Title: PSI 17/18 - Quantum Field Theory III - Lecture 16

Date: Feb 16, 2018 11:30 AM

URL: http://pirsa.org/18020036

Abstract:

Today 11:30 - 12:45

Resummation

Theta Vacua

$$f(g) \sim \varphi(g)$$

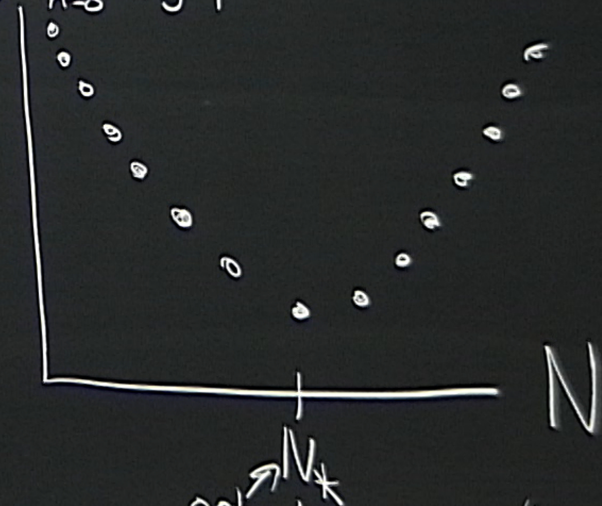
$$f(g) \underset{\text{Resummation w/ instanton}}{\overset{\text{pert}}{\rightleftarrows}} \varphi(g)$$

Typically in QM + QFT

$$\varphi(g) = \sum_{n\geq 0}^{\infty} a_n g^n$$

$$a_n \sim A^{-n}\, n! \qquad n \to \infty$$

$$\left| f(g) - \sum_{n=0}^{N} a_n g^n \right|$$



optimal truncation (typical than $a_{N_*} g^{N_*}$ is the s[illegible]st t[illegible]) minimize $|a_N g^N| \underset{N\gg 1}{\approx} c\, N! \left|\frac{g}{A}\right|^N$

$$\overset{\text{Stirling}}{\approx} c \exp\left[N\left(\log N - 1 - \log\left|\frac{A}{g}\right|\right)\right]$$

