Title: Strong Quantum energy inequality and the Hawking singularity theorem

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Abstract: <p>Singularities, boundary points of spacetime beyond which no extension is possible, continue to intrigue both mathematicians and physicists since they are places where our current understanding of physical law breaks down. The question of whether they exist in physical situations is still an open one. Fifty years ago, Hawking and Penrose developed the first general model independent singularity theorems. These theorems showed that singularities have to exist in any spacetime that satisfies certain properties. Some of these properties are mild assumptions but others, called energy conditions, depend on matter content and are more problematic. For both classical and quantum fields, violations of these conditions can be observed in some of the simplest of cases. Therefore there is a need to develop theorems with weaker restrictions, namely energy conditions averaged over an entire geodesic and quantum inequalities, weighted local averages of energy densities. In this work we investigate the strong energy condition in the presence of both classical and quantum non-minimally coupled scalar fields and derive bounds in each case. In the quantum case these bounds take the form of a set of state-dependent quantum energy inequalities valid for the class of Hadamard states. Finally, we show how the quantum inequalities derived can be used as an assumption to a modified Hawking singularity theorem.</p>

Strong quantum energy inequality and the Hawking singularity theorem

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Outline

Introduction

Are there singularities in our universe?

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- Intuitive definition: A "place" where the curvature diverges.

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Are there singularities in our universe?

What is a singularity?

- Intuitive definition: A "place" where the curvature diverges.
- Problems: Except in highly symmetrical cases (e.g Schwarzschild) we cannot represent the singularity as "place" since the metric is not defined there. The divergence of curvature scalars doesn't cover all singularity cases.

Definition

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- Initial efforts to specify whether our universe has singularities used particular spacetimes or stress-energy tensors
- The first breakthrough was the singularity theorems of Hawking and Penrose in the late 60's
- They proved that a spacetime is singular if it satisfies certain general properties
- They don't give any information about the nature of singularities (e.g. if they are curvature singularities)

Singularity theorems structure (Senovilla 1998)

1. **Causality condition**
   e.g. There is a Cauchy surface $\mathscr{H}$: complete spacelike C^∞ hypersurface that intersects every null and timelike line once only
2. **The initial or boundary condition**
   e.g. There exists a trapped surface: spacelike hypersurface for which two null normals have negative expansion
3. **The energy condition**
   e.g. Null Energy Condition (Penrose)

   $$R_{ab}\ell^a\ell^b \geq 0 \qquad \text{with} \qquad \ell^a\text{:null}$$

   Strong Energy Condition (Hawking)

   $$R_{ab}U^aU^b \geq 0 \qquad \text{with} \qquad U^a\text{:timelike}$$

$\Rightarrow$ Then the spacetime is geodesically incomplete.

Raychaudhuri equation

- Expansion

$$\dot{\theta} = -\frac{1}{n-1}\theta^2 - 2\sigma^2 - R_{ab}U^a U^b$$

- Shear scalar
- Curvature

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Proof structure:

- Initial condition: Geodesics start focusing
- Energy condition: Focusing continues
- Causality condition: No focal points

$\Rightarrow$ Geodesic incompleteness

Energy conditions and quantum inequalities

$\Rightarrow$ Pointwise energy conditions are violated!

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SEC violation

Minimally coupled real scalar field

$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) + \frac{1}{2} g_{\mu\nu}(m^2\phi^2 - (\nabla\phi)^2)$$

$$U^\mu U^\nu (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} U^\mu U_\mu m^2 \phi^2 \geq 0$$

Violation

- For large field mass m
- For vanishing derivatives of the scalar field

Average Energy Conditions

Average energy conditions bound the energy along an entire geodesic

e.g. Averaged null energy condition: $\int_{-\infty}^{\infty} T_{ab}\ell^a \ell^b d\lambda \geq 0$

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Quantum Inequalities

Quantum Inequalities bound the total energy when averaging over a time period

$\Rightarrow$ Originally introduced to prevent the violation of second law of thermodynamics in black holes

e.g. Null Quantum Energy Inequality in flat spacetime (Ford, Roman, 1995)

$$\int_{-\infty}^{\infty} \frac{\tau_0^2}{\tau^2+\tau_0^2}\langle T_{00}\rangle d\tau \geq -\frac{3}{32\pi\tau_0^3}$$

- Efforts to develop singularity theorems with weakened energy conditions (Tipler 1977, Borde 1986, Roman 1987)
- Singularity theorems with hypotheses inspired from QEI (Fewster, Galloway 2011)

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$$\int_{-\infty}^{\infty}(r(t)-r_0(t))f(t)^2dt \geq -|||f|||^2$$

- $r(t)$: Energy density of some state Ψ
- $r_0(t)$: Energy density of a reference state
- $|||f|||$: Sobolev norm

$$|||f|||^2 = \sum_{\ell=0}^{L} Q_\ell ||f^{(\ell)}||^2 \,.$$

Proof structure:

- Show that the Raychaudhuri equation and has no solution ($\theta \to -\infty$) if
  - The previous hypothesis holds
  - The geodesic is complete
- But the geodesic cannot have a focal point (causality condition).
  $\Rightarrow$ The geodesic is incomplete.

