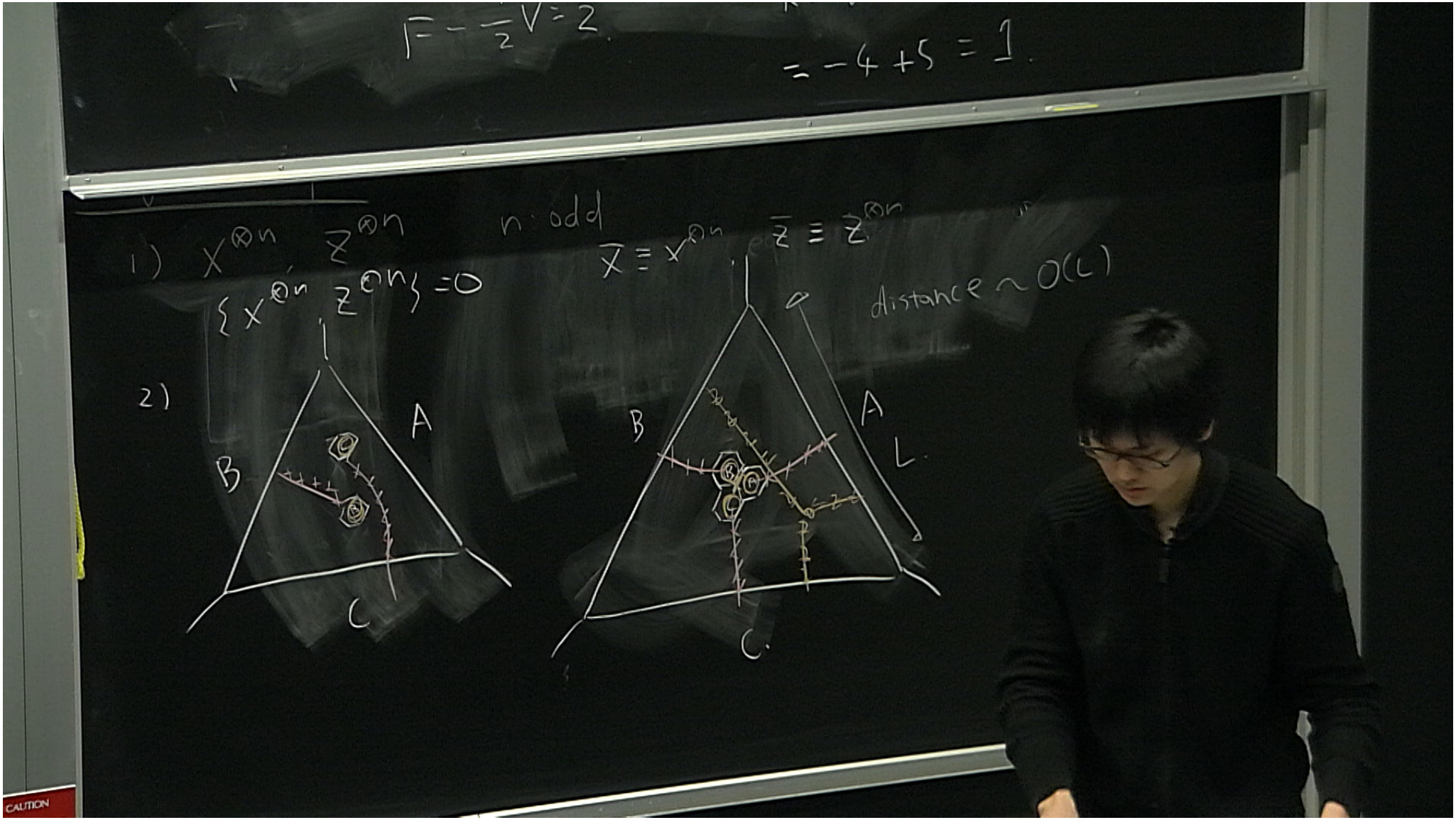
$F - \frac{1}{2} V = 2$

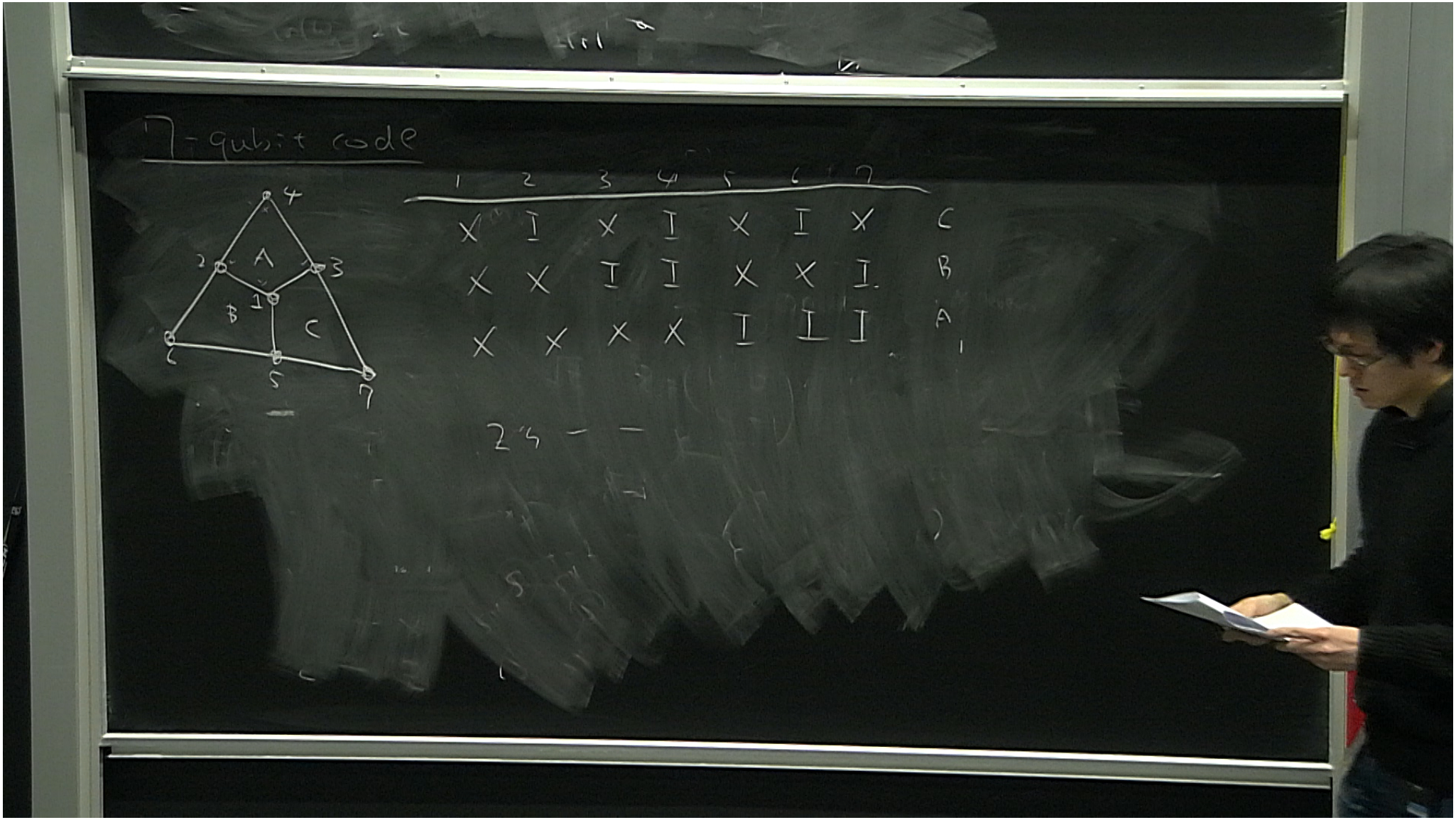

1 : 1 + spatial infin
V, E, F
vertex edge face
V - E + F = 2
E = 3/2 V
1 : vertex V - 1
faces
= K
(V - 1)
total qubit
K = V - 2
= -4
F - 1/2 V = 2
A
B
C

