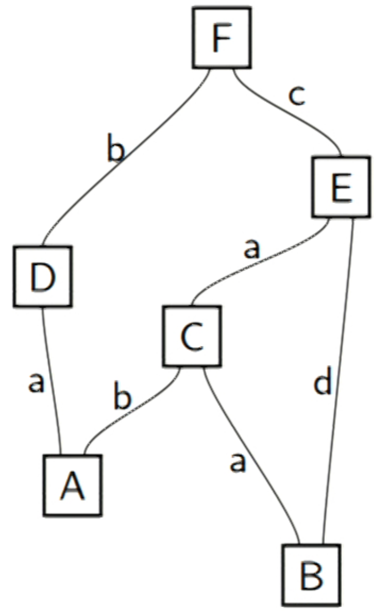
Title: PSI 17/18 - Foundations of Quantum Mechanics - Lecture 9

Date: Feb 08, 2018 10:15 AM

URL: http://pirsa.org/18020068

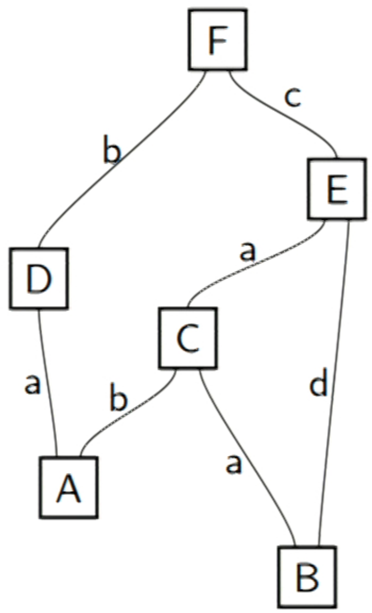
Abstract:

Consider a circuit



How do we calculate the probability for this circuit in standard framework of QT?

Consider again the circuit



$$A^{a_1 b_2} B^{a_3 d_4} C^{a_5}_{b_2 a_3} D^{b_6}_{a_1} E^{c_7}_{a_5 d_4} F_{b_6 c_7}$$

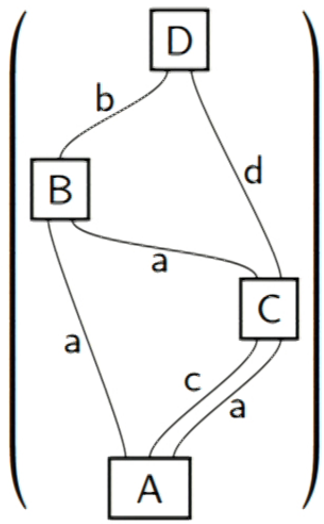
In the operator tensor formulation

$$\text{Prob}(A^{a_1 b_2} B^{a_3 d_4} C^{a_5}_{b_2 a_3} D^{b_6}_{a_1} E^{c_7}_{a_5 d_4} F_{b_6 c_7}) = \hat{A}^{a_1 b_2} \hat{B}^{a_3 d_4} \hat{C}^{a_5}_{b_2 a_3} \hat{D}^{b_6}_{a_1} \hat{E}^{c_7}_{a_5 d_4} \hat{F}_{b_7 c_7}$$

We will explain what the RHS means later.

Introducing probabilities

Assump 1 *We can associate a probability with any given circuit (the probability that the circuit "happens"), and this probability depends only on the specification of the given circuit (the knob settings and outcome sets at the operations, and the wiring).*

Prob 

is well conditioned

The $p(\cdot)$ function

We define the function $p(\cdot)$ as follows

$$p(\alpha \mathsf{A} + \beta \mathsf{B} + \dots) := \alpha \mathsf{Prob}(\mathsf{A}) + \beta \mathsf{Prob}(\mathsf{B}) + \dots$$

for *circuits* A, B, ... and real numbers α, β, ...(these can be negative).

Will use this to define a notion of equivalence.

Example of equivalence

Have

$$\alpha A^{a_1} + \beta B^{a_1} \equiv \gamma C^{a_1} + \delta D^{a_1}$$

if

$$p([\alpha A^{a_1} + \beta B^{a_1}]E_{a_1}) = p([\gamma C^{a_1} + \delta D^{a_1}]E_{a_1}) \quad \text{for all} \quad E_{a_1}$$

General definition of equivalence

We consider expressions like

$$\text{expression} = \alpha + \beta \mathsf{A} + \gamma \mathsf{B} + \ldots$$

where A, B, ... are fragments.

Equivalence: *We write*

$$\textit{expression}_1 \equiv \textit{expression}_2$$

if

$$p(\textit{expression}_1 \mathsf{E}) \equiv p(\textit{expression}_2 \mathsf{E})$$

for any fragment E *that makes the contents of the argument on both sides of this equation into a linear sum of circuits.*

Equivalence is a weaker notion than equality.

Another example of equivalence

In general, we have

$$\mathsf{A} \equiv \mathrm{Prob}(\mathsf{A}) \quad \text{for any circuit} \quad \mathsf{A}$$

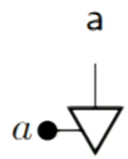
Proof: For any circuit E

$$p(\mathsf{AE}) = p(\mathsf{A})p(\mathsf{E}) = p(\mathrm{Prob}(\mathsf{A})\mathsf{E})$$

This example illustrates how equivalence is a weaker notion than equality.

