

Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 14

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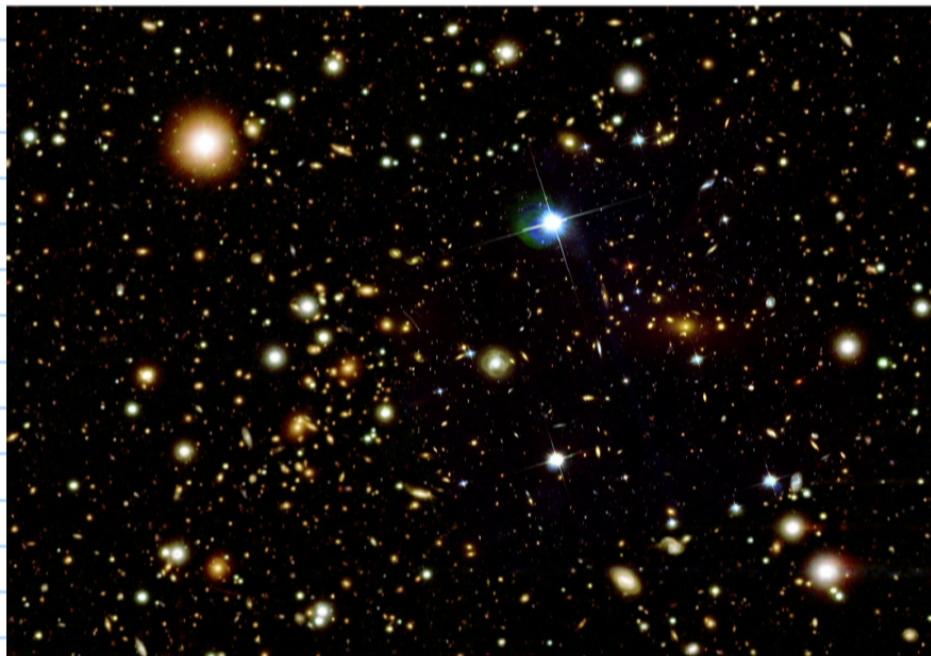
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Abstract:

QFT for Cosmology, Achim Kempf, Lecture 14

Note Title

Quantum field theory on FRW spacetimes.



Observations:

On scales $> 1 \text{ Gly}$:

□ The universe is roughly flat and Λ -dominated



Observations:

On scales $> 1 \text{ GLy}$:

- The universe is spatially very flat
- The cosmic expansion is very isotropic .

Friedmann Robertson Walker (FRW) spacetimes:

- Simplifying approximation:

Friedmann Robertson Walker (FRW) spacetimes:

◻ Simplifying approximation:

④

Spacetime is modeled as having

◻ no spatial curvature at all.

◻ entirely isotropic expansion

Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Wainwright & Ellis.

There are even solutions that only temporarily

Simplifying approximation:

Spacetime is modeled as having

no spatial curvature at all.

entirely isotropic expansion

Remark: It is known that the Einstein equations allow for highly nontrivial evolutions of non-isotropic spacetimes, see, e.g., the text by Wainwright & Ellis.

There are even solutions that only temporarily get very close to flatness. The Einstein equs are nonlinear!

With these assumptions, we choose convenient coordinates:

* Time coordinate t :



Definition: The motion of galaxies due to the cosmic expansion is called the Hubble flow.

Definition: The peculiar velocity is the "small" extra random velocity that galaxies can possess relative to the general Hubble flow.

Definition: As the time coordinate, t , let us use the proper time, t , of a freely streaming observer who has no peculiar velocity.



* Space coordinates:

It is convenient to use "comoving coordinates", x_1, x_2, x_3 :

- o At one time, t_0 , (say today) we set up an ordinary rectangular coordinate system.
- o Then, we let our spatial coordinate system shrink or grow to past or future, to match the Hubble flow.

Advantages:

- In the comoving coordinate system, galaxies have constant coordinates, except for possible peculiar motion.
- Waves keep their wave lengths monotonically

* The metric:

Recall that $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$

is the invariant 4-distance.

