

Title: PSI 17/18 - Quantum Field Theory III - Lecture 14

Date: Feb 14, 2018 11:30 AM

URL: <http://pirsa.org/18020034>

Abstract:

Instantons

solutions to Euclidean EOM w/ finite, non-zero action
non-trivial configuration

non-trivial saddle points of functional integral

typically $\propto e^{-A/g}$
↑ small coupling
all terms in Taylor series about $g=0$ vanish

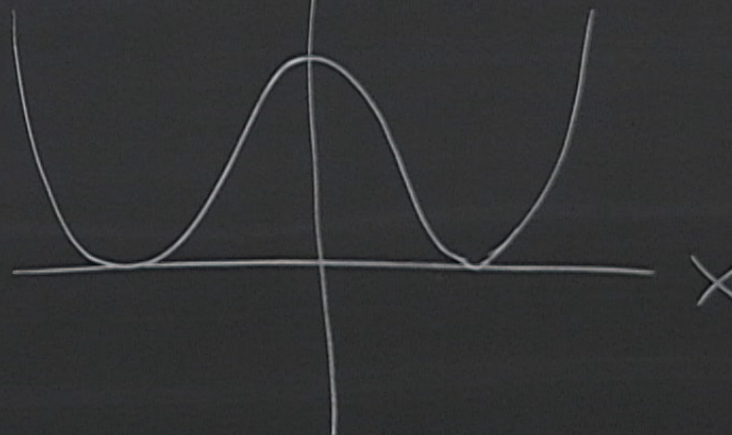
finite, non-zero action

non-trivial
configuration

integral

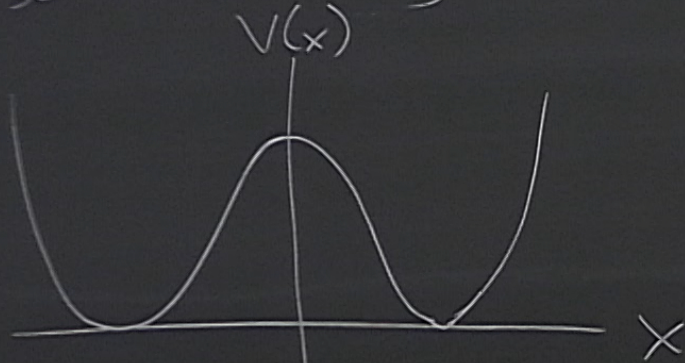
vanish

Sometimes leading contributions
 $V(x)$

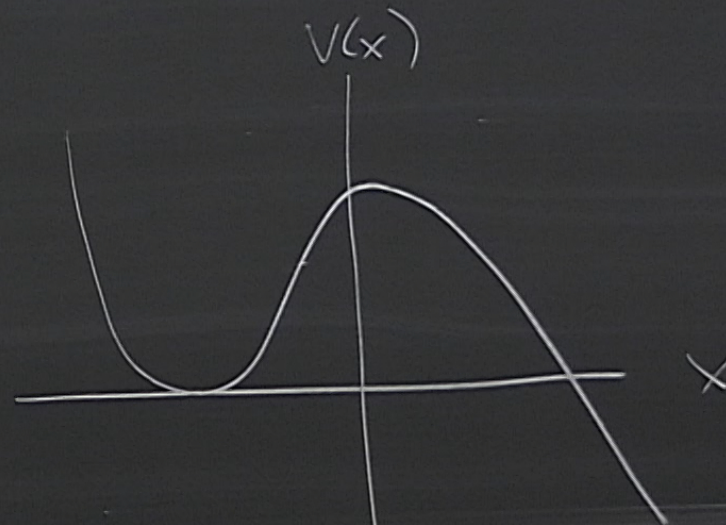


$$E_1(g) - E_0(g) \propto e^{-A/g}$$

Sometimes leading contributions



$$E_1(g) - E_0(g) \propto e^{-A/g}$$



$$\Pi \propto e^{-A/g}$$

$$f(g) \sim \sum a_n g^n + e^{-A/g}$$

↑
exact
quantity

Outline

- ① Instanton in QM
- ② Decay of False Vacuum in QFT
Resummation of Asymptotic Series + Instantons
- ③ Theta Vacua