Fiducial preparations

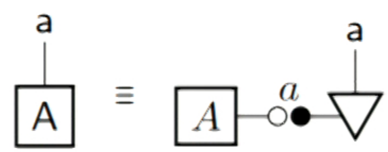
Fiducial preparations



$\Longleftrightarrow \quad {}_{a_1}\mathsf{X}^{\mathsf{a}_1}$ where $a_1 = 1$ to K_{a}

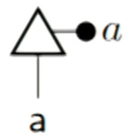
For any preparation $\mathsf{A}^{\mathsf{a}_1}$ (summation over a_1 implicit below)

$\mathsf{A}^{\mathsf{a}_1} \equiv {}^{a_1}A \; {}_{a_1}\mathsf{X}^{\mathsf{a}_1} \quad \Longleftrightarrow$



Fiducial results

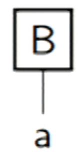
Fiducial results



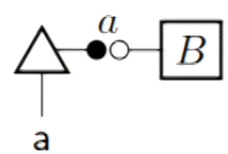
$\Longleftrightarrow \quad \mathsf{X}_{\mathsf{a}_1}^{a_1}$ where $a_1 = 1$ to K_{a}

For any result for a system of type a

$\mathsf{B}_{\mathsf{a}_1} \equiv B_{a_1} \mathsf{X}_{\mathsf{a}_1}^{a_1} \quad \Longleftrightarrow$

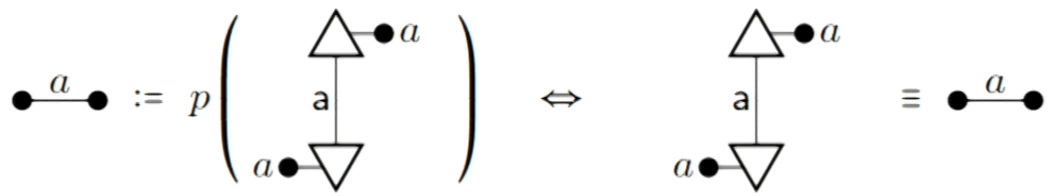


$\equiv$



The hopping metric

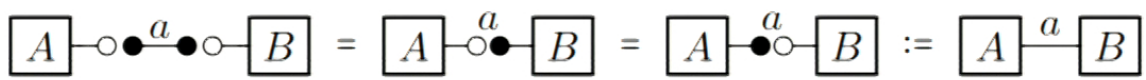
We define *the hopping metric*



Black and white dots

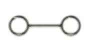
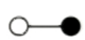
We define

Hence

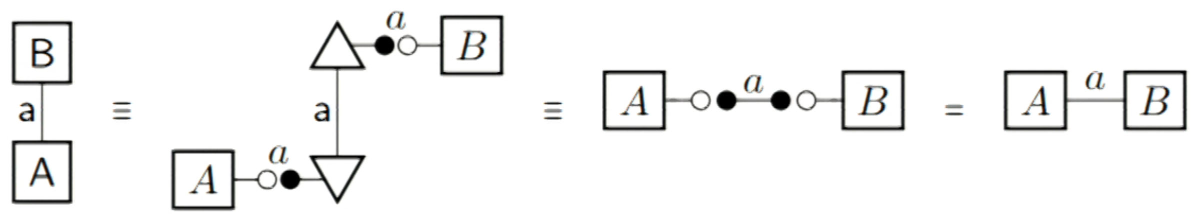


We have

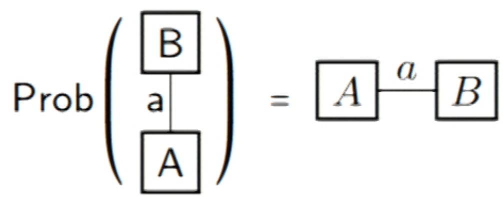
Hence, we can insert and delete pairs of black and white dots as we like.
Consistency requires

-  to be the inverse of 
-  to be equal to the identity
-  to be equal to the identity

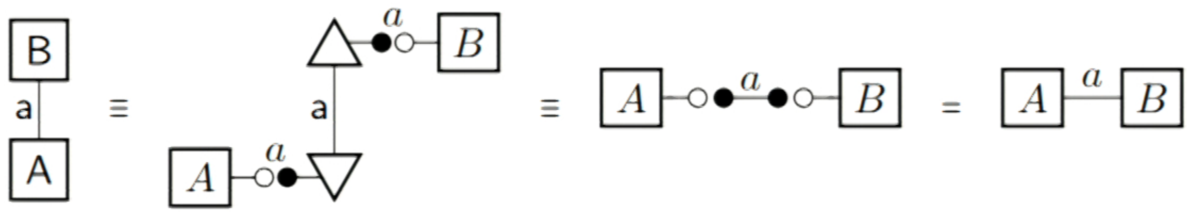
The steps for a simple circuit



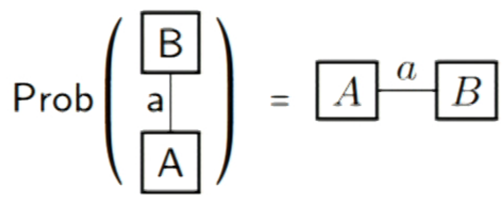
Hence



The steps for a simple circuit

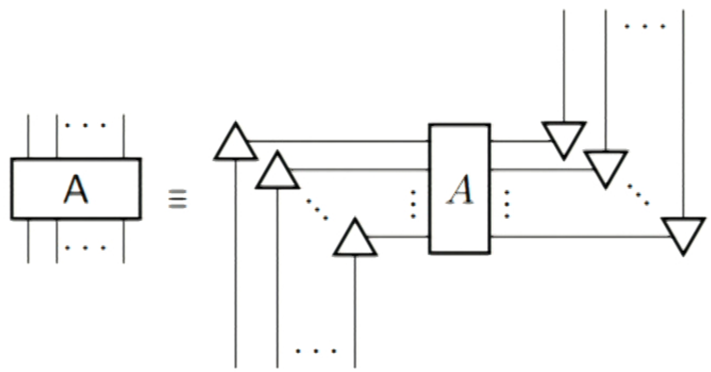


Hence



Assumption 2

Assumption 2: Operations are fully decomposable. We assume that any operation can be written as

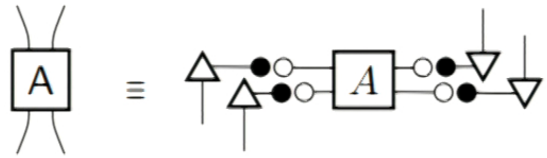


In words we will say that any operation is equivalent to a linear combination of operations each of which consists of an result for each input and a preparation for each output.