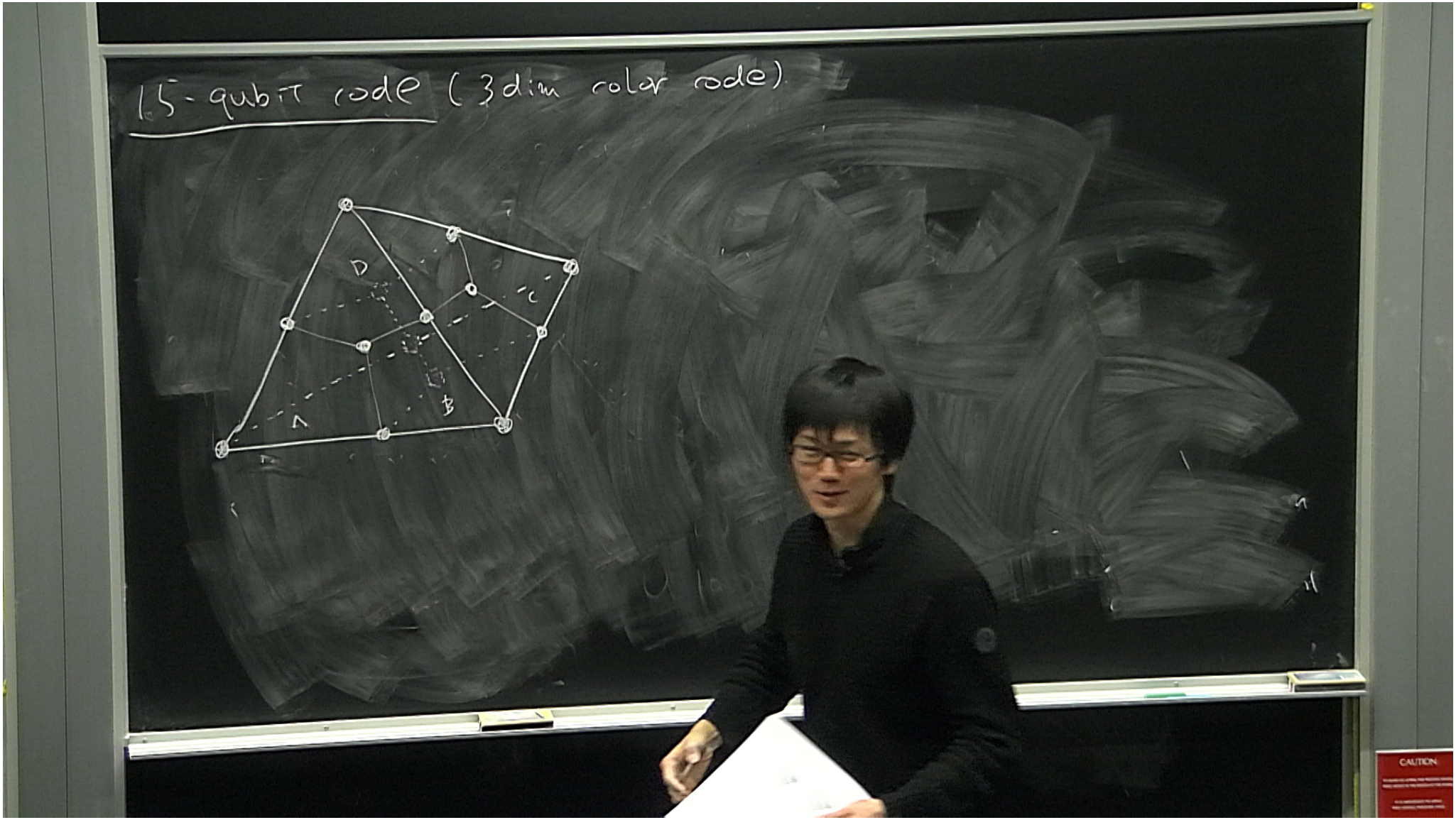

F = -1/2 V = 2
= -4 + 5 = 1
1) X⊗n Z⊗n n odd
{X⊗n Z⊗n} = 0
X̄ = X⊗n
Z̄ = Z⊗n
2)
A
B
C
CAUTION


