

Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 13

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Abstract:

QFT for Cosmology, Achim Kempf, Lecture 13

Note Title

Recall: The free Klein Gordon quantumfield in a generic curved space-time must obey:

$$\hat{\phi}^+(x,t) = \hat{\phi}(x,t), \quad \hat{\pi}^+(x,t) = \hat{\pi}(x,t) \quad (\text{HC})$$

$$i\dot{\hat{\phi}}(x,t) = [\hat{\phi}(x,t), \hat{H}(t)], \quad i\dot{\hat{\pi}}(x,t) = [\hat{\pi}(x,t), \hat{H}(t)] \quad (\text{EoM})$$

which can be written in this form:

$$\left(\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} g^{\mu\nu} \sqrt{|g|} \frac{\partial}{\partial x^\nu} + m^2 \right) \hat{\phi}(x,t) = 0, \quad \hat{\pi}(x,t) = \sqrt{|g|} g^{0\nu} \frac{\partial}{\partial x^\nu} \hat{\phi}(x,t) \quad (\text{EoM})$$

And: On all spacelike hypersurfaces, Σ , the CCRs must hold:

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0, \quad [\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0, \quad [\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar \delta^3(x-x') \quad (\text{CCR})$$

Conservation of the CCRs? This is implied by the self-adjointness of \hat{H} :

□ As always in quantum theory, the time evolution operator $\hat{U}(t, t_0) = T e^{i \int_{t_0}^t \hat{H}(t') dt'}$ is unitary: $\hat{U}^\dagger = \hat{U}^{-1}$.

□ It allows one to express the time evolution of the observables, such as field operators, through:

$$\hat{\phi}(x, t) = \hat{U}(t, t_0) \hat{\phi}(x, t_0) \hat{U}^\dagger(t, t_0)$$

$$\hat{\pi}(x, t) = \hat{U}(t, t_0) \hat{\pi}(x, t_0) \hat{U}^\dagger(t, t_0)$$

□ Thus: $[\hat{\phi}(x, t), \hat{\pi}(x', t)] = [\hat{U} \hat{\phi}(x, t_0) \hat{U}^{-1}, \hat{U} \hat{\pi}(x', t_0) \hat{U}^{-1}]$
 $= \hat{U} [\hat{\phi}(x, t_0), \hat{\pi}(x', t_0)] \hat{U}^{-1}$
 $= \hat{U} i \delta^3(x - x') \hat{U}^{-1} = i \delta^3(x - x')$

Problem: Is the quantization coordinate system independent?

Assume we solve the theory as above.

Now if we change coordinate system, and therefore the choices of $\{\Sigma\}$, would the CCRs still hold on every spacelike hypersurface Σ' ?

Proposition: Yes: if CCRs hold in one coordinate system, then they hold in all: The CCRs keep holding when deforming a Σ to a Σ' .

Proof: Rewrite the symplectic form (f, h) more abstractly:

Recall: $f, g \in V$ are solutions of KG eqn.

$$(f, h) := \int_{\Sigma} d\Sigma_r \sqrt{g} g^{rv} (f \partial_v h - h \partial_v f)$$

$$= \int_{\Sigma} \tilde{j}$$

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$$j^\mu(x,t) := g^{\mu\nu} (f \partial_\nu h - h \partial_\nu f)$$

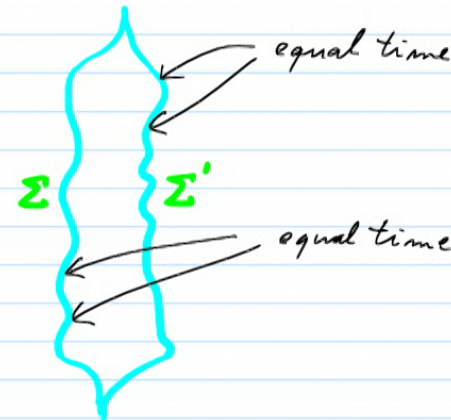
and from it the differential 3-form:

