Title: PSI 17/18 - Foundations of Quantum Mechanics - Lecture 13

Date: Feb 14, 2018 10:15 AM

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Abstract:

# Final comments on Bell's theorem and Contextuality

Bell inequality violations as a resource

Magic is a natural force that can be used to override the usual laws of nature.
-- Harry Potter entry in wikipedia

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-- Harry Potter entry in wikipedia

Bell–inequality violations are natural phenomena that can be used to override the usual (classical-like) laws of nature

# Quantum Spellcraft

*Based on Bell-inequality violation*

Reduction in communication complexity

Buhrman, Cleve, van Dam, SIAM J.Comput. 30 1829 (2001)

Brassard, Found. Phys. 33, 1593 (2003)

Device-independent secure key distribution

Barrett, Hardy, Kent, PRL 95, 010503 (2005)

Acin, Gisin, Masanes, PRL. 97, 120405 (2006)

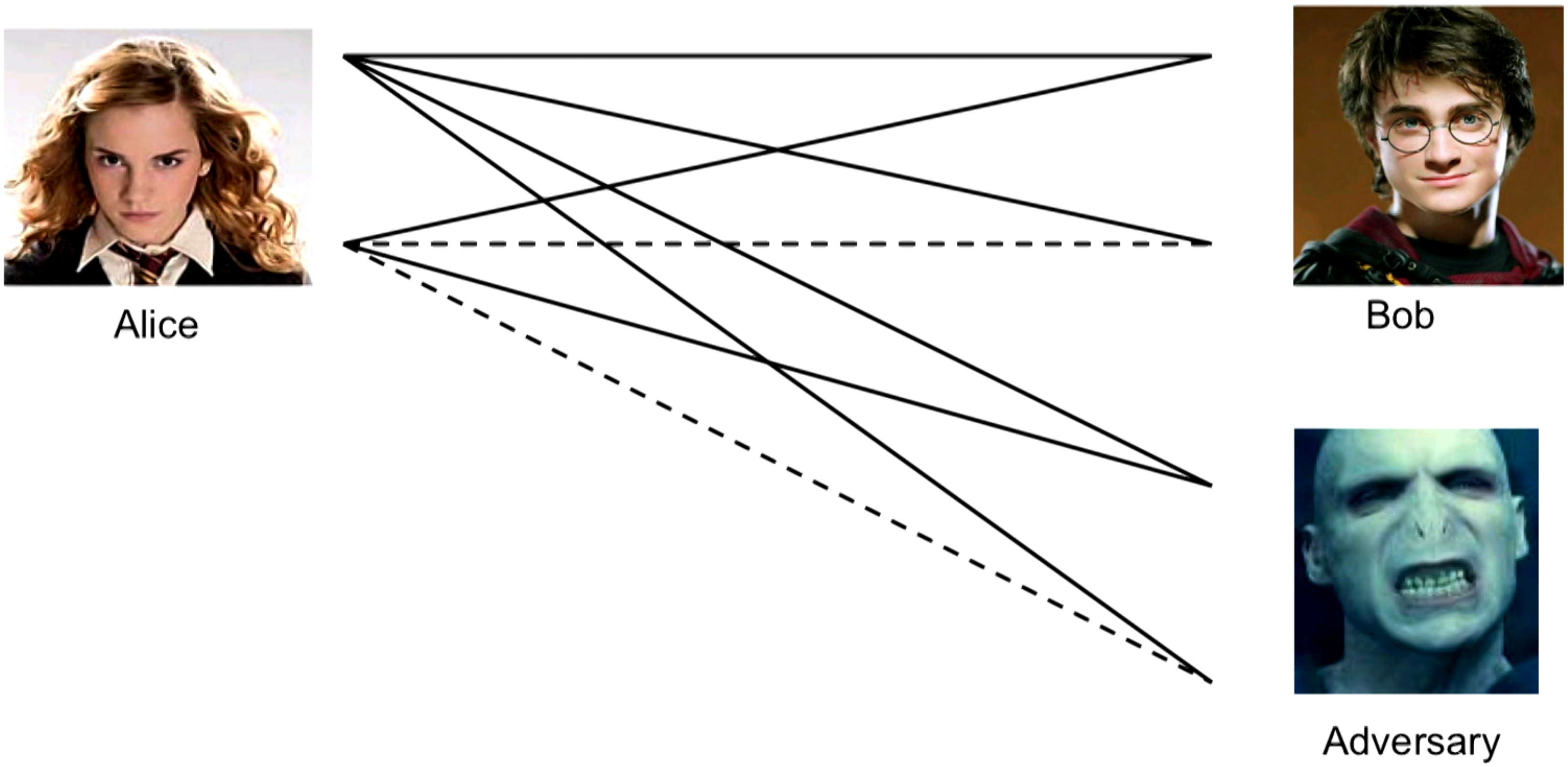
Randomness expansion

Colbeck, Kent, J. Phys. A, 44, 095305 (2011).

Enhancing zero-error channel capacity

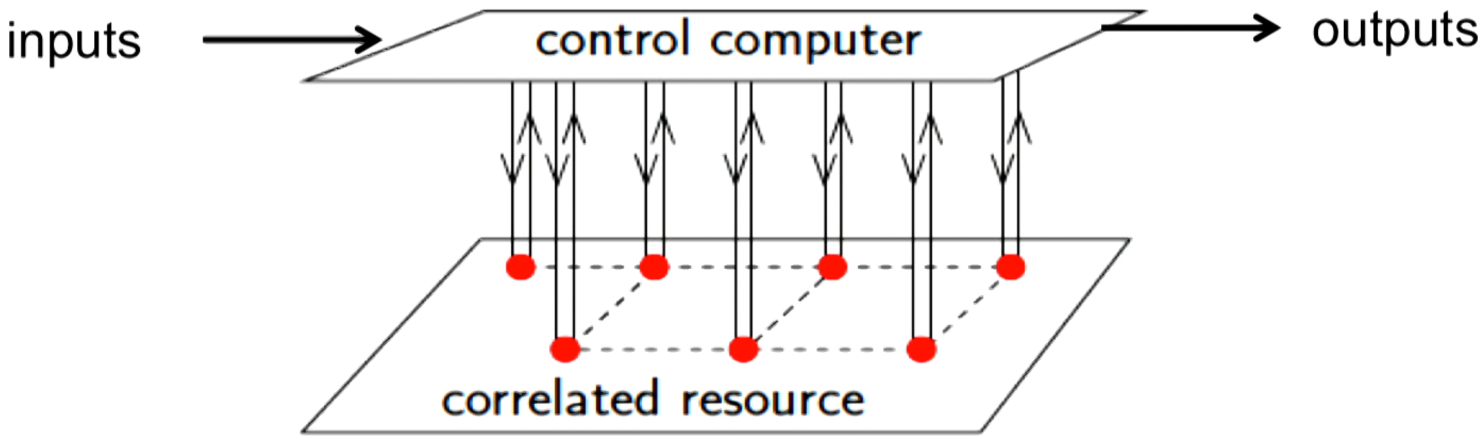
Cubitt, Leung, Matthews, Winter, arXiv:0911.5300

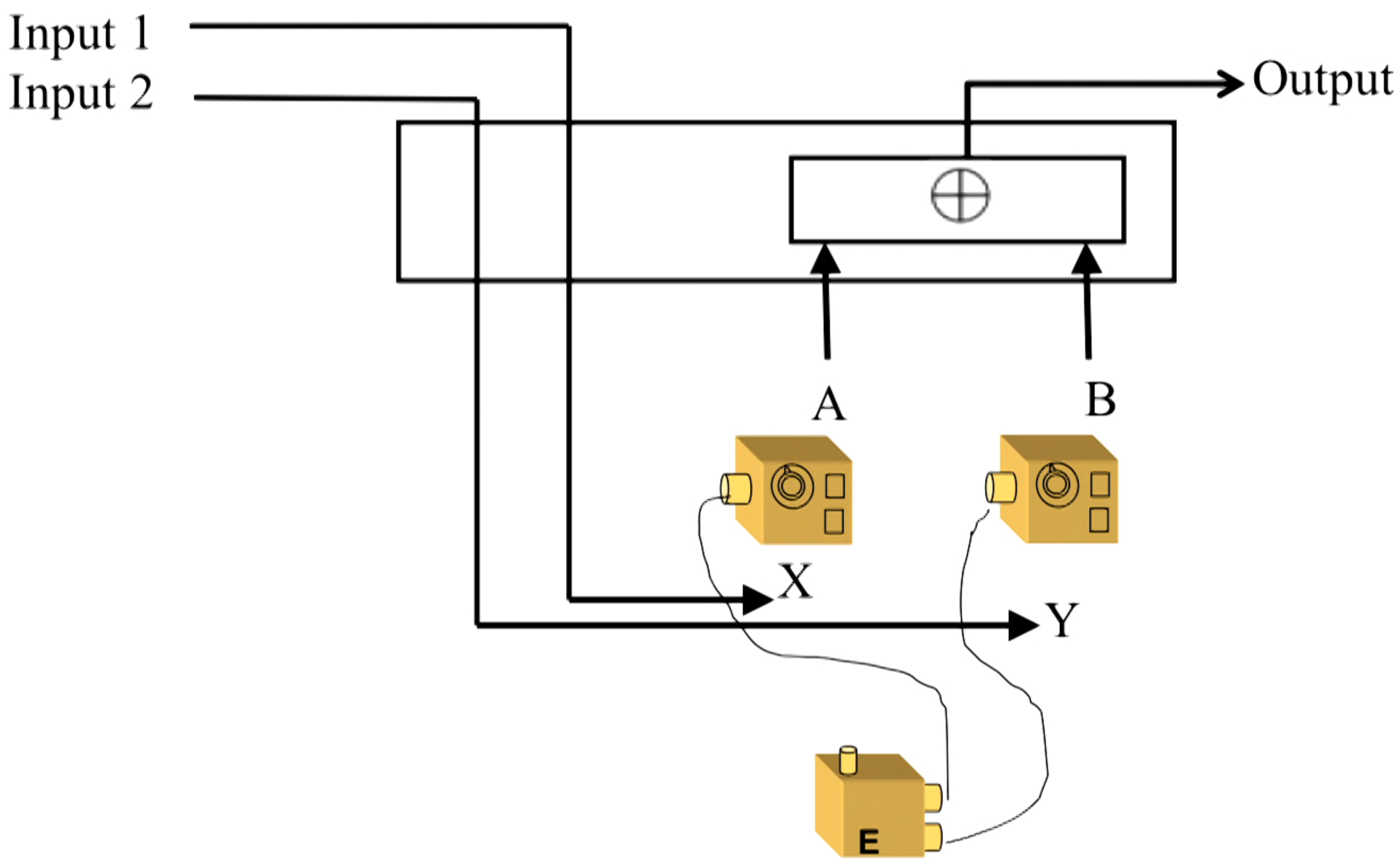
# Monogamy of Bell-inequality violating correlations