F = -1/2 V = 2
= -4 + 5 = 1
1) X^⊗n Z^⊗n n odd
{X^⊗n, Z^⊗n} = 0
X̄ = X^⊗n
Z̄ = Z^⊗n
2)
A
B
C
A
B
C


F = −½V = 2
= −4 + 5 = 1
1) X^⊗n Z^⊗n n odd
{X^⊗n, Z^⊗n} = 0
X̄ = X^⊗n
Z̄ = Z^⊗n
2)
A
B
C
A
B
C

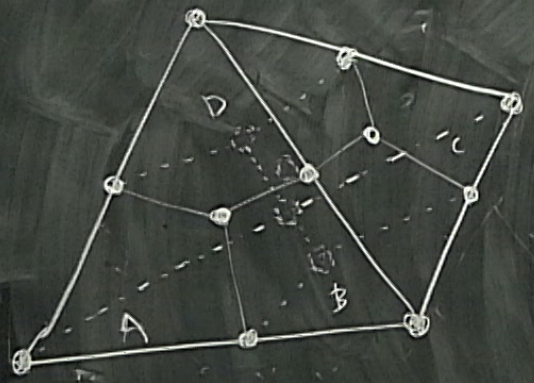

F = -1/2 V = 2
= -4 + 5 = 1
1) X^⊗n Z^⊗n n: odd
{X^⊗n, Z^⊗n} = 0
X̄ = X^⊗n, Z̄ = Z^⊗n
distance ~ O(L)
2)
A
B
C
L
CAUTION


7-qubit code
1 2 3 4 5 6 7
X I X I X I X C
X X I I X X I B
X X X X I I I A
2^4 - -
A
B
C
1
2
3
4
5
6
7


15-qubit code (3dim color code)
D
C
A
B
CAUTION

15-qubit code (3dim color code)

• 4 color labels A, B, C, D

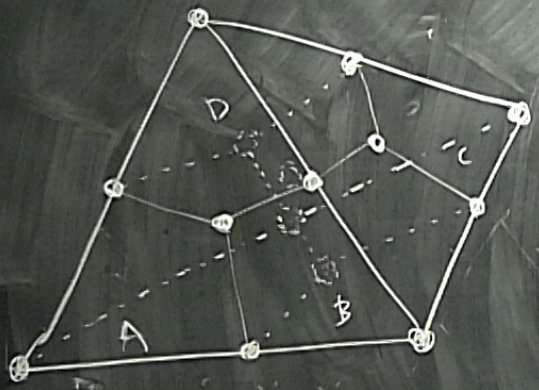


15-qubit code (3dim color code).

- 4 color labels A, B, C, D
- 3 copies of 3d toric code

≠ D-dim color code

= D copies of D-dim toric code

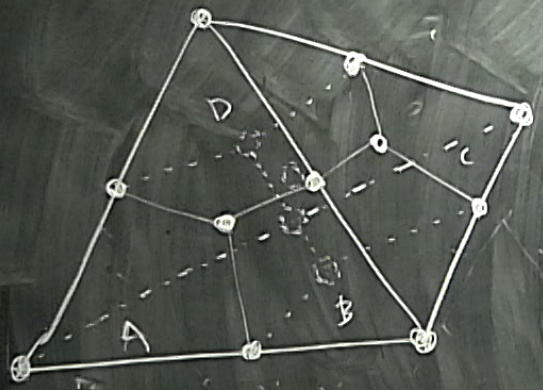


15-qubit code (3 dim color code).

