

Title: Modifying the quantum measurement postulates (and more)

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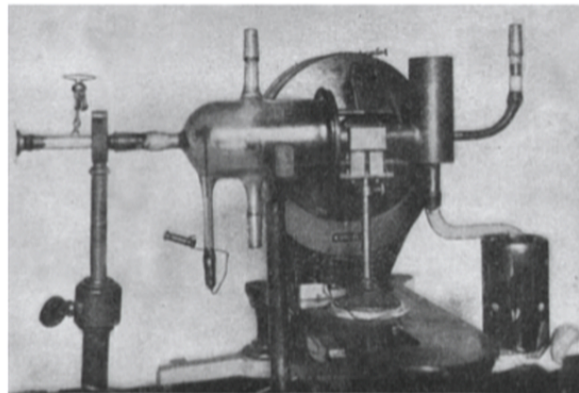
URL: <http://pirsa.org/18020086>

Abstract: <p>In this talk I show how to systematically classify all possible alternatives to the measurement postulates of quantum theory. All alternative measurement postulates are in correspondence with a representation of the unitary group. I will discuss composite systems in these alternative theories and show that they violate two operational properties: purification and local tomography. This shows that one can derive the measurement postulates of quantum theory from either of these properties. I will discuss the relevance of this result to the field of general probabilistic theories. In a second part of the talk I will discuss work in progress and directions for future research. I will show how to generalise the framework used to theories which have different pure states and dynamics than quantum theory. I will discuss two types of theories which can be studied in this framework: Grassmannian theories (same dynamical group and different pure states to quantum theory) and non-linear modifications to the Schrodinger equation (same pure states and different dynamical group).</p>

Modifying measurement (and more)

Thomas D. Galley

February 8, 2018



Contents

Classification of all alternatives to the Born rule in terms of informational properties

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June 14, 2017

Impossibility of mixed-state purification in any alternative to the Born Rule

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(Dated: January 22, 2018)



Contents

Modifying measurement (single systems):

- ▶ Consistently modify measurement postulates of quantum theory
- ▶ Classify all these alternative postulates

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Modifying measurement (single systems):

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Modifying measurement (bipartite systems):

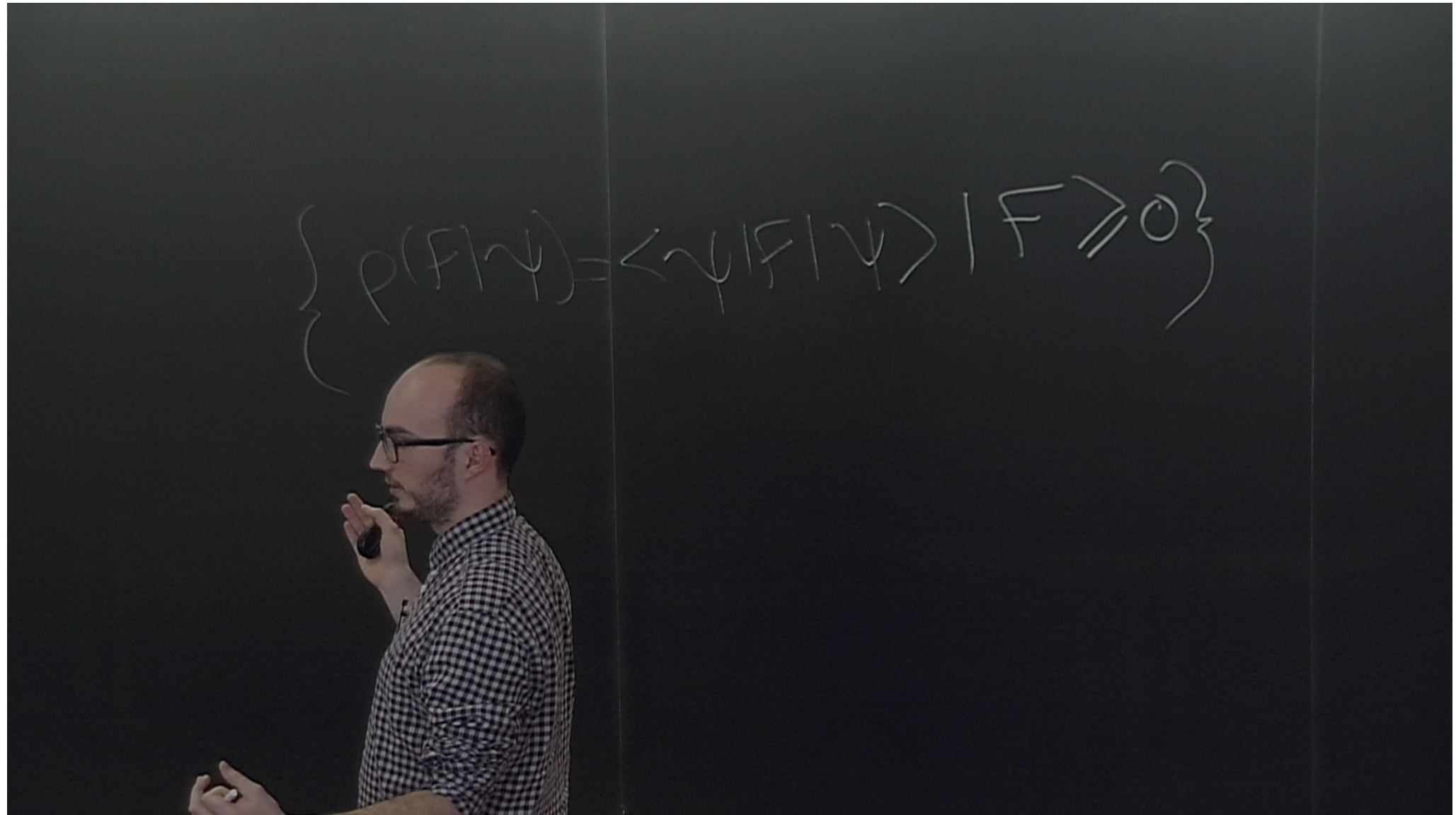
- ▶ Study composition in these alternative theories
- ▶ Study informational properties of these theories: purification and local tomography

