

Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 12

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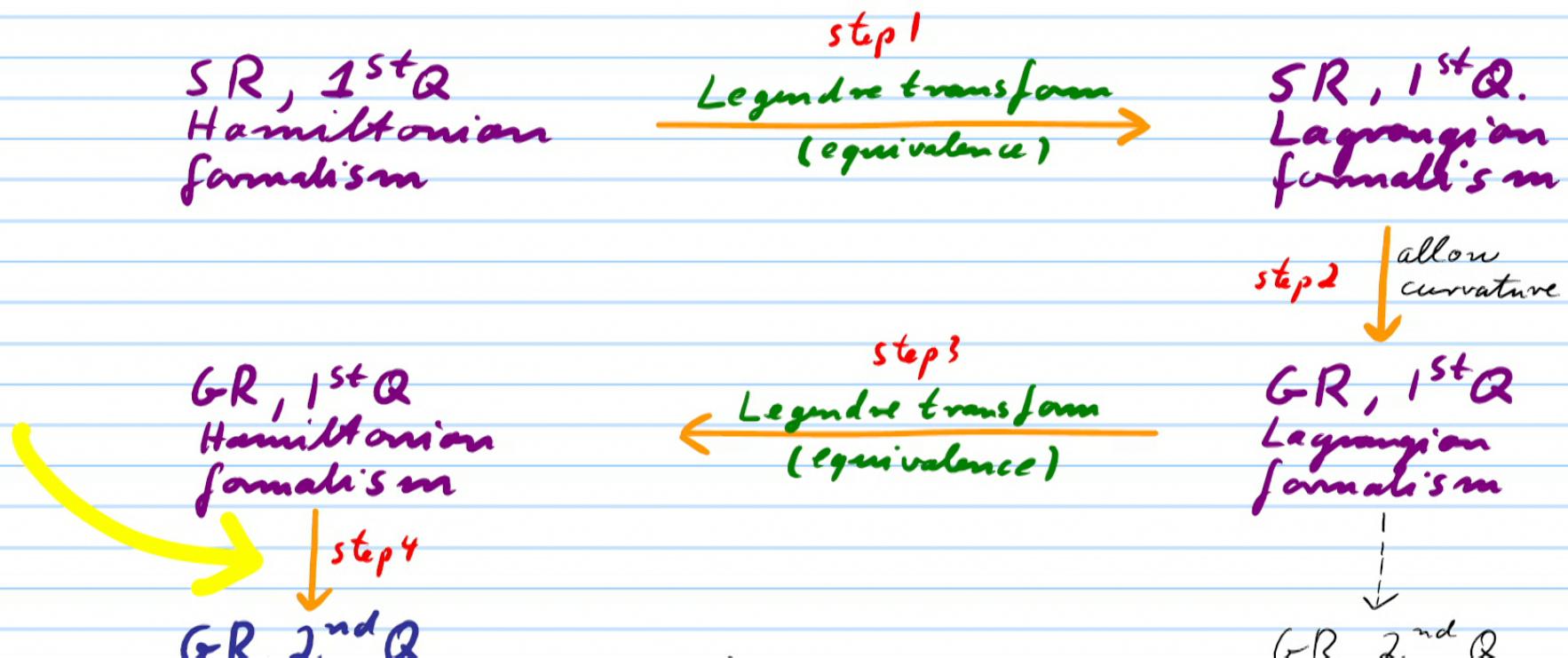
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Abstract:

QFT for Cosmology, Achim Kempf, Lecture 12

Note Title

We are now ready to 2nd quantize:



Solving the quantized theory is to solve:

1.) Commutation relations:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar \delta^3(x-x')$$

$$[\hat{\phi}(x,t), \hat{\phi}(x',t)] = 0$$

$$[\hat{\pi}(x,t), \hat{\pi}(x',t)] = 0$$

} (CCRs)

2.) Hermiticity:

$$\hat{\phi}^\dagger(x,t) = \hat{\phi}(x,t), \quad \hat{\pi}^\dagger(x,t) = \hat{\pi}(x,t) \quad (\text{HC})$$

which is needed so that the expectation values are real.