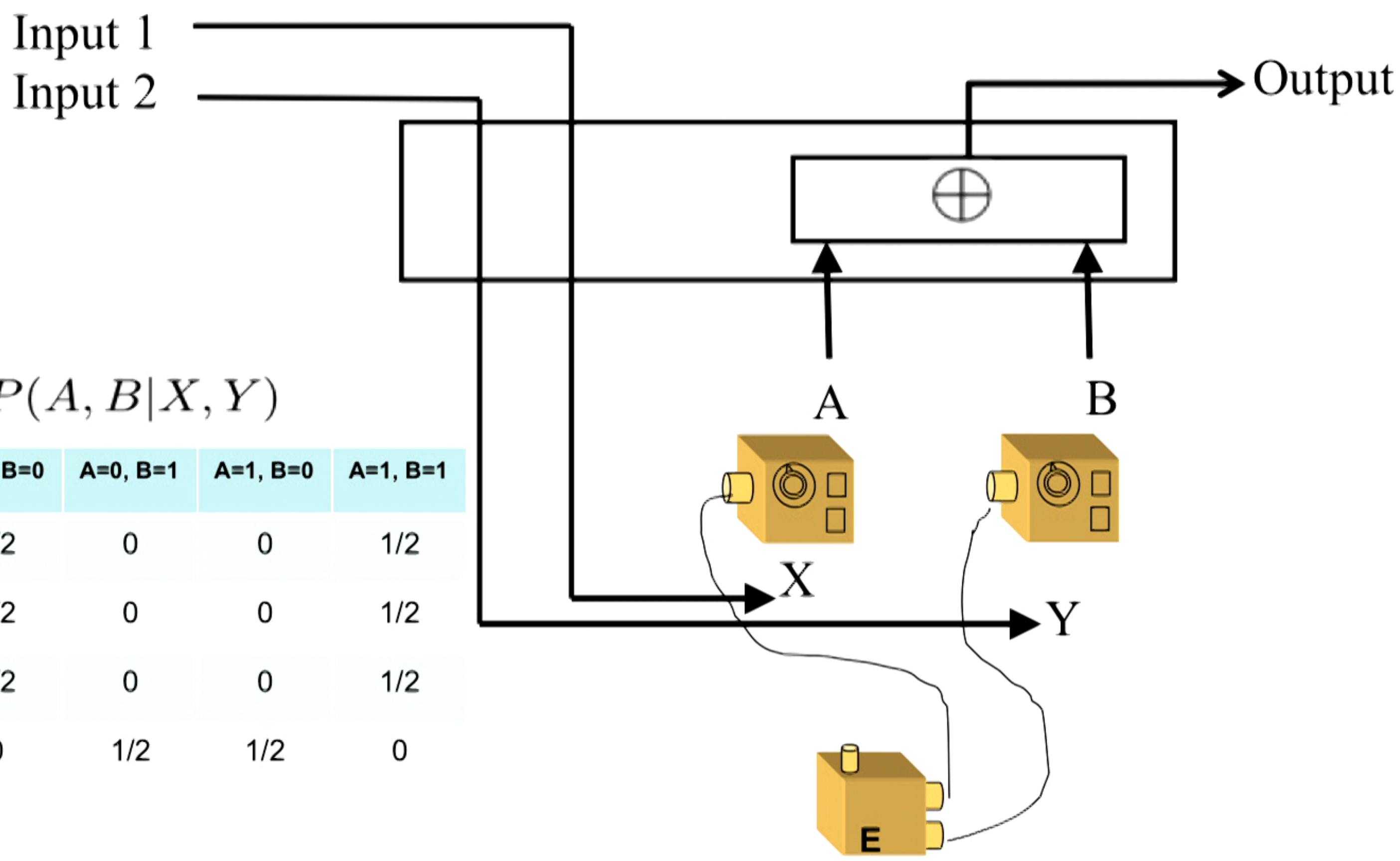


# Quantum advantages for computation

Anders and Browne. PRL 102, 050502 (2009).

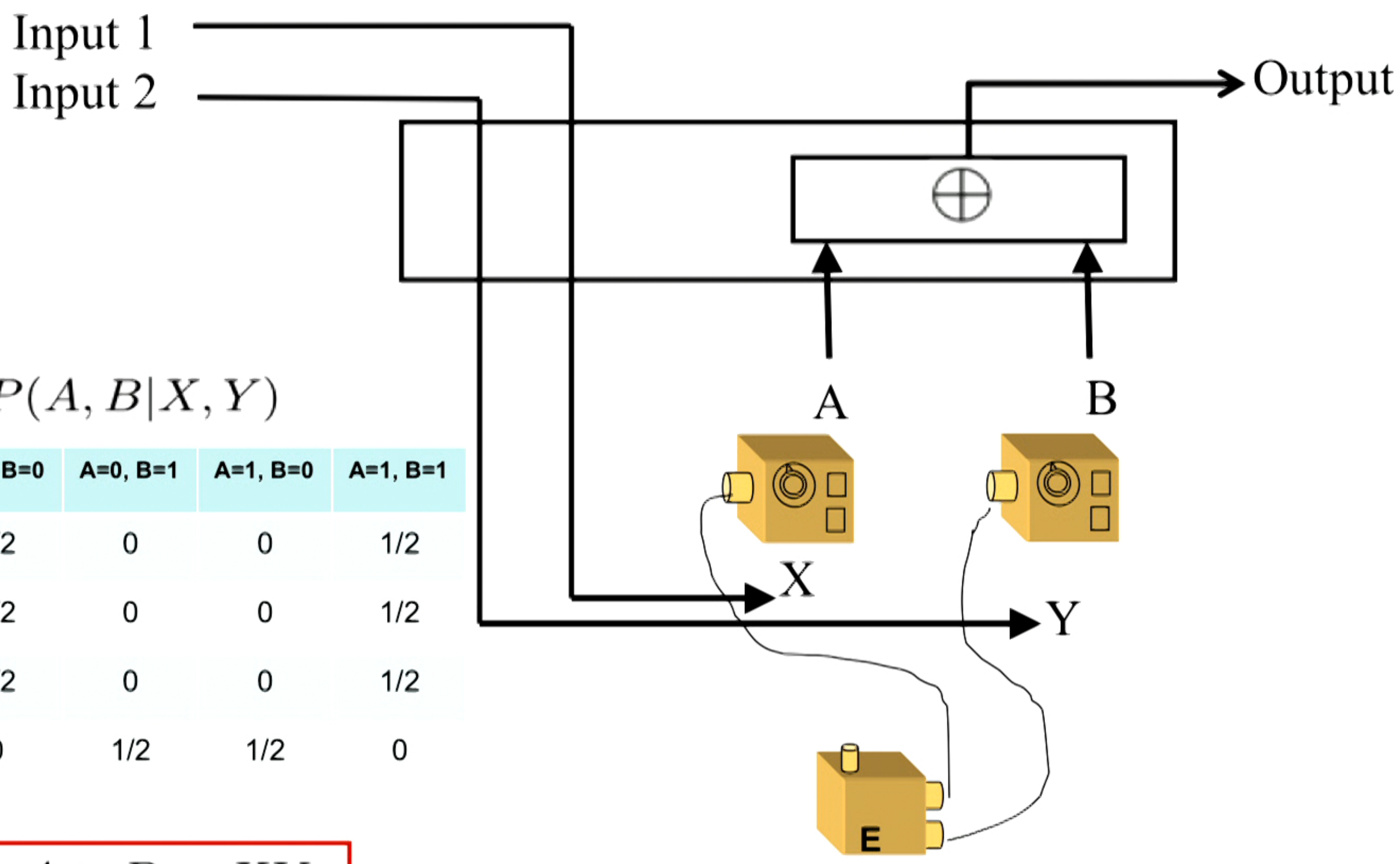



Input 1
Input 2
Output
⊕
A
B
X
Y
E




$$P(A, B|X, Y)$$

| | A=0, B=0 | A=0, B=1 | A=1, B=0 | A=1, B=1 |
|---|---|---|---|---|
| X=0, Y=0 | 1/2 | 0 | 0 | 1/2 |
| X=0, Y=1 | 1/2 | 0 | 0 | 1/2 |
| X=1, Y=0 | 1/2 | 0 | 0 | 1/2 |
| X=1, Y=1 | 0 | 1/2 | 1/2 | 0 |