Modifying more than measurement:

- ▶ General framework
- ▶ Non-linear dynamics and Grassmann manifolds

Axioms of quantum theory

1. $\psi \in \mathbb{C}^d$
2. $\psi \rightarrow U\psi, U \in \text{SU}(d)$
3. Probability of outcome F for state ψ : $p(F|\psi) = \langle \psi | F | \psi \rangle$
4. Joint pure states given by rays on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \simeq \mathbb{C}^{d_A d_B}$



$$\left\{ p(F|\psi) = \langle \psi | F | \psi \rangle \mid F \geq 0 \right\}$$

$$\{F_1, \dots, F_n\} \sum_i p(F_i|\psi) = 1$$

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Proposal

Modify 3. (probabilitic/measurement structure)

Keep 1. 2. and 4. (dynamical and compositional structure)

Alternative measurement postulates

- ▶ Outcome probabilities are arbitrary functions
 $P(F|\psi) = F(\psi)$
- ▶ $F : \mathbb{P}\mathbb{C}^d \rightarrow [0, 1]$
- ▶ Measurement (F_1, \dots, F_n) where $\sum_i F_i(\psi) = 1, \forall \psi \in \mathbb{P}\mathbb{C}^d$.
- ▶ To specify the measurement structure of a system \mathbb{C}^d we just specify the set of all outcome probability functions (OPFs) \mathcal{F}
- ▶ Consistency constraint: $F \in \mathcal{F} \implies F \circ U \in \mathcal{F}$.

$$F(\psi)$$

$$\{ p(F|\psi) = \langle \psi | F | \psi \rangle / F \geq 0 \}$$

$$\{ F_1, \dots, F_n \} \sum_i p(F_i|\psi) = 1$$

$$\Box \vdash D := \neg D'$$

$$F(\psi)$$

$$\left\{ p(F|\psi) = \langle \psi | F | \psi \rangle \mid F \geq 0 \right\}$$

$$\{F_1, \dots, F_n\} \sum_i p(F_i|\psi) = 1$$

Example

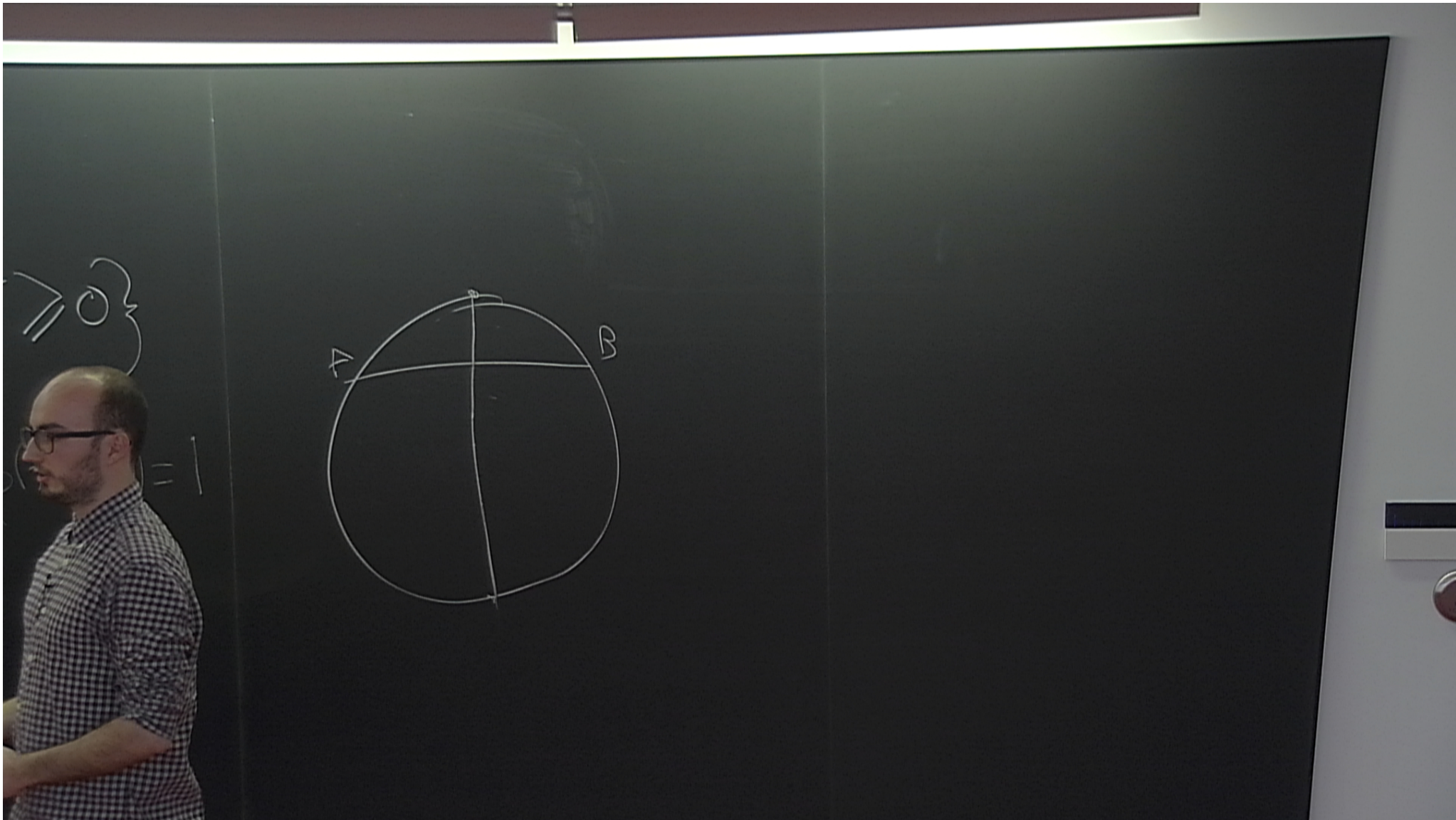
\mathcal{F} contains all OPFs of the form:

$$F(\psi) = \text{Tr}(|\psi\rangle\langle\psi|^{\otimes 2} F) \quad (1)$$

Where F is Hermitian and $F(\psi) \in [0, 1]$. F does not need to be a positive operator.