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- Show that the Raychaudhuri equation and has no solution ($\theta \to -\infty$) if
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There was no SEQI to show if it satisfies the hypothesis of the theorem

The classical strong energy condition
The non-minimally coupled field

Field equation for non-minimally coupled scalar fields is

$$(\Box_g + m^2 + \xi R)\phi = 0$$

where ξ is the coupling constant. The Langrangian is

$$L[\phi] = \frac{1}{2}[(\nabla\phi)^2 - (m^2 + \xi R)\phi^2]$$

We can get the stress energy tensor by varying the action of the Lagrangian with respect to the metric

$$T_{\mu\nu} = (\nabla_\mu\phi)(\nabla_\nu\phi) + \frac{1}{2}g_{\mu\nu}(m^2\phi^2 - (\nabla\phi)^2) + \xi(g_{\mu\nu}\Box_g - \nabla_\mu\nabla_\nu - G_{\mu\nu})\phi^2$$

$$\rho_\xi = (1-2\xi)U^\mu U^\nu (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1-2\xi}{n-2} U^\mu U_\mu m^2 \phi^2 - \frac{2\xi}{n-2} (\nabla \phi)^2 U^\mu U_\mu$$

$$-2\xi U^\mu U^\nu \phi \nabla_\mu \nabla_\nu \phi - \xi U^\mu U^\nu R_{\mu\nu} \phi^2 + \frac{2\xi^2}{n-2} U^\mu U_\mu R \phi^2 - \frac{2\xi}{n-2} (\phi P_\xi \phi) U^\mu U_\mu$$

$$\rho = U^\mu U^\nu (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{n-2} U^\mu U_\mu m^2 \phi^2 \text{ for } \xi = 0$$

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We can get an alternative expression if we use the field equation to replace the mass term

$$\rho_\xi = (1-2\xi)U^\mu U^\nu(\nabla_\mu\phi)(\nabla_\nu\phi) + \frac{1-2\xi}{n-2}(\phi\Box_g\phi)U^\mu U_\mu - \frac{2\xi}{n-2}(\nabla\phi)^2U^\mu U_\mu$$
$$-2\xi U^\mu U^\nu\phi\nabla_\mu\nabla_\nu\phi - \xi U^\mu U^\nu R_{\mu\nu}\phi^2 + \frac{1}{n-2}\xi R\phi^2U^\mu U_\mu - \frac{1}{n-2}(\phi P_\xi\phi)U^\mu U_\mu$$

$$\rho = U^\mu U^\nu(\nabla_\mu\phi)(\nabla_\nu\phi) + \frac{1}{n-2}(\phi\Box_g\phi)U^\mu U_\mu \text{ for } \xi = 0$$

Averaging the SED over a timelike geodesic γ parametrised by proper time τ and $U^\mu = f(\tau)\dot{\gamma}^\mu$

$$\begin{aligned}\int_\gamma d\tau \rho_\xi &= \int_\gamma d\tau - \frac{1-2\xi}{n-2} m^2 f^2(\tau) + \left(1 - 2\xi \frac{n-1}{n-2}\right) (\nabla_{\dot{\gamma}}\phi)^2 \\ &+ \frac{2\xi}{n-2} h^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + 2\xi [\nabla_{\dot{\gamma}}(f(\tau))\phi]^2 - 2\xi\phi^2 (f'(\tau))^2 \\ &- \xi R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu f^2(\tau) + \frac{2\xi^2}{n-2} R\phi^2\end{aligned}$$

$$\xi \in \left[0, \frac{n-2}{2(n-1)}\right]$$

Averaging the SED over a timelike geodesic γ parametrised by proper time τ and $U^\mu = f(\tau)\dot{\gamma}^\mu$

$$\int_\gamma d\tau \rho_\xi = \int_\gamma d\tau \underbrace{\left[-\frac{1-2\xi}{n-2} m^2 f^2(\tau)\right]}_{-} \underbrace{\left[+\left(1-2\xi\frac{n-1}{n-2}\right)(\nabla_{\dot{\gamma}}\phi)^2\right]}_{+}$$

$$\underbrace{\left[+\frac{2\xi}{n-2} h^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right]}_{+} \underbrace{\left[+2\xi[\nabla_{\dot{\gamma}}(f(\tau))\phi]^2\right]}_{+} \underbrace{\left[-2\xi\phi^2(f'(\tau))^2\right]}_{-}$$

$$\left[-\xi R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu f^2(\tau)\right] \left[+\frac{2\xi^2}{n-2}R\phi^2\right]$$