$P(A, B|X, Y)$

| | A=0, B=0 | A=0, B=1 | A=1, B=0 | A=1, B=1 |
|---|---|---|---|---|
| X=0, Y=0 | 1/2 | 0 | 0 | 1/2 |
| X=0, Y=1 | 1/2 | 0 | 0 | 1/2 |
| X=1, Y=0 | 1/2 | 0 | 0 | 1/2 |
| X=1, Y=1 | 0 | 1/2 | 1/2 | 0 |

$$A \oplus B = XY$$

# Categorizing quantum phenomena

| Those arising in a restricted statistical classical theory | Those not arising in a restricted statistical classical theory |
|---|---|
| Noncommutativity | Bell inequality violations |
| Entanglement | Contextuality |
| Ambiguity of mixtures | Computational speed-up |
| EPR Steering | Certain aspects of items on the left |
| Collapse | Others… |
| Coherent superposition | |
| Teleportation | |
| No cloning | |
| Others… | |
| **Weak Nonclassicality** | **Strong Nonclassicality** |

## What we want in a notion of nonclassicality

| Subject to direct experimental test | Constitutes a resource | Applicable to a broad range of physical scenarios |
| --- | --- | --- |

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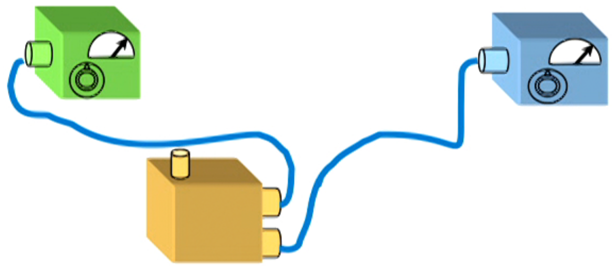
|---|---|---|---|
| Failure to admit a locally causal model | ✓ | ✓ | ✗ |

# What is needed to witness the failure of local causality



## What we want in a notion of nonclassicality

| | Subject to direct experimental test | Constitutes a resource | Applicable to a broad range of physical scenarios |
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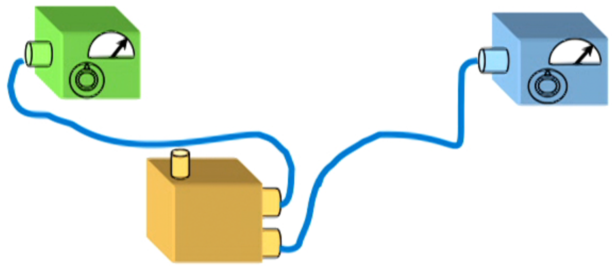
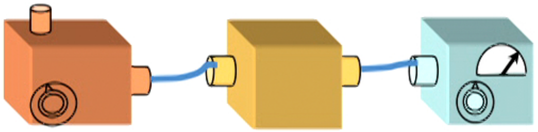
| Failure to admit a locally causal model | ✓ | ✓ | ✗ |
| Failure to admit a noncontextual model | | | ✓ |

# What is needed to witness the failure of local causality



# What is needed to witness the failure of noncontextuality



## What we want in a notion of nonclassicality

| | Subject to direct experimental test | Constitutes a resource | Applicable to a broad range of physical scenarios |
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| Failure to admit a locally causal model | ✓ | ✓ | ✗ |
| Failure to admit a noncontextual model | ? | ? | ✓ |

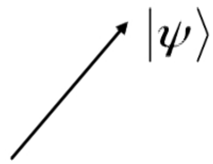
# What we want in a notion of nonclassicality

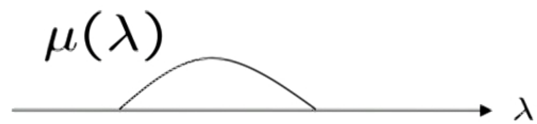
| | Subject to direct experimental test | Constitutes a resource | Applicable to a broad range of physical scenarios |
| --- | --- | --- | --- |
| Failure to admit a locally causal model | ✓ | ✓ | ✗ |
| Failure to admit a noncontextual model | ✓ | ✓ | ✓ |

# The traditional notion of noncontextuality in quantum theory

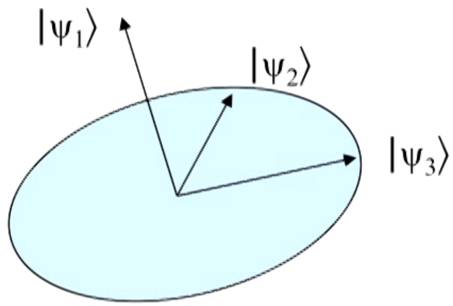
# Outcome-deterministic hidden variable model for pure states and projective measurements

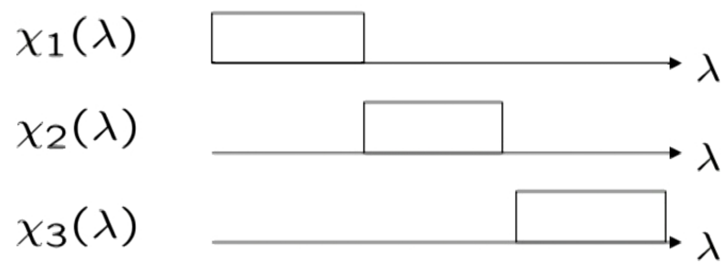
$\int \mu(\lambda) d\lambda = 1$




$\sum_k \chi_k(\lambda) = 1$ for all $\lambda$



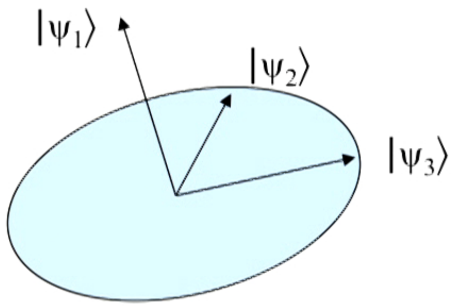



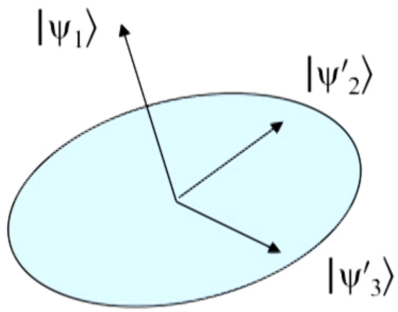

Note: the outcomes are deterministic given $\lambda$

$$|\langle\psi|\psi_k\rangle|^2 = \int d\lambda \mu(\lambda) \chi_k(\lambda)$$

# Traditional notion of noncontextuality

A given vector may appear in many different measurements



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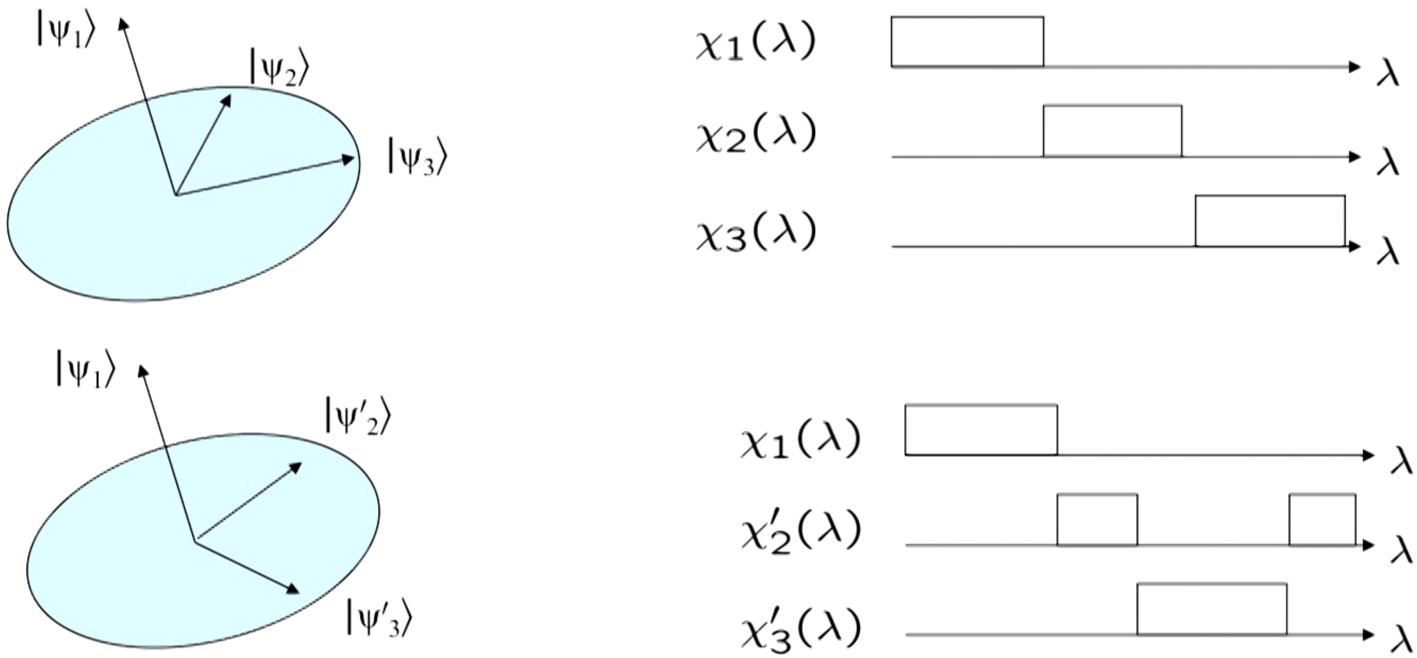

