Title: Symplectic resolutions of quiver varieties

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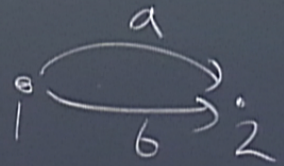
Abstract: <p>Quiver varieties, as introduced by Nakaijma, play a key role in representation theory. They give a very large class of symplectic singularities and, in many cases, their symplectic resolutions too. However, there seems to be no general criterion in the literature for when a quiver variety admits a symplectic resolution. In this talk, I will give necessary and sufficient conditions for a quiver variety to admit a symplectic resolution. This result builds upon work of Crawleyâ€“Boevey and of Kaledin, Lehn and Sorger. The talk is based on joint work with T. Schedler.</p>

Symplectic resolutions of quiver varieties.

(jt w. Schedler).

Quivers $Q = (\underset{\substack{\uparrow \\ \text{vertices}}}{Q_0}, \underset{\substack{\uparrow \\ \text{arrows}}}{Q_1})$ e.g.



$t(a)=1$ $h(a)=2$ etc

We want a root system $R \subseteq \mathbb{Z}^{Q_0}$.

Ringel form is $\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}$

$\rightsquigarrow (\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$

For each loop free vertex i we get

$$s_i : \alpha \longmapsto \alpha - (\alpha, e_i) e_i .$$

$$\rightsquigarrow \quad W = \langle s_i \mid i \text{ loop free} \rangle \curvearrowright \mathbb{Z}^{Q_0}.$$

$$R_{re} = \{ w(e_i) \mid w \in W ,\ i \text{ is loop free} \}.$$

$R_{im} = \{ w(\alpha) \mid \alpha \in F, w \in W \}$ where $F = \{ \alpha \in \mathbb{Z}^{Q_0}_{\geq 0} \mid (\alpha, e_i) \leq 0 \ \forall i$ and $\operatorname{Supp}(\alpha)$ connected $\}$.

$$p(\alpha) = 1 - \frac{1}{2}(\alpha, \alpha)$$

<u>Key fact</u> if $\alpha \in R$ then

$p(\alpha) = 0 \iff \alpha \in R_{re}$

$p(\alpha) = 1$ if α is "isotropic imaginary"

if $p(\alpha) > 1$ then α is "non isotropic imaginary".

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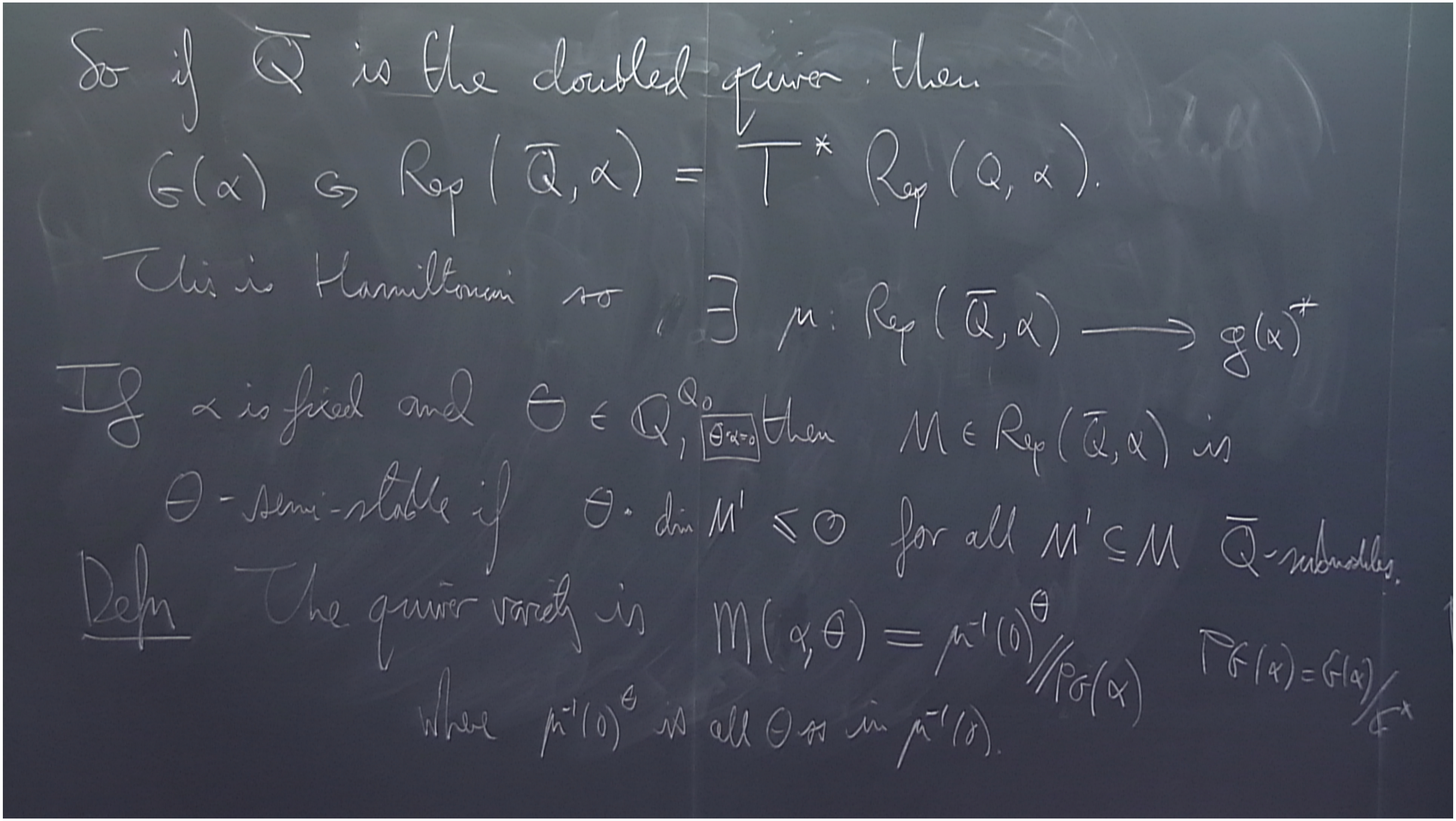
and $\}$.