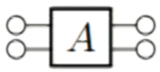
This is equivalent to tomographic locality.

Duotensor with all white dots

Inserting black and white dots (with black next to the fiducial elements)

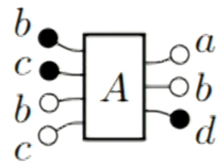


Therefore



(with all white dots) provides the weights in the sum over fiducial elements.
This is an example of a duotensor.

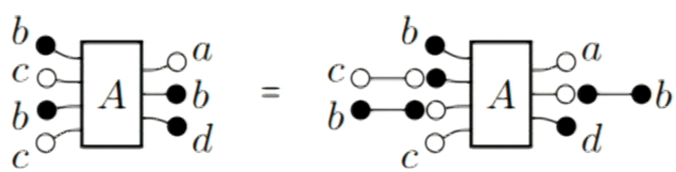
What are duotensors?

- Like tensors except that each index is associated with two bases.
- They transform like tensors but with respect to two bases.
- 

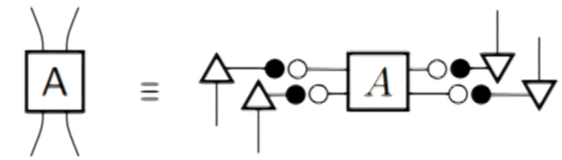
  corresponds to ${}_{b_3 c_4}^{a_1 b_2}A_{b_6 c_7}^{d_5}$
- Have map

$${}^{\circ}_{\bullet}A^{\bullet}_{\circ}$$

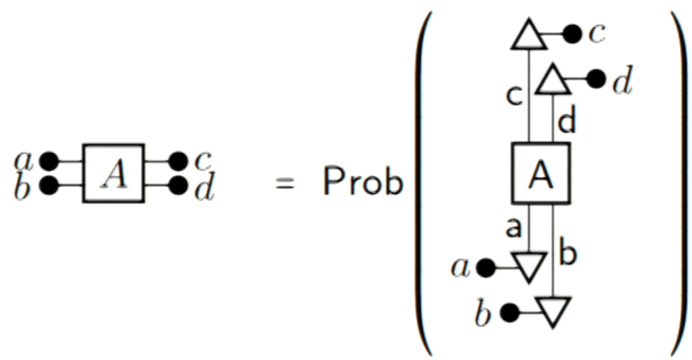
- Can change colours of dots using $\bullet\!\!-\!\!\bullet$ and $\circ\!\!-\!\!\circ$



All white dots gives coefficients in sum over fiducials



All black dots gives fiducial probabilities



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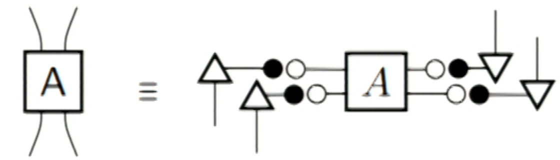

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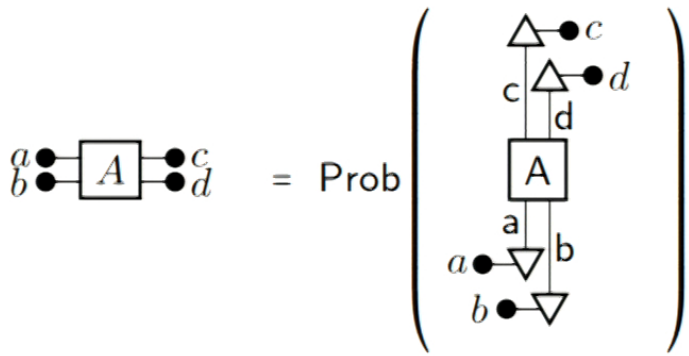
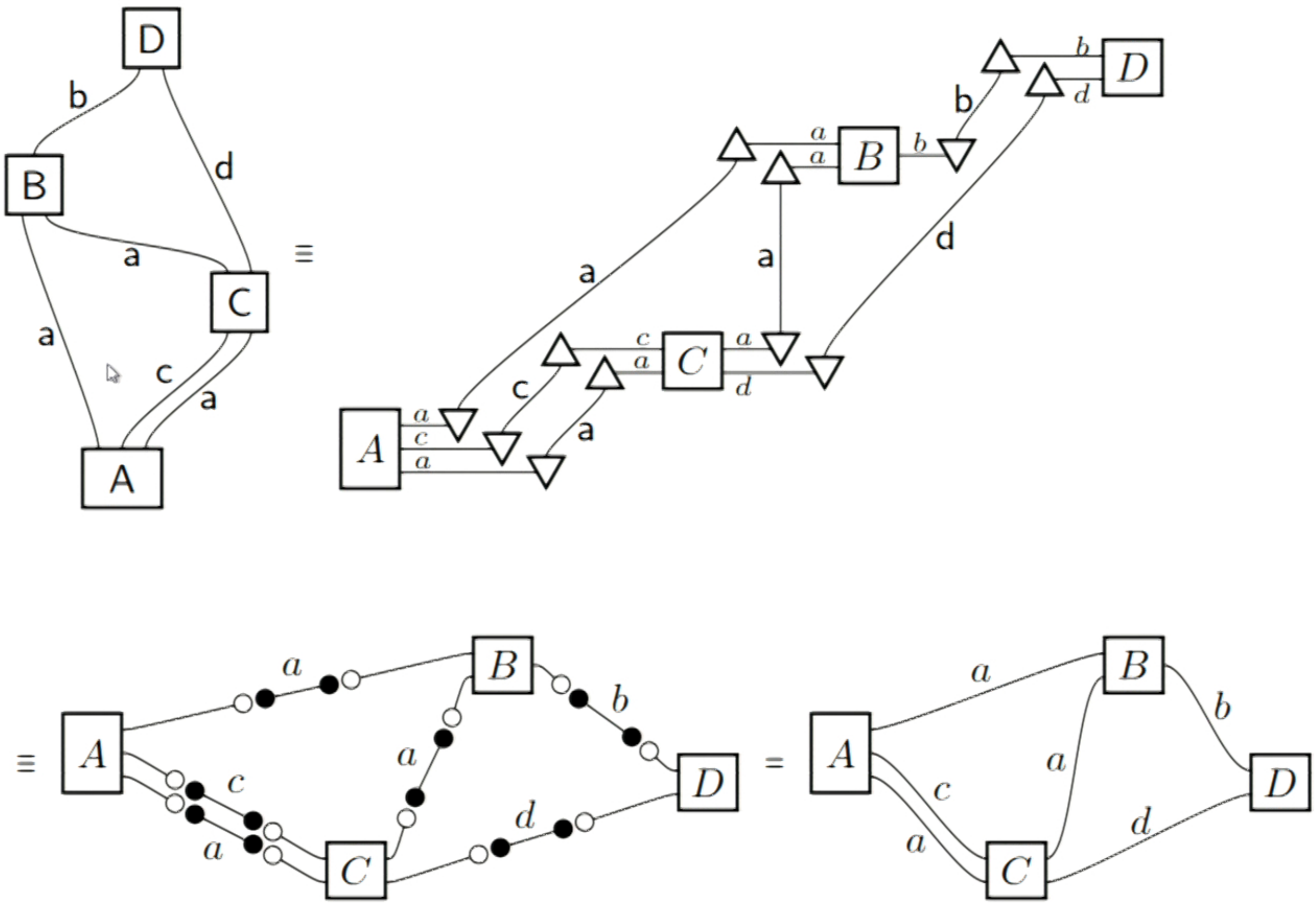