$$\tilde{j} := i_j \Omega$$

\downarrow inner derivation
 \uparrow Volume 4-form $|g| d^4x$

\uparrow 3-form

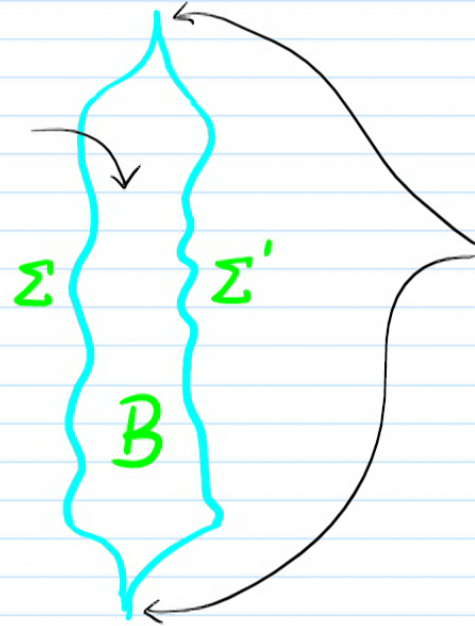
We need to show that the value of the symplectic form stays the same when deforming Σ :



□ Now integrate over both Σ and Σ' :

Σ and Σ' enclose
the 4-dim. volume B

$$\Sigma \cup \Sigma' = \partial B$$



We close the hypersurfaces arbitrarily far out; in the limit at spatial infinity.

□ Use Stokes' theorem:

$$\int_{\partial B} \tilde{j} = \int_B d\tilde{j}$$

Notice:

If we can show $d\tilde{j} = 0$ we are done!

That's because then:

$$0 = \int_{\Sigma \cup \Sigma'} \tilde{j} = \int_{\Sigma} \tilde{j} + \int_{\Sigma'} \tilde{j} = - \int_{\Sigma} \tilde{j} + \int_{\Sigma'} \tilde{j}$$

Both j pointing
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□ Indeed:

$$d\tilde{j} = d(i_j \Omega) = \text{div}_\Omega j = (\sqrt{|g|} j^\mu)_{,\mu} d^4x$$

Here:

$$(\sqrt{|g|} j^\mu)_{,\mu} = \left(\sqrt{|g|} g^{\mu\nu} \overbrace{(f \partial_\nu h - h \partial_\nu f)}^{(\text{is definition of } j)} \right)_{,\mu}$$

Recall:

$(\square + m^2)\phi = 0$ reads:

$$\frac{1}{\sqrt{|g|}} (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi)_{,\mu} + m^2 \phi = 0$$

and the f and h are

solutions of the K. G. eqn!

$$= \cancel{\sqrt{|g|} g^{\mu\nu} \partial_\nu h \partial_\mu f} + f \overbrace{(\sqrt{|g|} g^{\mu\nu} \partial_\nu h)_{,\mu}}^{=-m^2 h \sqrt{|g|}}$$

$$- \cancel{\sqrt{|g|} g^{\mu\nu} \partial_\mu h \partial_\nu f} - h \overbrace{(\sqrt{|g|} g^{\mu\nu} \partial_\nu f)_{,\mu}}^{=-m^2 f \sqrt{|g|}}$$

$$= \sqrt{|g|} (-f m^2 h + h m^2 f) = 0 \quad \checkmark$$

\rightsquigarrow We finally proved that, for globally hyperbolic spacetimes, there always exist mode functions $\{u_k(x,t)\}$ so that our ansatz for $\hat{\phi}$ and $\hat{\pi}$ also obeys the CCRs at all time and indeed $\forall \Sigma$:

$$\sqrt{|g|} g^{0\nu} \int \left(u_k(x,t) \frac{\partial}{\partial x^\nu} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x^\nu} u_k(x',t) \right) d^3k = i\delta^3(x-x') \quad (W)$$

Example:

For Minkowski space, we had found this solution for the noninteracting Klein Gordon field:

$$\hat{\phi}(x,t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{-i\omega_k t + ikx} + a_k^\dagger e^{i\omega_k t - ikx} \right) d^3k$$

We read off:
$$u_k(x,t) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k t + ikx}$$

Now: Verify the CCR condition, (W):

□ Here: $\sqrt{|g|} = 1$ and $g^{0\nu} = \delta_{0\nu}$.

