Title: The wonderful compactication and the universal centralizer

Date: Feb 26, 2018 02:00 PM

URL: http://pirsa.org/18020085

Abstract: <p>Let be a complex semisimple algebraic group of adjoint type and the wonderful compacti<br clear="all" />
cation. We show that the closure in \overline{G} of the centralizer of a regular nilpotent is isomorphic to the Peterson variety. We generalize this result to show that for any regular , the closure of the centralizer in is isomorphic to the closure of a general -orbit in the flag variety. We consider the family of all such centralizer closures, which is a partial compactication of the universal centralizer. We show that it has a natural log-symplectic Poisson structure that extends the usual symplectic structure on the universal centralizer.</p>

The wonderful compactification

& the universal centralizer

G semisimple conn alg grp / $\mathbb{C}$

$$Z(G) = 1$$

$\check{G}$ simply-connected cover

$$\check{G} \supset \check{B} \supset \check{T}$$

λ

...ctification

...centralizer

... alg grp / $\mathbb{C}$

... cover

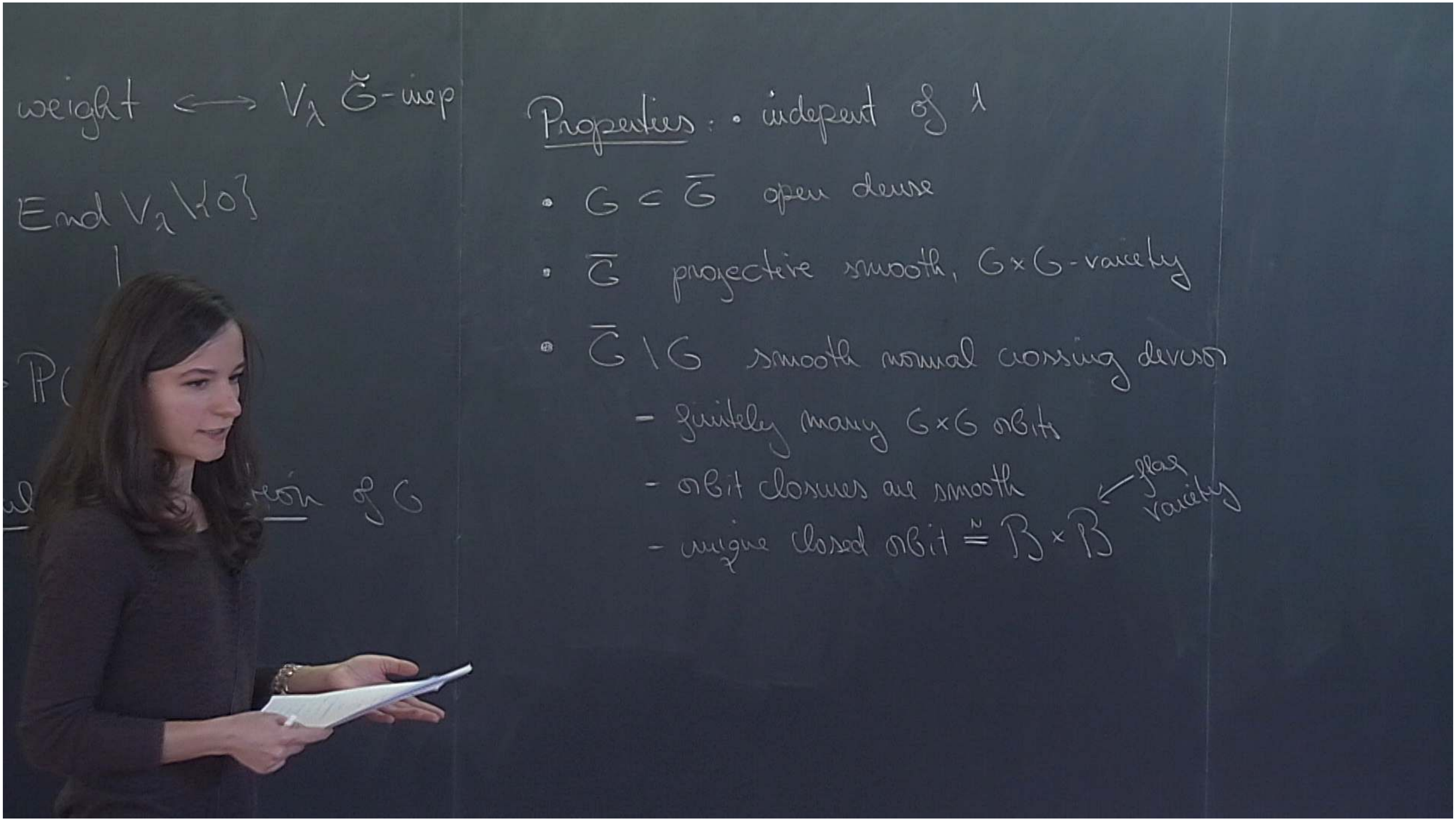
λ regular dom weight $\longleftrightarrow$ V_λ $\breve{G}$-irrep