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$$\overset{+}{\boxed{+\frac{2\xi}{n-2} h^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi}} \overset{+}{\boxed{+2\xi[\nabla_{\dot{\gamma}}(f(\tau))\phi]^2}} \overset{-}{\boxed{-2\xi\phi^2(f'(\tau))^2}}$$

$$\left. \boxed{-\xi R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu f^2(\tau)} \boxed{+\frac{2\xi^2}{n-2}R\phi^2} \right]$$

$$\xi \in \left[0, \frac{n-2}{2(n-1)}\right]$$

$$\int_\gamma d\tau\, \rho_\xi \geq -\int_\gamma d\tau \left\{ \frac{1-2\xi}{n-2} m^2 f^2(\tau) + \xi\left(2(f'(\tau))^2 + R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu f^2(\tau) - \frac{2\xi^2}{n-2} R f^2(\tau)\right)\right\}\phi^2$$

Worldvolume

$$\int dVol\left(U^\mu U^\nu T_{\mu\nu} - \frac{1}{n-2} T U^\mu U_\mu\right) \geq -min\{\mathcal{B}_1, \mathcal{B}_2\}\,,$$

where

$$\mathcal{B}_1 = \int dVol\Bigg\{\frac{1-2\xi}{n-2} m^2 f^2(x) + \xi\Bigg[-\frac{2\xi}{n-2} R f^2(x) + (\nabla_\mu U^\mu)^2 + (\nabla_\mu U^\nu)(\nabla_\nu U^\mu)\Bigg]\Bigg\}\phi^2$$

and

$$\mathcal{B}_2 = \int dVol\Bigg\{-\frac{1-2\xi}{2(n-2)}(\Box_g f^2(x)) - \frac{\xi R}{n-2} f^2(x) + \xi[(\nabla_\mu U^\mu)^2 + (\nabla_\mu U^\nu)(\nabla_\nu U^\mu)] f^2(x)\Bigg\}\phi^2$$

Let ϕ_{max} be the maximum amplitude of the field

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Rescaling the sampling function

$$f_\lambda(x) = \frac{f(x/\lambda)}{\lambda^{n/2}} .$$

$$\begin{aligned}\int dVol\, \rho[f_\lambda] \quad \geq \quad & -\frac{1}{4}\int dVol \frac{1}{\lambda^{n+2}}\Big(- \frac{1-2\xi}{2(n-2)} \Box f^2(x/\lambda) + \xi[(V^\mu[\nabla_\mu f(x/\lambda)])^2 \\ & +(V^\nu[\nabla_\mu f(x/\lambda)])(V^\mu[\nabla_\nu f(x/\lambda)])]\Big)\phi^2_{\text{max}}\end{aligned}$$

Changing variables $x \to \lambda x$ gives

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So in the limit of large λ the bound goes to zero and we have

$$\lim_{\lambda\to\infty}\int dVol\, \rho[f_\lambda] \geq 0 \,.$$

$\Rightarrow$ Recover ASEC for flat spacetimes

Quantization

- Introduction of a unital *-algebra $\mathscr{A}(M)$ on our manifold M
- Generated by the objects $\Phi(f)$, $f \in \mathscr{D}(M)$ where $\mathscr{D}(M)$ is the space of complex-valued, compactly-supported, smooth functions on M
- The objects $\Phi(f)$ must obey the following relations:
    1. **Linearity**
       The map $f \to \Phi(f)$ is complex-linear,
    2. **Conjugation**
       $\Phi(f)^* = \Phi(\overline{f}) \qquad \forall f \in C_0^\infty(M),$
    3. **Field Equation**
       $\Phi(P_\xi f) = 0$,
    4. **Canonical Commutation Relations**
       $[\Phi(f), \Phi(h)] = iE_\xi(f, h)\mathbb{1}.$

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We only consider Hadamard states on our algebra $\omega_2^\Psi(x, y) = \langle\Phi(x)\Phi(y)\rangle_\Psi : \mathscr{D}(M) \times \mathscr{D}(M) \to \mathbb{C}$ has a prescribed singularity structure so that the difference between two states is smooth.