minimize at $N_* \approx \left|\frac{A}{g}\right|$

error typically $\approx$ next term

$$\approx e^{-|A/g|}$$

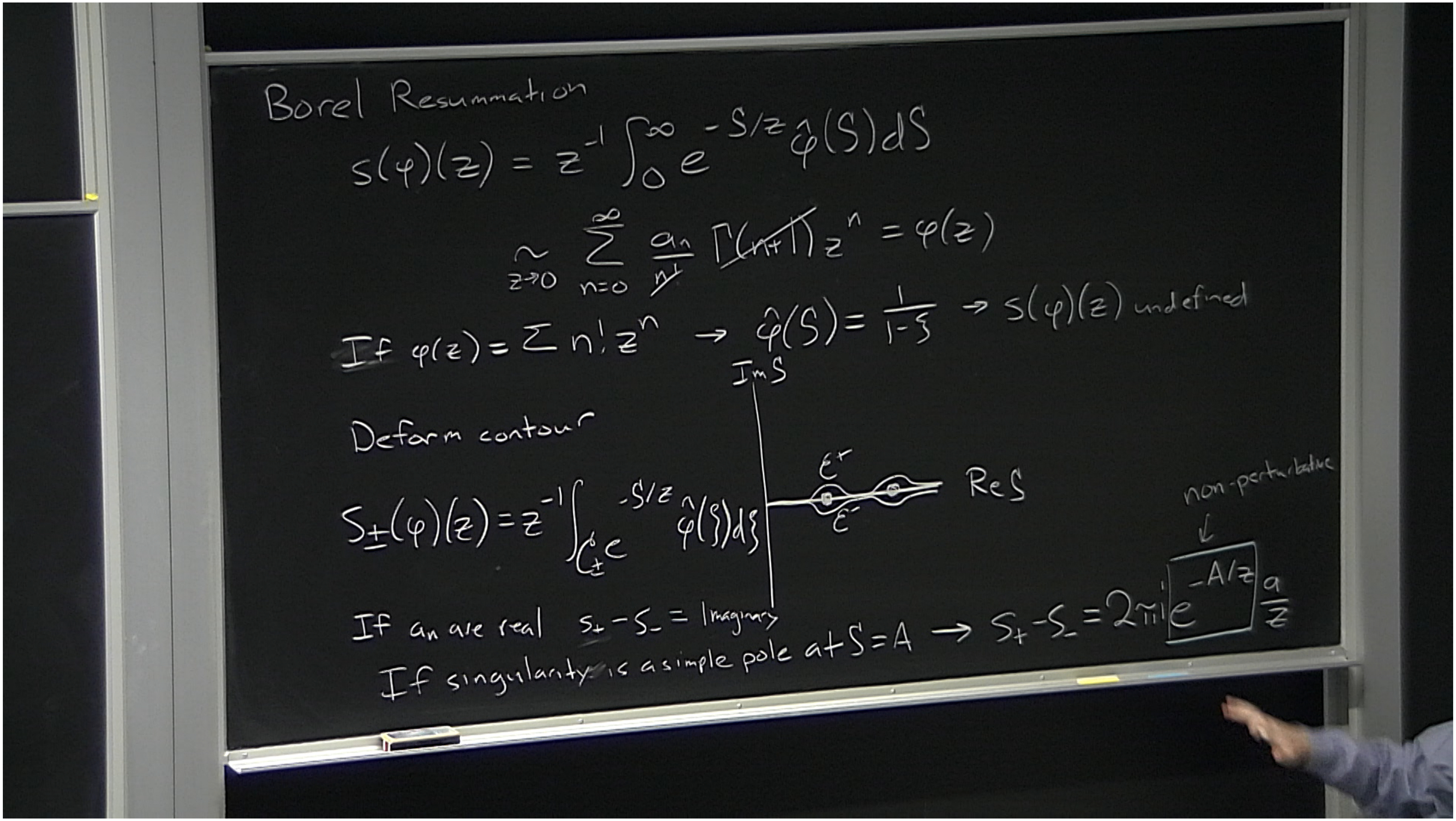
Borel transform $\hat{\varphi}(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n$

$\uparrow$ improves convergence

Example: $\varphi(z) = \sum_{n=0}^{\infty} (-1)^n n!\, z^n$

$$\hat{\varphi}(\zeta) = \sum_{n=0}^{\infty} (-1)^n \zeta^n \underset{|\zeta|<1}{=} \frac{1}{1+\zeta}$$

$\leftarrow$ analytically continue


Borel Resummation
$s(\varphi)(z) = z^{-1}\int_0^\infty e^{-S/z}\hat{\varphi}(S)\,dS$
$\underset{z\to 0}{\sim} \sum_{n=0}^{\infty} \frac{a_n}{\cancel{n!}} \cancel{\Gamma(n+1)}\, z^n = \varphi(z)$
If $\varphi(z) = \sum n!\, z^n \to \hat{\varphi}(S) = \frac{1}{1-S} \to s(\varphi)(z)$ undefined
Im S
Deform contour
$S_\pm(\varphi)(z) = z^{-1}\int_{C_\pm} e^{-S/z}\hat{\varphi}(S)\,dS$
ε^+
Re S
ε^-
non-perturbative
If a_n are real $S_+ - S_- =$ Imaginary
If singularity is a simple pole at $S = A \to S_+ - S_- = 2\pi i \boxed{e^{-A/z}} \frac{a}{z}$

$$\Gamma \propto \frac{1}{\sqrt{\det' S''}}\, e^{-A/g} \quad \leftrightarrow \text{Borel sum for } E_0 \text{ has ambiguit}$$

