Title: Analytic conformal bootstrap in 1D

Date: Feb 13, 2018 02:30 PM

URL: http://pirsa.org/18020081

Abstract: <p>All physical constraints of the conformal bootstrap in principle arise by applying linear functionals to the conformal bootstrap equation. An important goal of the bootstrap program is to identify a suitable basis for the space of functionals -- one that would allow us to solve crossing analytically. In my talk, I will describe two particularly convenient choices of the basis for the 1D conformal bootstrap. The two bases manifest the crossing symmetry of the four-point function of a generalized free boson and generalized free fermion respectively. I will use the bases to study small deformations of the two theories. Assuming no new operators appear in the OPE, the generalized free fermion allows no small deformation, and the generalized free boson allows a one-parameter deformation, which coincides with the AdS_2 four-point contact interaction at the leading order. Time allowing, I will discuss the connection of this work to the conformal bootstrap a la Polyakov.</p>

Analytic Conformal Bootstrap in 1D

Perimeter Institute, Feb 13 2018

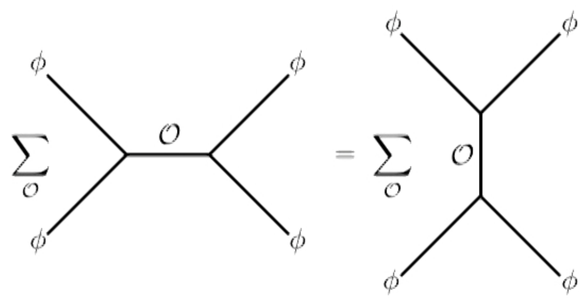
Dalimil Mazáč

Simons Center & C.N. Yang Institute, Stony Brook

work with M. Paulos (to appear)
see also DM: arXiv:1611.10060

Motivation 1

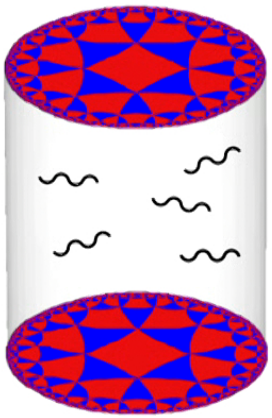
The conformal bootstrap equations place strong constraints on the CFT data.



Motivation 2: QFT in AdS

Every UV-complete QFT in AdS_{d+1} with an $SO(2,d)$-invariant boundary condition defines a consistent conformally-invariant d-dimensional theory on the boundary.

[Paulos, Penedones, Toledo, van Rees, Vieira, '16]



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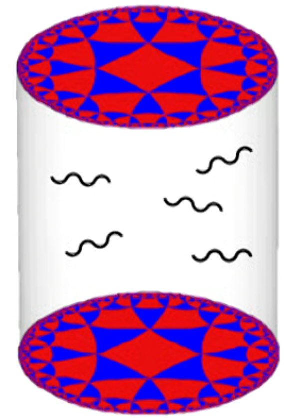

local operators on the boundary	$\leftrightarrow$	states in the Hilbert space in AdS
conformal bootstrap in d dimensions	$\leftrightarrow$	S-matrix bootstrap of QFT in AdS_{d+1}

AdS acts as a convenient IR regulator: maximal spacetime symmetry, and continuum of multi-particle states resolved into a discrete set.

Bonus: if there happens to be a stress-tensor on the boundary, we are describing a theory including quantum gravity.

[Heemskerk, Penedones, Polchinski, Sully, '09]

Does every boundary theory arise from some bulk theory?
Can we use boundary bootstrap to solve bulk theories?

Analytic conformal bootstrap to date

2D CFT with $0 < c < 1$ ✓ [Belavin, Polyakov, Zamolodchikov, '84]

In general D, analytic progress has generally followed from constraints in the Lorentzian regime.

← some pairs of operators nearly null- or time-like separated

An important result in general D: a Lorentzian inversion formula [Caron-Huot, '17]

"The CFT data of exchanged operators with spin ≥ 2 are fixed by the singularity in the crossed channel."

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Unification of many results obtained using the large spin expansion

[Komargodski, Zhiboedov; Fitzpatrick, Kaplan, Poland, Simmons-Duffin; Alday, Bissi, Lukowski, Aharony, Perlmutter, ...]

Related results using a Regge-like limit

[Hartman, Jain, Kundu, Afkhami-Jeddi, Tajdini, Hofman, Li, Meltzer, Poland, Rejon-Barrera, Perlmutter, Costa, Hansen, Penedones, ...]

The Euclidean bootstrap program

Expecation:

All bootstrap constraints in principle encoded in the Euclidean regime.

all pairs of operators space-like separated →

Numerical bootstrap

=

a machine to extract optimal constraints from the Euclidean regime

clearly knows about the Lorentzian bootstrap: [Simmons-Duffin, '17]

Natural task: Analytically extract a complete set of optimal constraints from the Euclidean bootstrap equations.

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Formalism unifying numerical bootstrap bounds and analytic Lorentzian bootstrap

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Formalism unifying numerical bootstrap bounds and analytic Lorentzian bootstrap

Closely related to a recent approach using Polyakov's "unitary blocks"

[Polyakov, '74; Gopakumar, Kaviraj, Sen, Sinha, Dey, Ghosh '16,17]

Conformal symmetry in 1D

Spacetime acted on by the symmetry algebra $so(1,2) = sl(2,\mathbb{R})$

Generators $P,\ D,\ K$ (translation, dilatation, special conformal transformation)

Primary operators $\mathcal{O}_j(x)$: $[K, \mathcal{O}_j(0)] = 0\,,\ [D, \mathcal{O}_j(0)] = \Delta_j \mathcal{O}_j(0)$

A theory completely specified by the CFT data $\{\Delta_j\}\ , \{c_{ijk}\}$

structure constants

$T_{11} = 0 \Rightarrow$ no stress tensor $\Rightarrow$ the theory must be non-local

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- conformal boundaries, interfaces, line defects in higher-D CFTs
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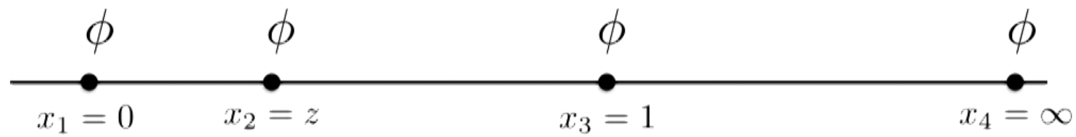
- cSYK model **[Gross, Rosenhaus, '17]**
- non-gravitational (1+1)D QFTs placed in AdS_2
- higher-D CFTs restricted to a line in spacetime