3.) Equations of motion:

In the Heisenberg picture, they are formally unchanged:

$$\frac{d}{dt} \hat{f}(\phi, \pi) = \frac{i}{\hbar} [\hat{f}, \hat{H}] \quad \text{for } \hat{f} = \hat{\phi}, \hat{f} = \hat{\pi}, \text{ etc}$$

Namely:

$$\left(\frac{1}{\sqrt{V_{\phi}} \frac{\partial}{\partial x^r}} \frac{\partial}{\partial V_{\phi}^r} \sqrt{V_{\phi}} + m^2 \right) \hat{\phi}(x, t) = 0 \quad (\text{EoM1})$$

and:

$$\hat{\pi}(x, t) = \sqrt{V_{\phi}} \frac{\partial}{\partial x^r} \hat{\phi}(x, t) \quad (\text{EoM2})$$

3.) Equations of motion:

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Namely:

$$\left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} g^{rv} \sqrt{g} \frac{\partial}{\partial x^v} + m^2 \right) \hat{\phi}(x, t) = 0 \quad (EoM1)$$

and:

$$\hat{\pi}(x, t) = \sqrt{g} g^{rv} \frac{\partial}{\partial x^v} \hat{\phi}(x, t) \quad (EoM2)$$

How to solve the CCR, HC and EoM equations?

Recall: the solution we obtained on Minkowski space:

$$\hat{\phi}(x, t) = \int_{\mathbb{R}^3} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(e^{-i\omega_k t + ikx} a_k + e^{i\omega_k t - ikx} a_k^+ \right) dk$$

Number-valued solutions to the K.G. equation

and $\hat{\pi}(x, t) = \dot{\hat{\phi}}(x, t)$

Strategy: *** ensure hermiticity, HC, by construction**
*** separate the CCR and EoM problems:**

Ansatz:

$$\hat{\phi}(x, t) := \sum_k u_k(x, t) a_k + u_k^*(x, t) a_k^+$$

$$\hat{\pi}(x, t) := \sqrt{-m^2 \omega_k^2} \partial_t u_k(x, t)$$

The a_k, a_k^+ take care of the CCRs

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$$\hat{\phi}(x, t) := \sum_k u_k(x, t) a_k + u_k^*(x, t) a_k^+$$

$$\hat{\pi}(x, t) := \sqrt{g} g^{00} \frac{\partial}{\partial x^0} \hat{\phi}(x, t)$$

The a_k, a_k^+ take care of the CCRs

(k need not be a "momentum"!)

□ Here, we use the easy-to-construct operators that obey

$$[a_k, a_{k'}^+] = \delta_{k,k'}$$

□ And, we use some number-valued functions $u_h(x, t)$

which are some yet-to-be-determined solutions to

the first eqn. of motion, EoM1, i.e. to the Klein

Gordon equation, called the Mode Functions.

□ Try out the ansatz:

* Hermiticity: ✓

$$(u_h, v_h) = \int u_h v_h dx$$

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* Hermiticity: ✓

(HC) holds by construction.

* The 1st equation of motion: ✓

$$\left(\frac{1}{\sqrt{V_{g1}}} \frac{\partial}{\partial x^r} g^{rv} \sqrt{V_{g1}} \frac{\partial}{\partial x^v} + m^2 \right) \hat{\phi}(x,t) = 0 \quad (\text{EOM1})$$

This eqn holds because in our ansatz,

(HC) holds by construction.

* The 1st equation of motion: ✓

$$\left(\frac{1}{V_{g1}} \frac{\partial}{\partial x^r} \delta^{rv} V_{g1} \frac{\partial}{\partial x^v} + m^2 \right) \hat{\phi}(x,t) = 0 \quad (\text{EoM})$$

This eqn holds because in our ansatz,

$$\hat{\phi}(x,t) := \sum u_k(x,t) a_k + u_k^*(x,t) a_k^*$$

the a_k are constant operators while the

* The 2nd equation of motion: ✓

This equation holds by the way we define $\hat{\pi}(x,t)$.

□ Checking the CCRs:

$$[\hat{\phi}(x,t), \hat{\pi}(x',t)] = i\hbar \delta^3(x-x')$$

□ Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum u_k(x,t) a_k + u_k^*(x,t) a_k^*$$