- 4 color labels A, B, C, D
- 3 copies of 3d toric code

$\neq$ D-dim color code

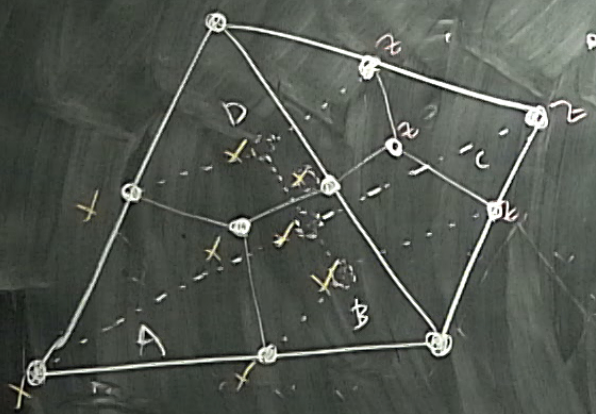
$=$ D copies of D-dim toric code



$$T = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{i} \end{pmatrix}$$

multi-qubit control-Z gate

15-qubit code (3 dim color code).



- 4 color labels A, B, C, D
- 3 copies of 3d toric code

$\neq$ D-dim color code

= D copies of D-dim toric code

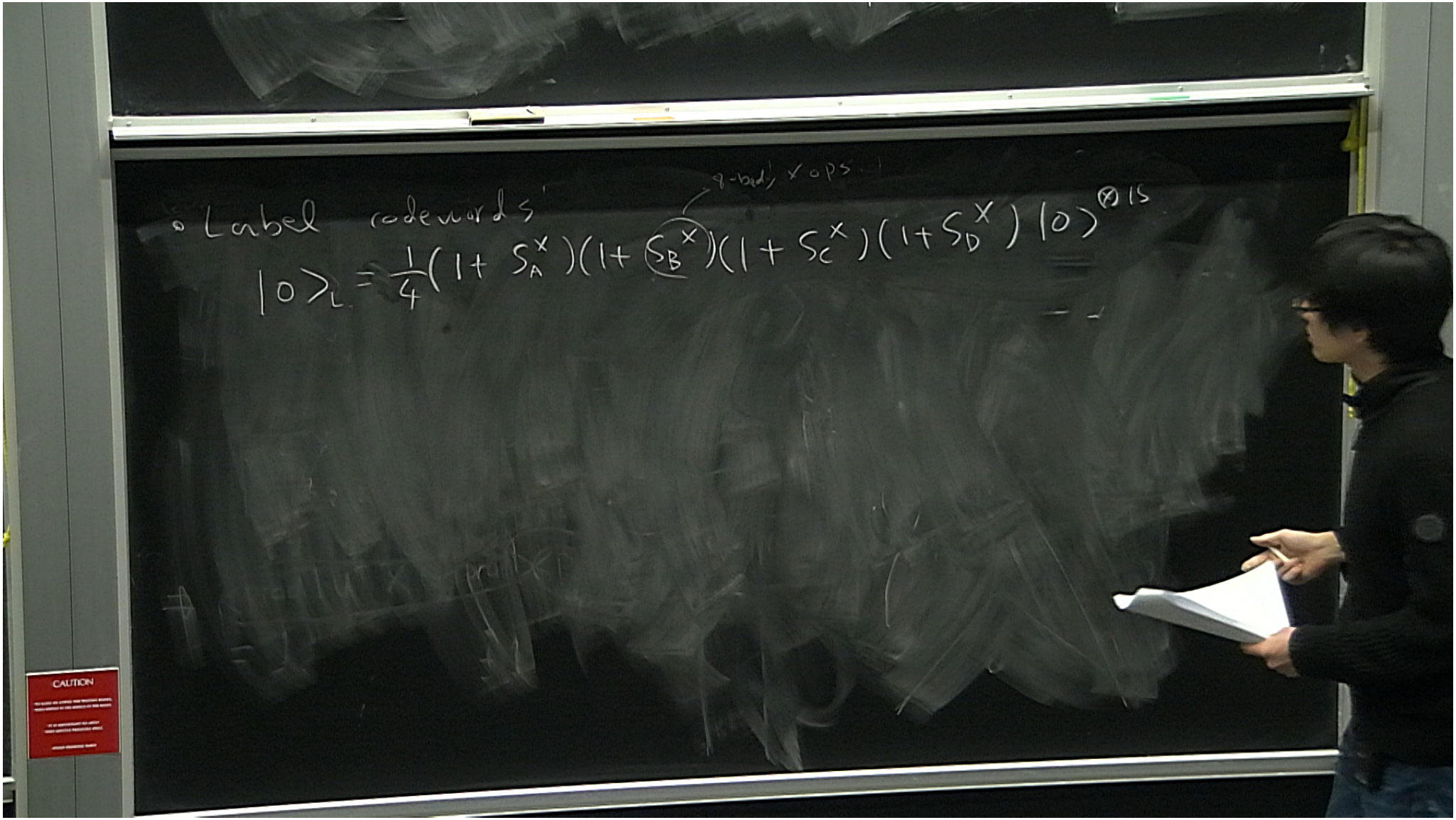
X-op on volume (8-body)