General circuits



Hilbert space notation

- We define $\mathcal{H}_{\mathsf{a}_1}$, $\mathcal{H}_{\mathsf{b}_2}$, . . . having dimensions N_{a}, N_{b}, . . .
- We define $\mathcal{H}^{\mathsf{a}_1}$, $\mathcal{H}^{\mathsf{b}_2}$, . . . having dimensions N_{a}, N_{b}, ...
- We define

$$\mathcal{H}_{\mathsf{a}_1\mathsf{b}_2\ldots\mathsf{c}_3}^{\mathsf{d}_4\mathsf{e}_5\ldots\mathsf{f}_6} := \mathcal{H}_{\mathsf{a}_1} \otimes \mathcal{H}_{\mathsf{b}_2} \otimes \cdots \otimes \mathcal{H}_{\mathsf{c}_3} \otimes \mathcal{H}^{\mathsf{d}_4} \otimes \mathcal{H}^{\mathsf{e}_5} \otimes \cdots \otimes \mathcal{H}^{\mathsf{f}_6}$$

These are all taken to be complex Hilbert spaces.

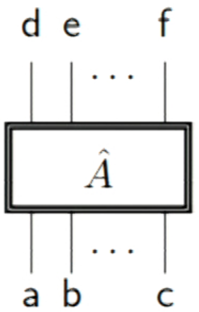
Space of operators

We define

$$\mathcal{V}_{a_1 b_2 \ldots c_3}^{d_4 e_5 \ldots f_6}$$

as the space of Hermitian operators acting on $\mathcal{H}_{a_1 b_2 \ldots c_3}^{d_4 e_5 \ldots f_6}$.
We write

$$\hat{A}_{a_1 b_2 \ldots c_3}^{d_4 e_5 \ldots f_6} \quad \Leftrightarrow$$



for elements of $\mathcal{V}_{a_1 b_2 \ldots c_3}^{d_4 e_5 \ldots f_6}$.

Notation

- We write $\hat{A}^{a_1}\hat{B}^{b_2}$ instead of $\hat{A}^{a_1} \otimes \hat{B}^{b_2}$ (in $\mathcal{V}^{a_1 b_2} = \mathcal{V}^{a_1} \otimes \mathcal{V}^{b_2}$).
- We write $\hat{A}_{a_1}\hat{C}^{b_2}$ instead of $\hat{A}_{a_1} \otimes \hat{C}^{b_2}$.
- We write $\hat{A}_{a_1}\hat{D}^{a_2}$ instead of $\hat{A}_{a_1} \otimes \hat{D}^{a_2}$.
- Order not important (information in the integers). $\hat{A}_{a_1}\hat{C}^{b_2} = \hat{C}^{b_2}\hat{A}_{a_1}$

When we have a repeated integer:

- We write $\hat{A}^{a_1}\hat{B}_{a_1}$ for trace$(\hat{A}^{a_1}\hat{B}_{a_1})$, denoted graphically by

  A wire or a repeated integer means we take the trace of the product of the given operators.

- When have a more complicated example like

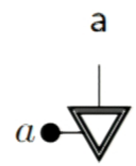
$$\hat{A}^{b_3 c_4}_{a_1 b_2}\hat{B}^{d_6 c_7}_{a_5 b_3}$$

  we take product in appropriate (b_3) subspace and then take *partial trace* in that subspace. This is accomplished very naturally using full decomposability of operators.