Now check CCR:

$$[\vec{\phi}(x,t), \vec{\pi}(x,t)]$$

$$= \left[\sum_k u_k(x,t) \alpha_k + u_k^*(x,t) \alpha_k^+, \sqrt{g} g^{vv} \sum_{k'} \left(\left(\frac{\partial}{\partial x^v} u_{k'}(x,t) \right) \alpha_{k'} + \left(\frac{\partial}{\partial x^v} u_{k'}^*(x,t) \right) \alpha_{k'}^+ \right) \right]$$

$$= \sqrt{g} g^{vv}(x,t) \sum_{k,k'} \left(u_k(x,t) \frac{\partial}{\partial x^v} u_{k'}^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x^v} u_{k'}(x,t) \right) \delta_{k,k'}$$

$$= \sqrt{g} g^{vv} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x^v} u_k^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x^v} u_k(x,t) \right) \stackrel{!}{=} i \delta^3(\vec{x} - \vec{x}')$$

$$[\hat{\phi}(x,t), \hat{\pi}(x,t)]$$

$$= \left[\sum_k u_k(x,t) \alpha_k + u_k^*(x,t) \alpha_k^+, T \overline{g} g^{ov} \sum_{k'} \left(\left(\frac{\partial}{\partial x} u_{k'}(x,t) \right) \alpha_{k'} + \left(\frac{\partial}{\partial x} u_{k'}^*(x,t) \right) \alpha_{k'}^+ \right) \right]$$

$$= T \overline{g} g^{ov}(x,t) \sum_{k,k'} \left(u_k(x,t) \frac{\partial}{\partial x} u_{k'}^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x} u_{k'}(x,t) \right) \delta_{k,k'},$$

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□ Express $\hat{\phi}$ in terms of the ansatz:

$$\hat{\phi}(x,t) := \sum u_k(x,t) a_k + u_k^*(x,t) a_k^*$$

□ Express $\hat{\pi}$ in terms of the ansatz:

$$\hat{\pi}(x,t) := \sqrt{g} g^{vv} \frac{\partial}{\partial x^v} \hat{\phi}(x,t)$$

$$\Rightarrow \hat{\pi}(x,t) = \sqrt{g} g^{vv} \sum_k \left[\left(\frac{\partial}{\partial x^v} u_k(x,t) \right) a_k + \left(\frac{\partial}{\partial x^v} u_k^*(x,t) \right) a_k^* \right]$$

Conclusion so far:

Our ansatz

$$\hat{\phi}(x,t) := \sum u_n(x,t) a_n + u_n^*(x,t) a_n^*$$

solves the QFT, i.e., HC, EoM and CCR

if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that always:

(1)

$$[\Box - m^2] u_n(x,t) = 0$$

solves the W+L , i.e., HC , EOM and CCR

if we can find a set of number-valued solutions

$$\{u_n(x,t)\}$$

of the Klein Gordon equation that obeys:

(w)

$$\sqrt{g} g^{00} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x^0} u_k^*(x,t) - u_k^*(x,t) \frac{\partial}{\partial x^0} u_k(x,t) \right) = i \delta^3(x-x')$$

When do such $\{u_n(x,t)\}$ exist? I.e., when does the ansatz succeed?

Proposition: \square Assume spacetime is "globally hyperbolic",

Proposition: □ Assume spacetime is "globally hyperbolic",

i.e., that it possesses a solution by Cauchy surfaces,

i.e., that it is topologically of the form:

$$\mathbb{R} \times M$$

↑ any 3-dim differentiable manifold

□ In this case, spacetime possesses no closed timelike curves (no travel into the past), i.e., initial conditions set on the Cauchy surfaces determine the solution everywhere.

□ Then, such a set of functions $\{u_n\}$ can be shown to exist.

□ In fact there are many such sets $\{\tilde{u}_n\}$ obeying (W)!

(And we will have to address which set to choose to solve the theory.)

Proof:

- Consider the vector space, V , of all real-valued solutions of the Klein Gordon equation.
- We define a bi-linear form (\cdot, \cdot) on V . For all $f, h \in V$:

$$(f, h) := \int_{\Sigma} d\Sigma_p \sqrt{g} g^{\mu\nu} (f \partial_\mu h - h \partial_\mu f)$$

any spacelike hypersurface
i.e. set of points of equal time.

- Proposition: (f, h) is independent of choice of Σ .
Proof: Later (uses Stokes' theorem and K.G. equation)
- (f, h) is a symplectic form i.e.: $(f, h) = - (h, f)$

solutions of the Klein Gordon equation.

□ We define a bi-linear form $(,)$ on V . For all $f, h \in V$:

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□ (f, h) is a symplectic form, i.e.: $(f, h) = -(h, f)$.

↑
easy to see

Q What can we do with (\cdot, \cdot) ? No diagonalization?