In our coordinates, $g_{\mu\nu}(x)$ must read:

because we use wrist watch "proper" time

$$g_{\mu\nu}(t, \vec{x}) = \begin{pmatrix} 1 & & & \\ -a^2(t) & -a^2(t) & -a^2(t) & \\ & -a^2(t) & -a^2(t) & \\ & & -a^2(t) & \end{pmatrix}$$

because our coordinate system's unit of length means over time a larger and larger proper length.

* The "scale factor":

- o The scale factor function $a(t)$ is needed to take into account the expansion when calculating distances.
- o Example: The proper distance d between two galaxies with comoving distance $(\Delta x_1, \Delta x_2, \Delta x_3)$ at proper time t is:

$$d = \sqrt{|g_{\mu\nu}(t_0) \Delta x^\mu \Delta x^\nu|}$$
$$= a(t) \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2}$$

* Dynamics of $a(t)$:

The function $a(t)$ is determined by all equations of motion:

1. Calculate the energy momentum tensor $T_{\mu\nu}(t, \vec{x})$

contributions of at least the most important fields, say $E_i(v, t)$.

2. Solve, simultaneously:

* The equations of motion for the fields E_i

* The Einstein equation for $g_{\mu\nu}$,

while setting $g_{\mu\nu}(t, \vec{x}) = \begin{pmatrix} 1 & a^2 \\ -a^2 & a^2 \end{pmatrix}$:

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while setting $g_{\mu\nu}(t, \vec{x}) = \begin{pmatrix} 1 & -a^2 \\ -a^2 & a^2 \end{pmatrix}$:

$$R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) + \Lambda g_{\mu\nu}(x) = 8\pi G T_{\mu\nu}(x)$$

* Semi-classical approximation

We can solve these classically, but not quantum mechanically:

Can quantize only \mathcal{L}_i , not $g_{\mu\nu}$.

\Rightarrow need to "make quantum $T_{\mu\nu}(t, \vec{x})$ classical" for Einstein eqn!

\rightsquigarrow One uses: $\bar{T}_{\mu\nu}(x) := \langle S | T_{\mu\nu}(t, \vec{x}) | \Omega \rangle$

Problem: Energy & momentum are naturally nonlocal because of uncertainty principle.

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Convenient Definition: The conformal time coordinate, η .

□ Recall that:

$$g_{\mu\nu}(t, \vec{x}) = \begin{pmatrix} 1 & -a^2(t) & 0 \\ -a^2(t) & -a^2(t) & 0 \\ 0 & 0 & -a^2(t) \end{pmatrix}$$

□ It would be convenient if $g_{\mu\nu}$ were proportional to $\eta_{\mu\nu} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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□ It would be convenient if $g_{\mu\nu}$ were proportional to $\eta_{\mu\nu} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$.

□ This can be achieved by choosing a new time coordinate η , so that time also has a prefactor a^2 , i.e., so that:

$$(1+\eta)^2 = a^2(t)(1-\eta)^2$$

□ Using conformal time and comoving spatial coordinates the metric reads:

$$g_{\mu\nu}(\eta, \vec{x}) = a^2(\eta) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = a^2(\eta) \eta_{\mu\nu}$$

do not mix up



□ This also implies:

$$g^{\mu\nu}(\eta, \vec{x}) = \bar{a}^{-2}(\eta) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \bar{a}^{-2}(\eta) \eta^{\mu\nu}$$

The Klein Gordon field in FRW spacetimes

□ Neglecting a potential $V(\phi)$ for now, we obtain

the

action of the "free K.G. field on the FRW background":

$$S_{KG} = \int \left(\frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) \sqrt{|g|} d^4x$$

$$= \int \left(\frac{1}{2} a^{-2}(\eta) \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4 d\eta d^3x$$

□ Thus, from the general Euler Lagrange equation

$$\partial_\mu \frac{\partial S}{\partial \dot{\phi}_\mu} - \frac{\partial S}{\partial \phi} = 0$$