Fiducial operators

We introduce a fiducial (spanning) set of operators for $\mathcal{V}^{\mathsf{a}}$

$${}_{a_1}\hat{X}^{\mathsf{a}_1} \iff$$



where $a_1 = 1$ to K_{a}

Similarly, we introduce a fiducial (spanning) set of operators for the space $\mathcal{V}_{\mathsf{a}_1}$

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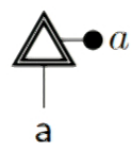
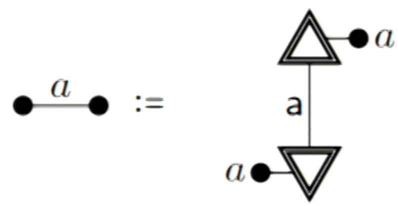


where $a_1 = 1$ to K_{a}

The hopping metric

In the context of operator tensors the hopping metric is given by



and its inverse is represented by ○—○.

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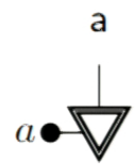

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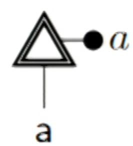

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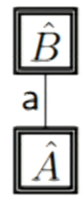

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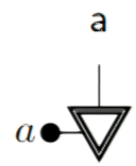
$${}_{a_1}\hat{X}^{\mathsf{a}_1} \iff$$ 

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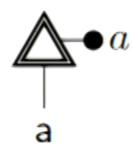
$$\hat{X}^{a_1}_{\mathsf{a}_1} \iff$$ 

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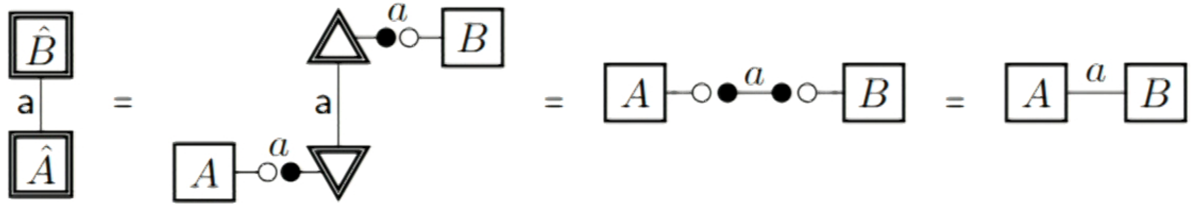
The steps for a simple operator circuit



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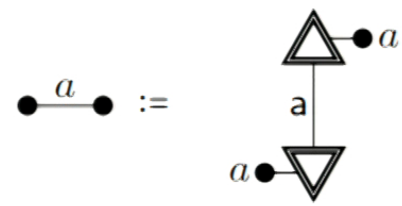
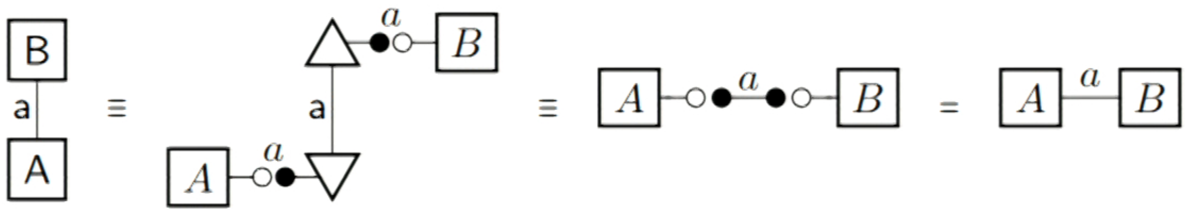


and its inverse is represented by ○—○.

Compare with a simple (operation) circuit



Since Hilbert spaces are complex ....

$$\mathcal{V}^{d_4 e_5 \ldots f_6}_{a_1 b_2 \ldots c_3} = \mathcal{V}_{a_1} \otimes \mathcal{V}_{b_2} \otimes \cdots \otimes \mathcal{V}_{c_3} \otimes \mathcal{V}^{d_4} \otimes \mathcal{V}^{e_5} \otimes \cdots \otimes \mathcal{V}^{f_6}$$

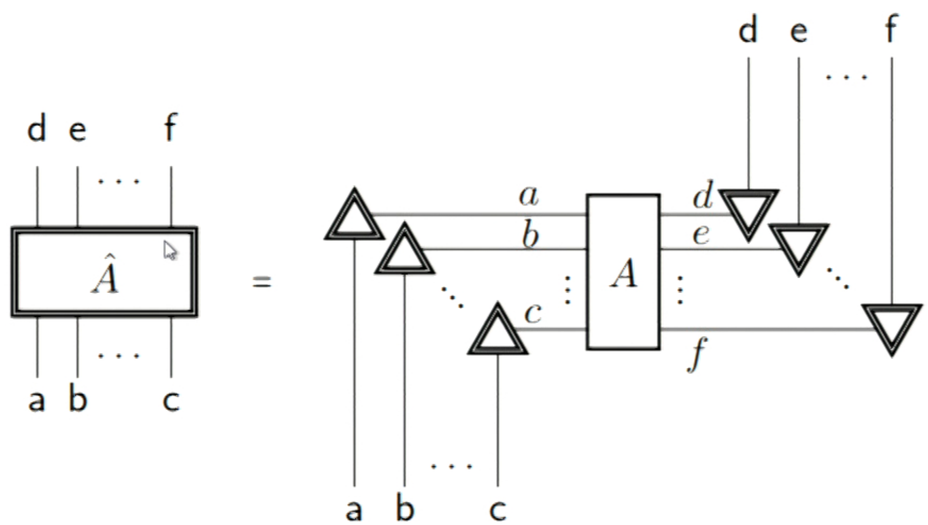
we have ....