$E_0(g)$ is real and $\varphi(g)$ is not Borel summable

double well ground state energy

ambiguity of imaginary part $\leftarrow$ cancels with ambiguity in Im of analytically continued instantons

Theta Vacuum

$$S_E = \frac{1}{2g^2}\int d^4x \, \mathrm{tr}\, F^{\mu\nu} F_{\mu\nu}$$

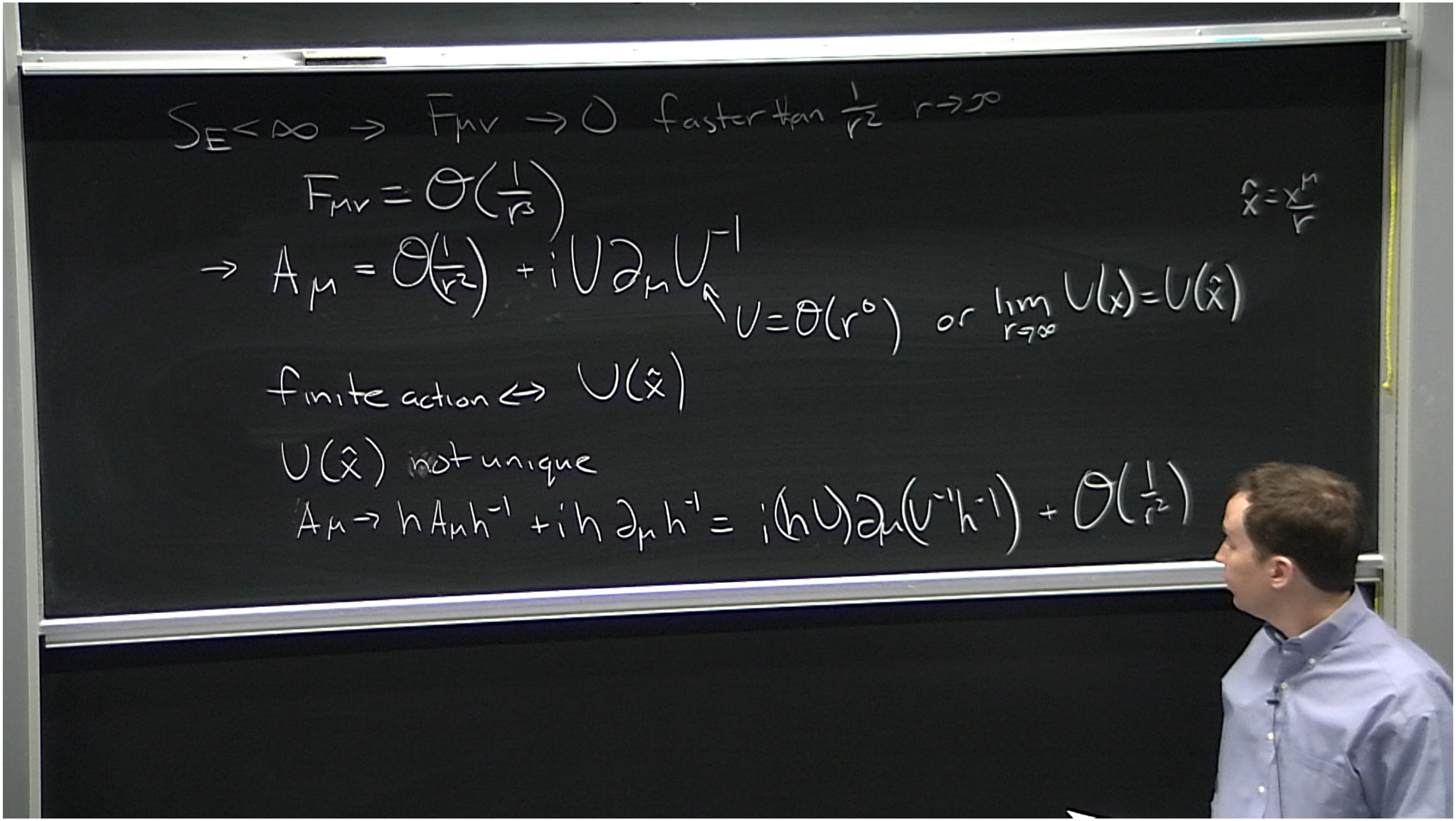
$$S_E < \infty \Rightarrow F_{\mu\nu} \to 0 \text{ faster than } \frac{1}{r^2} \quad r \to \infty$$