Theorem (Darboux):

For any nondegenerate symplectic form (\cdot, \cdot) , there exists a basis $\{v_m\}$ such that, in this basis, (\cdot, \cdot) takes the matrix form:

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & 0 & 1 & \\ & -1 & 0 & \\ 0 & & 0 & 1 \\ & & & -1 \end{pmatrix}$$

i.e., such that $(v_{2m}, v_{2m+1}) = 1$, $(v_{2m+1}, v_{2m}) = -1$

and all other pairings vanish.

Q Thus, if we expand $a, b \in V$ as: $a = a_n v_n, b = b_n v_n$

$$\begin{matrix} v & R & v \\ \psi & \psi & \psi \\ & R & \psi \end{matrix}$$

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and all other pairings vanish.

B Thus, if we expand $a, b \in V$ as: $a = a_n v_n$, $b = b_n v_n$

$$\text{Then: } (a, b) = \sum_{n=0}^{\infty} a_{2n} b_{2m+1} - a_{2m+1} b_{2n}$$

$$\begin{matrix} v & & R & v \\ \cup & & \cup & \cup \\ & & v & v \\ & & \cup & \cup \\ & & & R \end{matrix}$$

$$(f, h) := \int_{\Sigma} d\Sigma_r \sqrt{g} g^{\mu\nu} (f \partial_\nu h - h \partial_\nu f)$$

Σ any spacelike hypersurface
i.e. set of points of equal time.

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Proof: Later (uses Stokes' theorem and K.G. equation)

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easy to see

□ What can we do with $(,)$? No diagonalization?

Theorem (D. L...).

□ Now assume we picked such a basis $\{v_n\}$ in V .

□ Recall: $V = \text{space of real-valued solutions of K.G. eqn.}$

□ Definition:

$\bar{V} := \text{space of complex-valued solutions of K.G. eqn.}$

□ We easily find a basis of \bar{V} , namely $\{u_n\} \cup \{u_n^*\}$ where:

$$u_n := \frac{1}{\sqrt{2}} (v_{2n} + i v_{2n+1}), \quad u_n^* = \frac{1}{\sqrt{2}} (v_{2n} - i v_{2n+1})$$

□ What is a natural product \langle , \rangle on \bar{V} ?

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□ On \bar{V} we define:

$$\langle f, h \rangle = i \int_{\Sigma} d\Sigma_r \sqrt{g} g^{\mu\nu} (f^\star \partial_\nu h - (\partial_\nu f^\star) h)$$

□ Then, $(,)$ yields:

$$\langle u_n, u_m \rangle = -\delta_{n,m}, \quad \langle u_n^*, u_m^* \rangle = +\delta_{n,m}, \quad \langle u_n, u_m^* \rangle = 0 \quad (I)$$

Exercise: verify this.

□ On \bar{V} we define:

$$\langle f, h \rangle = i \int_{\Sigma} d\Sigma_r \sqrt{g} g^{\mu\nu} (f^* \partial_\nu h - (\partial_\nu f^*) h)$$

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Exercise: verify this.

Thus, $\langle \cdot, \cdot \rangle$ is an indefinite inner product on \bar{V} : $\langle \cdot, \cdot \rangle = \begin{pmatrix} \dots & -1 & \dots \\ \vdots & \ddots & \vdots \\ \dots & 1 & \dots \end{pmatrix}$

Proposition: A resolution of the identity on \bar{V} is given by:

Thus, \langle , \rangle is an indefinite inner product on \bar{V} : $\langle , \rangle = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Proposition: A resolution of the identity on \bar{V} is given by:

$$1\mathbb{I} = \sum_n -|u_n\rangle\langle u_n| + |u_n^+\rangle\langle u_n^+|$$

Remark: One can also turn \bar{V} into a Hilbert space, namely the Krein space: Let P^+ and P^- be the projectors on the spaces spanned by the u_n^+ and the u_n^- respectively.

Then, $\langle\langle f, g \rangle\rangle := \langle f, P^+ g \rangle - \langle f, P^- g \rangle$ is a positive definite inner product, and the Krein space $(\bar{V}, \langle\langle , \rangle\rangle)$ is a Hilbert space.