$$\mathrm{Rep}(Q, \alpha) = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{t(a)}, \mathbb{C}^{h(a)})$$ space of rep^n.

$$G(\alpha) = \prod_{i \in Q_0} GL(\mathbb{C}^{\alpha_i})$$ acts here.



So if $\overline{Q}$ is the doubled quiver, then

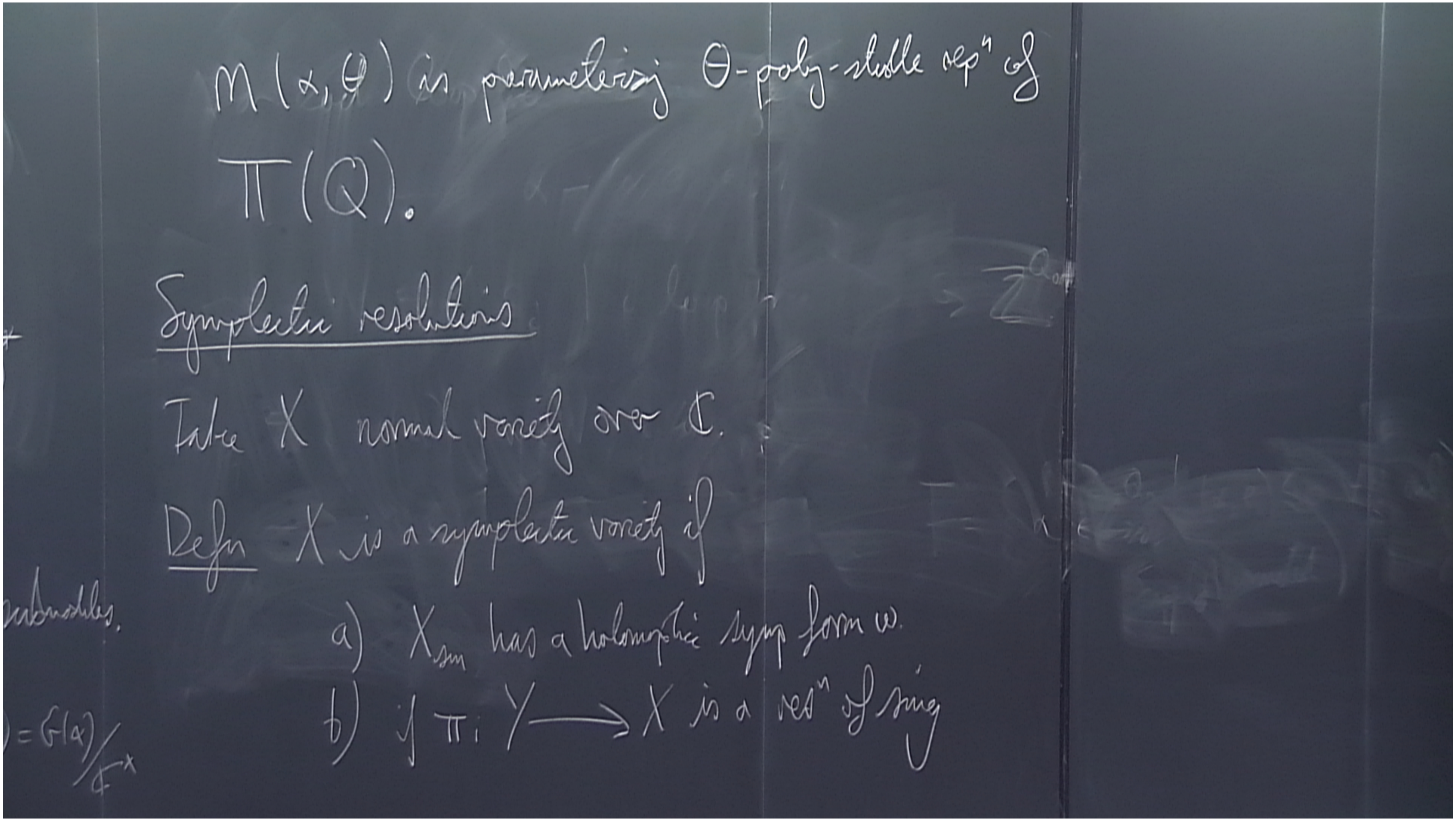
$$G(\alpha) \curvearrowright \mathrm{Rep}(\overline{Q}, \alpha) = T^* \mathrm{Rep}(Q, \alpha).$$