$$F_{\mu\nu} = O\left(\frac{1}{r^3}\right)$$

$$\hat{x} = \frac{x^\mu}{r}$$

$$\to A_\mu = O\left(\frac{1}{r^2}\right) + i U \partial_\mu U^{-1}$$

$$U = O(r^0) \text{ or } \lim_{r\to\infty} U(x) = U(\hat{x})$$


$S_E < \infty \to F_{\mu\nu} \to 0$ faster than $\frac{1}{r^2}$ $r \to \infty$
$F_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^3}\right)$
$\hat{x} = \frac{x^\mu}{r}$
$\to A_\mu = \mathcal{O}\left(\frac{1}{r^2}\right) + iU\partial_\mu U^{-1}$
$U = \mathcal{O}(r^0)$ or $\lim_{r\to\infty} U(x) = U(\hat{x})$
finite action $\leftrightarrow U(\hat{x})$
$U(\hat{x})$ not unique
$A_\mu \to hA_\mu h^{-1} + ih\partial_\mu h^{-1} = i(hU)\partial_\mu(U^{-1}h^{-1}) + \mathcal{O}\left(\frac{1}{r^2}\right)$

If a_n are real $S_+ - S_- = $ Imaginary

If singularity is a simple pole at $S = A \rightarrow S_+ - S_- = 2\pi i \boxed{e^{-A/z}} \frac{a}{z}$

If $hU = 1$ at $r = \infty$ then we could remove $U\partial_\mu U^{-1}$ term

<u>NOT</u> possible in general

<u>Simplest case</u> $U(1)$ in $D=2$ Euclidean

$$F_{\mu\nu} = \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$A_\mu = \mathcal{O}\left(\frac{1}{r}\right) + U\partial_\mu U^{-1} \quad \text{with } U(\theta) = e^{i\alpha(\theta)} \quad \text{as } r \to \infty$$

$$U : \underset{\substack{\text{circle}\\ \text{at } \infty\\ \text{in } D=2\\ \text{Euclidean}\\ \text{space}}}{S^1} \rightarrow \underset{U(1)}{S^1}$$

$\hat{x} = \frac{x^\mu}{r}$

Standard mappings

trivial: $U^{(0)}(\theta) = \underline{1}$

identity $U^{(1)}(\theta) = e^{i\theta}$

winding ν times $U^{(\nu)}(\theta) = e^{i\nu\theta}$

Theorem: any $f: S^1 \to S^1$ is $\overbrace{\text{continuously deformable}}^{\text{homotopic}}$ to one + only one $U^{(\nu)}$

$$\nu = \frac{i}{2\pi}\int_0^{2\pi} d\theta \left(U \frac{d}{d\theta} U^{-1}\right) = \frac{i}{2\pi}\int_0^{2\pi} d\theta\, \cancel{e^{i\nu\theta}}\,(-i\nu)\,\cancel{e^{-i\nu\theta}}$$

$$= \nu$$

If singularity is a simple pole ...