Full decomposability of operators

We can write any operator as a linear sum over fiducial operators for the inputs and outputs.

$$\hat{A}^{\mathsf{d}_4 \mathsf{e}_5 \dots \mathsf{f}_6}_{\mathsf{a}_1 \mathsf{b}_2 \dots \mathsf{c}_3} = {}^{d_4 e_5 \dots f_6} A_{a_1 b_2 \dots c_3} \, \hat{X}^{a_1}_{\mathsf{a}_1} \hat{X}^{b_2}_{\mathsf{b}_2} \cdots \hat{X}^{c_3}_{\mathsf{c}_3} \; {}_{d_4}\hat{X}^{\mathsf{d}_4} \, {}_{e_5}\hat{X}^{\mathsf{e}_5} \cdots {}_{f_6}\hat{X}^{\mathsf{f}_6} \qquad (1)$$

in symbolic notation, or



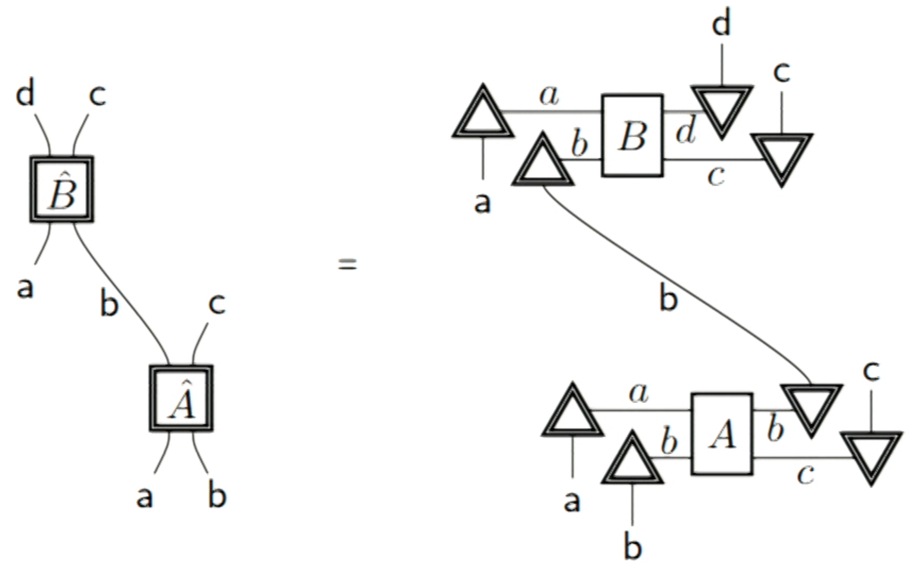
(2)

in diagrammatic notation.

Using full decomposability

$$\hat{A}^{b_3 c_4}_{a_1 b_2} \hat{B}^{d_6 c_7}_{a_5 b_3} = A^{b_3 c_4}_{a_1 b_2} \hat{X}^{a_1}_{a_1} \hat{X}^{b_2}_{b_2} {}_{b_3}\hat{X}^{b_3} {}_{c_4}\hat{X}^{c_4} B^{d_6 c_7}_{a_5 b_{3'}} \hat{X}^{a_5}_{a_5} \hat{X}^{b_{3'}}_{b_3} {}_{d_6}\hat{X}^{d_6} {}_{c_7}\hat{X}^{c_7}$$