Intuition

Mixed states tell us which ensembles are indistinguishable.



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Changing the measurements will mean that ensembles that were previously indistinguishable will be distinguishable.

This entails there will be a different set of mixed states (\neq density operators)

Single system

Pure states $\mathbb{P}\mathbb{C}^d$, transformations $SU(d)$, measurements \mathcal{F} .

- ▶ \mathcal{F} is just the set of all outcome probability functions (OPFs).
- ▶ Set of mixed states depend on \mathcal{F} .
- ▶ Need the convex linear representation.

Bloch sphere: states and effects

States $\mathbb{P}\mathbb{C}^2$, transformations $SU(2)$ and outcome probabilities:

$$P(a|\psi) = |\langle a|\psi\rangle|^2.$$

$$\omega_\psi = \begin{pmatrix} P(+X|\psi) \\ P(+Y|\psi) \\ P(+Z|\psi) \end{pmatrix} \quad (2)$$

$$\square \vdash D := \neg D'$$

$$F(\psi)$$

$$P(\mathbb{C}^2)$$

$$(U \in \text{SU}(d))$$

$$p(a|\psi) = |K a|\psi\rangle|^2$$



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These fiducial outcome functions provide a basis for the whole space of OPFs.

Every outcome probability is an affine function of these 3:

$$P(a|\psi) = \alpha_1 P(+X|\psi) + \alpha_2 P(+Y|\psi) + \alpha_3 P(+Z|\psi) + \alpha_0 \quad (3)$$

$$D := -D'$$

$$\psi \in \mathbb{P}^2$$

$$\downarrow$$

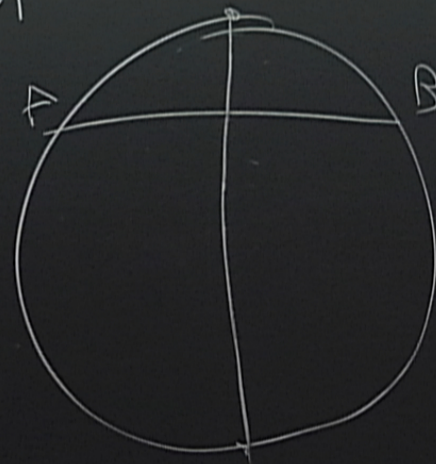
$$W$$

$$F(\psi)$$

$$U \in \text{SU}(d)$$

$$\rho(\alpha\psi) = K|\alpha\psi|^2$$

$$(\alpha_1, \alpha_2, \alpha_3) \cdot \begin{pmatrix} p(i) \\ p(i+1) \\ p(i+2) \end{pmatrix}$$



Bloch sphere: states and effects

States \mathcal{PC}^2 , transformations $SU(2)$ and outcome probabilities:

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Outcome probabilities are affine functions of the states:

$$P(a|\psi) = \mathbf{e}_a \cdot \omega_\psi + \alpha_0 \quad (4)$$

Bloch sphere: Transformations

$$\psi \rightarrow U\psi , \quad (5)$$

$$\omega_\psi \rightarrow R_U \omega_\psi \quad (6)$$

There is a map $R : U \mapsto R_U$: 3 dimensional representation of $SU(2)$.

$$R_{U_1 U_2} = R_{U_1} R_{U_2} \quad (7)$$

Mixed states: general case

Standard construction using fiducial outcomes. Finite number of OPFs $\{F_1(\psi), \dots, F_n(\psi)\}$ such that

$$F(\psi) = \sum_i c_i F_i(\psi) + c_0 \quad (8)$$

$$\omega_\psi = \begin{pmatrix} F_1(\psi) \\ F_2(\psi) \\ \vdots \\ F_n(\psi) \end{pmatrix} \quad (9)$$

Different measurement postulates correspond to different sets of functions. As vector spaces these sets have different dimensions. We have adopted the **finiteness principle**.