- The smeared local Wick polynomials of the form

$$\langle : \nabla^{(r)}\Phi\nabla^{(s)}\Phi : (f)\rangle_\Psi$$

are part of an extended algebra
- The dependence of these normal-ordered expressions on Ψ is unsatisfactory, because there is no canonical choice of a Hadamard state in a general curved spacetime.
- We need a prescription for finding algebra elements that qualify as local and covariant Wick powers. This might be done in various ways, expressing finite renormalisation freedoms. Hollands and Wald (2014) set out a list of axioms that we follow

In particular their form of Leibniz' rule gives e.g.

$$\frac{1}{2}(\nabla_\mu\nabla_\nu(\Phi^2))(f^{\mu\nu}) = (\nabla_\mu\Phi\nabla_\nu\Phi)(f^{\mu\nu}) + (\Phi\nabla_{(\mu}\nabla_{\nu)}\Phi)(f^{\mu\nu}),$$

where the left-hand side is understood distributionally, i.e.,

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where the left-hand side is understood distributionally, i.e.,

$$(\nabla_\mu \nabla_\nu (\Phi^2))(f^{\mu\nu}) = (\Phi^2)(\nabla_\nu \nabla_\mu f^{\mu\nu}).$$

$\Rightarrow$ While the quadratic normal ordered expressions obey Leibniz' rule, but not generally the field equation, the differences in their expectation values obey both.

$$((\nabla_\mu \Phi) P_\xi \Phi)_H = \frac{n}{2(n+2)} \nabla_\mu Q \mathbb{1}$$

$$(\nabla \Phi \nabla \Phi) = (\nabla \Phi \nabla \Phi)_H + \frac{n}{n^2 - 4} g_{\mu\nu} Q \mathbb{1}$$

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$$(\nabla\Phi\nabla\Phi) = (\nabla\Phi\nabla\Phi)_H + \frac{n}{n^2-4}g_{\mu\nu}Q\mathbb{1}$$

$\Rightarrow$ When taking differences of expectation values, multiples of the unit cancel

$$\langle(\nabla^{(r)}\Phi P_\xi\Phi)\rangle_0 - \langle(\nabla^{(r)}\Phi P_\xi\Phi)\rangle_\Psi = 0.$$

Any classical expression constructed from the stress energy tensor such as the strong energy density for the non-minimally coupled scalar field

$$\rho(x) = [\hat{\rho}(\phi \otimes \phi)]_c(x)\,,$$

has a quantized form in Hadamard state Ψ defined by

$$\langle \rho \rangle_\Psi(x) = [\hat{\rho}\colon \omega_2^\Psi \colon]_c(x)\,.$$

Here we used $:\omega_2^\Psi: \; = \omega_2^\Psi - \omega_2^0$, the normal ordering of the two-point function.

Strong quantum energy inequality

$$\rho_\xi = \ = (1-2\xi)(\nabla_{\dot\gamma}\phi)^2 - \frac{1-2\xi}{n-2}m^2\phi^2 - \frac{2\xi}{n-2}(\nabla\phi)^2 - 2\xi\phi(\nabla^2_{\dot\gamma}\phi)$$
$$-\xi\dot\gamma^\mu\dot\gamma^\nu R_{\mu\nu}\phi^2 + \frac{2\xi^2}{n-2}R - \frac{2\xi}{n-2}(\phi P_\xi\phi)\,.$$

We introduce the operators

$$\hat\rho_1 = \left(1-2\xi\frac{n-1}{n-2}\right)(\nabla_{v_0}\otimes\nabla_{v_0}) + \frac{2\xi}{n-2}\sum_{i=1}^{n-1}(\nabla_{v_i}\otimes\nabla_{v_i})\,,$$
$$\hat\rho_2 = -\frac{1-2\xi}{n-2}m^2(\mathbb{1}\otimes\mathbb{1}) - 2\xi(\mathbb{1}\otimes_{\mathfrak{s}}\nabla^2_{v_0})\,,$$
$$\hat\rho_3 = -\xi(\mathbb{1}\otimes_{\mathfrak{s}} R_{\mu\nu}v_0^\mu v_0^\nu\mathbb{1}) + \frac{2\xi^2}{n-2}(1\otimes_{\mathfrak{s}} R\mathbb{1}) + \frac{2\xi}{n-2}(\mathbb{1}\otimes_{\mathfrak{s}} P_\xi)\,,$$

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$$\hat\rho_2 = -\frac{1-2\xi}{n-2}m^2(\mathbb{1}\otimes\mathbb{1}) - 2\xi(\mathbb{1}\otimes_{\mathfrak{s}}\nabla^2_{v_0}),$$

$$\hat\rho_3 = -\xi(\mathbb{1}\otimes_{\mathfrak{s}} R_{\mu\nu}v_0^\mu v_0^\nu\mathbb{1}) + \frac{2\xi^2}{n-2}(\mathbb{1}\otimes_{\mathfrak{s}} R\mathbb{1}) + \frac{2\xi}{n-2}(\mathbb{1}\otimes_{\mathfrak{s}} P_\xi),$$