Z-op on face (4-body)

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{i} \end{pmatrix}$$

multi-qubit control-Z gate

codewords

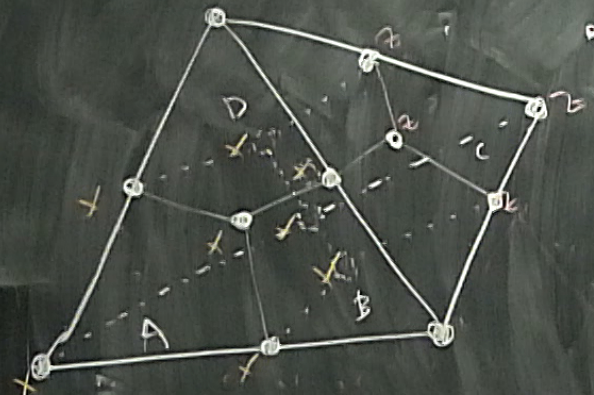

• Label codewords
9-body X ops
$|0\rangle_L = \frac{1}{4}(1+S_A^X)(1+S_B^X)(1+S_C^X)(1+S_D^X)|0\rangle^{\otimes 15}$
CAUTION

15-qubit code (3dim color code).

- 4 color labels A, B, C, D
- 3 copies of 3d toric code

$\equiv$ D-dim color code

= D copies of D-dim toric code



X-op on volume (8-body)