$$\square \vdash D := \neg D'$$

$$F_e \quad \psi \in P(\mathbb{C}^2)$$

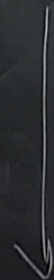
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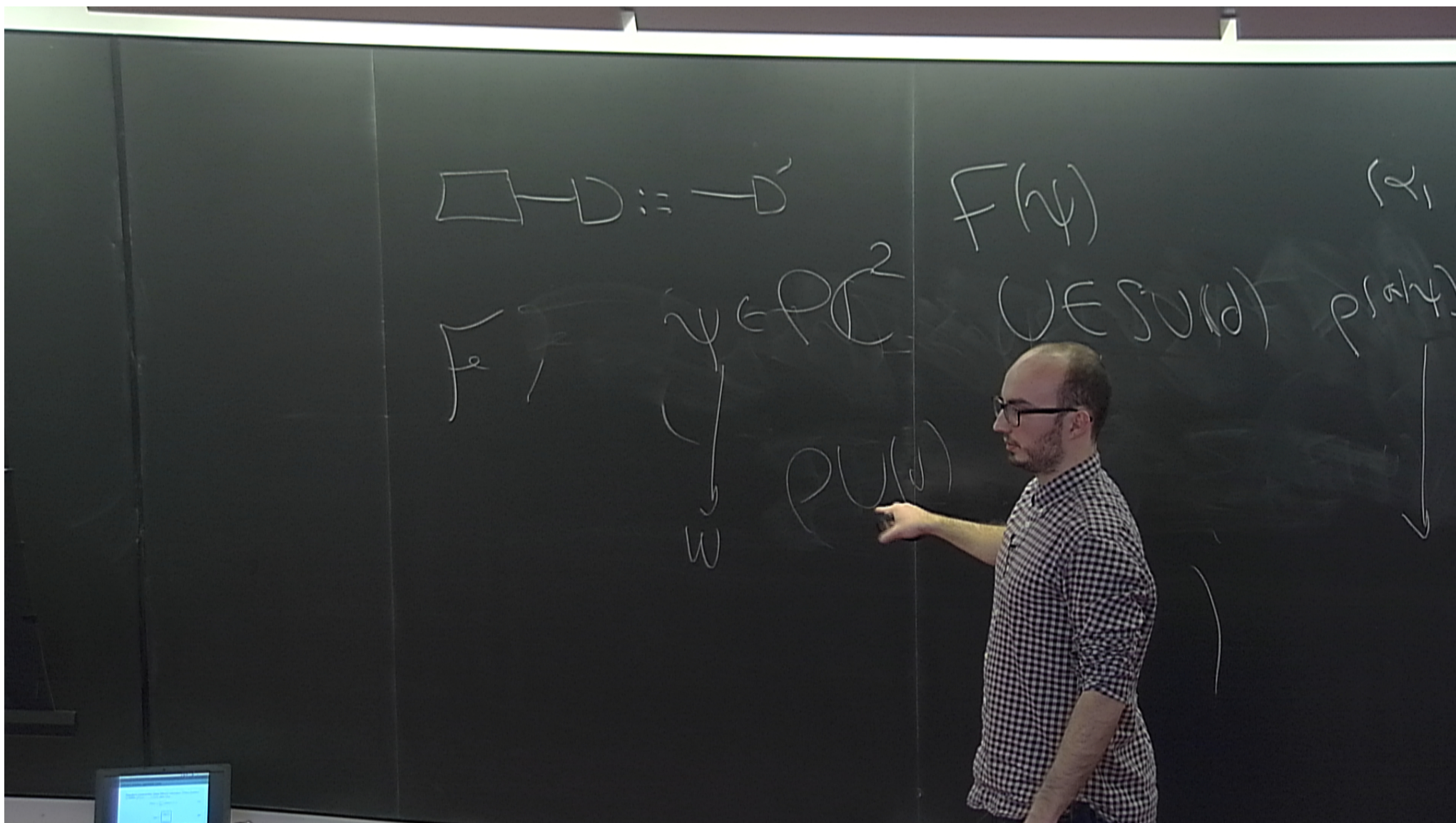
$$w$$

$$F(\psi)$$

$$(\forall \in \text{SU}(d))$$

$$p(\alpha, \psi)$$





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F_e

$$\psi \rightarrow U\psi$$

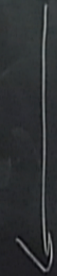
$U\psi$

$$F(\psi)$$

$$(\psi \in \text{PU}(d))$$

$$(\alpha_1, \alpha_n, \alpha_3)$$

$$\rho(\alpha\psi) = K\alpha\psi?$$



Mixed states: Transformations

$$\psi \rightarrow U\psi , \quad (10)$$

$$\omega_\psi \rightarrow \Gamma_U \omega_\psi \quad (11)$$

There is a map $\Gamma : U \mapsto \Gamma_U$: n dimensional representation of $SU(d)$.

Each set \mathcal{F} is in correspondence with a representations of $SU(d)$.

Brief summary

Change measurements \rightarrow change set of mixed states.

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Change measurements \rightarrow change set of mixed states.

Transformations of these mixed states correspond to representations of the dynamical group.

Different measurements correspond to different representation.

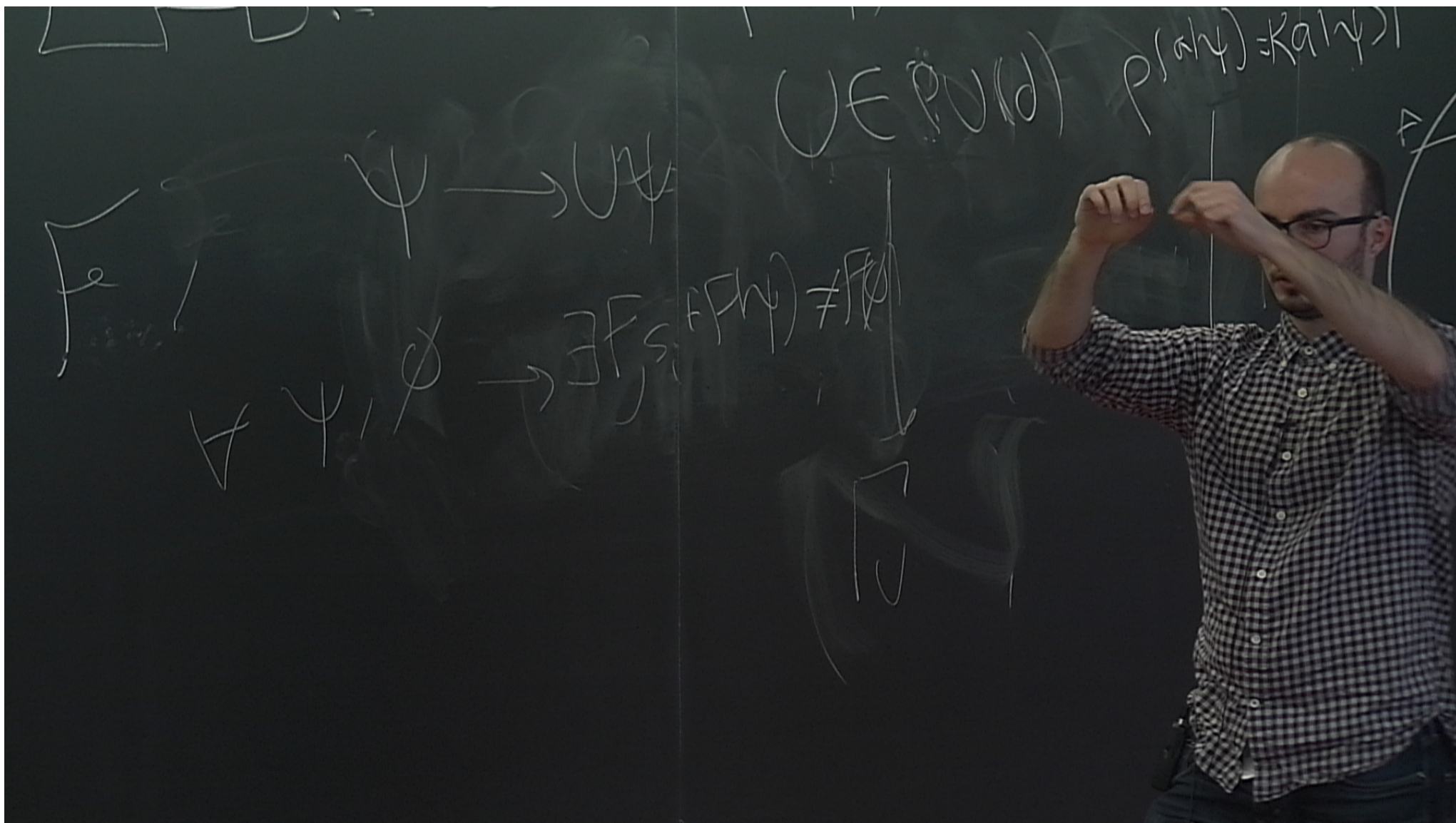
Which representations?