This is Hamiltonian so $\exists\ \mu : \mathrm{Rep}(\overline{Q}, \alpha) \longrightarrow \mathfrak{g}(\alpha)^*$

If α is fixed and $\theta \in \mathbb{Q}^{Q_0}$, $\boxed{\theta \cdot \alpha = 0}$ then $M \in \mathrm{Rep}(\overline{Q}, \alpha)$ is θ-semi-stable if $\theta \cdot \dim M' \leq 0$ for all $M' \subseteq M$ $\overline{Q}$-submodules.

<u>Defn</u> The quiver variety is $\mathcal{M}(\alpha, \theta) = \mu^{-1}(0)^{\theta} /\!/ PG(\alpha)$ $\quad PG(\alpha) = G(\alpha)/\mathbb{C}^*$

where $\mu^{-1}(0)^{\theta}$ is all θ ss in $\mu^{-1}(0)$.



$M(\alpha, \theta)$ is parameterising θ-poly-stable repⁿ of $\Pi(Q)$.

Symplectic resolutions

Take X normal variety over $\mathbb{C}$.

Defn X is a symplectic variety if

a) X_{sm} has a holomorphic symp form ω.

b) if $\pi: Y \longrightarrow X$ is a resⁿ of sing

ubundles.

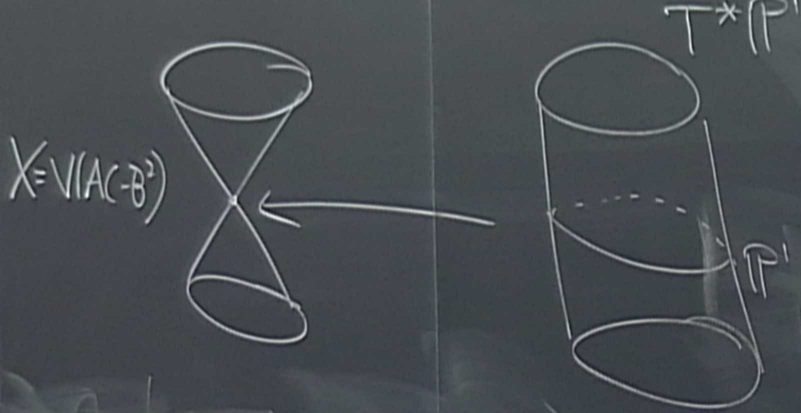
$) = G(\alpha)/\mathbb{C}^*$

stable rep^n of

orm ω.

^{n} of sing

then $\pi^*\omega$ is a regular 2-form on Y



<u>Refn</u> $\pi : Y \longrightarrow X$ is a symp res if $\pi^*\omega$ is a non-deg (hence symp) 2-form on Y.