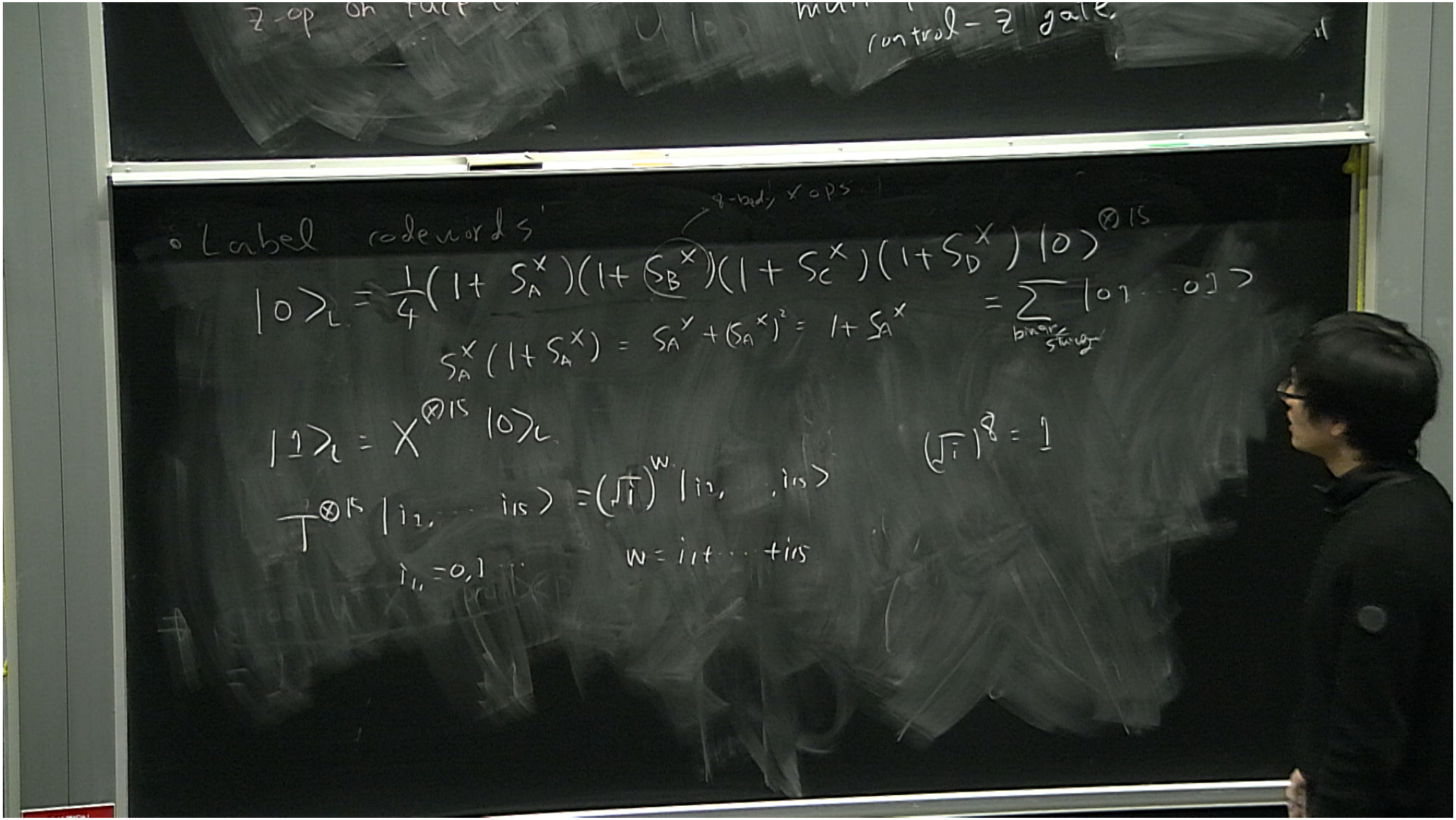
Z-op on face (4-body)