□ Thus, the LHS of Eq. (W) reads:

$$\int u_k(x,t) \frac{\partial}{\partial x^0} u_k^\dagger(x',t) - u_k^\dagger(x,t) \frac{\partial}{\partial x^0} u_k(x',t) d^3k$$

$$= \frac{1}{(2\pi)^3} \int \frac{1}{2\omega_k} \left[e^{-i\omega_k t + ikx} (i\omega_k) e^{i\omega_k t - ikx'} - e^{i\omega_k t - ikx} (-i\omega_k) e^{-i\omega_k t + ikx'} \right] d^3k$$

$$= \frac{1}{(2\pi)^3} \int \frac{2i\omega_k}{2\omega_k} e^{ik(x-x')} d^3k \stackrel{\text{Fourier}}{=} i\delta^3(x-x') \checkmark$$

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□ Example: 2-dim. solution subspace.

Assume: $v_1, v_2 \in V$ are a Darboux basis, so
so that $(,)$ reads $\begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix}$.

Q: Can we change basis, $v_i = Q\bar{v}_i$, so that $(,)$ keeps
that matrix form? Is there a matrix Q so that

$$Q^t \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} Q = \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} ?$$

A: Yes, any change of basis $Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

with $ad - bc = 1$ will do. (Exercise: check)

□ More generally: Also different boxes $\begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix}$ can get mixed!

$$\left(\begin{array}{l} \text{Since } (v, w) = v^t \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} w \text{ we require} \\ v^t \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} w = \bar{v}^t \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} \bar{w} \quad \forall \bar{v}, \bar{w} \\ (B\bar{v})^t \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} (B\bar{w}) = \bar{v}^t \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} \bar{w} \quad \forall \bar{v}, \bar{w} \\ \text{i.e.: } B^t \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} B = \begin{pmatrix} 0, 1 \\ -1, 0 \end{pmatrix} \end{array} \right) \rightarrow$$

Exercise: □ Recall that with respect to the hermitian bi-linear form \langle, \rangle of Lecture 12, the mode functions $\{u_n\}$ obey:

$$\left. \begin{aligned} \langle u_n, u_m \rangle &= \delta_{nm} \\ \langle u_n^*, u_m^* \rangle &= -\delta_{nm} \\ \langle u_n, u_m^* \rangle &= 0 = \langle u_n^*, u_m \rangle \end{aligned} \right\} (*)$$

□ Now consider an invertible change of basis (in the space of complex-number-valued solutions of the K.G. eqn) to new mode functions:

$$\bar{u}_n := \sum_m (A_{nm} u_m + B_{nm} u_m^*)$$

□ Show that for the $\{\bar{u}_n\}$ to qualify as mode functions, i.e., for them to obey $(*)$, i.e., $\langle \bar{u}_n, \bar{u}_m \rangle = \delta_{nm}$ etc, A, B must obey:

$$A^+ A - B^t B^* = 1 \text{ and } A^+ B - B^t A^* = 0$$

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Non-uniqueness of the solution (A):

- Clearly, this means that there are infinitely many solutions of the form of (A) to HC, EoM and CCRs:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t) a_k + u_k^*(x,t) a_k^\dagger$$

$$\hat{\phi}(x,t) := \sum_k \bar{u}_k(x,t) \bar{a}_k + \bar{u}_k^*(x,t) \bar{a}_k^\dagger$$

$$\hat{\phi}(x,t) := \sum_k \bar{\bar{u}}_k(x,t) \bar{\bar{a}}_k + \bar{\bar{u}}_k^*(x,t) \bar{\bar{a}}_k^\dagger \quad \text{etc, etc...}$$