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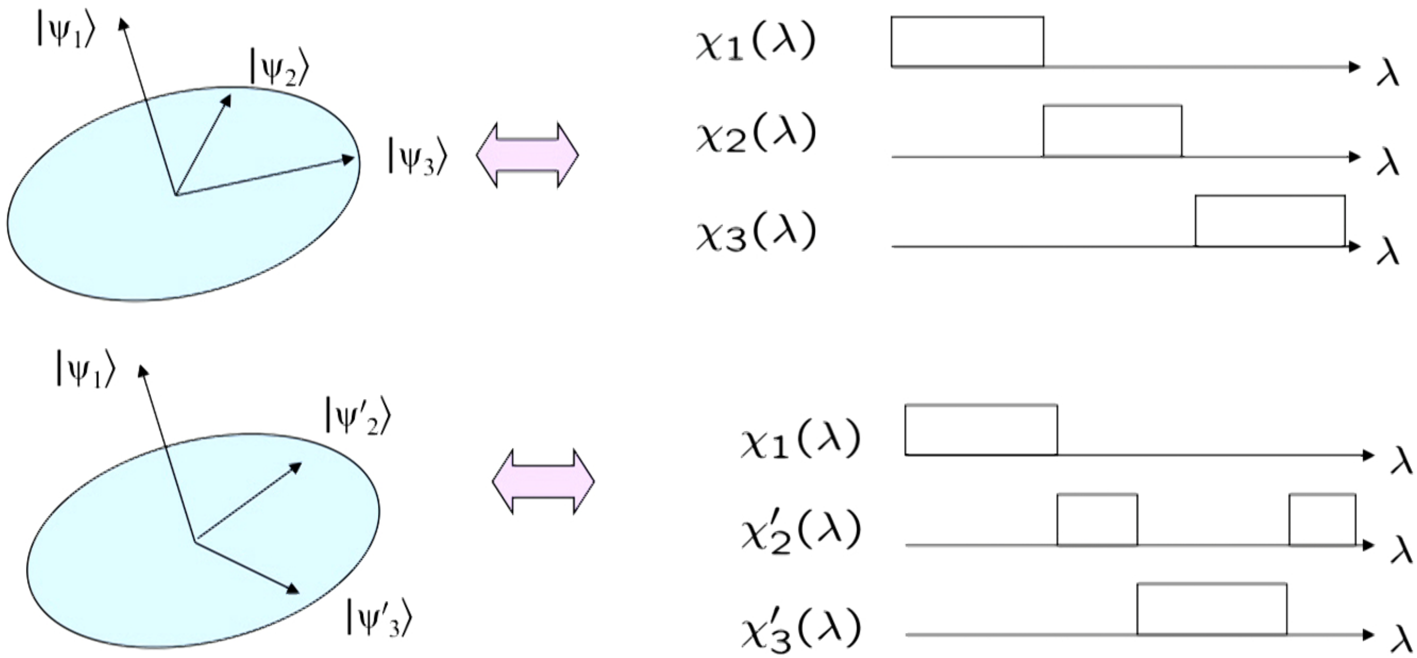
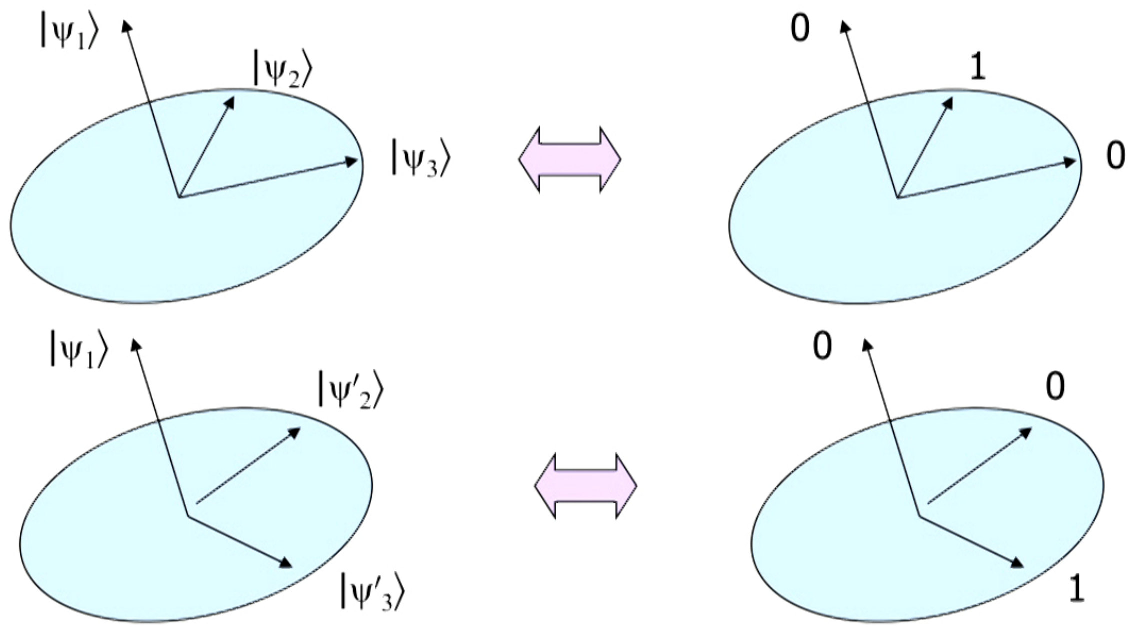
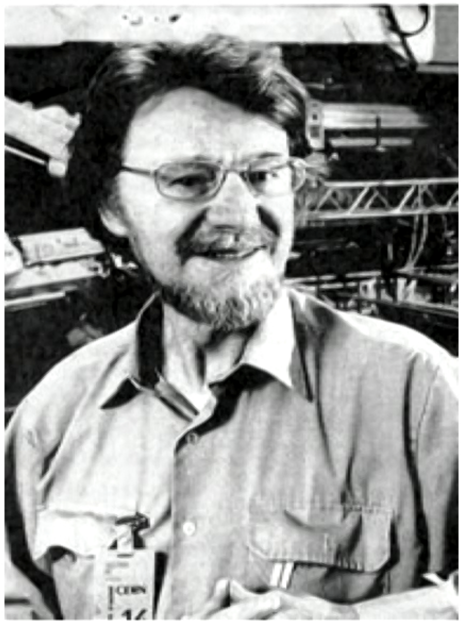



The traditional notion of noncontextuality:
Every vector is associated with the same $\chi(\lambda)$
regardless of how it is measured (i.e. the context)

The traditional notion of noncontextuality (take 2):
For every $\lambda$, every basis of vectors receives a 0-1 valuation, wherein exactly one element is assigned the value 1 (corresponding to the outcome that would occur for $\lambda$), and every vector is assigned the same value regardless of which basis it is considered a part (i.e. the context).