Today

E_0 from Z

Double Well

Periodic Potential

Unstable Potential

E_0 from

$Z = \langle x$

E_0 from Euclidean Path Integral

$$Z = \langle x_f | e^{-HT/\hbar} | x_i \rangle = \int_{x(-T/2)=x_i}^{x(T/2)=x_f} \mathcal{D}[x(t)] e^{-S_E[x]/\hbar}$$

$$H|n\rangle = E_n|n\rangle$$

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = \sum_n e^{-E_n T/\hbar} \langle x_f | n \rangle \langle n | x_i \rangle$$

$$E_0 = - \lim_{T \rightarrow \infty} \frac{\hbar}{T} \ln Z$$

E_0 from Euclidean Path Integral

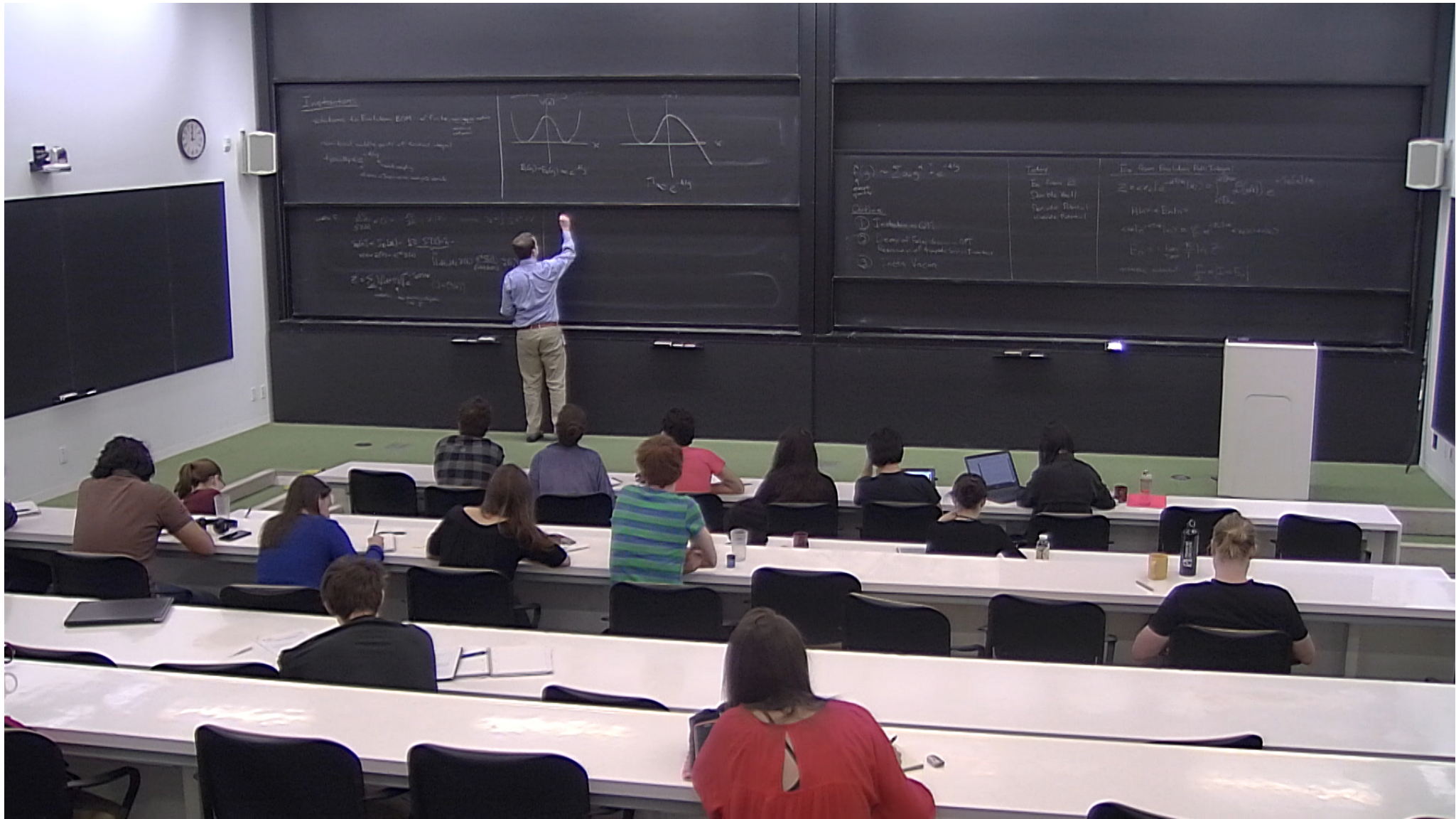
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$$E_0 = - \lim_{T \rightarrow \infty} \frac{\hbar}{T} \ln Z$$