If $hU = 1$ at $r = \infty$ then we could remove $U\partial_\mu U^{-1}$ term

<u>NOT</u> possible in general

<u>Simplest case</u> $U(1)$ in $D=2$ Euclidean

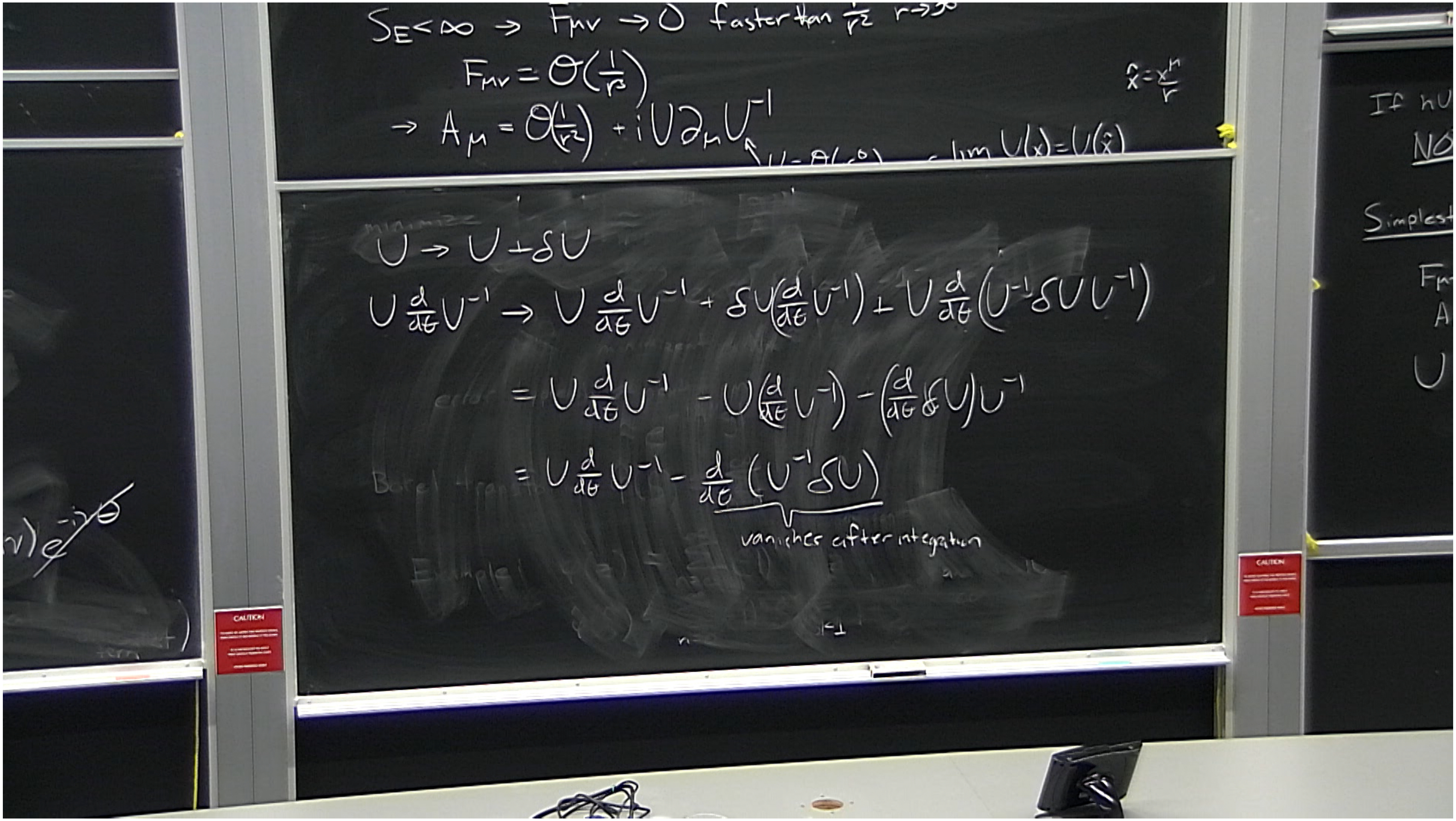
$F \propto \frac{1}{r}$ $\quad S_E = \int \frac{r\,dr}{r^2} \propto \log r$

$$F_{\mu\nu} = \Theta\left(\tfrac{1}{r^2}\right)$$

$$A_\mu = \Theta\left(\tfrac{1}{r}\right) + U\partial_\mu U^{-1} \quad \text{with } U(\theta) = e^{i\alpha(\theta)} \quad \text{as } r \to \infty$$