The 1D conformal bootstrap: Part I

Study the four-point function of a primary operator $\phi, \quad a \equiv \Delta_\phi$



Single cross-ratio $z = \dfrac{x_{12}x_{34}}{x_{13}x_{24}} \in (0,1) \qquad x_{ij} = x_i - x_j$

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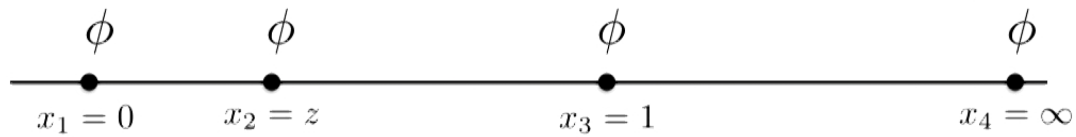

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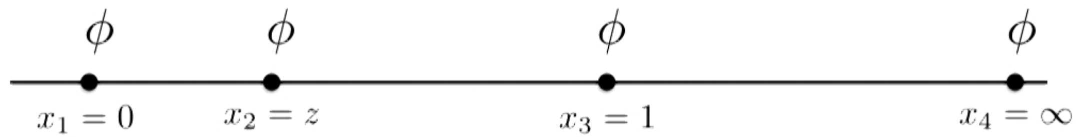

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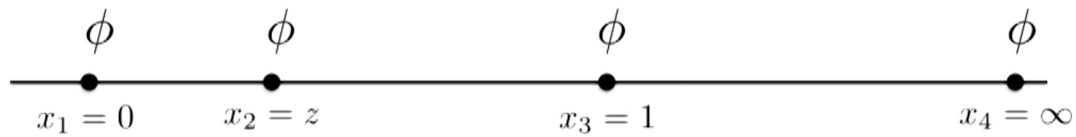

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conformal blocks

$G_\Delta(z) = z^\Delta {}_2F_1(\Delta, \Delta; 2\Delta; z)$

t-channel expansion: $z \to 1 \qquad \mathcal{G}(z) = \left(\dfrac{z}{1-z}\right)^{2a} \sum_{\mathcal{O}\in\phi\times\phi} c^2_{\phi\phi\mathcal{O}}\, G_{\Delta_\mathcal{O}}(1-z)$

The 1D conformal bootstrap: Part II

s- and t-channel expansions must be equal

$$\sum_{\mathcal{O}\in\phi\times\phi} c^2_{\phi\phi\mathcal{O}} \frac{G_{\Delta_\mathcal{O}}(z)}{z^{2a}} = \sum_{\mathcal{O}\in\phi\times\phi} c^2_{\phi\phi\mathcal{O}} \frac{G_{\Delta_\mathcal{O}}(1-z)}{(1-z)^{2a}}$$

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Rewrite this as a sum of vectors with positive coefficients (unitarity)

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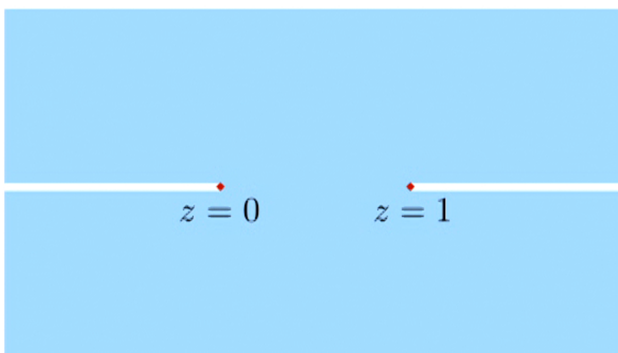
Let z be complex. The equation holds in the common region of convergence of the s- and t-channel OPE



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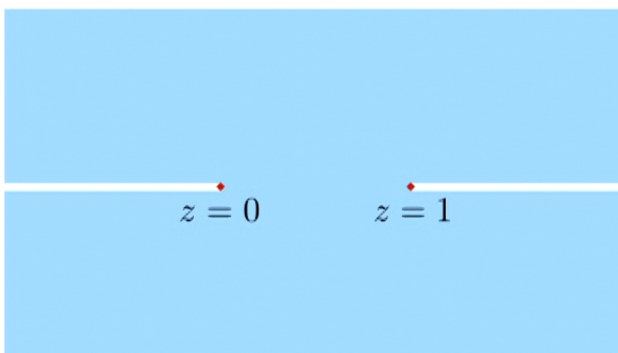

The 1D bootstrap equation lives in the vector space $\mathcal{V}$ of functions $\mathcal{F}(z)$ holomorphic in the blue region, such that

$$\mathcal{F}(z) = -\mathcal{F}(1-z)$$

Behaviour for $z \to 0, 1, \infty$
the same as physical 4-pt function

The simplest solutions of the 1D bootstrap

Free fields in AdS_2

1) Free massive scalar: $m_\phi^2 R_{\mathrm{AdS}}^2 = a(a-1)$

$$\mathcal{G}_{\mathrm{b}}(z) = 1 + \left(\frac{z}{1-z}\right)^{2a} + z^{2a} = 1 + \sum_{n=0}^{\infty} \lambda_n G_{\Delta_n}(z)$$

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$$\Delta_n = 2a + 2n$$

$$\lambda_n = \frac{2\Gamma(2a+2n)^2\Gamma(4a+2n-1)}{\Gamma(2a)^2\Gamma(2n+1)\Gamma(4a+4n-1)}$$

2) Free massive fermion: $\Delta_\psi = a$

$$\mathcal{G}_{\mathrm{f}}(z) = 1 + \left(\frac{z}{1-z}\right)^{2a} - z^{2a} = 1 + \sum_{n=0}^{\infty} \tilde{\lambda}_n G_{\tilde{\Delta}_n}(z)$$

$$\tilde{\Delta}_n = 2a + 2n + 1$$

$$\tilde{\lambda}_n = \frac{2\Gamma(2a+2n+1)^2\Gamma(4a+2n)}{\Gamma(2a)^2\Gamma(2n+2)\Gamma(4a+4n+1)}$$

We will see that in these cases the bootstrap fixes OPE coefficients after we have fixed the spectrum.

The dual space

We get one constraint for every element ω of the dual space $\mathcal{V}^*$.