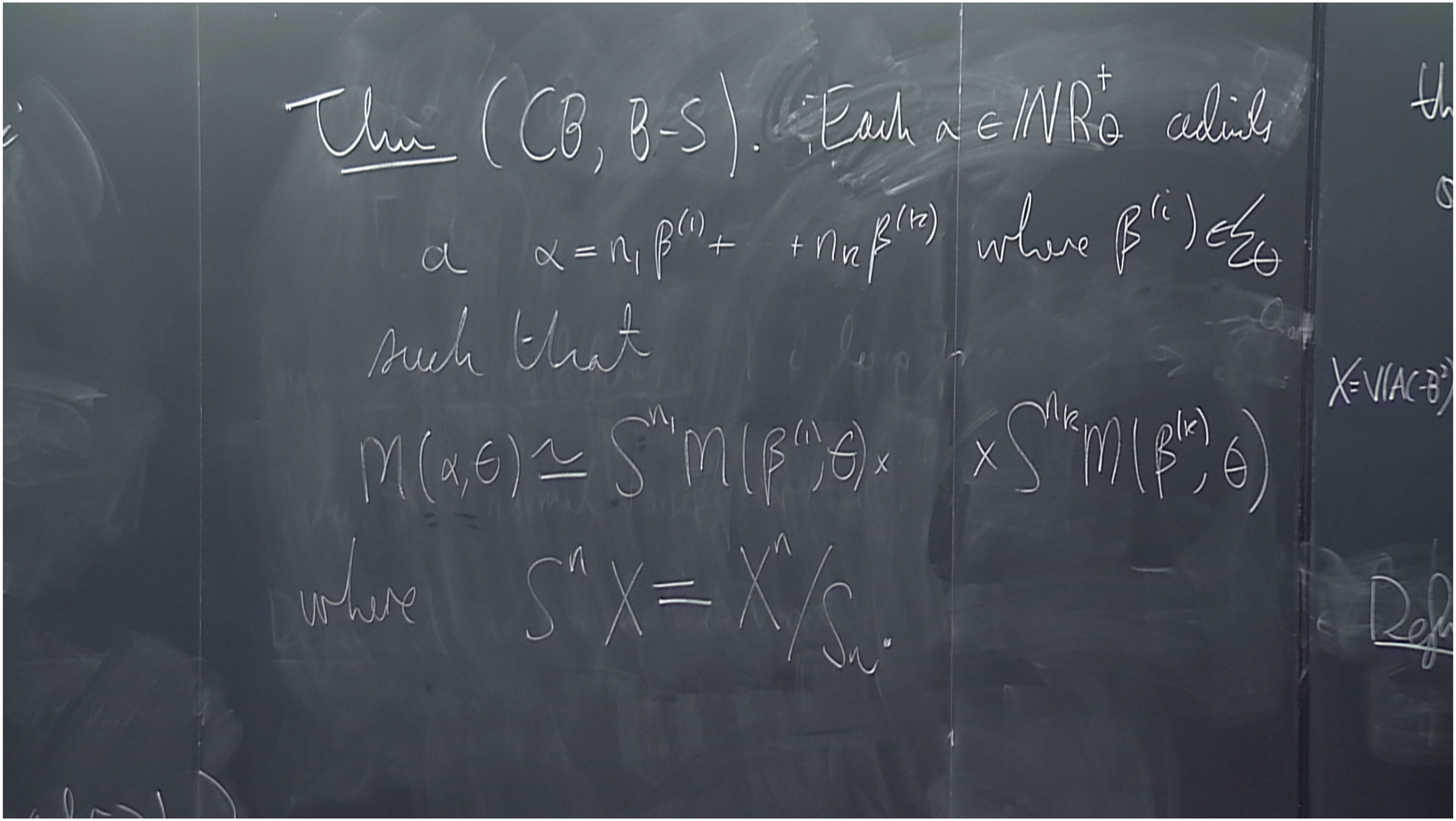
__Prop__ If $\alpha \in \mathbb{N}R^+_\Theta$ then $M(\alpha,\theta)$ is a symplectic variety.
$\neq \emptyset$.

$(R^+_\theta = \{\alpha \in R^+ \mid \theta(\alpha) = 0\}$.

__Q.__ When does $M(\alpha,\epsilon)$ admit a res^n?

__Decomposition__

$$\Sigma_\Theta = \left\{ \alpha \in R^+_\theta \;\middle|\; \begin{array}{l} \text{if } \alpha = \beta^{(1)} + \ + \beta^{(r)} \text{ where } \beta^{(i)} \in R^+_\theta \text{ and } r > 1 \\ \text{then } p(\alpha) > p(\beta^{(1)}) + \cdots + p(\beta^{(r)}) \end{array} \right\}.$$



Thm (CB, B-S). Each $\alpha \in NR^+_\Theta$ admits

$a \quad \alpha = n_1\beta^{(1)} + \quad + n_k\beta^{(k)}$ where $\beta^{(i)} \in \Sigma_\Theta$

such that

$$M(\alpha,\theta) \simeq S^{n_1}M(\beta^{(1)};\theta) \times \quad \times S^{n_k}M(\beta^{(k)};\theta)$$