$$\begin{tikzcd} \breve{G} \arrow[r] \arrow[d] & \operatorname{End} V_\lambda \setminus \{0\} \arrow[d] \\ G \arrow[r, hook, "X"] & \mathbb{P}(\operatorname{End} V_\lambda) \end{tikzcd}$$

<u>Def</u>. The <u>wonderful compactification</u> of G is $\overline{G} = \overline{X(G)}$.

Propert...

- $G \subset$ ...
-


weight ⟷ V_λ Ǧ-irrep
End V_λ \ {0}
Properties: • indepent of λ
• G ⊂ Ḡ open dense
• Ḡ projective smooth, G×G-variety
• Ḡ \ G smooth normal crossing divisor
- finitely many G×G orbits
- orbit closures are smooth
- unique closed orbit ≅ B×B
flag variety
of G

ly

levision

flag variety

<u>Ex</u>: $G = PGL_2 \quad \tilde{G} = SL_2$

std rep $V = \mathbb{C}^2$

$X: G \hookrightarrow \mathbb{P}(M_{2\times 2}) \mid \mathrm{Im}\, X = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, ad - bc \neq 0 \right\}$

$\overline{G} = \mathbb{P}(M_{2\times 2}) \cong \mathbb{P}^3$

$\cup$

$\partial \overline{G} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, ad - bc = 0 \right\} \cong \mathbb{P}^1 \times \mathbb{P}^1$ ← flag variety