Define $\omega(\Delta) \equiv \omega(F_\Delta)$. Acting with ω on the bootstrap equation implies

$$\boxed{\sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}\,\omega(\Delta_{\mathcal{O}}) = 0} \qquad \text{where} \quad \lambda_{\mathcal{O}} \equiv c^2_{\phi\phi\mathcal{O}}$$

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But what does "suitable" mean? Simply evaluating the equation at various values of z is not very revealing. Numerical bootstrap instead typically uses odd-order derivatives evaluated at $z = 1/2$, also not too useful for analytics.

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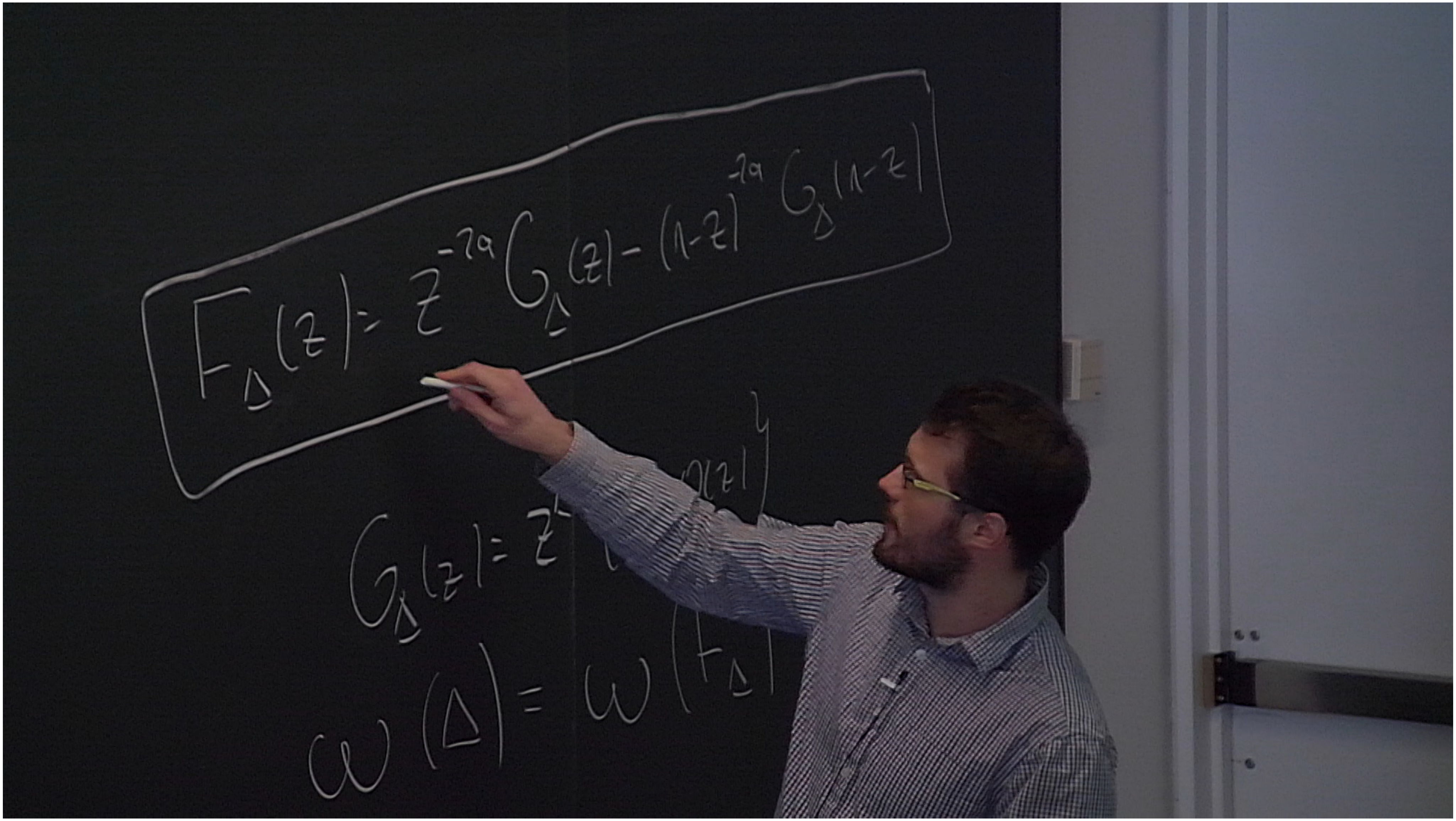
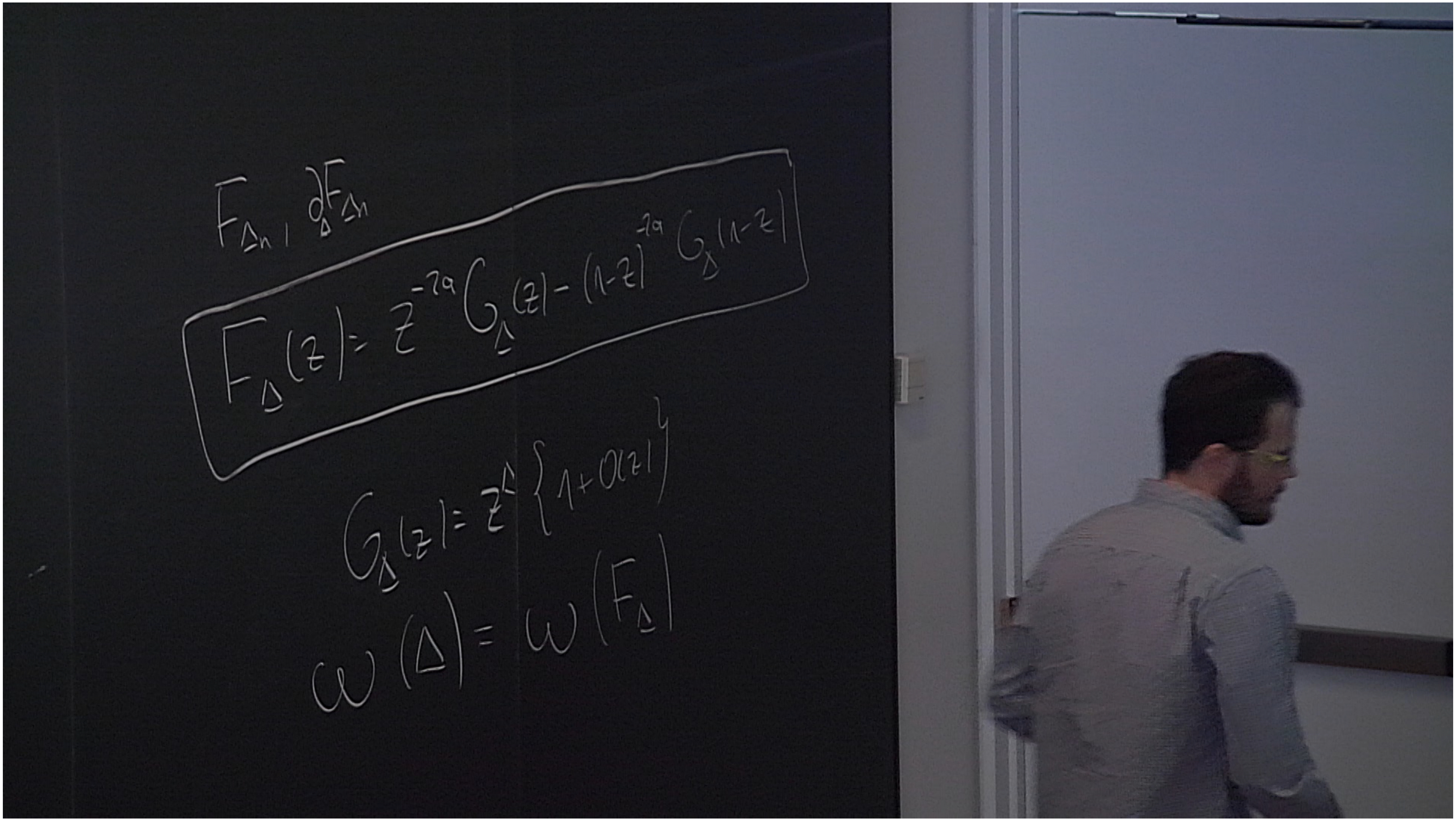
We will construct bases dual to the bases of $\mathcal{V}$ consisting of $F_{\Delta_n}, \partial_\Delta F_{\Delta_n}$ of the free fermion and boson theories.