Which representations correspond to these \mathcal{F} ?

These representations preserve the dynamical structure:

$$\psi \rightarrow U\psi.$$

There is a one-to-one correspondence between \mathcal{F} (up to restriction of effects) and representations Γ .



Restricted representation

Consider the 3 dimensional representation of $SO(3)$. Consider a $SO(2)$ subgroup:

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & o \end{pmatrix} o \in SO(2) , \quad (12)$$

We have **restricted** the representation of $SO(3)$ to a $SO(2)$ subgroup. We obtain a reducible representation of $SO(2)$.

Dynamical properties of pure states

$$U = \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & u & \\ 0 & & & \end{array} \right), \quad u \in \mathrm{SU}(d-1). \quad (13)$$

$$\psi = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (14)$$

$$\psi = U\psi, \quad \forall U \in \mathrm{S}(\mathrm{U}(d-1) \times \mathrm{U}(1)) \quad (15)$$

$$\mathbb{P}\mathbb{C}^d \simeq \mathrm{SU}(d)/\mathrm{S}(\mathrm{U}(d-1) \times \mathrm{U}(1)) \quad (16)$$

Hence $\Gamma_U \omega_\psi = \omega_\psi$, $\forall U \in S(U(d-1) \times U(1))$.

$$\omega_\psi = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (17)$$

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \Gamma' \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \quad (18)$$

where Γ' a representation of $S(U(d-1) \times U(1))$

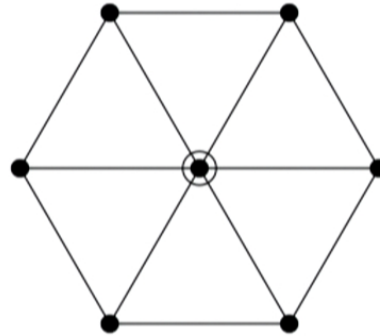
Branching rule

Hence need representations Γ which are such that they have a trivial component when restricted to $S(U(d-1) \times U(1))$.

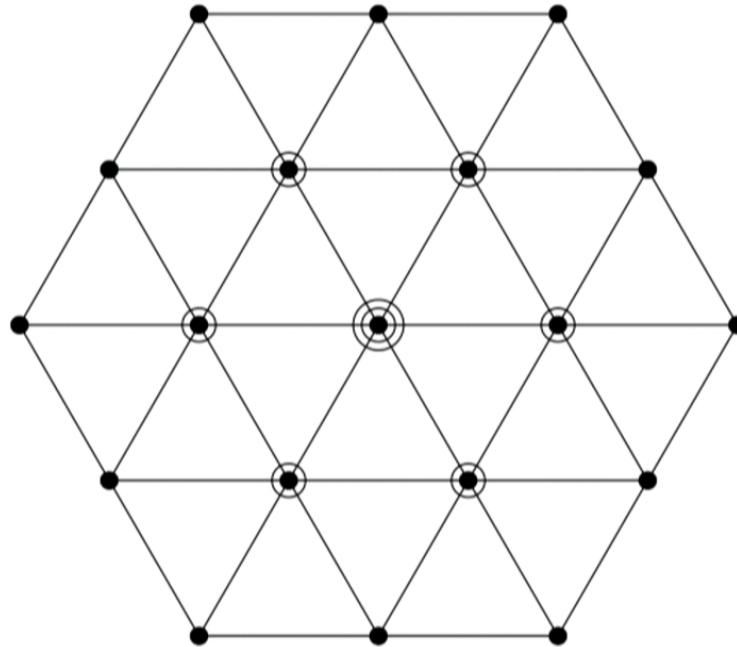
Obtained using branching rules.

Representations with Dynkin index $(j, 0, 0, \dots, 0, 0, j)$ with $j \in \mathbb{N}$.

Representations which preserve the dynamical structure



Representations which preserve the dynamical structure



Pause

Observation: representation theory of $SU(d)$ well studied. Can use existing techniques to obtain fully general results

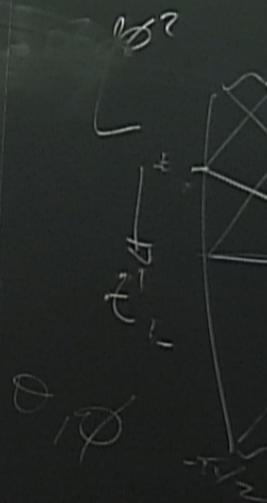
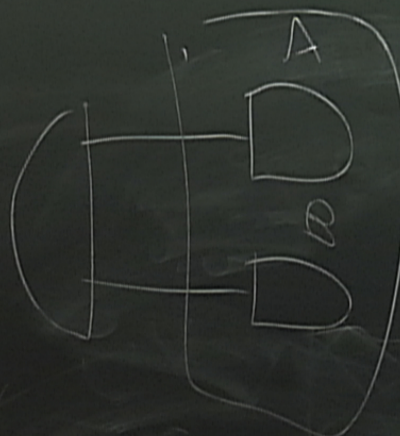
Any questions before continuing?