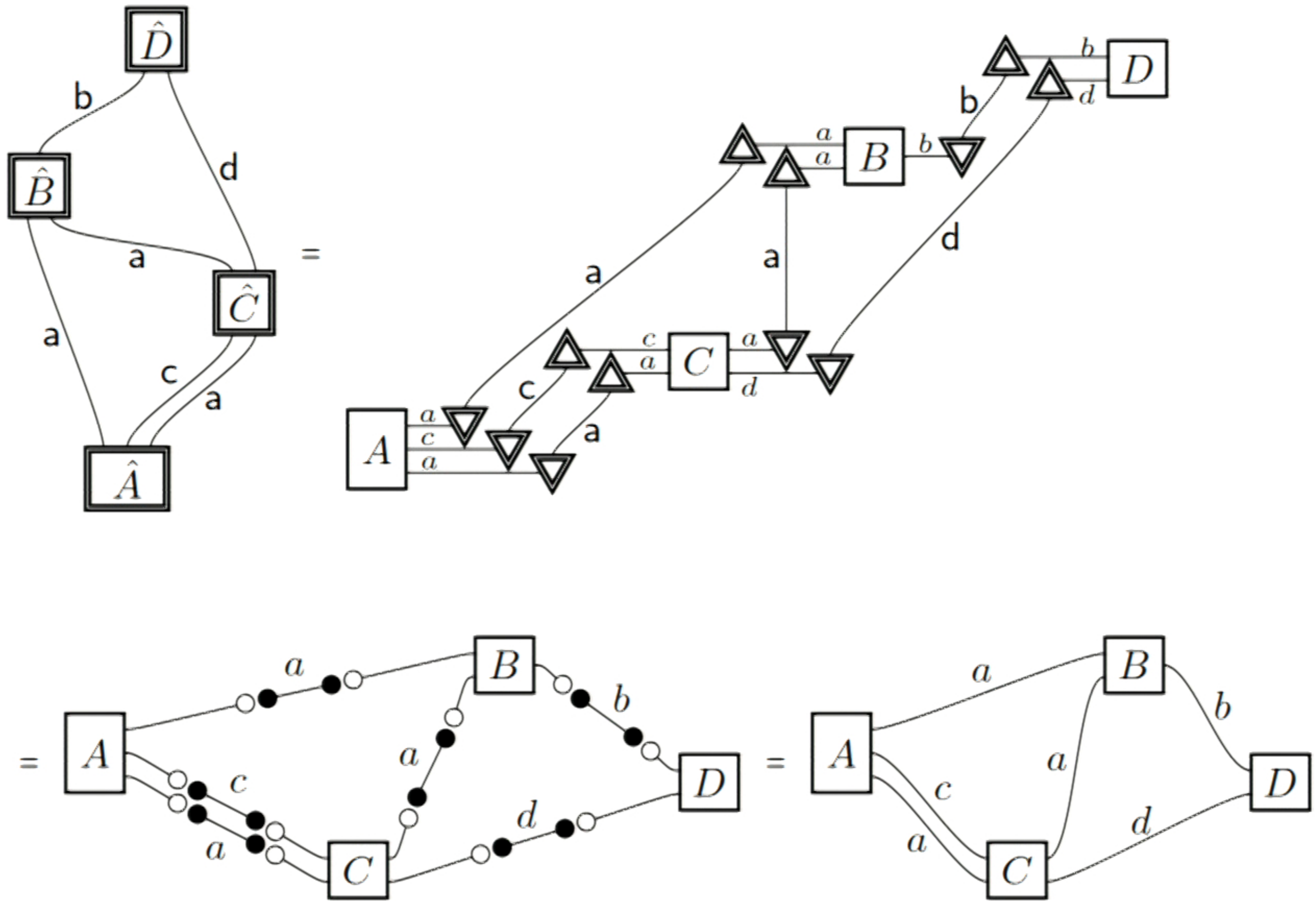
or, in diagrammatic notation,



This operator is in the space

$$\mathcal{V}^{c_4 d_6 c_7}_{a_1 b_2 a_5} = \mathcal{V}_{a_1} \otimes \mathcal{V}_{b_2} \otimes \mathcal{V}_{a_5} \otimes \mathcal{V}^{c_4} \otimes \mathcal{V}^{d_6} \otimes \mathcal{V}^{c_7}$$

General operator circuit



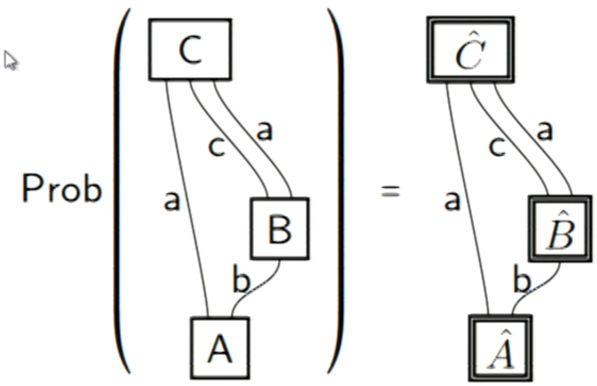
Operation-operator correspondence.

We will say that operations correspond to operators if there is a mapping from operations, $A_{a_1 b_2 \ldots c_3}^{d_4 e_5 \ldots f_6}$*, to operators,* $\hat{A}_{a_1 b_2 \ldots c_3}^{d_4 e_5 \ldots f_6}$*, such that the probability for any circuit comprised of operations is equal to the trace of the circuit operator obtained under this mapping.*

If operations correspond to operators then, for example

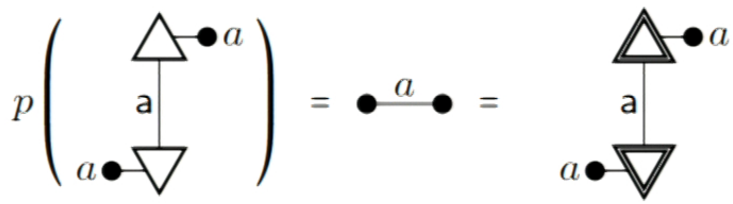
$$\text{Prob}(A^{a_1 b_2} B_{b_2}^{c_3 a_4} C_{a_1 c_3 a_4}) = \hat{A}^{a_1 b_2} \hat{B}_{b_2}^{c_3 a_4} \hat{C}_{a_1 c_3 a_4}$$

Same example in diagrammatic notation

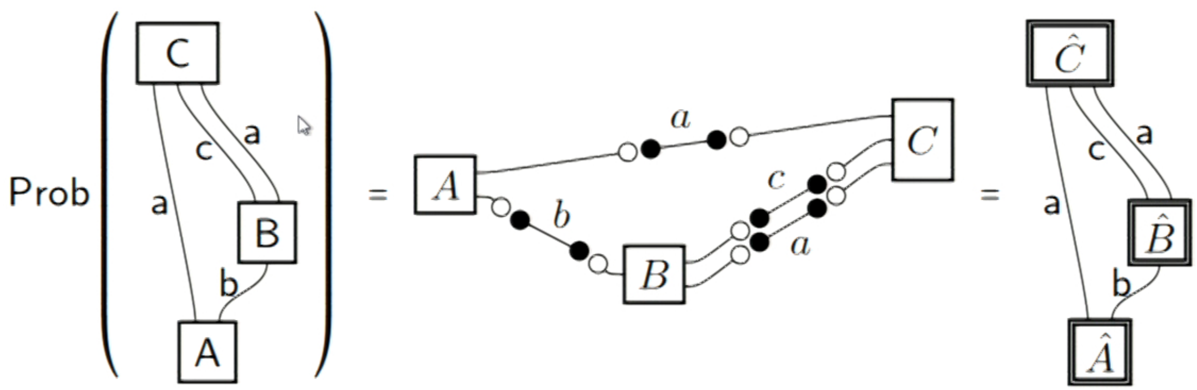


Condition for correspondence - KEY IDEA!