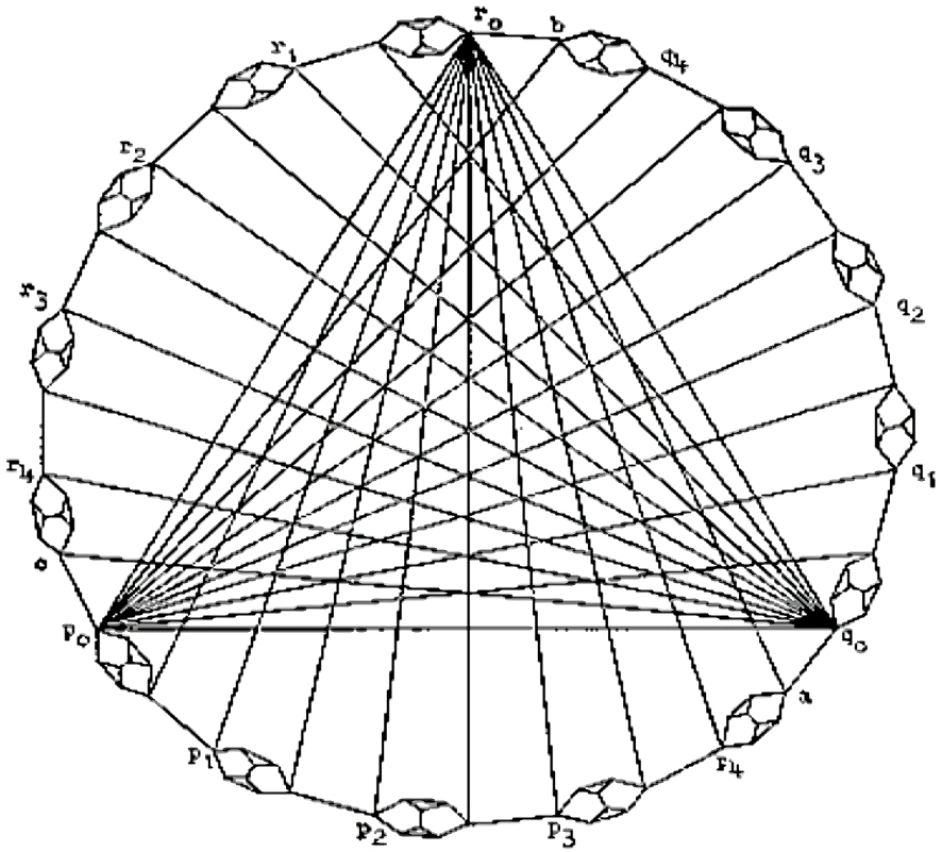


John S. Bell

Ernst Specker (with son) and Simon Kochen

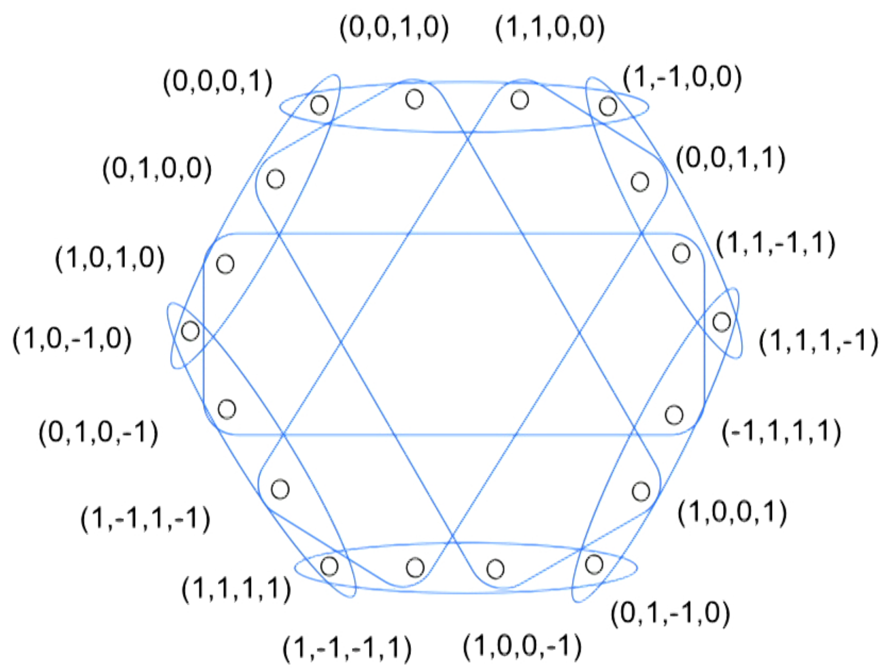
Bell-Kochen-Specker theorem: A traditional noncontextual hidden variable model of quantum theory for Hilbert spaces of dimension 3 or greater is impossible.

## Example: Kochen and Specker's original 117 ray proof in 3d

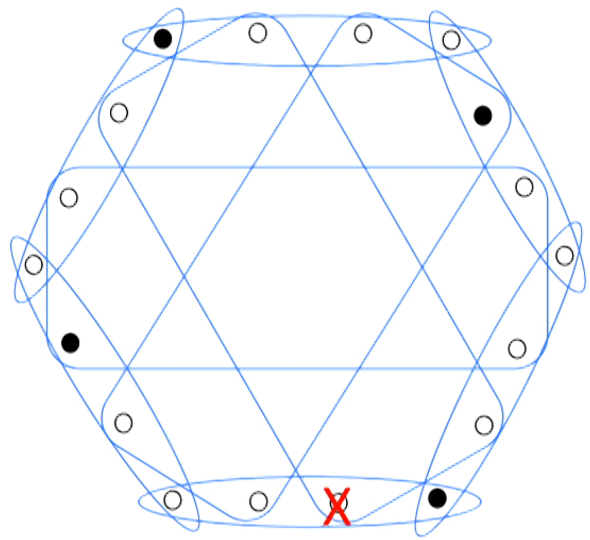


# 18 ray proof in 4d

Cabello, Estebaranz, Garcia-Alcaine, Phys. Lett. A 212, 183 (1996)



# No traditional noncontextual assignments




○ : value 0
● : value 1

If we list all 9 orthogonal quadruples, each ray appears twice in the list

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0,0,0,1 | 0,0,0,1 | 1,-1,1,-1 | 1,-1,1,-1 | 0,0,1,0 | 1,-1,-1,1 | 1,1,-1,1 | 1,1,-1,1 | 1,1,1,-1 |
| 0,0,1,0 | 0,1,0,0 | 1,-1,-1,1 | 1,1,1,1 | 0,1,0,0 | 1,1,1,1 | 1,1,1,-1 | -1,1,1,1 | -1,1,1,1 |
| 1,1,0,0 | 1,0,1,0 | 1,1,0,0 | 1,0,-1,0 | 1,0,0,1 | 1,0,0,-1 | 1,-1,0,0 | 1,0,1,0 | 1,0,0,1 |
| 1,-1,0,0 | 1,0,-1,0 | 0,0,1,1 | 0,1,0,-1 | 1,0,0,-1 | 0,1,-1,0 | 0,0,1,1 | 0,1,0,-1 | 0,1,-1,0 |

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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0,0,0,1 | 0,0,0,1 | 1,-1,1,-1 | 1,-1,1,-1 | 0,0,1,0 | 1,-1,-1,1 | 1,1,-1,1 | 1,1,-1,1 | 1,1,1,-1 |
| 0,0,1,0 | 0,1,0,0 | 1,-1,-1,1 | 1,1,1,1 | 0,1,0,0 | 1,1,1,1 | 1,1,1,-1 | -1,1,1,1 | -1,1,1,1 |
| 1,1,0,0 | 1,0,1,0 | 1,1,0,0 | 1,0,-1,0 | 1,0,0,1 | 1,0,0,-1 | 1,-1,0,0 | 1,0,1,0 | 1,0,0,1 |
| 1,-1,0,0 | 1,0,-1,0 | 0,0,1,1 | 0,1,0,-1 | 1,0,0,-1 | 0,1,-1,0 | 0,0,1,1 | 0,1,0,-1 | 0,1,-1,0 |

In each of the 9 quadruples, one ray is assigned 1, the other three 0
Therefore, 9 rays must be assigned 1