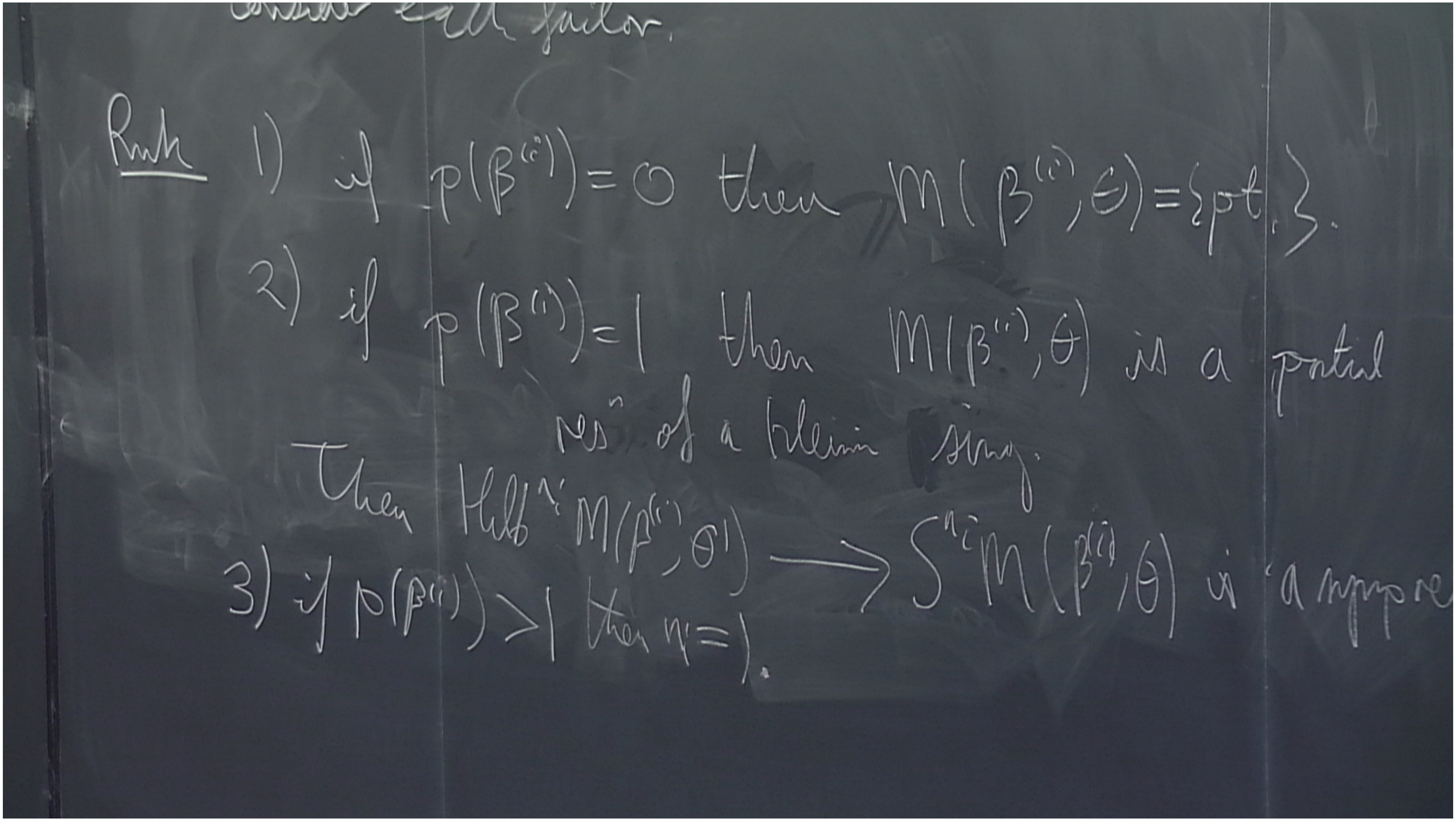
where $S^nX = X^n/S_n$.

$X = V(AC-B^2)$

Turns out that its enough to
consider each factor.

Rmk 1) if $p(\beta^{(i)}) = 0$ then $M(\beta^{(i)}, \theta) = \{pt.\}$.

2) if $p(\beta^{(i)}) = 1$ then $M(\beta^{(i)}, \theta)$ is a partial res^{n} of a Kleinian sing.