So the classical ρ can be expressed as

$$\rho = [\hat{\rho}(\phi \otimes \phi)]_c \,,$$

where

$$\hat{\rho} = \hat{\rho}_1 + \hat{\rho}_2 + \hat{\rho}_3 \,.$$

Then the weighted average of the quantum SED on γ is

$$\langle \rho^{\text{quant}} \circ \gamma \rangle_\Psi (f^2) = \int d\tau f^2(\tau) \langle \rho^{\text{quant}} \rangle_\Psi (\gamma(\tau)) \,,$$

$$\langle \rho^{\text{quant}} \circ \gamma \rangle_\Psi (f^2) = \left(\left[\hat{\rho}_1 : \omega_2^\Psi : \right]_c \circ \gamma \right)(f^2) + \left(\left[\hat{\rho}_2 : \omega_2^\Psi : \right]_c \circ \gamma \right)(f^2) + \left(\left[\hat{\rho}_3 : \omega_2^\Psi : \right]_c \circ \gamma \right) .$$

Point-splitting technique

$$\left(\left[\hat{\rho}_1 : \omega_2^\Psi :\right]_c \circ \gamma\right)(f^2)$$
$$= \int d\tau f^2(\tau)\phi^*(\hat{\rho}_1 : \omega_2^\Psi :)(\tau,\tau)$$
$$= \int d\tau d\tau' \delta(\tau-\tau')f(\tau)f(\tau')\phi^*(\hat{\rho}_1 : \omega_2^\Psi :)(\tau,\tau')$$
$$= \int_0^\infty \frac{d\alpha}{\pi}\int d\tau d\tau' e^{i\alpha(\tau-\tau')}f(\tau)f(\tau')\phi^*(\hat{\rho}_1 : \omega_2^\Psi :)(\tau,\tau')$$
$$= \int_0^\infty \frac{d\alpha}{\pi}\phi^*(\hat{\rho}_1 : \omega_2^\Psi :)(\bar{f}_\alpha, f_\alpha)$$
$$= \int_0^\infty \frac{d\alpha}{\pi}\phi^*(\hat{\rho}_1\, \omega_2^\Psi)(\bar{f}_\alpha, f_\alpha) - \int_0^\infty \frac{d\alpha}{\pi}\phi^*(\hat{\rho}_1\, \omega_2^0)(\bar{f}_\alpha, f_\alpha)\,,$$

where $f_\alpha(\tau) = e^{i\alpha\tau}f(\tau)$. Here ϕ^* is the distributional pull-back from $M \times M$ to $\mathbb{R}^2$ by the map $\phi(\tau,\tau') = (\gamma(\tau),\gamma(\tau'))$.

Theorem

Let ω_2^0 be the two-point function of a reference Hadamard state for the non-minimally coupled scalar field with coupling constant $\xi \in \left[0, \frac{n-2}{2(n-1)}\right]$ defined on a globally hyperbolic spacetime M with smooth metric g. Let γ be a timelike geodesic parametrised in proper time τ and let $f \in \mathcal{D}(\mathbb{R}, \mathbb{R})$. On the set of Hadamard states

$$\langle \rho^{quant} \circ \gamma \rangle_\Psi (f^2) \geq - \left[\mathfrak{D}_A(f) \mathbb{1} + (:\Phi^2: \circ \gamma)_\Psi(\mathfrak{D}_B(f)) + (:\Phi^2: \circ \gamma)_\Psi(\mathfrak{D}_C(f)) \right]$$

where

$$\mathfrak{D}_A(f) = \int_0^\infty \frac{d\alpha}{\pi} \left(\phi^*(\hat{\rho}_1 \, \omega_2^0)(\bar{f}_\alpha, f_\alpha) + 2\xi\alpha^2 \phi^* \omega_2^0(\bar{f}_\alpha, f_\alpha) \right) ,$$

$$\mathfrak{D}_B[f](\tau) = \frac{1-2\xi}{n-2} m^2 f^2(\tau) + 2\xi (f''(\tau))^2 ,$$

and

$$\mathfrak{D}_C[f](\tau) = f^2(\tau) \xi \left(R_{\mu\nu} \gamma^\mu \gamma^\nu - \frac{2\xi}{n-2} R \right)(\tau) .$$