Operational constraints

In the above the sets \mathcal{F} were not completely arbitrary: subject to an operational constraint: $F \in \mathcal{F} \implies F \circ U \in \mathcal{F}$.

Composition will impose further constraints on the allowed sets \mathcal{F} .

$$\int \frac{2}{x^2} dx$$



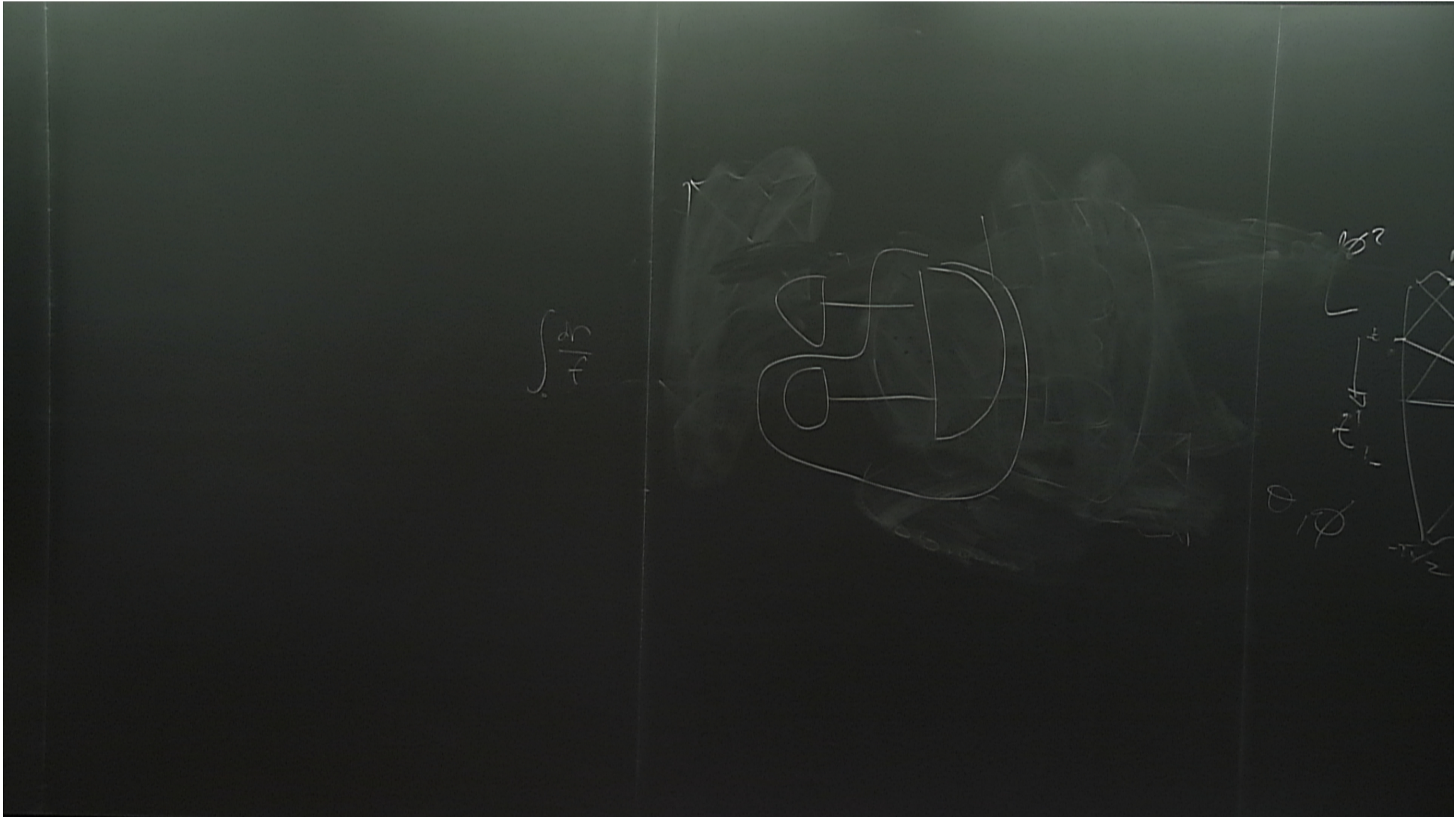
Composition

Consider systems $\mathbb{C}^{d_A d_B}, \mathbb{C}^{d_A}$ and \mathbb{C}^{d_B} with sets of OPFs $\mathcal{F}_{d_A d_B}$, \mathcal{F}_{d_A} and \mathcal{F}_{d_B} .

There exists a bilinear associative product

$\star : \mathcal{F}_{d_A} \times \mathcal{F}_{d_B} \rightarrow \mathcal{F}_{d_A d_B}$ satisfying:

$$(F_A \star F_B)(\psi_A \otimes \phi_B) = F_A(\psi_A)F_B(\phi_B) , \quad (19)$$



Steering and measuring with ancilla

For each $\phi_{AB} \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ and $F_B \in \mathcal{F}_{d_B}$ there is an ensemble (ψ_A^i, p_i) in \mathbb{C}^{d_A} such that

$$\frac{(F_A \star F_B)(\phi_{AB})}{(\mathbf{u}_A \star F_B)(\phi_{AB})} = \sum_i p_i F_A(\psi_A^i) , \quad (20)$$

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For any ancillary state $\phi_B \in \mathbb{C}^{d_B}$ and any OPF in the composite $F_{AB} \in \mathcal{F}_{d_A d_B}$ there exists an OPF on the system $F'_A \in \mathcal{F}_{d_A}$ such that

$$F'_A(\psi_A) = F_{AB}(\psi_A \otimes \phi_B) \quad (21)$$

for all ψ_A .