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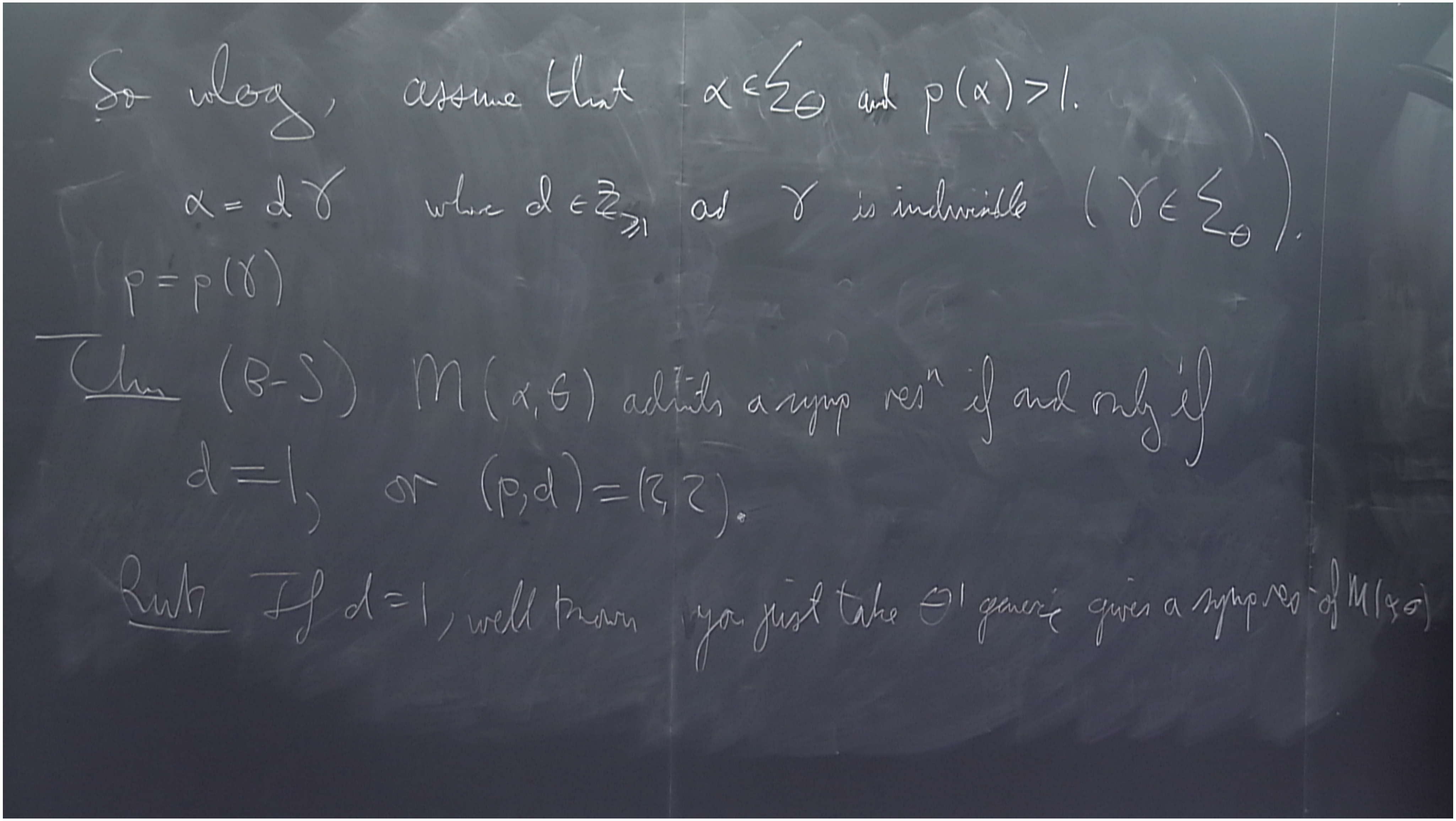
2) if $p(\beta^{(i)})=1$ then $M(\beta^{(i)},\theta)$ is a partial res^n of a Klein sing.

Then $\mathrm{Hilb}^{n_i} M(\beta^{(i)},\theta) \longrightarrow S^{n_i} M(\beta^{(i)},\theta)$ is a symp re

3) if $p(\beta^{(i)})>1$ then $n_i=1$.