F_Δ(z) = z^{-2a} G_Δ(z) - (1-z)^{-2a} G_Δ(1-z)
G_Δ(z) = z
ω(Δ) = ω(F_Δ)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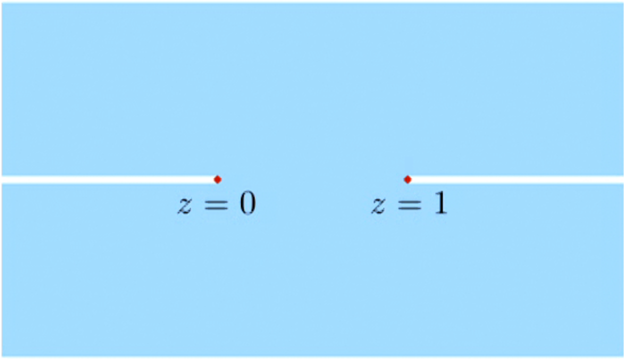
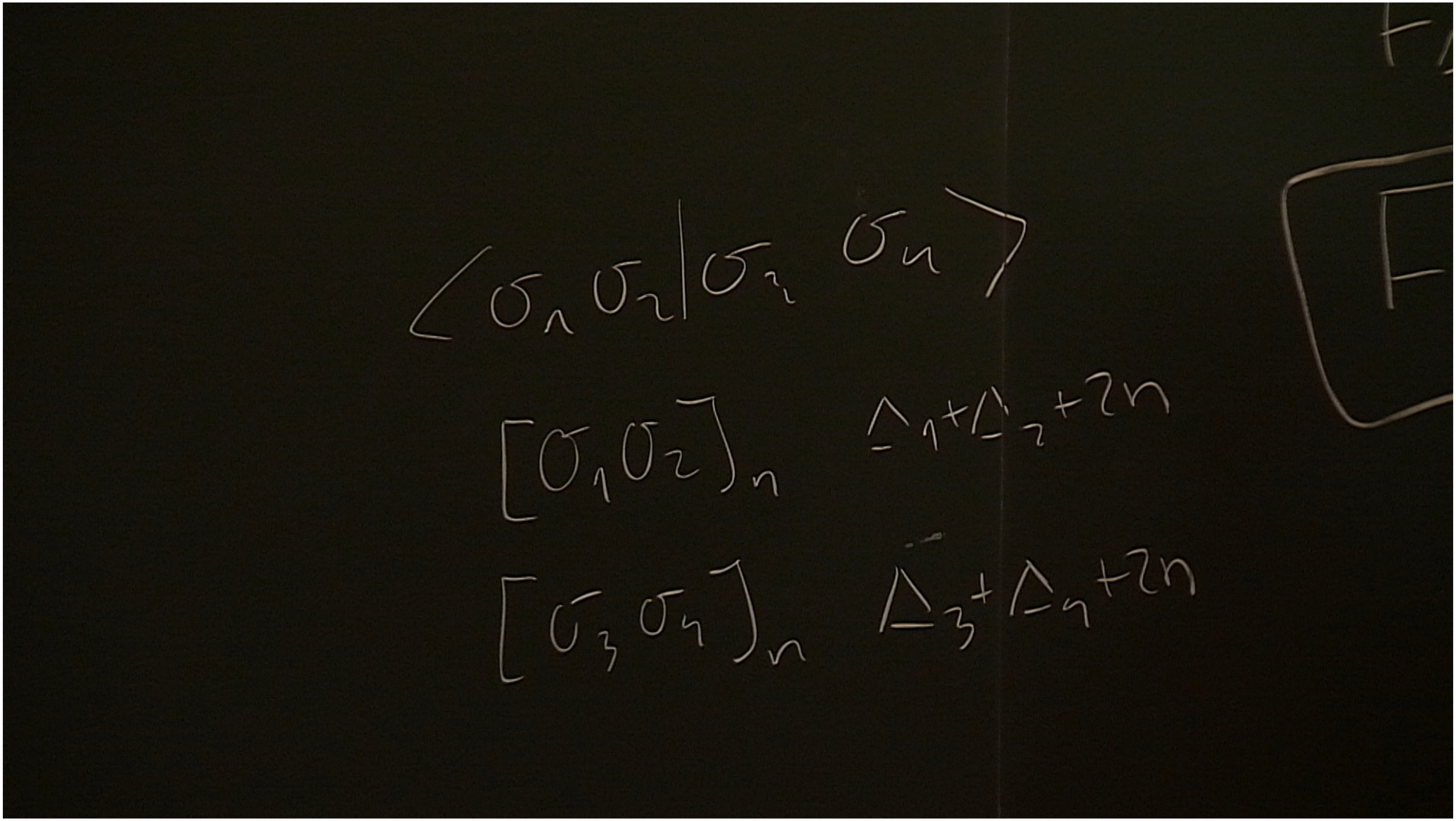
The elements of these bases are extremal functionals. **[el-Showk, Paulos '12]**