Representation theoretic features of bipartite systems

There is a necessary (but not sufficient) feature all representations must obey. Every composite state space has a locally tomographic part.

$$\omega_{AB} = \begin{pmatrix} \lambda_{AB} \\ \eta_{AB} \end{pmatrix} \quad (22)$$

What constraints do this impose on the global representation $\Gamma^{d_A d_B}$?

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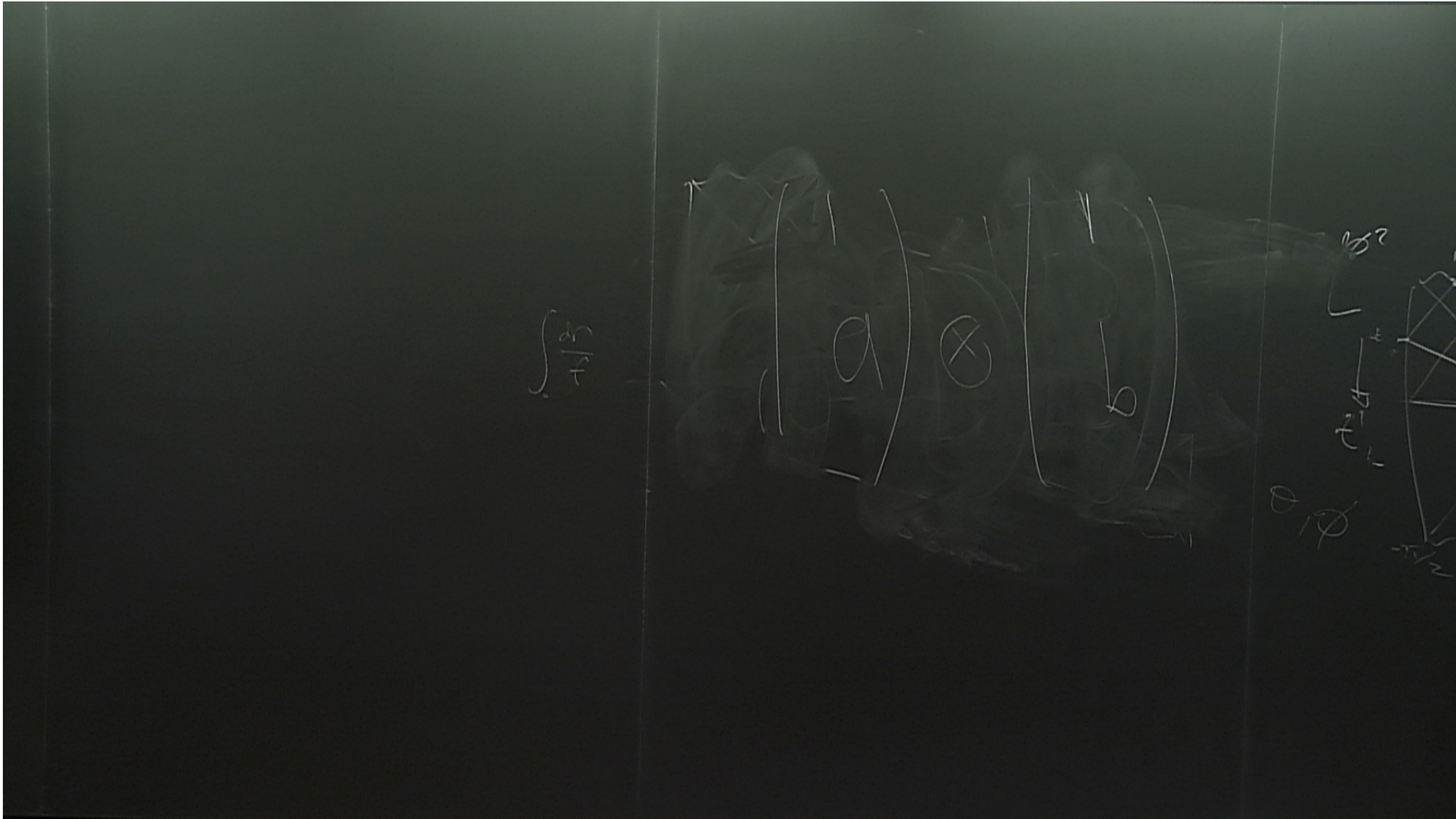
Representation theory and local tomography: an example

Consider the 2-qubit state space. We have two reduced Bloch vectors \mathbf{a} and \mathbf{b} and a correlation vector \mathbf{c} (9 dimensional). An arbitrary state is:

$$\omega_{AB} = \begin{pmatrix} 1 \\ \mathbf{b} \\ \mathbf{a} \\ \mathbf{c} \end{pmatrix} \quad (23)$$

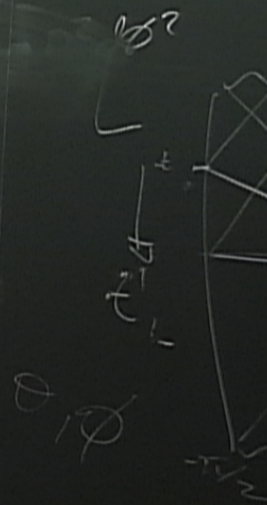
For a product state this is just:

$$\omega_{AB} = \begin{pmatrix} 1 \\ \mathbf{b} \\ \mathbf{a} \\ \mathbf{a} \otimes \mathbf{b} \end{pmatrix} \quad (24)$$



$$\left(\begin{array}{c|c} 1 & \\ \hline & A \end{array} \right) \otimes \left(\begin{array}{c|c} 1 & \\ \hline & B \end{array} \right)$$

$$\left(\begin{array}{c|c} 1 & \\ \hline & a \end{array} \right) \otimes \left(\begin{array}{c|c} 1 & \\ \hline & b \end{array} \right)$$



Action of local subgroup

$$\begin{pmatrix} 1 \\ \mathbf{b} \\ \mathbf{a} \\ \mathbf{a} \otimes \mathbf{b} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{B} & 0 & 0 \\ 0 & 0 & \mathbf{A} & 0 \\ 0 & 0 & 0 & \mathbf{A} \otimes \mathbf{B} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{b} \\ \mathbf{a} \\ \mathbf{a} \otimes \mathbf{b} \end{pmatrix} \quad (25)$$