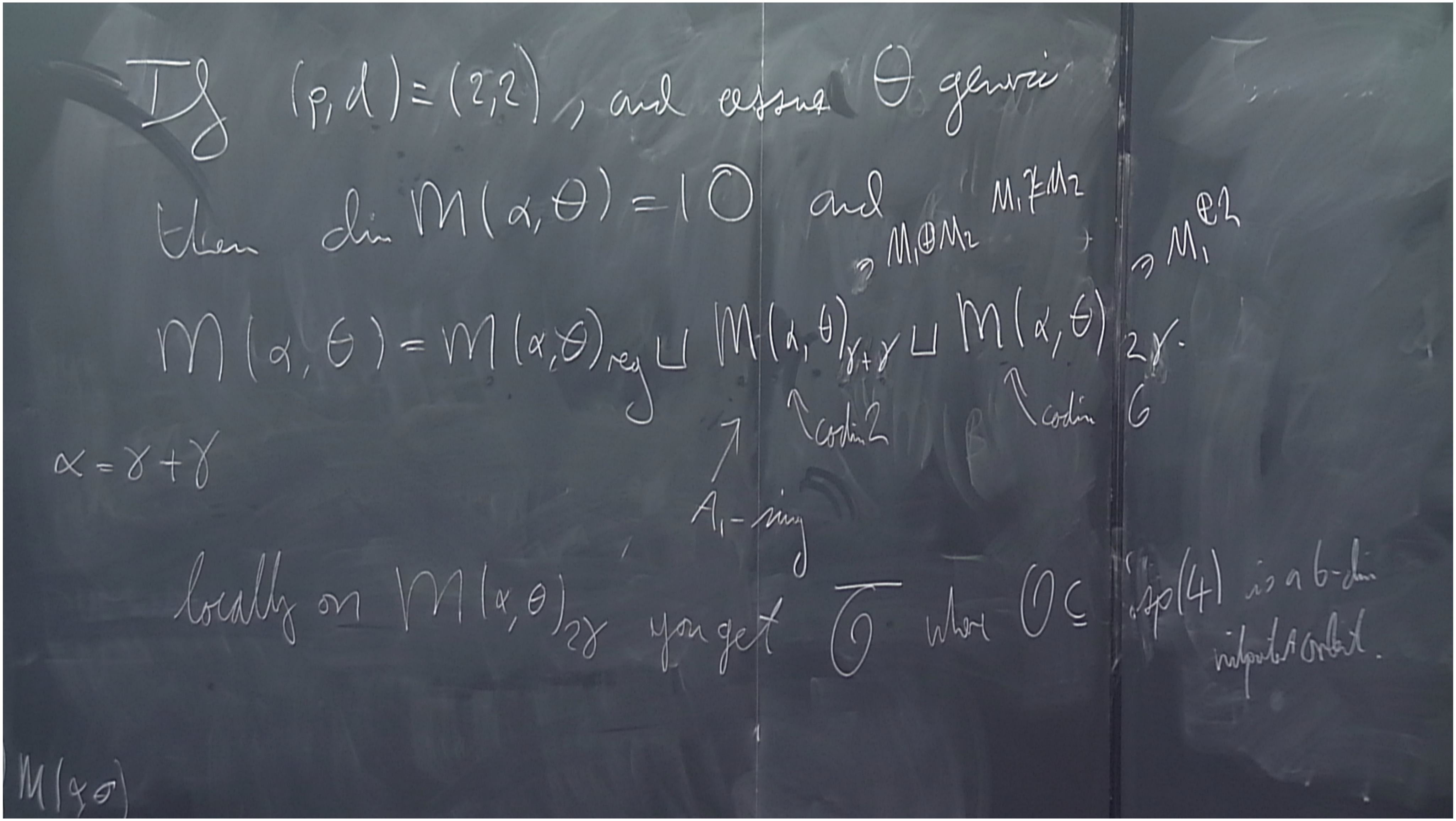
So wlog, assume that $\alpha \in \Sigma_\theta$ and $p(\alpha) > 1$.

$\alpha = d\gamma$ where $d \in \mathbb{Z}_{\geq 1}$ and γ is indivisible $(\gamma \in \Sigma_\theta)$.

$p = p(\gamma)$

<u>Thm</u> (B-S) $M(\alpha, \theta)$ admits a symp res^n if and only if $d = 1$, or $(p, d) = (2, 2)$.