unstable potential $\frac{\Gamma}{2} = |\text{Im } E_0|$



saddle $\bar{x}$

$$\frac{\delta S}{\delta \bar{x}(t)} = 0 = -\frac{d^2 \bar{x}}{dt^2} + V'(\bar{x})$$

assume $S_E = \int \frac{1}{2} \dot{x}^2 + V$

$$S_E[x] = S_E[\bar{x}] + \frac{1}{2} \tilde{x} \cdot S''[\bar{x}] \cdot \tilde{x} + \dots$$

$$x(t) = \bar{x}(t) + \hbar^{1/2} \tilde{x}(t)$$

$$\iint dt_1 dt_2 \tilde{x}(t_1) \frac{\delta^2 S[x]}{\delta x(t_1) \delta x(t_2)} \tilde{x}(t_2)$$

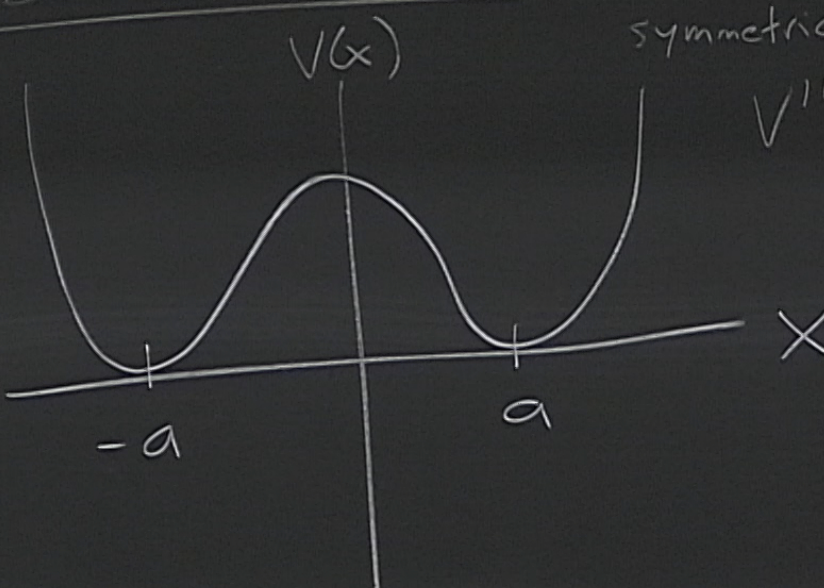
$$Z = \sum_{\text{saddles}} N \left[\det S''[\bar{x}] \right]^{-1/2} e^{-S_E[\bar{x}]/\hbar} (1 + O(\hbar))$$