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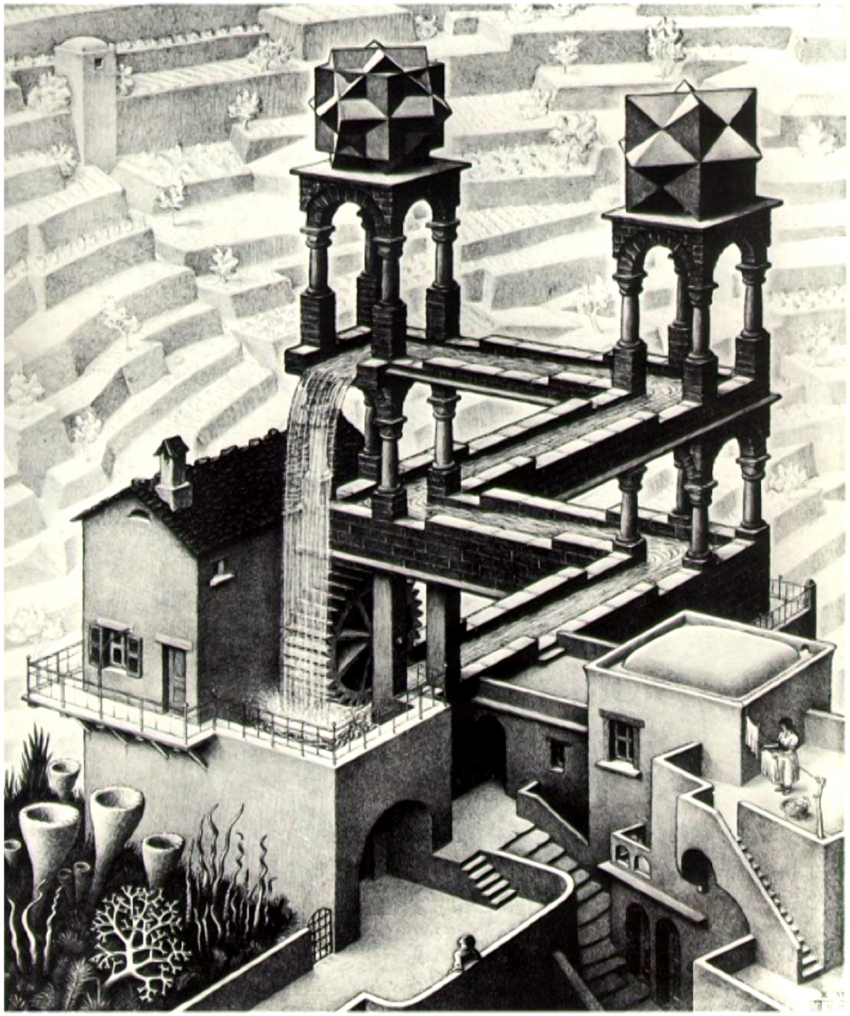
| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0,0,0,1 | 0,0,0,1 | 1,-1,1,-1 | 1,-1,1,-1 | 0,0,1,0 | 1,-1,-1,1 | 1,1,-1,1 | 1,1,-1,1 | 1,1,1,-1 |
| 0,0,1,0 | 0,1,0,0 | 1,-1,-1,1 | 1,1,1,1 | 0,1,0,0 | 1,1,1,1 | 1,1,1,-1 | -1,1,1,1 | -1,1,1,1 |
| 1,1,0,0 | 1,0,1,0 | 1,1,0,0 | 1,0,-1,0 | 1,0,0,1 | 1,0,0,-1 | 1,-1,0,0 | 1,0,1,0 | 1,0,0,1 |
| 1,-1,0,0 | 1,0,-1,0 | 0,0,1,1 | 0,1,0,-1 | 1,0,0,-1 | 0,1,-1,0 | 0,0,1,1 | 0,1,0,-1 | 0,1,-1,0 |

In each of the 9 quadruples, one ray is assigned 1, the other three 0
Therefore, 9 rays must be assigned 1

But each ray appears twice and so there must be an even number of rays assigned 1

CONTRADICTION!



The traditional notion of noncontextuality (take 3):
For every $\lambda$, every projector $\Pi$ is assigned a value 0 or 1 regardless of which basis it is a coarse-graining of (i.e. the context)

$v(\Pi) = 0$ or $1$ for all $\Pi$

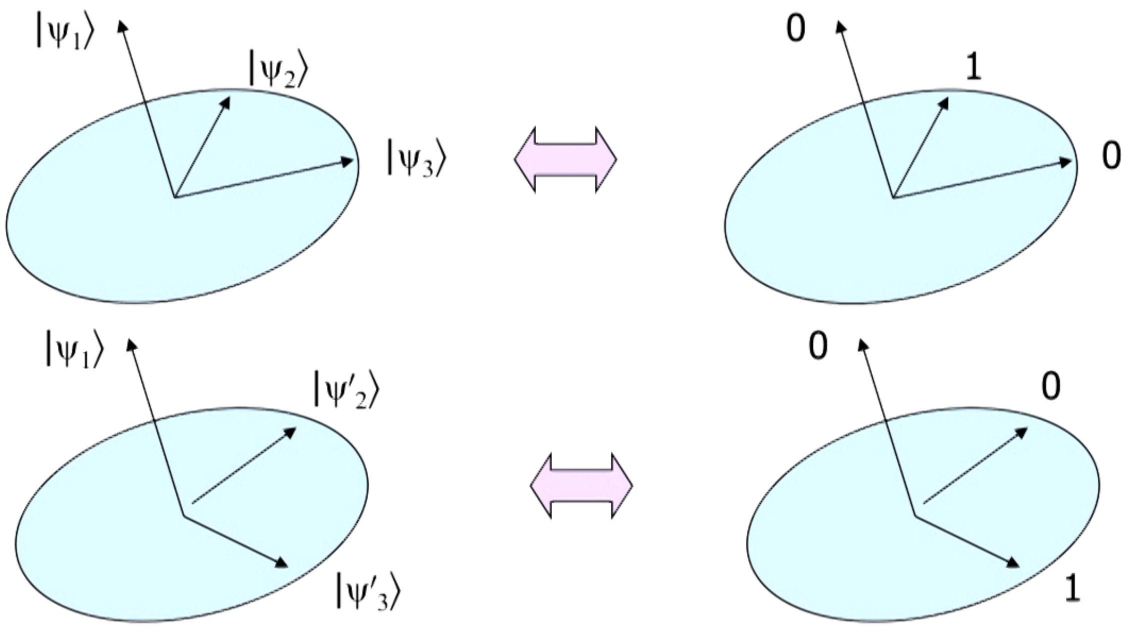
Coarse-graining of a measurement implies a coarse-graining of the value (because it is just post-processing)

$v(\sum_k \Pi_k) = \sum_k v(\Pi_k)$

Every measurement has *some* outcome

$v(I) = 1$

The traditional notion of noncontextuality (take 2):
For every $\lambda$, every basis of vectors receives a 0-1 valuation, wherein exactly one element is assigned the value 1 (corresponding to the outcome that would occur for $\lambda$), and every vector is assigned the same value regardless of which basis it is considered a part (i.e. the context).



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$$v(\sum_k \Pi_k) = \sum_k v(\Pi_k)$$

Every measurement has *some* outcome

$$v(I) = 1$$

The traditional notion of noncontextuality (take 4):

For Hermitian operators A, B, C satisfying

$$[A,B]=0 \qquad [A,C]=0 \qquad [B,C]\neq 0$$

the value assigned to A should be independent of whether it is measured together with B or together with C (i.e. the context)

Measure A = measure projectors onto eigenspaces of A, $\{\Pi_a\}$

$$A=\sum_a a\,\Pi_a \quad \rightarrow \quad v(A)=\sum_a a\, v(\Pi_a)$$

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For Hermitian operators A, B, C satisfying

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Measure A = measure projectors onto eigenspaces of A, $\{\Pi_a\}$

$$A = \sum_a a\, \Pi_a \quad \rightarrow \quad v(A) = \sum_a a\, v(\Pi_a)$$

Measure A in context of B
= measure projectors onto joint eigenspaces of A and B, $\{\Pi_{ab}\}$
then coarse-grain over B outcome $\Pi_a = \sum_b \Pi_{ab}$

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then coarse-grain over B outcome $\Pi_a=\sum_b \Pi_{ab}$

Measure A in context of C
= measure projectors onto joint eigenspaces of A and C, $\{\Pi_{ac}\}$
Then coarse-grain over C outcome $\Pi_a=\sum_c \Pi_{ac}$

The traditional notion of noncontextuality (take 4):

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$$[A, B] = 0 \qquad [A, C] = 0 \qquad [B, C] \neq 0$$

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Measure A in context of C
= measure projectors onto joint eigenspaces of A and C, $\{\Pi_{ac}\}$
Then coarse-grain over C outcome $\Pi_a = \sum_c \Pi_{ac}$

$v(\Pi_a)$ is independent of context → $v(A)$ is independent of context

Functional relationships among commuting Hermitian operators must be respected by their values

$$\text{If} \quad f(L, M, N, \ldots) = 0$$

$$\text{then} \quad f(v(L), v(M), v(N), \ldots) = 0$$

## Example: Mermin's magic square proof in 4d

| | | | |
|---|---|---|---|
| $X_1$ | $X_2$ | $X_1 X_2$ | $I$ |
| $Y_2$ | $Y_1$ | $Y_1 Y_2$ | $I$ |
| $X_1 Y_2$ | $Y_1 X_2$ | $Z_1 Z_2$ | $I$ |
| $I$ | $I$ | $-I$ | |

$$X_1\ X_2\ (X_1 X_2) = I$$
$$Y_1\ Y_2\ (Y_1 Y_2) = I$$
$$(X_1 Y_2)\ (Y_1 X_2)\ (Z_1 Z_2) = I$$
$$X_1\ Y_2\ (X_1 Y_2) = I$$
$$Y_1\ X_2\ (Y_1 X_2) = I$$
$$(X_1 X_2)\ (Y_1 Y_2)\ (Z_1 Z_2) = -I$$