A new class of functionals

Consider functionals taking the form of an integral transform




⟨σ1 σ2 | σ3 σ4⟩
[σ1 σ2]n Δ1+Δ2+2n
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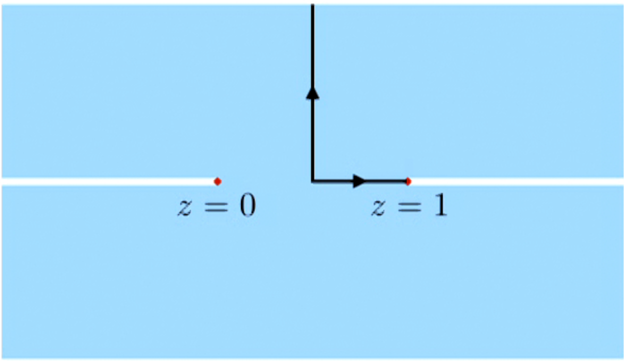
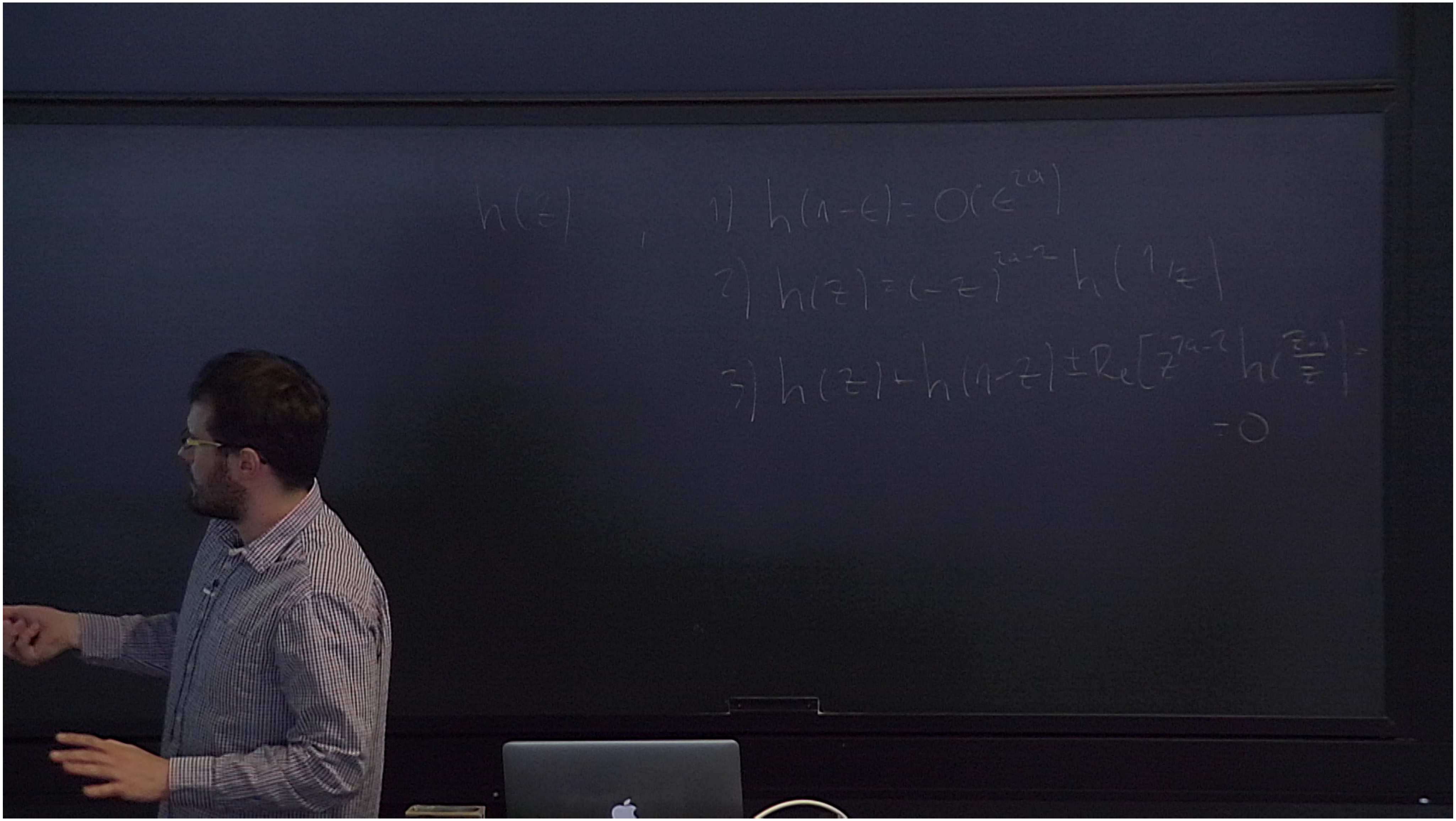
$$\omega(\mathcal{F}) = \int_{\frac{1}{2}}^{1} dz\, h(z)\mathcal{F}(z) \pm$$

boson / fermion

$$\pm \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} dz\, z^{2a-2}\, h(1-1/z)\, \mathcal{F}(z)$$




h(z)
1) h(1-ε) = O(ε^{2a})
2) h(z) = (-z)^{2a-2} h(1/z)
3) h(z) + h(1-z) ± Re[z^{2a-2} h((z-1)/z)] = 0

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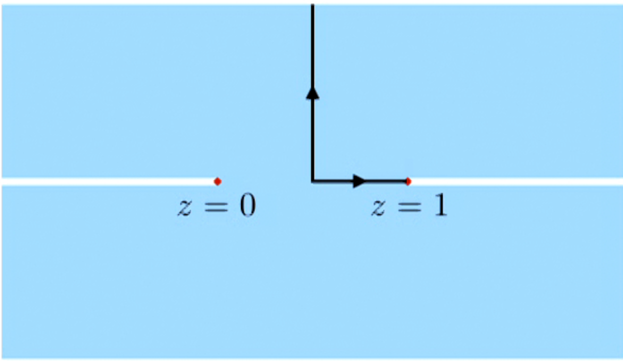