<u>Non-ex</u>: $G = PGL_n$

$n \geq 3$

$\overline{G} \neq \mathbb{P}^{n^2-1}$

$\mathfrak{g} = \mathrm{Lie}\, G$

$\ell = \mathrm{rk}\, \mathfrak{g}$

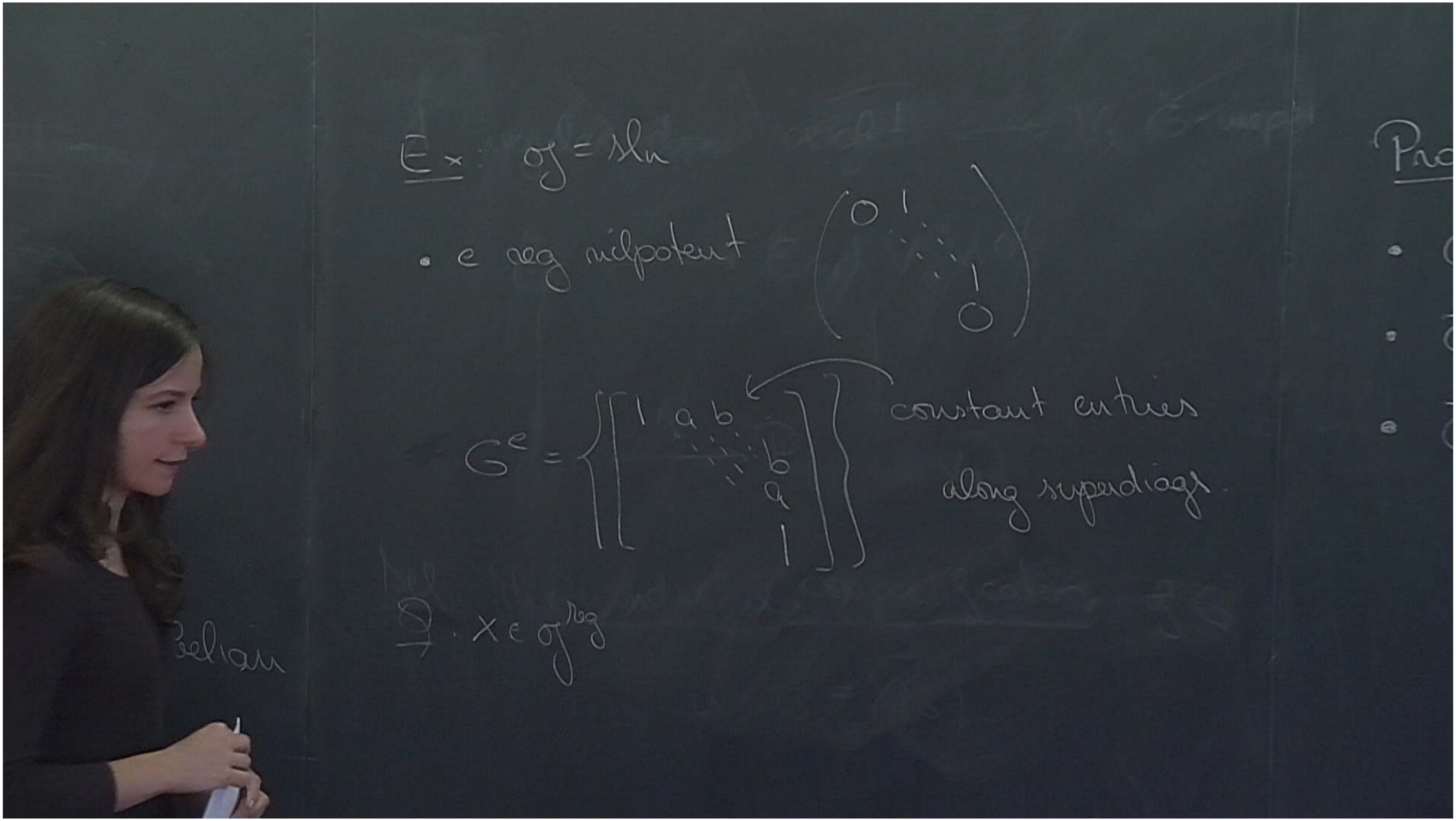
<u>Def</u>: $x \in \mathfrak{g}$ is <u>regular</u> if G^x has min dim ℓ.

- $s \in \mathfrak{g}$ reg ss $\iff$ G^s max torus
- $e \in \mathfrak{g}$ reg nilp $\iff$ G^e unipotent abelian

<u>Ex</u>:

- e re

G^e


Ex: g = sl_n
e reg nilpotent
0 1
1
0
G^e =
1 a b
b
a
1
constant entries
along superdiags.
Q. X ∈ g^reg
belian
Pro

$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$

constant entries along superdiags.

?

1) $s \in \mathfrak{g}$ regular semisimple

$G^s = T$ max torus

$\overline{T}$ projective toric variety

$\cong$ fan of Weyl chambers. (smooth).

closure of a $\underline{\text{generic}}$ T-orbit on $\mathcal{B}$.

$\rightsquigarrow$ suff. many Plücker coords $\neq 0$.

(Dabrowski)

(Klyachko).

2) $e \in$

e

semisimple

max torus

projective toric variety

fan of Weyl chambers.