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$$\left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\nu} g^{\mu\nu} \sqrt{g} \frac{\partial}{\partial x^\nu} + m^2 \right) \phi(x) = 0$$

$$\left(\frac{1}{a^4(\eta)} \frac{\partial}{\partial x^\nu} \gamma^{\mu\nu} a^2 \frac{\partial}{\partial x^\nu} + m^2 \right) \phi(x) = 0$$

$$\left(\frac{1}{a^4(\eta)} \gamma^{\mu\nu} a^2(\eta) \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} + \frac{1}{a^4(\eta)} 2a' a \frac{\partial}{\partial x^0} + m^2 \right) \phi(x) = 0$$

$$\boxed{\phi''(\eta, \vec{x}) + 2 \frac{a'(\eta)}{a(\eta)} \phi'(\eta, \vec{x}) - \Delta \phi(\eta, \vec{x}) + a^2(\eta) m^2 \phi(\eta, \vec{x}) = 0}$$

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$$\phi''(\eta, \vec{x}) + 2 \frac{a'(\eta)}{a(\eta)} \phi'(\eta, \vec{x}) - \Delta \phi(\eta, \vec{x}) + a^2(\eta) m^2 \phi(\eta, \vec{x}) = 0$$

This is the K.G. eqn. in FRW spacetimes!

Problem: the equation above has this general form:

$$\phi'' + \cancel{\alpha} \phi' + \cancel{\beta} \phi = 0$$

\nearrow
a time-dependent
friction-like term
that is entirely new.

\nwarrow a term that also occurs in the usual
harmonic oscillator. Notice though
that it is now time-dependent.

Strategy: Use a new, re-scaled, field variable χ :

We try to change from $\phi(\eta, \vec{x})$ to a new field variable, say $\chi(\eta, \vec{x})$, so that the equation of motion for χ has no "friction"-type term.

This simple ansatz succeeds:

$$\chi(\eta, \vec{x}) := a(\eta) \phi(\eta, \vec{x})$$

Namely:

$$\text{we have: } \dot{\phi} = \frac{\partial}{\partial \eta} \frac{1}{a} \chi = -\frac{a'}{a^2} \chi + \frac{1}{a} \chi'$$

$$x(\eta, \vec{r}) := a(\eta) \phi(\eta, \vec{r})$$

Namely:

$$\text{we have: } \phi' = \frac{\partial}{\partial \eta} \frac{1}{a} x = -\frac{a'}{a^2} x + \frac{1}{a} x'$$

$$\text{and: } \phi_{,i} = \frac{\partial}{\partial x^i} \frac{1}{a(\eta)} x(\eta, \vec{r}) = \frac{1}{a} x_{,i} \text{ for } i=1,2,3$$

Using these, the action in terms of x becomes:

$$S_{kg} = \int \frac{1}{2} \left(x'^2 - \sum_{i=1}^3 x_{,i}^2 - \underbrace{\left(m^2 a^2 - \frac{a''}{a} \right) x^2}_{\text{constant}} \right) dy d^3x$$

Using these, the action in terms of x becomes:

$$S_{x_0} = \int \frac{1}{2} \left(\dot{x}^2 - \sum_{i=1}^3 \dot{x}_i^2 - \underbrace{\left(m^2 a^2 - \frac{a''}{a} \right) x^2}_{\text{Note that this term is like a time-dependent mass term } m_{\text{eff}}^2(\eta)} \right) dy d^3x$$

Exercise: verify

□ Equation of motion:

* Do

$$\frac{\delta S}{\delta \phi(y, \vec{x})} = 0 \quad \text{and} \quad \frac{\delta S}{\delta x(y, \vec{x})} = 0$$

wield equivalent equations of motion?

□ Equation of motion:

* Do

$$\frac{\delta S'}{\delta \phi(y, z)} = 0 \quad \text{and} \quad \frac{\delta S'}{\delta x(y, z)} = 0$$

yield equivalent equations of motion?

* Yes, because:

$$0 = \frac{\delta S'}{\delta \phi} = \frac{\delta S}{\delta x} \frac{\delta x}{\delta \phi}$$

directly in terms of x from $S[x]$, to obtain:

Exercise: verify!

$$x'' - \Delta x + \left(m^2 a^2 - \frac{a''}{a}\right)x = 0 \quad (\text{EoM!})$$

Remark:

We could have obtained this equation of motion directly from that of ϕ by change of variable. But finding the action for x was still worthwhile, namely to get the conjugate to x !