$\omega \in \mathcal{V}^*$ provided $h(1-\epsilon) = O(\epsilon^{2a})$ and $h(z) = (-z)^{2a-2}h(1/z)$

If, in addition $h(z) + h(1-z) \pm \mathrm{Re}[z^{2a-2}h(1-1/z)] = 0$, a contour deformation gives

$$\omega(F_\Delta) = 2\left\{1 \mp \cos\left[\pi(\Delta - 2a)\right]\right\} \int_0^1 dz\, h(z) z^{-2a} G_\Delta(z)$$

Fermionic functionals

$$\omega(\Delta) = 4\cos^2\left[\frac{\pi}{2}(\Delta - 2a)\right]\int_0^1 dz\, h(z) z^{-2a} G_\Delta(z)$$

The solutions of the functional equations classified by the behaviour as $z \to 0$

Two infinite classes of solutions $h_m = O(z^{-2m-2})$ for $m = 0, 1, \ldots$

$\hat{h}_m = O(\log(z) z^{-2m-2})$ for $m = 0, 1, \ldots$

The singularity at $z = 0$ cancels the double pole at $\Delta = 2a + 2m + 1$

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We find the following behaviour at the free fermion spectrum $\Delta_n = 2a + 2n + 1$

$$\omega_m(\Delta_n) = 0 \qquad \omega'_m(\Delta_n) = \delta_{mn}$$

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We find the following behaviour at the free fermion spectrum $\Delta_n = 2a + 2n + 1$

$$\omega_m(\Delta_n) = 0 \qquad \omega'_m(\Delta_n) = \delta_{mn}$$
$$\hat{\omega}_m(\Delta_n) = \delta_{mn} \qquad \hat{\omega}'_m(\Delta_n) = 0$$

It can be shown these functionals form a basis for the dual space.

Fermionic functionals

$$\omega(\Delta) = 4\cos^2\left[\frac{\pi}{2}(\Delta - 2a)\right]\int_0^1 dz\, h(z) z^{-2a} G_\Delta(z)$$

The solutions of the functional equations classified by the behaviour as $z \to 0$

Two infinite classes of solutions $h_m = O(z^{-2m-2})$ for $m = 0, 1, \ldots$

$\hat{h}_m = O(\log(z) z^{-2m-2})$ for $m = 0, 1, \ldots$

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It can be shown these functionals form a basis for the dual space.

The free fermion crossing-symmetric $\Leftrightarrow$ $\hat{\omega}_m(0) = -\tilde{\lambda}_m$

More on the fermionic functionals

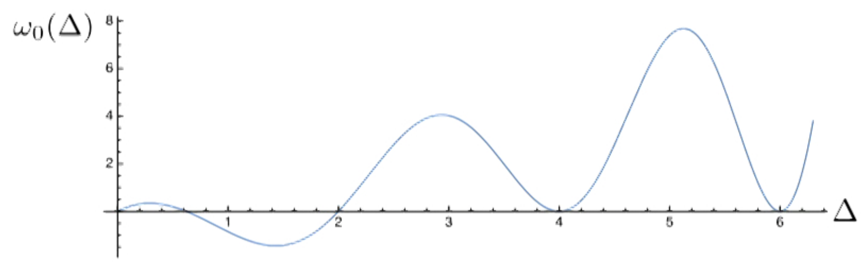
Closed formula for $h_m(z)$ when $a = 1/2$

Legendre polynomials

$$h_m(z) = z^{-1} P_{2m+1}(2/z - 1) - P_{2m+1}(2z - 1)$$

Closed formula for the action of ω_0 for $a = 1/2$

$$\omega_0(\Delta) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} \sin^2\left(\frac{\pi\Delta}{2}\right) \left\{ \frac{1}{(\Delta-2)(\Delta+1)} - \left[\Delta(\Delta-1) + \frac{1}{2}\right] \left[\psi'\left(\frac{\Delta+1}{2}\right) - \psi'\left(\frac{\Delta}{2}\right)\right] - 2 \right\}$$



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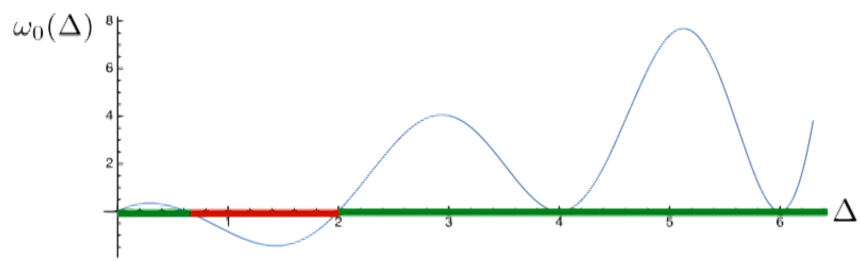

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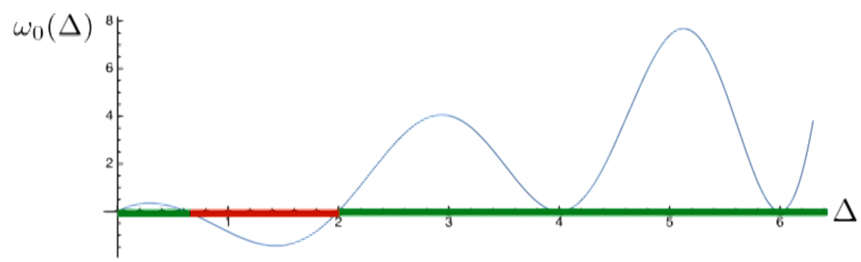

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Dream: are there analytic formulas for the functionals of interacting extremal CFTs?

Deformations of the free fermion solution

A deformation of the free fermion, such that no new operators appear in the OPE:

$$\Delta_n(g) = 2a + 2n + 1 + \gamma_n^{(1)} g + O(g^2)$$

$$\lambda_n(g) = \lambda_n^{(0)} + \lambda_n^{(1)} g + O(g^2)$$

Impose crossing symmetry

$$F_0(z) + \sum_{n=0}^{\infty} \lambda_n(g) F_{\Delta_n(g)}(z) = 0$$

Apply ω_m and expand to the first order in g $\quad\Rightarrow\quad \gamma_m^{(1)} = 0$

Apply $\hat{\omega}_m$ and expand to the first order in g $\quad\Rightarrow\quad \lambda_m^{(1)} = 0$

The free fermion admits no deformations
unless we introduce new states in the OPE!