Proof:

$$\text{Indeed, } 1\mathbb{I}|u_n\rangle = |u_n\rangle \text{ using } \langle u_n, u_n \rangle = -1$$

$$1\mathbb{I}|u_n^+\rangle = |u_n^+\rangle \text{ using } \langle u_n^+, u_n^+ \rangle = 1 .$$

$$1\!\!1 = \sum_n -|u_n\rangle\langle u_n| + |u_n^*\rangle\langle u_n^*|$$

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$$\text{Indeed, } 1\!\!1 |u_n\rangle = |u_n\rangle \text{ using } \langle u_n, u_n \rangle = -1$$

$$1\!\!1 |u_n^*\rangle = |u_n^*\rangle \text{ using } \langle u_n^*, u_n^* \rangle = 1 .$$

so that for any $|f\rangle \in \bar{V}$ we have:

$$-\sum |u_n\rangle\langle u_n|f\rangle + |u_n^*\rangle\langle u_n^*|f\rangle = |f\rangle \quad (\text{P})$$

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Indeed, $|u_n\rangle = |u_n\rangle$ using $\langle u_n, u_n \rangle = -1$

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so that for any $|f\rangle \in \mathcal{V}$ we have:

$$-\sum_n |u_n\rangle \langle u_n| f\rangle + |u_n^*\rangle \langle u_n^*| f\rangle = |f\rangle \quad (\text{P})$$

Writing this out, we will now show that it yields (W) , i.e.:

$$\sqrt{\hbar g} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'} u_k(x',t) \right) = i \delta^3(x-x')$$

B Indeed, using

$$\langle f, h \rangle = i \int_{\Sigma} d\Sigma_{\nu} \sqrt{g} g^{\mu\nu} (f^* \partial_{\nu} h - (\partial_{\nu} f^*) h)$$

which reads, in a suitable coordinate system :

$$= i \int d^3x' \sqrt{g(x')} g^{00}(x') \left(f^*(x') \partial_0 h(x') - \partial_0 f^*(x') h(x') \right)$$

we see from (P) that $\nabla f \in \bar{V}$:

$$-\sum_n |u_n\rangle \langle u_n| f + |u_n^*\rangle \langle u_n^*| f = |f\rangle$$

reads:

$$\sum_n u_n(x,t) i \int_{\Sigma} d^3x' \sqrt{g} g^{00} (u_n^* \partial_0 f - (\partial_0 u_n^*) f)$$

$$- \sum_n u_n^*(x,t) i \int_{\Sigma} d^3x' \sqrt{g} g^{00} (u_n \partial_0 f - (\partial_0 u_n) f) = f(x,t)$$

Now, interchanging \sum_n and \int_{Σ} yields $\forall f \in \bar{V}$:

$$i \int_{\Sigma} d^3x \sqrt{|g(x)|} \tilde{g}^{00} \sum_n \left(u_n(x,t) u_n^*(x',t') \partial_{x'^0} - u_n^*(x,t) \partial_{x'^0} u_n^*(x',t') \right. \\ \left. - u_n^*(x,t) u_n(x',t') \partial_{x'^0} + u_n^*(x,t) \partial_{x'^0} u_n(x',t') \right) f(x',t') = f(x,t) \quad (*)$$

Notice:



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$$-\sum_n |u_n\rangle \langle u_n| f + |u_n^*\rangle \langle u_n^*| f = |\bar{f}\rangle$$

reads:

$$\sum_n u_n(x,t) : \int d^3x' \sqrt{g} g^{00} (u_n^* \partial_0 f - (\partial_0 u_n^*) f)$$

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Now, interchanging \sum_n and \int_{Σ} yields $\forall f \in V$:

$$i \int_{\Sigma} d^3x \sqrt{|g(x)|} \tilde{g}^{00}(x) \sum_n \left(u_n(x,t) u_n^*(x,t) \partial_{x^0} - u_n(x,t) \partial_{x^0} u_n^*(x,t) \right.$$

$$\left. - u_n^*(x,t) u_n(x,t) \partial_{x^0} + u_n^*(x,t) \partial_{x^0} u_n(x,t) \right) f(x',t_0) = f(x,t) \quad (*)$$

Notice:

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On the left hand side of this equation, f is evaluated for all x' but only at the one time, say t_0 , of Σ .

□ Now choose an arbitrary function $g(x')$.

□ Then there exists a solution $f(x', t)$ of the Klein Gordon equation obeying :

$$1) \quad f(x', t_0) = g(x')$$

$$2) \quad g^{\mu\nu}(x', t_0) \partial_{x^\nu} f(x', t_0) = 0$$

(R... and ...)