$$U : S^1 \to S^1$$

circle at ∞ in $D=2$ Euclidean space $\qquad U(1)$


$S_E<\infty \to F_{\mu\nu}\to 0$ faster than $\frac{1}{r^2}$ $r\to\infty$
$F_{\mu\nu}=\mathcal{O}(\frac{1}{r^3})$
$\hat{x}=\frac{x^\mu}{r}$
$\to A_\mu=\mathcal{O}(\frac{1}{r^2})+iU\partial_\mu U^{-1}$
$\lim U(x)=U(\hat{x})$
$U\to U+\delta U$
$U\frac{d}{d\theta}U^{-1}\to U\frac{d}{d\theta}U^{-1}+\delta U(\frac{d}{d\theta}U^{-1})+U\frac{d}{d\theta}(U^{-1}\delta U U^{-1})$
$=U\frac{d}{d\theta}U^{-1}-U(\frac{d}{d\theta}U^{-1})-(\frac{d}{d\theta}\delta U)U^{-1}$
$=U\frac{d}{d\theta}U^{-1}-\underbrace{\frac{d}{d\theta}(U^{-1}\delta U)}_{\text{vanishes after integration}}$
If
NO
Simplest

$$\mathrm{Ind}\,\nu = \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[U \frac{d}{d\theta} U^{-1} \right]$$

$$= \lim_{r\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \underbrace{r A_\theta}_{r \hat{r}^\mu \epsilon_{\mu\nu} A^\nu} d\theta$$

$$= \frac{1}{2\pi}\int d^2x\, \partial^\mu \epsilon_{\mu\nu} A^\nu$$

$$= \frac{1}{4\pi}\int d^2x\, \underbrace{\epsilon_{\mu\nu} F^{\mu\nu}}_{\text{anomaly}}$$

In 4D similarly story for non-abelian gauge theories

not for abelian gauge theory

$f: S^3 \to S^1$ trivial homotopy

$\to$ discrete set of vacuum states $|n\rangle$

instanton w/ finite action $S_1 = \frac{8\pi^2}{g^2}$ that changes ν by 1

$$\lim_{T\to\infty} \langle n| e^{-HT} |m\rangle = \int [\mathcal{D}A_\mu]_{\nu=n-m}\, e^{-S_E}$$