Bosonic functionals

$$\omega(\Delta) = 4\sin^2\left[\frac{\pi}{2}(\Delta - 2a)\right]\int_0^1 dz\, h(z) z^{-2a} G_\Delta(z)$$

The structure of solutions to the functional equations is similar to the fermionic case except one functional is "missing".

$\Delta_n = 2a + 2n,\ n = 0, 1, \ldots$

Specifically, every functional vanishing on the whole spectrum has at least two simple zeros in the spectrum, rather than one.

The complete basis now consists of ω_m for $m = 1, 2, \ldots$

and $\hat{\omega}_m$ for $m = 0, 1, \ldots$

such that

$\omega_m(\Delta_n) = 0$ for $m \geq 1, n \geq 0$	$\omega'_m(\Delta_n) = \delta_{mn}$ for $m, n \geq 1$
$\hat{\omega}_m(\Delta_n) = \delta_{mn}$ for $m, n \geq 0$	$\hat{\omega}'_m(\Delta_n) = 0$ for $m \geq 0, n \geq 1$

but $\omega'_m(\Delta_0),\ \hat{\omega}'_m(\Delta_0) \neq 0$

Legendre polynomials

For example, for $a = 1$ $\quad h_m(z) = P_{2m+1}(2/z - 1) + P_{2m+1}(2z - 1) - 2\left(\frac{1}{z} + z - 1\right)$

Hence $h_0(z) = 0$ (missing functional)

Deformations of the free boson solution

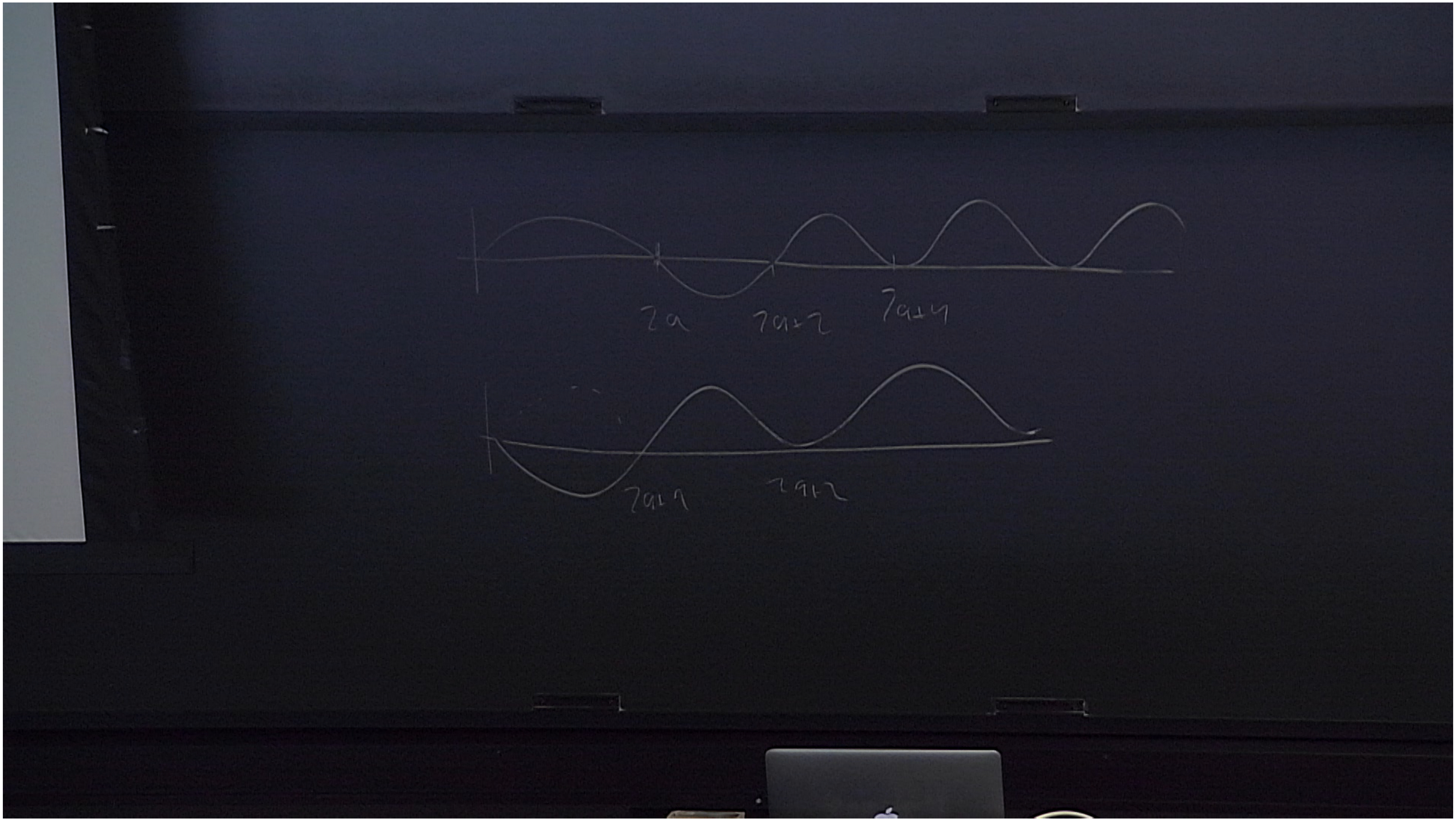
A deformation of the free boson, such that no new operators appear in the OPE:

$$\Delta_n(g) = 2a + 2n + \gamma_n^{(1)} g + \gamma_n^{(2)} g^2 + O(g^3)$$

$$\lambda_n(g) = \lambda_n^{(0)} + \lambda_n^{(1)} g + \lambda_n^{(2)} g^2 + O(g^3)$$

One functional missing $\Leftrightarrow$ A one-parameter deformation allowed

Define the coupling as the anomalous dimension of the first operator $g = \Delta_0(g) - 2a$


2a
2a+2
2a+4
2a+1
2a+2

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Tree-level: Apply $\omega_n, \hat{\omega}_n$ and expand to the first order in g

$$\gamma_n^{(1)} = -\frac{\lambda^{(0)}}{\lambda_n^{(0)}} \omega_n'(2a) \qquad \lambda_n^{(1)} = -\lambda_0^{(0)} \hat{\omega}_n'(2a)$$

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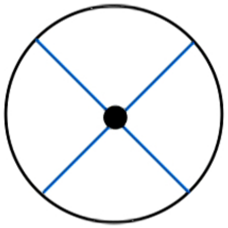
$$\gamma_n^{(1)} = \frac{(2n)!(a)_n^4 (4a-2)_{2n}}{(n!)^2 (2a)_n (2a-1)_n (2a)_{2n}^2} \qquad \lambda_n^{(1)} = \frac{\partial}{2\partial n}\left(\lambda_n^{(0)} \gamma_n^{(1)}\right)$$

We recover the only UV-complete leading-order deformation of the 4pt function of a free massive scalar in AdS, namely the tree-level ϕ^4 interaction.



d>1: [Heemskerk, Penedones, Polchinski, Sully, '09]

An integrable theory in AdS_2?