... " " " "

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□ Then there exists a solution $f(x', t)$ of the Klein Gordon equation obeying :

$$1) \quad f(x', t_0) = g(x)$$

$$2) \quad g^{(0)}(x', t_0) \partial_{x'^\mu} f(x', t_0) = 0$$

(Because the 2nd order K.G. equation on a globally hyperbolic spacetime has a well-defined Cauchy problem)

□ Therefore, (*) yields, for all choices of $g(x)$:

$$i \int_{\Sigma} d^3 x' \sqrt{|g(x)|} g^{(0)} \sum_n \left(-u_n^\ast(x', t) \partial_{x'^\mu} u_n^\ast(x', t) + u_n^\ast(x', t) \partial_{x'^\mu} u_n(x', t) \right) g(x) = g(x) \quad \forall g(x)$$

$$1) \quad f(x', t_0) = g(x')$$

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(Because the 2nd order K.G. equation on a globally hyperbolic spacetime has a well-defined Cauchy problem)

□ Therefore, (*) yields, for all choices of $g(x)$:

$$i \int_{\Sigma} d^3 x' \sqrt{|g(x')|} g^{00} \sum_n \left(-u_n^*(k, t) \partial_{x'^0} u_n^*(k', t) + u_n^*(k, t) \partial_{x'^0} u_n(k', t) \right) g(x') = g(x) \quad \forall g(x)$$

$$\Rightarrow \boxed{\sqrt{|g|} g^{00} \sum_k \left(u_k(x, t) \frac{\partial}{\partial x'^0} u_k^*(x', t) - u_k^*(x, t) \frac{\partial}{\partial x'^0} u_k(x', t) \right)} = i \delta^3(x - x') \quad (W) \quad \checkmark$$

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□ Therefore, (*) yields, for all choices of $g(x)$:

$$i \int_{\Sigma} d^3 x' \sqrt{|g(x')|} g^{00} \sum_n \left(-u_n^*(x', t) \partial_{x',0} u_n^*(x', t) + u_n^*(x', t) \partial_{x',0} u_n(x', t) \right) g(x') = g(x) \quad \forall g(x)$$

$$1) \quad f(x', t_0) = g(x)$$

$$2) \quad g^{vv}(x', t_0) \partial_{x',v} f(x', t_0) = 0$$

(Because the 2nd order K.G. equation on a globally hyperbolic spacetime has a well-defined Cauchy problem)

□ Therefore, (*) yields, for all choices of $g(x)$:

$$i \int_{\Sigma} d^3 x' \sqrt{|g(x')|} g^{vv} \sum_n \left(-u_n^*(k, t) \partial_{x',v} u_n^*(k, t) + u_n^*(k, t) \partial_{x',v} u_n(k, t) \right) g(x) = g(x) \quad \forall g(x)$$

$$\Rightarrow \boxed{\sqrt{|g|} g^{vv} \sum_k \left(u_k(x, t) \frac{\partial}{\partial x',v} u_k^*(x', t) - u_k^*(x, t) \frac{\partial}{\partial x',v} u_k(x', t) \right) = i \delta^3(x - x')} \quad (W) \quad \checkmark$$

can always be solved with this ansatz
on any globally hyperbolic spacetime:

$$\hat{\phi}(x,t) := \sum_k u_k(x,t) a_k + u_k^*(x,t) a_k^*$$

$$\hat{\pi}(x,t) := \sqrt{|g|} g^{0v} \frac{\partial}{\partial x^v} \hat{\phi}(x,t)$$

where $[a_k, a_{k'}^+] = \delta(k-k')$ and where the $u_k(x,t)$ are
number-valued solutions to (KG) which also obey (R1):

$$\sqrt{|g|} g^{0v} \sum_k \left(u_k(x,t) \frac{\partial}{\partial x'^v} u_k^*(x',t) - u_k^*(x,t) \frac{\partial}{\partial x'^v} u_k(x',t) \right) = i \delta^3(x-x')$$

Outlook:

Q: We showed that (W) ensures the CCRs at one time.

What guarantees conservation of the CCRs?

A: Stokes' theorem and unitarity.

Q: Are the $u_n(x,t)$ unique?

A: No! Math: Bogoliubov transformations



Q: What's the physics?

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What guarantees conservation of the CCRs?

A: Stokes' theorem and unitarity.

Q: Are the $\alpha_n(x,t)$ unique?

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↑

Q: What's the physics?

A: Vacuum ambiguity.