□ Preparation for quantization:

□ Preparation for quantization:

* We need the canonically conjugate field

$$\Pi^{(x)}(\eta, \vec{x})$$

to the field $\mathcal{X}(\eta, \vec{x})$, i.e., the Legendre transform of X' .

* To this end, we consider the Lagrangian:

$$L = \int \frac{1}{2} \left(\dot{x}'^2 - \sum_{i=1}^3 \dot{x}_i^2 - \left(m^2 a^2 - \frac{a''}{a} \right) x \right) d^3 x$$

* Thus, the Legendre transformed variable reads:

$$\underline{\underline{(x)}} \dots \underline{\underline{S}} \dots \underline{\underline{L}} \dots$$

* To this end, we consider the Lagrangian:

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* Thus, the Legendre transformed variable reads:

$$\pi^{(x)}(\eta, \vec{x}) := \frac{\delta L}{\delta \dot{x}'} = x'(\eta, \vec{x}) \quad (\text{Eq 1.2})$$

* Which is the field that is conjugate to ϕ ?

$$S_{\text{k.g.}} = \int \left(\frac{1}{2} a^{-2}(\eta) \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4 d\eta d^3x$$

* Which is the field that is conjugate to ϕ ?

$$S_{\text{K.G.}} = \int \left(\frac{1}{2} a^{-2}(\eta) \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} m^2 \phi^2 \right) a^4 d\eta d^3x$$

\Rightarrow The field $\pi^{(\phi)}$ which is conjugate to ϕ reads:

$$\pi^{(\phi)} := \frac{\delta L}{\delta \dot{\phi}} = a^2 \dot{\phi}'$$

* Compare:

$$\begin{aligned}\pi^{(x)} &= x' \\ &= (a \phi)' \\ &= a \phi' + a' \phi\end{aligned}$$

□ Proof: Only the first CCR is nontrivial to check:

$$\begin{aligned} [\vec{x}(\gamma, \vec{x}), \hat{\pi}^{(x)}(\gamma, \vec{x}')] &= [\alpha(\gamma) \hat{\phi}(\gamma, \vec{x}), \frac{1}{\alpha(\gamma)} \hat{\pi}^{(\phi)}(\gamma, \vec{x}') + \alpha'(\gamma) \hat{\phi}(\gamma, \vec{x}')] \\ &= [\hat{\phi}(\gamma, \vec{x}), \hat{\pi}^{(\phi)}(\gamma, \vec{x}')] \\ &= i\delta^3(\vec{x} - \vec{x}') \end{aligned}$$

□ Thus, the change from ϕ to x is fairly trivial.

Notice, however:

$$\begin{array}{ccc} L & \xrightarrow{\text{L.T. } \phi \text{ replaced by } \pi^\phi} & H^{(\phi)} := \int \phi' \pi^{(\phi)} d^3x - L \\ & \searrow \text{L.T. } x^\phi \text{ replaced by } \pi^{x^\phi} & \left. \right\} \text{they have no reason to be the same!} \\ & & L^{(\pi)} := \int \pi' \pi^{(x)} d^3x - L \end{array}$$

□ Question:

How can both be valid generators of time evolution,

i.e., how can we have:

$$i\dot{\hat{\phi}}' = [\hat{\phi}, \hat{H}^{(\phi)}] \quad \text{and} \quad i\dot{\hat{x}}' = [\hat{x}, \hat{H}^{(x)}]$$

and yet $\hat{H}^{(\phi)} \neq \hat{H}^{(x)}$?

□ Should there not be one Hamiltonian for all variables?

□ Answer: Yes, and it is, of course $\hat{H}^{(\phi)}$.

This extra term is due if the variable $\hat{\phi}$ has also explicit time-dependence, e.g., $\hat{\phi} = \cos(a\hat{t} + b\hat{t}^2)\hat{q} + c\hat{p}$, or here: $\hat{x} = \frac{1}{a}\hat{\phi}$.

i.e., how can we have:

$$i\dot{\hat{\phi}}' = [\hat{\phi}, \hat{H}^{(0)}] \quad \text{and} \quad i\dot{\hat{x}}' = [\hat{x}, \hat{H}^{(0)}]$$

and yet $\hat{H}^{(0)} \neq \hat{H}'^{(0)}$?