- Correspondingly, we obtain different Fock bases:

$$\text{Either: } a_k |0\rangle = 0 \quad |n_k\rangle := \frac{1}{\sqrt{n!}} (a_k^\dagger)^n |0\rangle$$

$$\text{Or: } \bar{a}_k |\bar{0}\rangle = 0 \quad |\bar{n}_k\rangle := \frac{1}{\sqrt{n!}} (\bar{a}_k^\dagger)^n |\bar{0}\rangle \quad \text{etc, etc...}$$

□ Q: Do these solutions of the QFT

○ describe different physics, or

○ do they differ by a mere change of basis in Fock space and so describe the same physics?

□ A: It depends!

1) Assume first we can impose IR and UV cutoffs with negligible consequences

* This means we truncate to a finite (though large) number of independent mode oscillators, a_n, a_n^\dagger .

* Then, the following theorem implies that all solutions to HC, EoM, CCR differ merely by a change of basis:

Theorem (Stone and von Neumann):

* Assume in a Hilbert space, \mathcal{H} , the operators \hat{x}_i, \hat{p}_j obey:

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \quad [\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j] \quad i, j \in \{1, \dots, N\}$$

* Assume that in a Hilbert space $\tilde{\mathcal{H}}$ other operators \tilde{x}_i, \tilde{p}_j also obey:

$$[\tilde{x}_i, \tilde{p}_j] = i\delta_{ij} \quad [\tilde{x}_i, \tilde{x}_j] = 0 = [\tilde{p}_i, \tilde{p}_j] \quad i, j \in \{1, \dots, N\}$$

* Assume that the representations are irreducible (i.e., no invariant subspace)

We can assume that $\tilde{\mathcal{H}} = \mathcal{H}$ because all separable Hilbert spaces (the usual: with countable bases) are unitarily equivalent.

Then: there exists a unitary operator \hat{U} so that:

$$\tilde{x}_i = U \hat{x}_i U^\dagger \quad \tilde{p}_j = U \hat{p}_j U^\dagger \quad (\text{i.e., a change of basis})$$

* **Remark:** Strictly speaking, there can be pathological cases.

The pathological cases can be avoided by requiring representations of the CCRs of the (bounded and therefore better behaved) operators:

$$e^{i\alpha \hat{x}_i}, e^{i\beta \hat{p}_i}$$

* **Application to QM and to UV&IR regularized QFT:**

$$\text{Consider } \hat{x}_n := \frac{1}{\sqrt{2}} (a_n + a_n^\dagger), \hat{p}_n := \frac{-i}{\sqrt{2}} (a_n - a_n^\dagger)$$

$$\text{and } \bar{x}_n := \frac{1}{\sqrt{2}} (\bar{a}_n + \bar{a}_n^\dagger), \bar{p}_n := \frac{-i}{\sqrt{2}} (\bar{a}_n - \bar{a}_n^\dagger) \text{ etc.}$$

The theorem of Stone & v. Neumann implies that

$$a_n = \hat{U} \bar{a}_n \hat{U} \text{ with } \hat{U} \text{ unitary.}$$

\Rightarrow All solutions are the same up to a mere change of basis.

2.) Consider now the possibility that we cannot truncate to a finite number of degrees of freedom.

Q: When would this happen?

A: E.g., phase transitions formally need systems with an infinite number of degrees of freedom.

Then: The QFT can have unitarily non-equivalent solutions, that differ physically: different "phases".

Underlying math of non-equivalent representations?

Assume $\langle a|b\rangle = d$ with $0 < d < 1$, i.e., not \perp

Then $(\langle a|\langle a|\langle a|\dots\langle a|) \overbrace{(|b\rangle|b\rangle|b\rangle\dots|b\rangle)}^N = d^N$, i.e., not \perp

But for $N = \infty$ have $|a\rangle|a\rangle\dots|a\rangle \perp |b\rangle|b\rangle\dots|b\rangle$, so that them can no longer use $|a\rangle|a\rangle\dots|a\rangle$ to help linearly combine, e.g., $|b\rangle|b\rangle\dots|b\rangle$.