Theorem

Let ω_2^0 *be the two-point function of a reference Hadamard state for the non-minimally coupled scalar field with coupling constant* $\xi \in \left[0, \frac{n-3}{2(n-2)}\right]$ *defined on a globally hyberbolic spacetime* M *with smooth metric* g*. Let* T *be a causal world tube,* $f(x)$ *a smearing function with compact support in* T *and* x *a spacetime point. On the set of Hadamard states*

$$\int_T dVol\langle\rho^{quant}\rangle_\Psi \geq -\min\left(\mathfrak{D}_1^\Psi, \mathfrak{D}_2^\Psi\right)$$

where

$$\mathfrak{D}_1^\Psi = \mathfrak{D}_1 +_\Psi \mathfrak{D}_1^A +_\Psi \mathfrak{D}_1^B\,, \qquad \mathfrak{D}_2^\Psi = \mathfrak{D}_2 +_\Psi \mathfrak{D}_2^A +_\Psi \mathfrak{D}_2^B\,,$$

$$\mathfrak{D}_1 = 2\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} \frac{d^n\alpha}{(2\pi)^n}(\tilde{\rho}_1^A\,\omega_2^0)(\bar{f_\kappa^\alpha}, f_\kappa^\alpha)$$

$$\mathfrak{D}_2 = 2\int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} \frac{d^n\alpha}{(2\pi)^n}(\tilde{\rho}_1^B\,\omega_2^0)(\bar{f_\kappa^\alpha}, f_\kappa^\alpha)\,,$$

$$
{}_{\Psi}\mathfrak{O}_1^A = \frac{1-2\xi}{n-2} m^2 \int_T dVol\, f^2(x) \langle :\omega_2: \rangle_\Psi + \mathfrak{A}_\Psi \,,
$$

$$
{}_{\Psi}\mathfrak{O}_2^A = \frac{1-2\xi}{2(n-2)} \int_T dVol (\Box_g f^2(x)) \langle :\omega_2: \rangle_\Psi + \mathfrak{A}_\Psi \,,
$$

$$
\begin{aligned}
\mathfrak{A}_\Psi \quad = \quad & -\frac{1}{2}\xi \int_T dVol\, \langle :\omega_2: \rangle_\Psi \nabla_\nu [(\nabla_\mu U^\mu) U^\nu] \\
& -\xi \int_T dVol\, \langle :\omega_2: \rangle_\Psi \nabla_\nu (U^\mu \nabla_\mu U^\nu)
\end{aligned}
$$

$$
{}_{\Psi}\mathfrak{O}_1^B = \xi \int_T dVol\, f^2(x) \left(V^\mu V^\nu R_{\mu\nu} - \frac{2\xi}{n-2} R \right) \langle :\omega_2: \rangle_\Psi \,,
$$

$$
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$$\mathfrak{O}_2 = 2 \int_{\mathbb{R}^+ \times \mathbb{R}^{n-1}} \frac{d^n \alpha}{(2\pi)^n} (\tilde{\rho}_1^B \, \omega_2^0)(\bar{f_\kappa^\alpha}, f_\kappa^\alpha) \,,$$

Minkowski space

In Minkowski we can use the vacuum state as the reference state

$$\omega_2^{\Omega}(t,\mathbf{x},t',\mathbf{x}') = \int d\mu(\mathbf{k}) e^{-i[(t-t')\omega(\mathbf{k})-(\mathbf{x}-\mathbf{x}')\mathbf{k}]},$$

$$d\mu(\mathbf{k}) = \int \frac{d^{n-1}\mathbf{k}}{(2\pi)^{n-1}} \frac{1}{2\omega(\mathbf{k})},$$

the measure, with $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$. The operator $\hat{\rho}_1$ simplifies

$$\hat{\rho}_1 = \left(1 - 2\xi\frac{n-1}{n-2}\right)(\partial_t \otimes \partial_{t'}) + \frac{2\xi}{n-2}\sum_{i=1}^{n-1}(\partial_i \otimes \partial_{i'}).$$

Theorem

In n-dimensional Minkowski space we have

$$\langle \rho^{quant} \circ \gamma \rangle_\Psi (f^2) \geq -\left[\mathfrak{D}_A(f) \mathbb{1} + (: \Phi^2 : \circ \gamma)_\Psi (\mathfrak{D}_B(f)) \right] ,$$

where

$$\mathfrak{D}_A(f) = \frac{S_{n-2}}{(2\pi)^n} \int_0^\infty d\alpha \int_0^\infty dk \frac{k^{n-2}}{\omega(k)} \left(\omega^2(k)(1-2\xi) - \frac{2\xi m^2}{n-2} + 2\xi\alpha^2 \right) \times |\hat{f}(\alpha + \omega(k))|^2 ,$$

and

$$\mathfrak{D}_B[f](\tau) = \frac{1-2\xi}{n-2} m^2 f^2(t) + 2\xi (f''(t))^2 ,$$

for $f \in \mathcal{D}(\mathbb{R} \times \mathbb{R})$ *and* $\xi \in \left[0, \frac{n-2}{2(n-1)}\right]$.