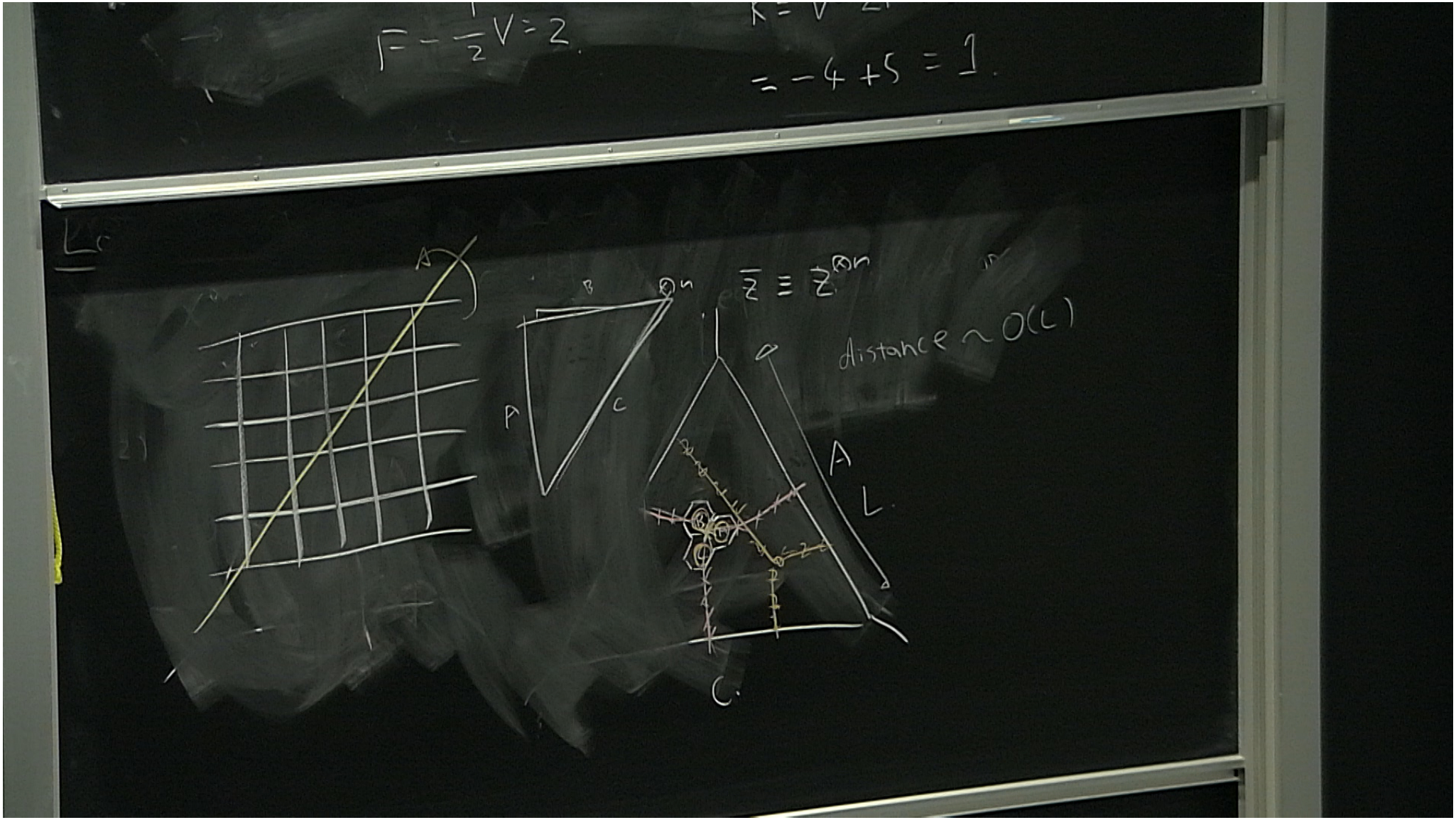
$$T = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{i} \end{pmatrix} = |0\rangle\langle 0| + \sqrt{i}\,|1\rangle\langle 1|$$

multi-qubit control-Z gate

$S_A^x (1 + S_A) = S_A$

$|1\rangle_L = X^{\otimes 15} |0\rangle_L$


Z-op on
control-Z gate
• Label codewords
8-body X ops
$|0\rangle_L = \frac{1}{4}(1+S_A^X)(1+S_B^X)(1+S_C^X)(1+S_D^X)|0\rangle^{\otimes 15}$
$= \sum_{\text{binary strings}} |0\,1\,\cdots\,0\,1\rangle$
$S_A^X(1+S_A^X) = S_A^Y + (S_A^X)^2 = 1+S_A^X$
$|1\rangle_L = X^{\otimes 15}|0\rangle_L$
$T^{\otimes 15}|i_1,\cdots i_{15}\rangle = (\sqrt{i})^w |i_1,\cdots,i_{15}\rangle$
$(\sqrt{i})^8 = 1$
$i_1 = 0,1$
$w = i_1 + \cdots + i_{15}$


$F = -\frac{1}{2}V = 2$
$= -4 + 5 = 1$
$\bar{z} \equiv z^{\oplus n}$
distance $\sim O(L)$
A
B
C
L