Have correspondence if can find fiducial sets such that have equal hopping metrics



since, for example,



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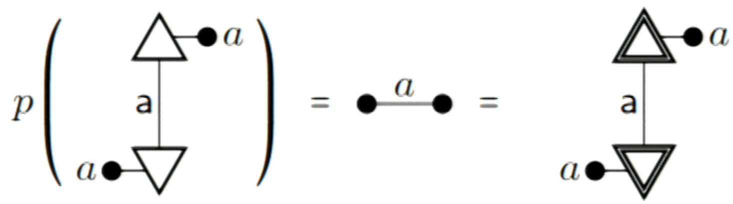

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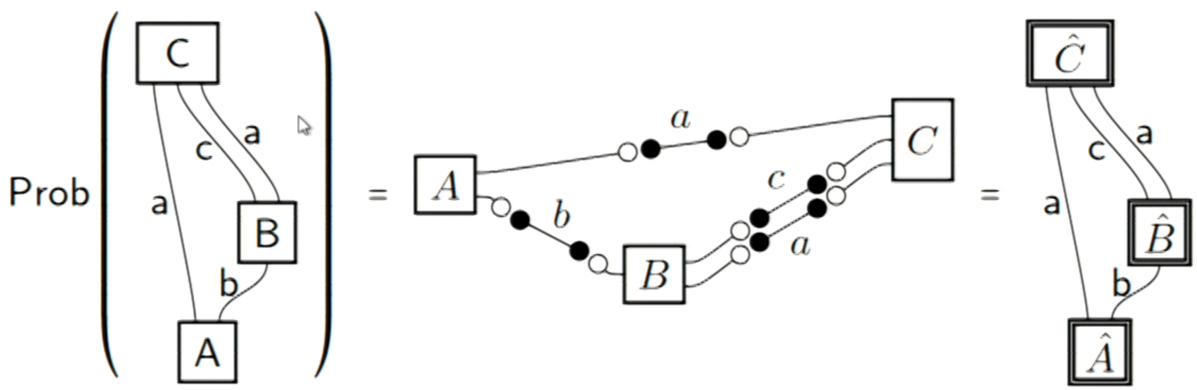
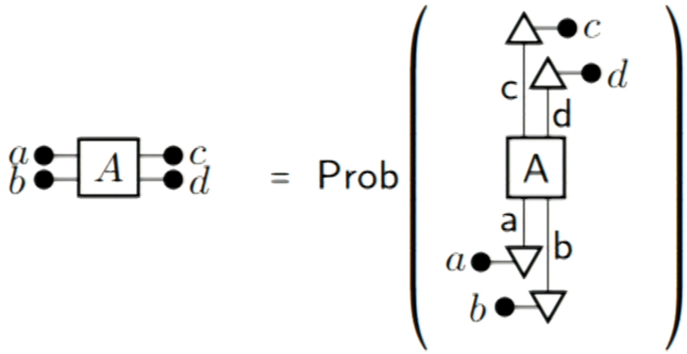
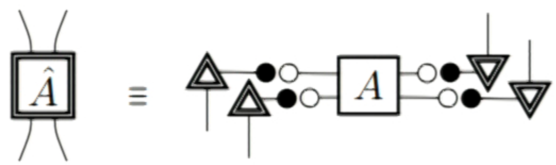


Determining operator corresponding to operation

Do local process tomography to get duotensor with all black dots:



Convert to duotensor with all white dots (using ○—○) and then operator given by



Physicality theorem

Assume operator circuits must take values between 0 and 1 and we allow at least

1. all rank one projectors, $\hat{A}^{\mathsf{a}_1}$ (for all systems a),
2. all rank one projectors, $\hat{C}_{\mathsf{b}_2}$ (for all systems b),
3. the identity effect, $\hat{I}_{\mathsf{b}_2}$ (for all systems b),

then all operators, $\hat{B}^{\mathsf{d}_4}_{\mathsf{c}_3}$, we must have

Positive input transpose

$$\hat{B}^{\mathsf{d}_4}_{\mathsf{c}_3^{\mathsf{T}}}$$

Output trace less than identity

$$\hat{B}^{\mathsf{d}_4}_{\mathsf{c}_3}\hat{I}_{\mathsf{d}_4} \leq \hat{I}_{\mathsf{c}_3}$$

We call operators satisfying these two conditions *physical*.

Complete sets

A complete set of operations, $\{\mathsf{B}_{\mathsf{a}_1}^{\mathsf{b}_2}[l] : l = 1 \text{ to } L\}$ *is a set of operations corresponding to the same apparatus use with disjoint outcome sets whose union is the set of all possible outcomes for this apparatus.*

A complete set of physical operators, $\{\hat{B}_{\mathsf{a}_1}^{\mathsf{b}_2}[l] : l = 1 \text{ to } L\}$, *is a set for which every operator positive input transpose and*

$$\sum_{l=1}^{L} B_{\mathsf{a}_1}^{\mathsf{b}_2}[l]\hat{I}_{\mathsf{b}_2} = \hat{I}_{\mathsf{a}_1} \qquad (5)$$

Elements of a complete set of physical operators satisfy

$$B_{\mathsf{a}_1}^{\mathsf{b}_2}[l]\hat{I}_{\mathsf{b}_2} \leq \hat{I}_{\mathsf{a}_1}$$

so are physical.

Mathematical axioms

Axiom 1 Operations correspond to operators.

Axiom 2 All complete sets of physical operators correspond to complete sets of operations.

Or, more glibly,

All complete sets of operations correspond to complete sets of physical operators and vice versa.

Equivalent to usual formulation of quantum theory.

End comments

- unification of all objects - states, effects, transformations, fragments in general all described by positive operators.
- simple way to combine them.
- Don't need to foliate.
- Curious time asymmetry in the physicality condition.
- formalism locality.
- played a crucial role in my recent reconstruction.
- relation with q-combs, multi-time, general boundary, and Leifer Spekkens work.
- embraces the quantum picturalism revolution!
- quantum field theory, quantum gravity, ...

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