## Example: Mermin's magic square proof in 4d

| | | | |
|---|---|---|---|
| $X_1$ | $X_2$ | $X_1 X_2$ | $I$ |
| $Y_2$ | $Y_1$ | $Y_1 Y_2$ | $I$ |
| $X_1 Y_2$ | $Y_1 X_2$ | $Z_1 Z_2$ | $I$ |
| $I$ | $I$ | $-I$ | |

$$X_1 \ X_2 \ (X_1 X_2) = I$$
$$Y_1 \ Y_2 \ (Y_1 Y_2) = I$$
$$(X_1 Y_2) \ (Y_1 X_2) \ (Z_1 Z_2) = I$$
$$X_1 \ Y_2 \ (X_1 Y_2) = I$$
$$Y_1 \ X_2 \ (Y_1 X_2) = I$$
$$(X_1 X_2) \ (Y_1 Y_2) \ (Z_1 Z_2) = -I$$

$$v(X_1) \ v(X_2) \ v(X_1 X_2) = 1$$
$$v(Y_1) \ v(Y_2) \ v(Y_1 Y_2) = 1$$
$$v(X_1 Y_2) \ v(Y_1 X_2) \ v(Z_1 Z_2) = 1$$
$$v(X_1) \ v(Y_2) \ v(X_1 Y_2) = 1$$
$$v(Y_1) \ v(X_2) \ v(Y_1 X_2) = 1$$
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## Example: Mermin's magic square proof in 4d

| | | | |
|---|---|---|---|
| $X_1$ | $X_2$ | $X_1X_2$ | $I$ |
| $Y_2$ | $Y_1$ | $Y_1Y_2$ | $I$ |
| $X_1Y_2$ | $Y_1X_2$ | $Z_1Z_2$ | $I$ |
| $I$ | $I$ | $-I$ | |

$$X_1\ X_2\ (X_1X_2) = I$$
$$Y_1\ Y_2\ (Y_1Y_2) = I$$
$$(X_1Y_2)\ (Y_1X_2)\ (Z_1Z_2) = I$$
$$X_1\ Y_2\ (X_1Y_2) = I$$
$$Y_1\ X_2\ (Y_1X_2) = I$$
$$(X_1X_2)\ (Y_1Y_2)\ (Z_1Z_2) = -I$$

$$v(X_1)\ v(X_2)\ v(X_1X_2) = 1$$
$$v(Y_1)\ v(Y_2)\ v(Y_1Y_2) = 1$$
$$v(X_1Y_2)\ v(Y_1X_2)\ v(Z_1Z_2) = 1$$
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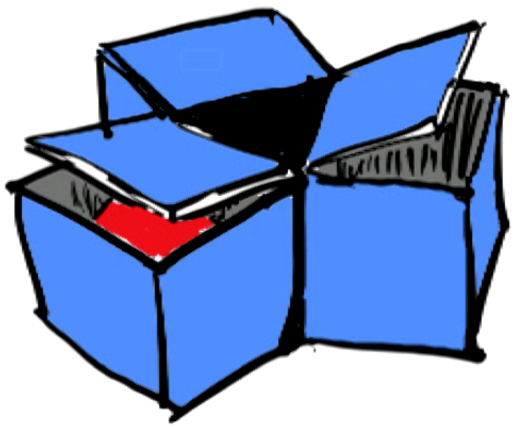
$$v(X_1X_2)\ v(Y_1Y_2)\ v(Z_1Z_2) = -1$$

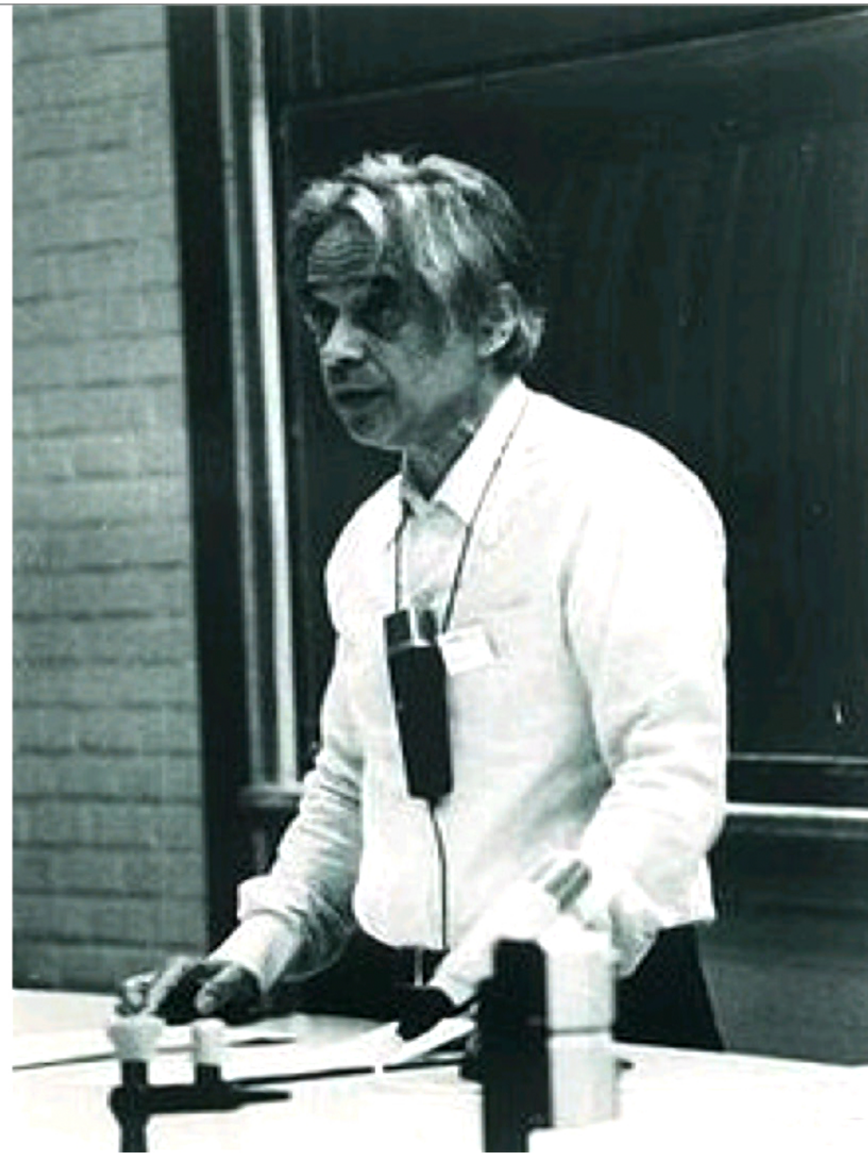
Product of LHSs = +1

Product of RHSs = -1

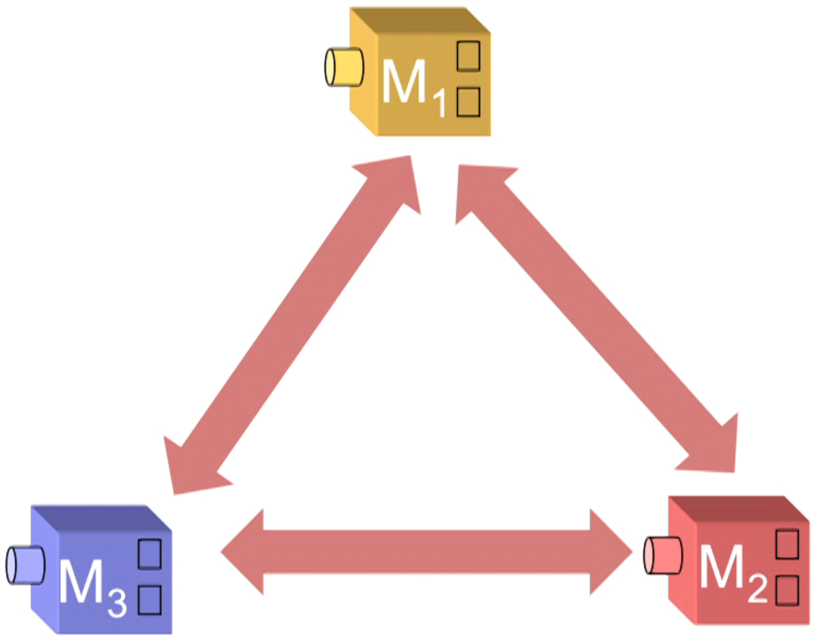
CONTRADICTION

Ernst Specker, "The logic of propositions which are not simultaneously decidable", Dialectica 14, 239 (1960).

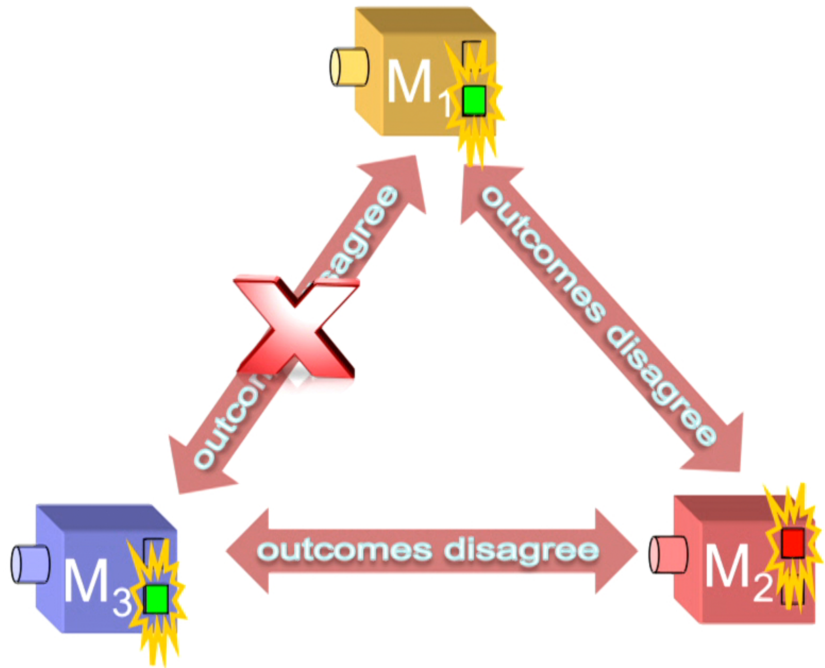




# Specker's example



# Specker's example




If the outcomes are fixed deterministically by the ontic state and are independent of the context in which the measurement is performed, then