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This extra term is there if the variable \hat{Q} has also explicit time-dependence, e.g., $\hat{Q} = \cos(\omega t + \theta) \hat{q} + c \hat{p}$, or here: $\hat{x} = \frac{1}{a} \hat{\phi}$.

$$\text{Recall that in QM: } i\dot{\hat{Q}}' = [\hat{Q}, \hat{H}] + i\frac{\partial}{\partial t} \hat{Q}$$

◻ Explicitly:

* From $\hat{x} = a\hat{\phi}$ and $i\hat{\phi}' = [\hat{\phi}, \hat{H}^{(b)}]$ we obtain:

$$i\left(\frac{1}{a}\hat{x}\right)' = \frac{1}{a} [\hat{x}, \hat{H}^{(b)}]$$

$$\Rightarrow i\frac{1}{a}\hat{x}' - i\frac{a'}{a^2}\hat{x} = \frac{1}{a} [\hat{x}, \hat{H}^{(b)}]$$

$$\Rightarrow i\hat{x}' = [\hat{x}, \hat{H}^{(b)}] + i\frac{a'}{a}\hat{x}$$

* But we also have:

$$i\hat{x}' = [\hat{x}, \hat{H}^{(x)}]$$

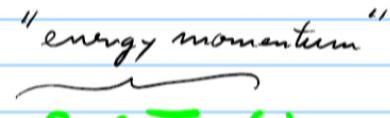
\Rightarrow We must have: $\hat{H}^{(x)} \neq \hat{H}^{(b)}$

Since there are multiple Hamiltonians, which, if anyone, is the energy?

◻ One usually defines the energy as the generator of time evolution. We saw that in the presence of gravity this is ambiguous: one can define many different Hamiltonians for the same theory (same action).

◻ Therefore, with Einstein, we define the energy (density) not as the generator of time evolution but as a generator of curvature:

◻ Recall: The Einstein equation



□ Recall: The K.G. field's energy-momentum tensor

$$T_{\mu\nu}^{\text{KG}}(\eta, \vec{x}) = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \left[\frac{1}{2} g^{\rho\sigma} \phi_{,\rho} \phi_{,\sigma} - \frac{1}{2} m^2 \phi^2 \right]$$

□ Consider $T_{00}(\eta, \vec{x})$, which is called the "energy density":

Note: In differential geometry, there is also another use of the term "density":

For every tensor, say $Q_{\mu\nu}$, there is a so-called "tensor density" $\tilde{Q}_{\mu\nu}$, defined as $\tilde{Q}_{\mu\nu} := Q_{\mu\nu} \sqrt{g}$, which absorbs the obligatory measure factor in integrations.

$$T_{00}(\eta, \vec{x}) = a^{-4} \frac{1}{2} \pi^{(\phi)}{}^2 + \frac{1}{2} \sum_{i=1}^3 \phi_{,i}{}^2 + \frac{a^2}{2} m^2 \phi^2 \quad (\text{T})$$

□ Exercises:

a) Verify (T).

b) Calculate $H^{(\phi)}$.

Notice that $H^{(\phi)}$ is not a scalar.

c) Show that $H^{(\phi)}(\eta) = \int T_0^0(\eta, \vec{x}) \sqrt{g} d^3x$.

$$T_{r\nu}(\eta, \vec{x}) = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^r} = \phi_{,r} \phi_{,\nu} - g_{\mu\nu} \left[\frac{1}{2} g^{\mu\rho} \phi_{,\nu} \phi_{,\rho} - \frac{1}{2} m^2 \phi^2 \right]$$

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b) Calculate $H^{(\phi)}$.

Notice that $H^{(\phi)}$ is not a scalar.

c) Show that $H^{(\phi)}(\eta) = \int_{\mathbb{R}^3} T_{00}(\eta, \vec{x}) \sqrt{g} d^3x$.

d) Calculate $H^{(0)}(\eta)$.