Having determined the CFT data at $O(g^k)$, apply ω_n, $\hat{\omega}_n$ to find $\lambda_n^{(k+1)}$, $\gamma_n^{(k+1)}$

$$a = 1$$

$$\Delta_0 = 2 + g$$

$$\Delta_1 = 4 + \frac{1}{6}g + \left(\frac{317}{144} - \frac{5}{3}\zeta(3)\right) g^2 + O(g^3)$$

$$\Delta_2 = 6 + \frac{1}{15}g + \left(\frac{25127}{10800} - \frac{28}{15}\zeta(3)\right) g^2 + O(g^3)$$

$$\lambda_0 = 2 - 2g + \left(\frac{5}{2} - 4\zeta(3) + \frac{\pi^4}{15}\right) g^2 + O(g^3)$$

$$\lambda_1 = \frac{6}{5} - \frac{37}{150}g + \left(-\frac{612119}{108000} + \frac{19\zeta(3)}{15} + \frac{\pi^4}{25}\right) g^2 + O(g^3)$$

$$\lambda_2 = \frac{5}{21} - \frac{1627}{79380}g + \left(-\frac{3889170127}{3000564000} + \frac{1177\zeta(3)}{2835} + \frac{\pi^4}{126}\right) g^2 + O(g^3)$$

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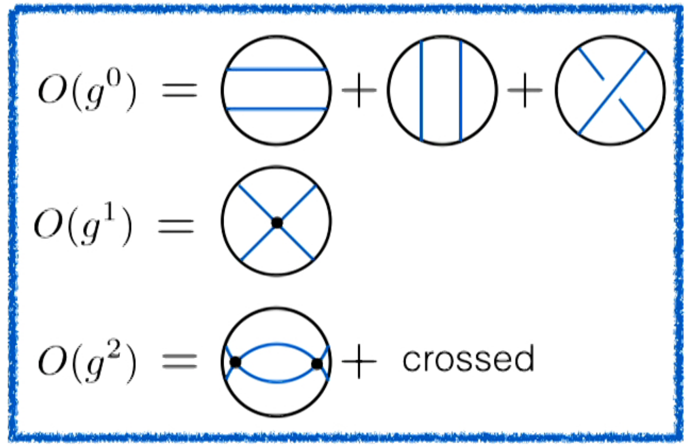

reproduce 1-loop results in AdS_2

1-loop in d>1: [Aharony, Alday, Bissi, Perlmutter, '16]

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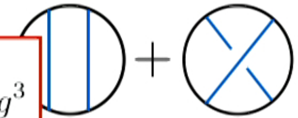
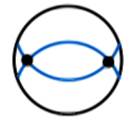


two-loop bootstrap result

$O(g^2)$ =  + crossed

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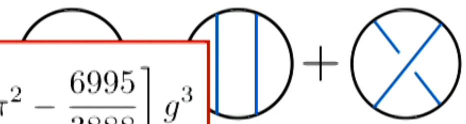

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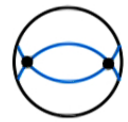
$O(g^2) =$  $+$ crossed

reproduce 1-loop results in AdS$_2$

1-loop in d>1: [Aharony, Alday, Bissi, Perlmutter, '16]

We are getting an interacting theory with "no particle production" in AdS.

A natural guess is that it is the sinh-Gordon theory in AdS background.

An explicit calculation of particle production in sG in AdS refutes this conjecture.

Bootstrapping Witten exchange diagrams 1

Introduce into the OPE a new "single-trace" operator of dimension Δ and OPE coefficient $\sqrt{g},\ g \ll 1$

Crossing symmetry implies the double-trace operators must acquire anomalous dimensions and OPE coefficients.

$$\mathcal{G}^{(1)}(z) = G_\Delta(z) + \sum_{n=0}^{\infty} \left[\lambda_n^{(1)}(\Delta) G_{2a+2n}(z) + \lambda_n^{(0)} \gamma_n^{(1)}(\Delta) \partial G_{2a+2n}(z) \right]$$

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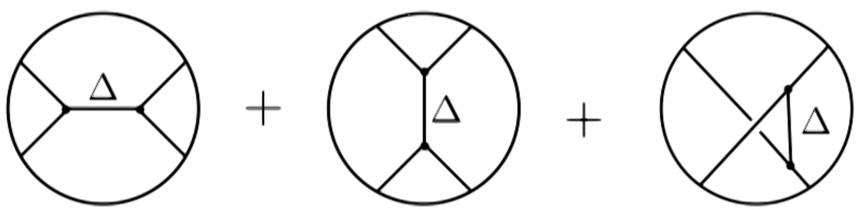
Expectation from the bulk: $\lambda_n^{(1)}(\Delta)$, $\gamma_n^{(1)}(\Delta)$ fixed to be those arising from the crossing-symmetric combination of exchange Witten diagrams (up to contact diagrams)

$\mathcal{G}^{(1)}(z) =$ 

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Bootstrapping Witten exchange diagrams 2

Crossing:

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Apply $\omega_n, \hat{\omega}_n$:

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up to the contact interaction

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The recent approach to the conformal bootstrap using Polyakov's crossing-symmetric blocks is equivalent to expressing crossing equations in the basis of $\omega_n, \hat{\omega}_n$ functionals. Clear hint for a d>1 generalization.

Conclusions and Outlook

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The bases can be used to see how a conformal block in the crossed channel influences the direct-channel CFT data, and compute Feynman diagrams in AdS_2.

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$$d > 1$$

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Thank you!