$$p(\text{success}) \leq \frac{2}{3}$$

# Frustrated Networks

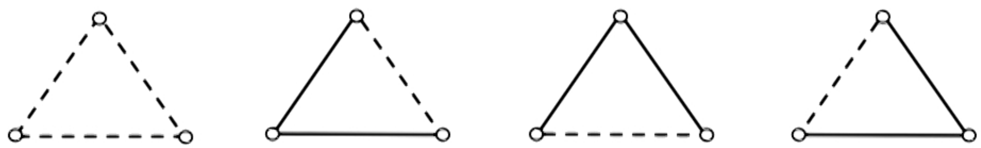
Nodes are binary variables

Edges imply joint measurability

 Outcomes agree

 Outcomes disagree

Frustration = no valuation satisfying all correlations



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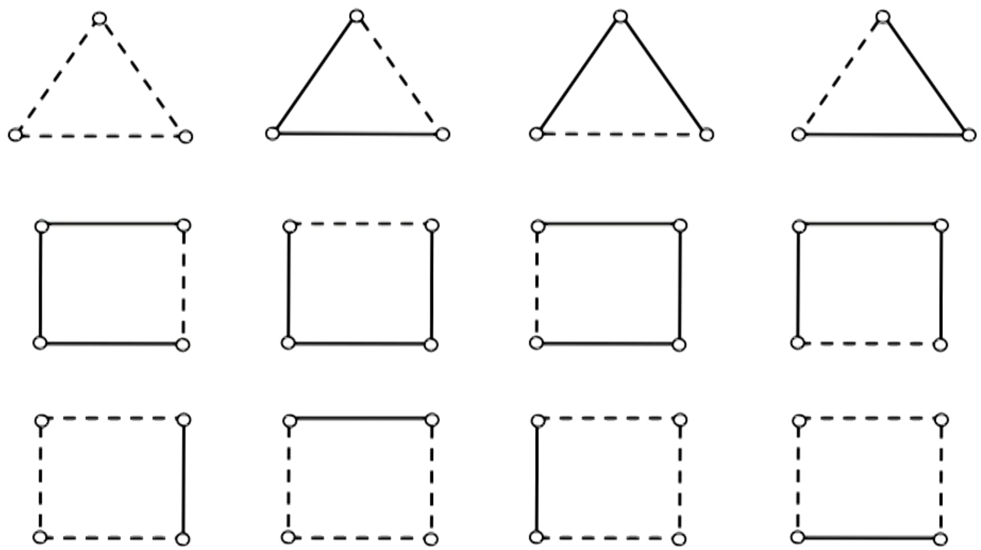
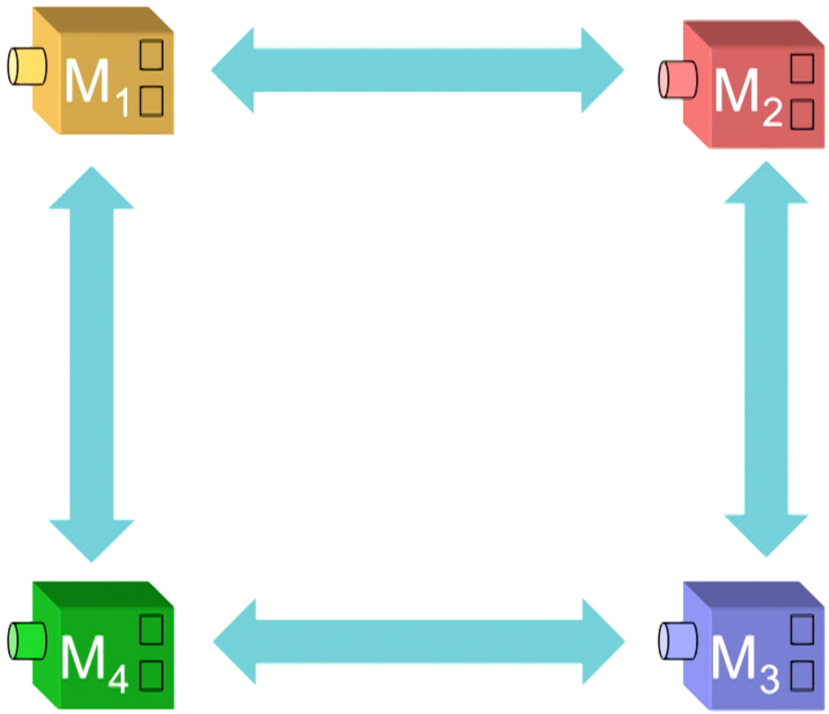
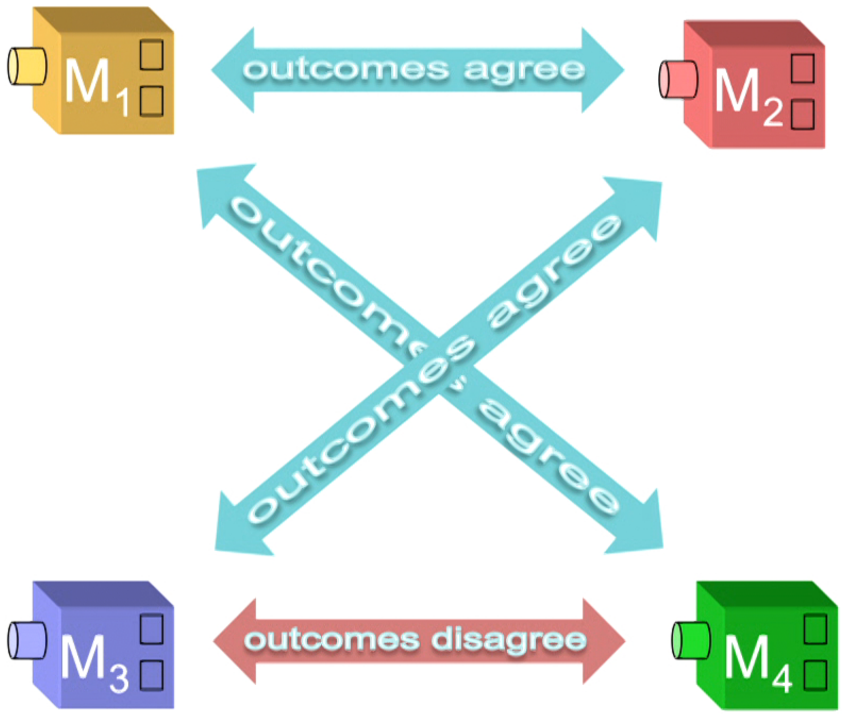
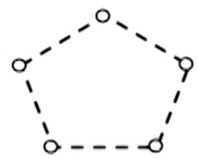




$M_1$
$M_2$
$M_4$
$M_3$


M1
outcomes agree
M2
outcomes agree
outcomes agree
M3
outcomes disagree
M4

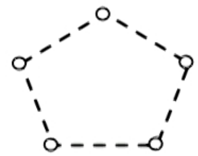
Klyachko, Can, Biniciolu, Shumovsky, PRL 101, 020403 (2008)



In a traditional noncontextual model

$$p(\text{success}) \leq \frac{4}{5}$$

## Klyachko, Can, Biniciolu, Shumovsky, PRL 101, 020403 (2008)



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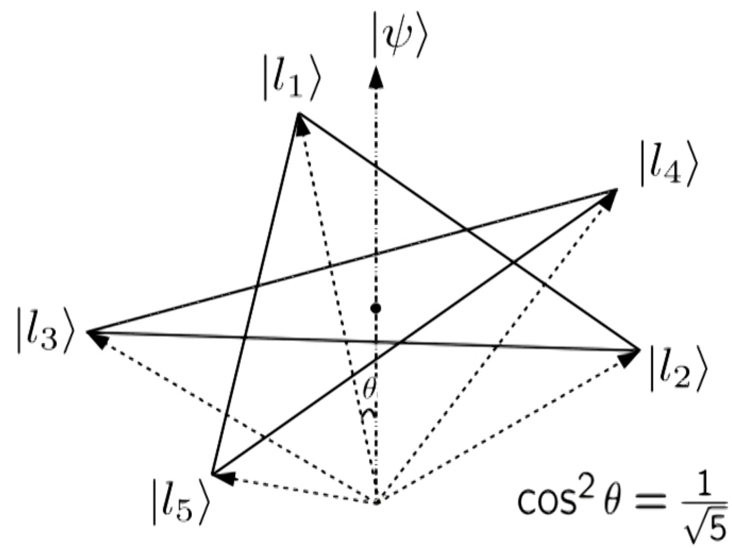


Quantum probability of success

$p(\text{success}) = \frac{2}{\sqrt{5}} \simeq 0.89 > \frac{4}{5}$