<u>Rmk</u> If $d = 1$, well known you just take θ' generic gives a symp res^n of $M(\alpha, \theta)$



If $(p,d)=(2,2)$, and assume θ generic

then $\dim \mathcal{M}(\alpha,\theta) = 10$ and

$\ni M_1 \oplus M_2$ $\quad M_1 \not\cong M_2$ $\quad \ni M_1^{\oplus 2}$

$$\mathcal{M}(\alpha,\theta) = \mathcal{M}(\alpha,\theta)_{reg} \sqcup \mathcal{M}(\alpha,\theta)_{\gamma+\gamma} \sqcup \mathcal{M}(\alpha,\theta)_{2\gamma}.$$

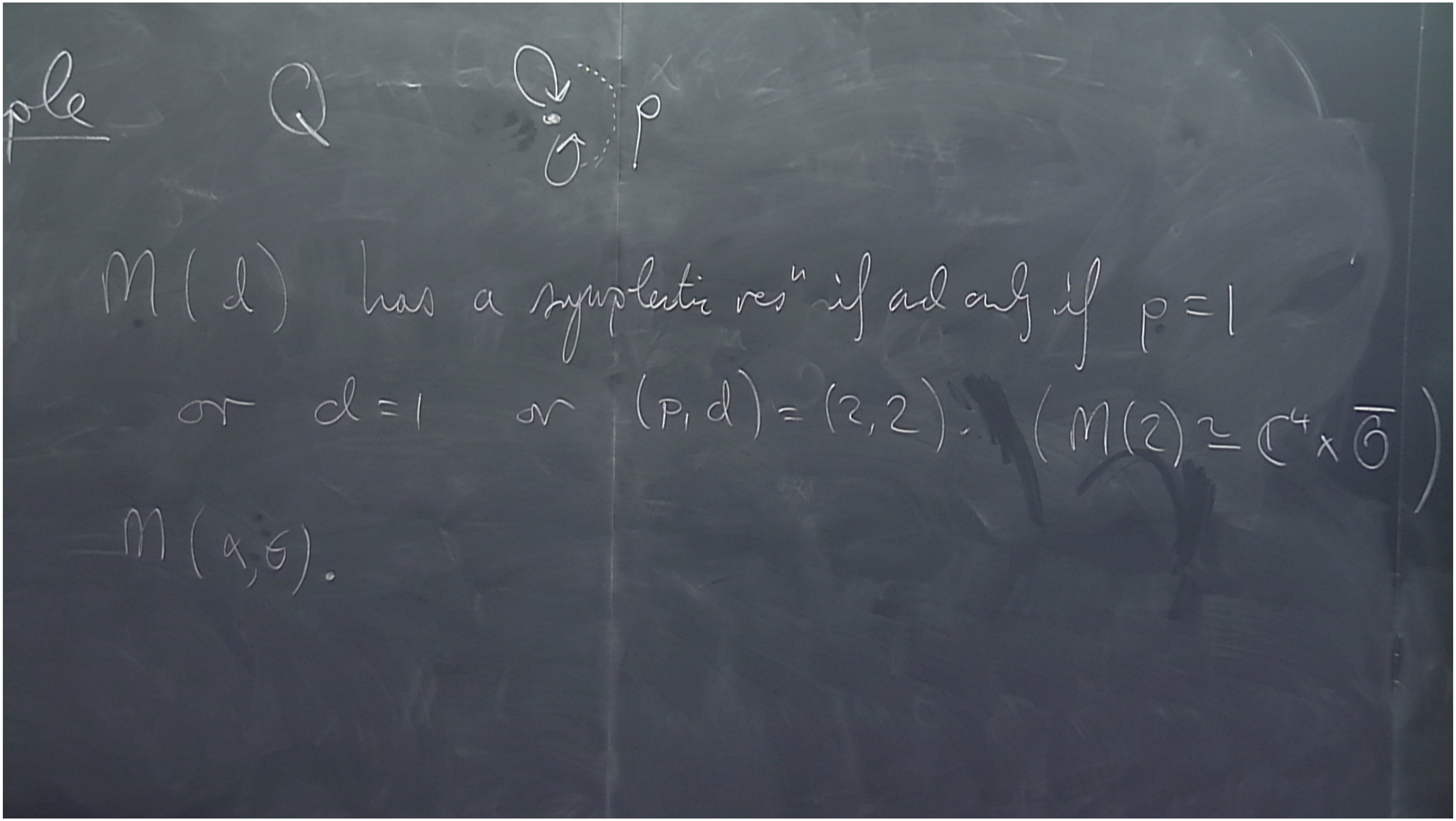
codim 2 $\qquad$ codim 6

A_1 - sing

$\alpha = \gamma + \gamma$

locally on $\mathcal{M}(\alpha,\theta)_{2\gamma}$ you get $\overline{\mathcal{O}}$ where $\mathcal{O} \subseteq \mathfrak{sp}(4)$ is a 6-dim nilpotent orbit.

$M(\gamma,\theta)$



$M(d)$ has a symplectic res^n if and only if $p=1$

or $d=1$ or $(p,d)=(2,2)$. $\left(M(2) \simeq \mathbb{C}^4 \times \overline{\mathcal{O}} \right)$

$M(4,6)$.