Theorem

In n-dimensional Minkowski space

$$\int_T dVol\langle\rho^{quant}\rangle_\Psi \geq -\min\left(\mathfrak{O}_1^\Psi, \mathfrak{O}_2^\Psi\right)$$

where

$$\mathfrak{O}_1^\Psi = \mathfrak{O}_1(f) +_\Psi \mathfrak{O}_1^A, \qquad \mathfrak{O}_2^\Psi = \mathfrak{O}_2(f) +_\Psi \mathfrak{O}_2^A,$$

and

$$\mathfrak{O}_1(f) = \frac{S_{n-2}}{(2\pi)^{2n-1}} \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} d^n\alpha \int_0^\infty dk \frac{k^{n-2}}{\omega(k)} \left[(1-2\xi)\omega(k)^2 - \frac{2\xi m^2}{n-2}\right] \times|\hat{f}_\kappa(\alpha+k)|^2$$

$$\mathfrak{O}_2(f) = \frac{S_{n-2}}{(2\pi)^{2n-1}} \int_{\mathbb{R}^+\times\mathbb{R}^{n-1}} d^n\alpha \int_0^\infty dk \frac{k^{n-2}}{\omega(k)} \left[(1-2\xi)\omega(k)^2 - \frac{m^2}{n-2}\right] \times|\hat{f}_\kappa(\alpha+k)|^2$$

$$ {}_\Psi\mathfrak{O}_1^A = \frac{1-2\xi}{n-2} m^2 \int_T dVol\, f^2(x) \langle :\omega_2:\rangle_\Psi + \mathfrak{A}_\Psi \,, $$

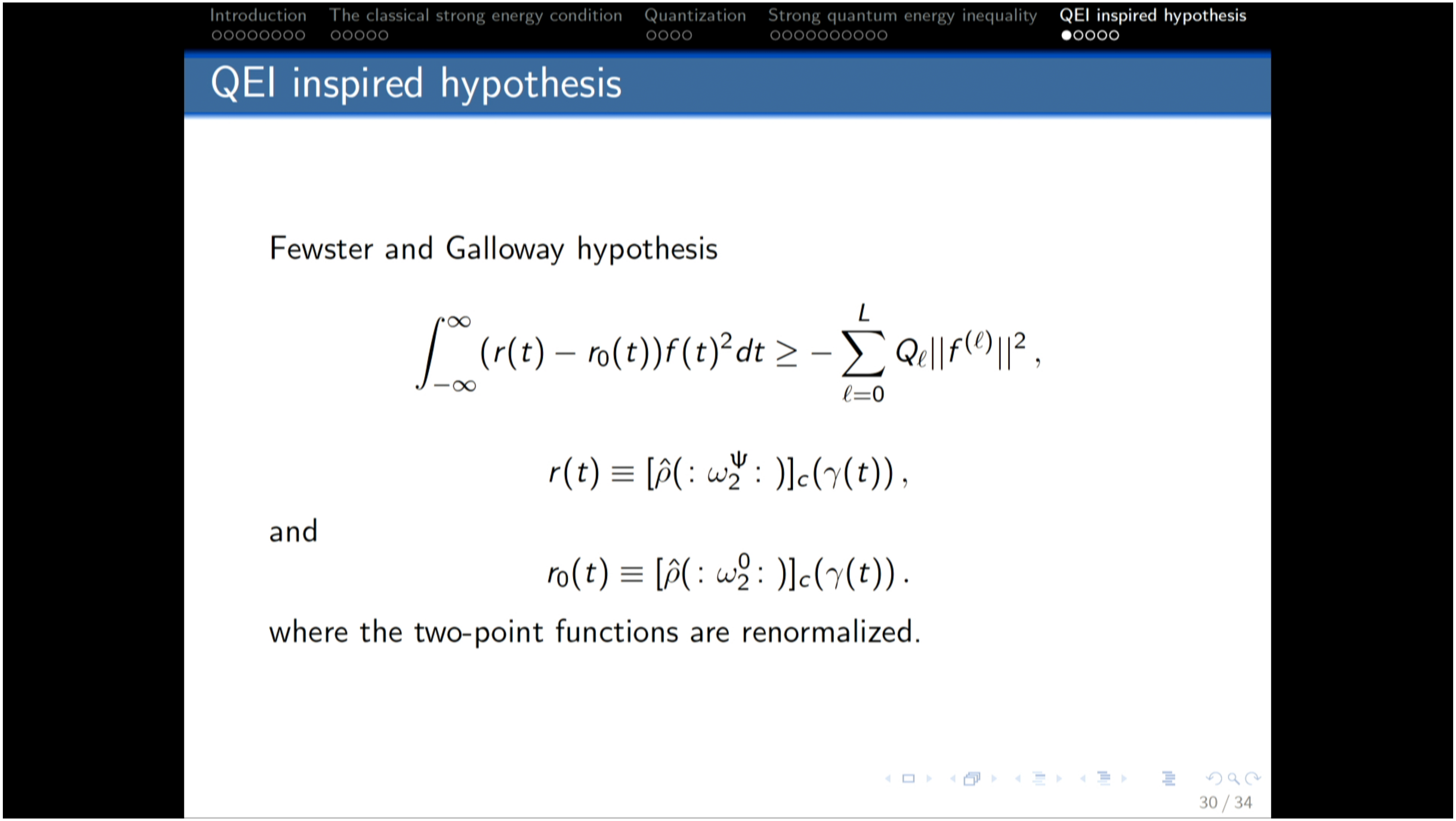
$$ {}_\Psi\mathfrak{O}_2^A = \frac{1-2\xi}{2(n-2)} \int_T dVol (\Box_g f^2(x)) \langle :\omega_2:\rangle_\Psi + \mathfrak{A}_\Psi \,, $$