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From now on: We will assume IR & UV cutoffs are possible and that Stone v. Neumann therefore applies.

Therefore:

□ No matter which set of suitable mode functions

$$\{u_n(x,t)\} \text{ or } \{\bar{u}_n(x,t)\} \text{ or } \{\bar{\bar{u}}_n(x,t)\}, \dots$$

we choose, we obtain the same solution

$$\begin{aligned}\hat{\phi}(x,t) &= \sum_{\vec{k}} u_{\vec{k}}(x,t) a_{\vec{k}} + u_{\vec{k}}^*(x,t) a_{\vec{k}}^\dagger \\ &= \sum_{\vec{k}} \bar{u}_{\vec{k}}(x,t) \bar{a}_{\vec{k}} + \bar{u}_{\vec{k}}^*(x,t) \bar{a}_{\vec{k}}^\dagger \\ &= \sum_{\vec{k}} \bar{\bar{u}}_{\vec{k}}(x,t) \bar{\bar{a}}_{\vec{k}} + \bar{\bar{u}}_{\vec{k}}^*(x,t) \bar{\bar{a}}_{\vec{k}}^\dagger = \dots\end{aligned}$$

with their Fock bases being different bases in the same Hilbert space.

□ For example, using the $\{u_k\}$, we are led to span the Hilbert space \mathcal{H} using this ON basis:

$$|0\rangle \text{ where } a_k |0\rangle = 0 \quad \forall k$$

$$a_k^+ |0\rangle, \frac{1}{\sqrt{n!}} (a_k^+)^n |0\rangle$$

$$\frac{1}{\sqrt{\dots!}} (a_k^+)^{\dots} \dots (a_{k'}^+)^{\dots} |0\rangle, \text{ etc } \dots$$

□ Or, using other mode functions, say $\{\bar{u}_k\}$, we may span the same Hilbert space, \mathcal{H} , using this ON basis:

$$|\bar{0}\rangle \text{ where } \bar{a}_k |\bar{0}\rangle = 0 \quad \forall k$$

$$\bar{a}_k^+ |\bar{0}\rangle, \frac{1}{\sqrt{n!}} (\bar{a}_k^+)^n |\bar{0}\rangle$$

$$\frac{1}{\sqrt{\dots!}} (\bar{a}_k^+)^{\dots} \dots (\bar{a}_{k'}^+)^{\dots} |\bar{0}\rangle, \text{ etc } \dots$$

Does the choice of mode functions matter?

□ In principle, it does not:

Any state of the system, say $|\Psi\rangle$, can be expanded in each basis.

□ In practice, however:

It is convenient, whenever we know which state is the no-particle (i.e., vacuum) state, say $|\Omega\rangle$, to choose the mode functions $\{u_k\}$ such that the corresponding $|0\rangle$ is $|\Omega\rangle$, i.e., such that

$$|0\rangle = |\Omega\rangle, \text{ i.e., such that } a_k |\Omega\rangle = 0$$

Then, conveniently, states like $\frac{1}{\sqrt{n!}} (a_k^\dagger)^n |0\rangle$ are the multi-particle states.

Outlook: (only a rough sketch)

- Say we know the system's state, $|\Psi\rangle$, is the vacuum initially.
- \rightsquigarrow We choose $\{u_n\}$ appropriately, so that $|0_{in}\rangle = |\Psi\rangle$.
- After some evolution (e.g. the universe expands) the vacuum state may be a different state, say $|X\rangle$.
- \rightsquigarrow We choose $\{\bar{u}_n\}$ appropriately, so that $|0_{out}\rangle = |X\rangle$
- At late times, since we work in the Heisenberg picture, the system is still in the state $|0_{in}\rangle$, but this is then an excited state!
 \rightsquigarrow Description of particle production due to cosmic expansion.
- Recall: We had an analogous situation with driven harmonic oscillators!