(smooth).

a generic T-orbit on $\mathcal{B}$.

↳ suff many Plücker coords $\neq 0$.

2) $e \in \mathfrak{g}$ reg nilpotent

$e \in \mathfrak{b}$ unique Borel

$\mathfrak{b}^- \in \mathcal{B}$ opposite Borel

<u>Def</u>: The <u>Peterson variety</u> is $\overline{G^e \cdot \mathfrak{b}^-} \subset \mathcal{B}$ (non-normal except in small rk)

<u>Thm</u> (B) There is a G^e-isomorphism

$$\overline{G^e} \xrightarrow{\sim} P_e$$

G^x - orbit on $\mathcal{B}$

dense

s w/ previous

in ss case

orbit on $\mathcal{B}$

morphism

$\{e, h, f\}$ principal $\mathfrak{sl}_2$-triple

$\mathfrak{g}^e = \mathrm{Lie}\, G^e$.

$\rightsquigarrow$ principal slice $f + \mathfrak{g}^e$

- intersects each regular G-orbit on $\mathfrak{g}$ exactly once
- section to adjoint quot $\mathfrak{g} \longrightarrow \mathfrak{g}/\!\!/G$.

ple.

$\mathfrak{g}^e$

lar

exactly one

quot

/G.

Def: The universal centralizer is

$$Z = \{(g, \xi) \in G \times (f + \mathfrak{g}^e) \mid g \in G^\xi\}$$

Natural symplectic struct:

$e \in \mathfrak{b} = \mathrm{Lie}\, B$

$N \subset B$ unipotent radical

$\mathfrak{n} = \mathrm{Lie}\, N$

$$
\begin{tikzcd}
N \times N \circlearrowright T^*G \cong G \times \mathfrak{g} \arrow[d, "\text{moment map } \nu"'] \arrow[dr, "\mu"] & \\
 & \mathfrak{g} \times \mathfrak{g} \arrow[d] \\
\mathfrak{n}^* \times \mathfrak{n}^* \cong \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b} \ni (f, f) &
\end{tikzcd}
$$

moment map for $G \times G$

- $\mu(g, \xi) = (g\xi, \xi)$
- $\mathrm{Im}\,\mu = \{\text{conjugate pairs}\}$
- $\mu^{-1}(\xi, \xi) \cong G^{\xi}$

(f, f) fixed by $N \times N$.

- (f, f) is a regular value of ν
- $N \times N \circlearrowright \nu^{-1}(f, f)$ free & proper
- $\nu^{-1}(f, f)/_{N \times N} \cong Z \longleftarrow$ symplectic struct.

Partial compactification

$$\overline{Z} = \left\{ (x, \xi) \in \overline{G} \times (f + \mathfrak{g}^e) \;\middle|\; x \in \overline{G^{\xi}} \right\}$$

$$\begin{array}{l} \overline{Z} \\ \big\downarrow \\ f + \mathfrak{g}^e \end{array} \quad \rightsquigarrow \text{ generic fiber } \overline{T} \text{ is smooth}$$

$\overline{G}^{\Xi}\Big\}$

$D = \overline{G} \setminus G$ boundary divisor

$T^*\overline{G}(-\log D)$ logarithmic cotangent bdle

- sections are log forms w/ poles along D

- restricts to T^*G on $G \subset \overline{G}$

- canonical log-symplectic struct

$$D = \{x_1 \ldots x_\ell = 0\} \qquad \omega = \sum_{i=1}^{\ell} \frac{dx_i}{x_i} \wedge d\xi_i + \sum_{i=\ell+1}^{n} dx_i \wedge d\xi_i$$