Where A and B are the adjoint representation of $SU(2)$. If we consider global representation (adjoint of $SU(4)$) and only take subgroup $SU(2) \times SU(2)$ (tensor product embedding) we obtain a reducible representation of $SU(2) \times SU(2)$:

$$(1 \otimes 1) \oplus (1 \otimes B) \oplus (A \otimes 1) \oplus (A \otimes B) \quad (26)$$

General case

$$\omega_{AB} = \begin{pmatrix} \lambda_{AB} \\ \eta_{AB} \end{pmatrix} \quad (27)$$

The action of the local subgroup $SU(d_A) \times SU(d_B)$ on the locally tomographic subspace has the same decomposition:

$$(1 \otimes 1) \oplus (1 \otimes \Gamma^{d_B}) \oplus (\Gamma^{d_A} \otimes 1) \oplus (\Gamma^{d_A} \otimes \Gamma^{d_B}) \quad (28)$$

Necessary criterion for composition

Pure states $\mathbb{P}\mathbb{C}^{d_A d_B}$, transformations $SU(d_A d_B)$, measurements \mathcal{F} .

If this system is a composite system then the representation $\Gamma^{d_A d_B}$ is such that:

$$\Gamma_{AB|SU(d_A) \times SU(d_B)}^{d_A d_B} = (1^{d_A} \otimes \Gamma^{d_B}) \oplus (\Gamma^{d_A} \otimes 1^{d_B}) \oplus (\Gamma^{d_A} \otimes \Gamma^{d_B}) \oplus \text{other terms}.$$

Violation of local tomography

All relevant representations of $SU(9)$ have a locally tomographic part and extra terms.

Hence all systems \mathbb{C}^9 with alternative measurement postulates violate local tomography.

All theories have systems which violate local tomography.

Lack of Purification

All systems with alternative Born rules violate purification.

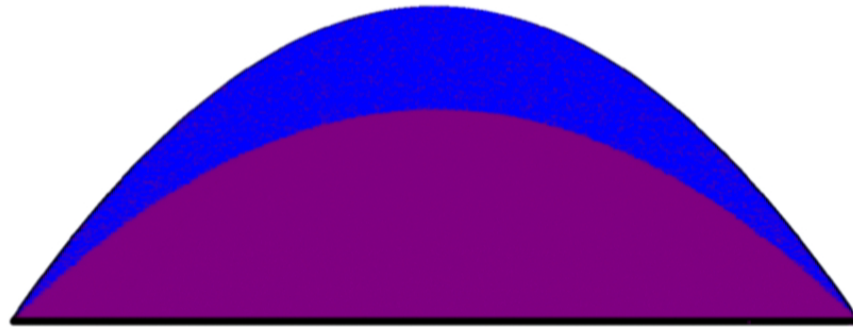


Figure 1: Lack of purification in toy theory for \mathbb{PC}^2

Implications

- ▶ Irreducible classicality

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- ▶ Quantum measurement postulates are the most non-classical

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- ▶ Irreducible classicality
- ▶ Quantum measurement postulates are the most non-classical
- ▶ Also follows from state space dimension argument (see Mielnik 1974)
- ▶ Can single out (derive) the Born rule from purification (or local tomography)

Limitations

The claims that all alternatives violate local tomography and purification are conditional. Have not shown that all systems compose.

We have one toy model, so at least one amongst all the possible measurement postulates composes.

In general for single (transitive) systems we can do everything using representation theory. For composite systems can only obtain necessary conditions for existence.

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The claims that all alternatives violate local tomography and purification are conditional. Have not shown that all systems compose.

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In general for single (transitive) systems we can do everything using representation theory. For composite systems can only obtain necessary conditions for existence.

Question: Do any alternative measurement postulates compose beyond bipartite systems?

Implications for work on GPTs

There are many results which show properties of all theories which obey one of these two assumptions.

These are natural/convenient assumptions, but here we see that there is an infinite family of theories which do not have these features.

Lee/Selby recent result: potential successor theories?

Pause

Going to generalise the framework and use it to study different modifications to QT.

Modify the pure states (and keep the dynamics) and modify the dynamics (but keep the pure states)

Before moving onto suggestions for future work, are there any questions?

Generalising the framework

Pure states: set X

Dynamical group G (and a group action): $x \mapsto gx$

Set of OPFs \mathcal{F} , $F : X \rightarrow [0, 1]$

Impose transitivity: $X := G/H$.

Further distortions of quantum theory

Pure states of quantum theory are obtained by taking all 1-dimensional subspaces of \mathbb{C}^d

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Pure states of quantum theory are obtained by taking all 1-dimensional subspaces of \mathbb{C}^d

But could have theories where the pure states are all k -dimensional subspaces.

This defines a Grassmann manifold:

$$\text{Gr}(k, \mathbb{C}^d) = \{W \subset \mathbb{C}^d, \dim(W) = k\} , \quad W \text{ a subspace} \quad (30)$$

$$\text{Gr}(m, \mathbb{C}^{m+n}) = \text{SU}(n+m)/\text{S}(\text{U}(n) \times \text{U}(m)) . \quad (31)$$

Dynamics are unitary.

Grassman manifolds

Have a complete classification of all measurements structures \mathcal{F} , using branching rule.

Composition: $\text{Gr}(k_A, \mathbb{C}^{d_A}), \text{Gr}(k_B, \mathbb{C}^{d_B}) \rightarrow \text{Gr}(k_A k_B, \mathbb{C}^{d_A d_B})$.

In the adjoint representation they do not compose (as corresponds to Quartic quantum theory of Zyczkowski)

Can generalise to Flag manifolds.

Summary

Results:

- ▶ Classified all possible measurement postulates using representation theory
- ▶ Studied composition in these alternative theories
- ▶ All alternatives violate purification and local tomography

Suggestions for future work:

- ▶ Presented general framework for dealing with transitive theories
- ▶ Suggested theories with modified pure states and dynamics

Thank you for your attention.