for $\xi \in \left[0, \frac{n-3}{2(n-2)}\right]$.

QEI inspired hypothesis

Fewster and Galloway hypothesis

$$\int_{-\infty}^{\infty}(r(t)-r_0(t))f(t)^2dt \geq -\sum_{\ell=0}^{L} Q_\ell||f^{(\ell)}||^2\,,$$


Introduction
The classical strong energy condition
Quantization
Strong quantum energy inequality
QEI inspired hypothesis
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Fewster and Galloway hypothesis
$$\int_{-\infty}^{\infty}(r(t)-r_0(t))f(t)^2dt \geq -\sum_{\ell=0}^{L} Q_\ell ||f^{(\ell)}||^2 ,$$
$$r(t) \equiv [\hat{\rho}(:\omega_2^{\Psi}:)]_c(\gamma(t)) ,$$
and
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where the two-point functions are renormalized.
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where the two-point functions are renormalized.

Does the bound satisfy the hypothesis?

Minkowski space and minimally coupled field

$$\langle \rho^{\text{quant}} \circ \gamma \rangle_\Psi (f^2) \geq -\left(\frac{S_{n-2}}{(2\pi)^n} \int_m^\infty du \frac{u^n}{n} |\hat{f}|^2(u) + \frac{1}{n-2} m^2 f^2(t) (: \Phi^2 : \gamma)_\Psi \right) .$$

Does the bound satisfy the hypothesis?

Minkowski space and minimally coupled field

$$\langle \rho^{\text{quant}} \circ \gamma \rangle_\Psi (f^2) \geq - \left(\frac{S_{n-2}}{(2\pi)^n} \int_m^\infty du \frac{u^n}{n} |\hat{f}|^2(u) + \frac{1}{n-2} m^2 f^2(t) (: \Phi^2 : \gamma)_\Psi \right) .$$

Maximum expectation value in M for set of Hadamard states Ψ

$$\left| (: \Phi^2 : \gamma)_\Psi \right| \leq \phi_* ,$$

$\Rightarrow$ The bound satisfies the hypothesis

What about curved space?

What about curved space?

- If the support of the sampling function is constrained to be small relative to local curvature length scales it is reasonable to assume that the bound would remain close to the Minkowski bound
- This is for finite curvature but divergent curvature indicates singular behavior

Conclusions

- The classical singularity theorems have in their hypotheses easily violated energy conditions
- Fewster and Galloway proved singularity theorems with weaker, QEI inspired hypotheses

Conclusions

- The classical singularity theorems have in their hypotheses easily violated energy conditions
- Fewster and Galloway proved singularity theorems with weaker, QEI inspired hypotheses
- We developed a strong quantum energy inequality for the non-minimally coupled scalar field
- The Minkowski bound obeys the Fewster-Galloway hypothesis for the Hawking singularity theorem and it seems reasonable that similar bound could be derived for curved spacetime

Future directions

- Prove an absolute strong QEI for spacetime with curvature and verify that it satisfies the hypothesis of a singularity theorem
- Study the effects of backreaction in the bounds of QEI and determine if a singularity theorem can be developed with that hypothesis