...ngent bdle

...s w/

...es along D

$...\subset \overline{G}$

...ic struct

$\sum_{i=l+1}^{n} dx_i \wedge d\xi_i$

$$T^*\overline{G}(-\log D) \hookrightarrow \overline{G} \times \mathfrak{g} \times \mathfrak{g}$$

fiber at $x \in \overline{G}$:

$$\Theta = (G \times G)\, x$$

$$\mathrm{Stab}_{G\times G}(x) \curvearrowright T_x \overline{G} \big/ T_x \Theta$$

$$/\!/ \quad \mathfrak{g} \times \mathfrak{g}(x) = \mathrm{Lie}(\text{ker of this action})$$

$$T_x^* \overline{G}(-\log D)$$

$\overline{G} \times \mathfrak{g} \times \mathfrak{g}$

$T_x \overline{G} / T_x O$

... of this action)

$g \in G \subset \overline{G}$,

$\mathfrak{g} \times \mathfrak{g}_{(g)} = \mathrm{Lie}(\mathrm{Stab}_{G\times G}(g))$

$\mathfrak{g} \times \mathfrak{g}_{(e)} = \mathrm{Lie}(G_\Delta) = \mathfrak{g}_\Delta$

$\mathfrak{g} \times \mathfrak{g}_{(g)} = g \cdot \mathfrak{g}_\Delta$

$G_\Delta \hookrightarrow G \times G$

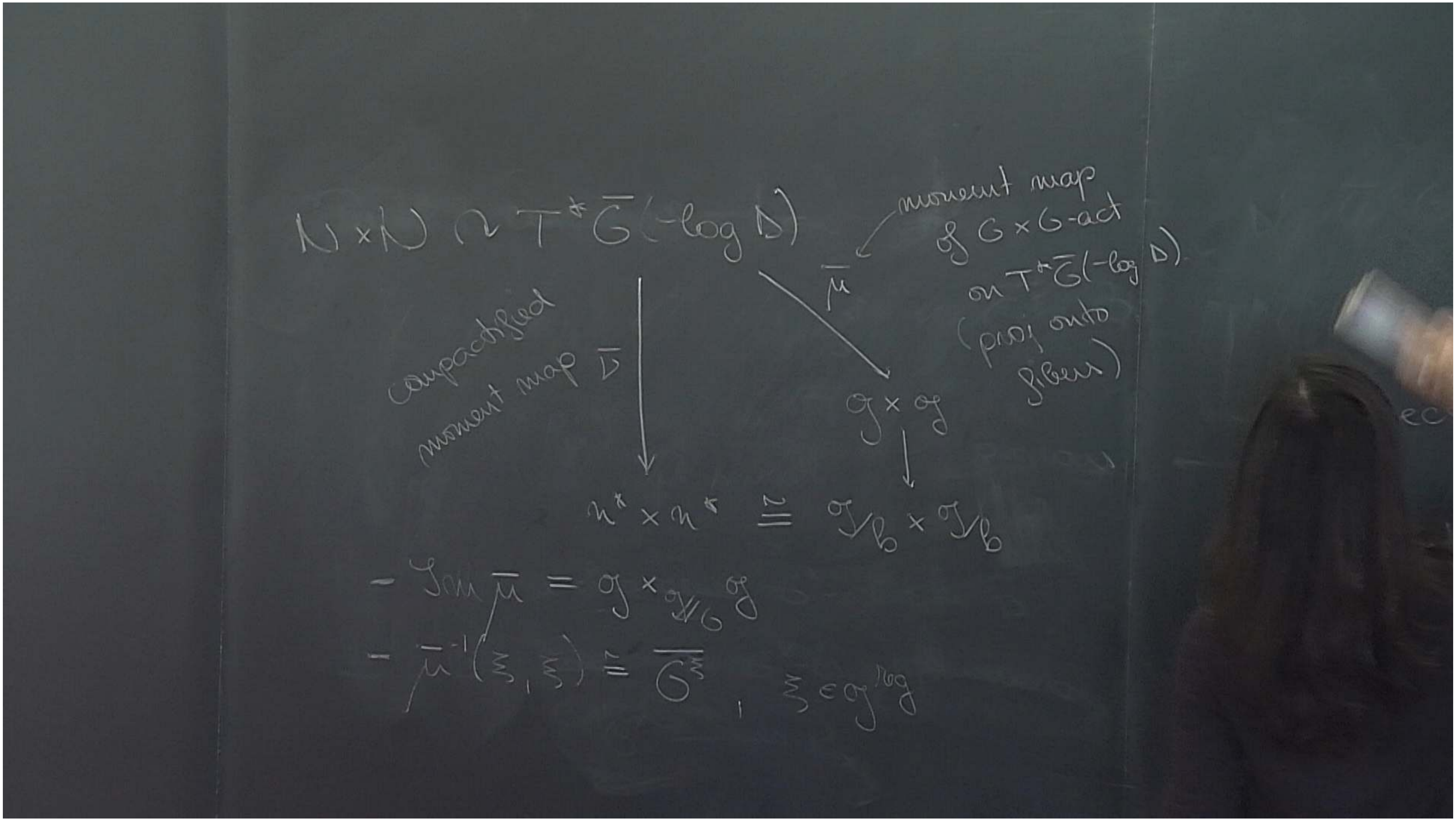
$g \longmapsto (g, g^{-1})$

Rk: Alternative constr: $\imath : G \hookrightarrow \mathrm{Gr}(\dim \mathfrak{g}, \mathfrak{g} \times \mathfrak{g})$

$g \longmapsto g \cdot \mathfrak{g}_\Delta$

$\overline{G} = \overline{\imath(G)}$

$\Rightarrow T^*G(-\log \partial)$ a restr. of taut bdle.


$N \times N \curvearrowright T^*\overline{G}(-\log D)$
moment map
of $G\times G$-act
on $T^*\overline{G}(-\log D)$
(proj onto
fibers)
$\overline{\mu}$
compactified