$$\langle n'|H|n\rangle \propto e^{-|n'-n|S_1}$$

$|n\rangle$ not energy eigenstates

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta}|n\rangle$$

θ are

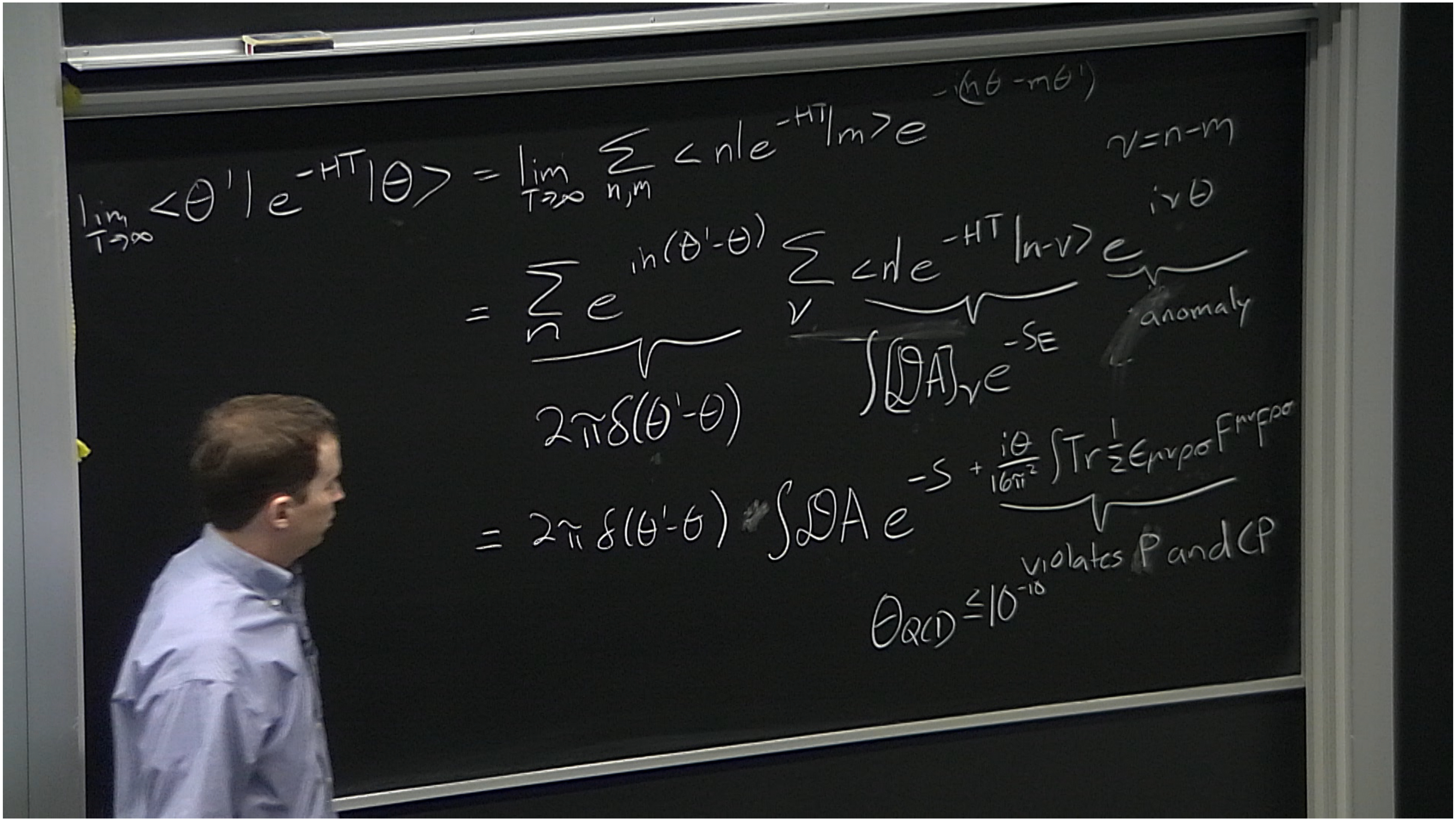
$$\langle n'|H|\theta\rangle = \sum_n e^{-in\theta}\langle n'|H|n\rangle$$

$$= \sum_n e^{-in\theta} e^{-|n-n'|S_1}$$

$n \to m+n'$

$$= \sum_m e^{-i(m+n')\theta} e^{-|m|S_1}$$

$$= \underbrace{e^{-in\theta}}_{\langle n'|\theta\rangle} \underbrace{\sum_m e^{-im\theta} e^{-|m|S_1}}_{E_\theta}$$


$\lim_{T\to\infty} \langle\theta'| e^{-HT} |\theta\rangle = \lim_{T\to\infty} \sum_{n,m} \langle n| e^{-HT} |m\rangle e^{-i(n\theta - m\theta')}$
$\nu = n-m$
$= \underbrace{\sum_n e^{in(\theta'-\theta)}}_{2\pi\delta(\theta'-\theta)} \sum_\nu \underbrace{\langle n| e^{-HT} |n-\nu\rangle}_{\int [\mathcal{D}A]_\nu e^{-S_E}} \underbrace{e^{i\nu\theta}}_{\text{anomaly}}$
$= 2\pi\delta(\theta'-\theta) \int \mathcal{D}A\, e^{-S + \underbrace{\frac{i\theta}{16\pi^2}\int \mathrm{Tr} \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}}_{\text{violates P and CP}}}$
$\theta_{QCD} \lesssim 10^{-10}$