moment map $\overline{\nu}$
$\mathfrak{g}\times\mathfrak{g}$
$\mathfrak{n}^* \times \mathfrak{n}^* \cong \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b}$
$-$ Im $\overline{\mu} = \mathfrak{g}\times_{\mathfrak{g}/\!/G}\mathfrak{g}$
$-$ $\overline{\mu}^{-1}(\xi,\xi) \cong \overline{G^{\xi}}$, $\xi\in\mathfrak{g}^{reg}$

alue of $\overline{\nu}$

prope.

Rk. $\overline{\mathcal{Z}}$ is smooth.

$$\begin{array}{ccc} \overline{\mathcal{Z}} & \supset & P_f \\ \downarrow & & \downarrow \\ f+\mathfrak{g}^e & \ni & f \end{array}$$

$\overline{G} = \coprod S_J$ strat into $G \times G$-orbits
indexed by $J \subset \{\text{simple roots}\}$.

induces

$\rightsquigarrow \overline{\mathcal{Z}} = \coprod \overline{\mathcal{Z}}_J$

$J \rightsquigarrow \mathfrak{p}_J \rightsquigarrow \mathfrak{p}_J / [\mathfrak{p}_J, \mathfrak{p}_J] \cong \mathfrak{a}_J$

Symplectic leaves in $\overline{\mathcal{Z}}_J \longleftrightarrow \mathfrak{a}_J$

of $\overline{\nu}$

of.

Rk. $\overline{\mathcal{Z}}$ is smooth.

$$\begin{array}{ccc} \overline{\mathcal{Z}} & \supset & P_f \\ \downarrow & & \downarrow \\ f+\mathfrak{g}^e & \ni & f \end{array}$$

$G \times G \curvearrowright T^*G \times X$

$\overline{G} = \coprod S_y$ strat

unde

induces

$\rightsquigarrow \overline{\mathcal{Z}} = \coprod \overline{\mathcal{Z}}$

$y \rightsquigarrow \mu_y \rightsquigarrow \mu_y / [\mu$

Symplectic leaves in $\overline{\mathcal{Z}}$