$\uparrow$ normalization $\uparrow$ from Gaussian integration over $\tilde{x}$

$$= \int \frac{1}{2} \dot{x}^2 + V$$

$$\tilde{x}(t_2)$$

Double Well



symmetric $x \rightarrow -x$
 $V''(\pm a) = \omega^2$

$$E_1 - E_0 = ?$$

compute $\langle \pm a | e^{-HT/\hbar} | \pm a \rangle$

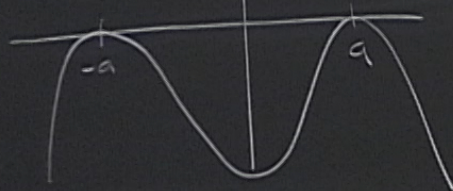
③ Theta Vacua

unstable potential

$$\langle a | e^{-HT/\hbar} | -a \rangle$$

$$-\frac{d^2\bar{x}}{dt^2} + V'(\bar{x}) = 0 \leftarrow \text{EOM for a particle of unit mass in potential } -V$$

$-V$



$$E = \frac{1}{2} \left(\frac{d\bar{x}}{dt} \right)^2 - V(\bar{x}) \quad \text{constant of motion}$$

there is a solution with $E=0$

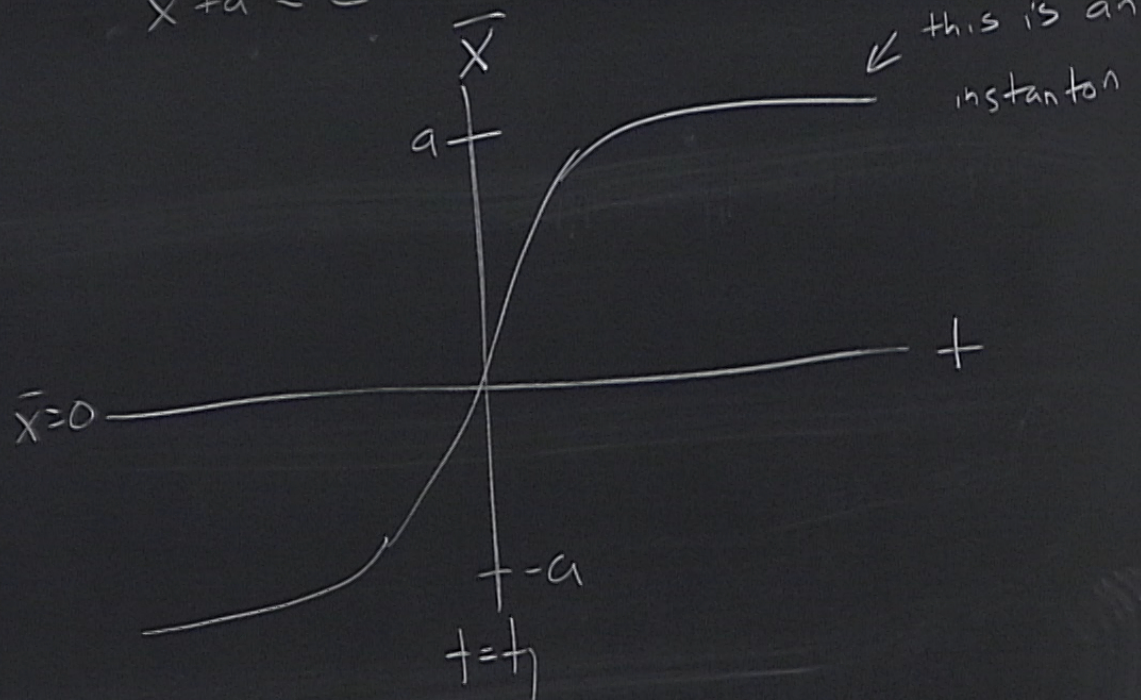
for large t $\bar{x}(t) \approx a$

$$\frac{d\bar{x}}{dt} = \sqrt{2V(\bar{x})} \approx \left(2 \left[V(a) + (a-\bar{x})V'(a) + \frac{1}{2}(a-\bar{x})^2 V''(a) + \dots \right] \right)^{1/2}$$

$$\rightarrow \boxed{a-\bar{x} \approx e^{-\omega t}}$$

Similarly for $t < 0$ and $|t|$ large
 $+wt$

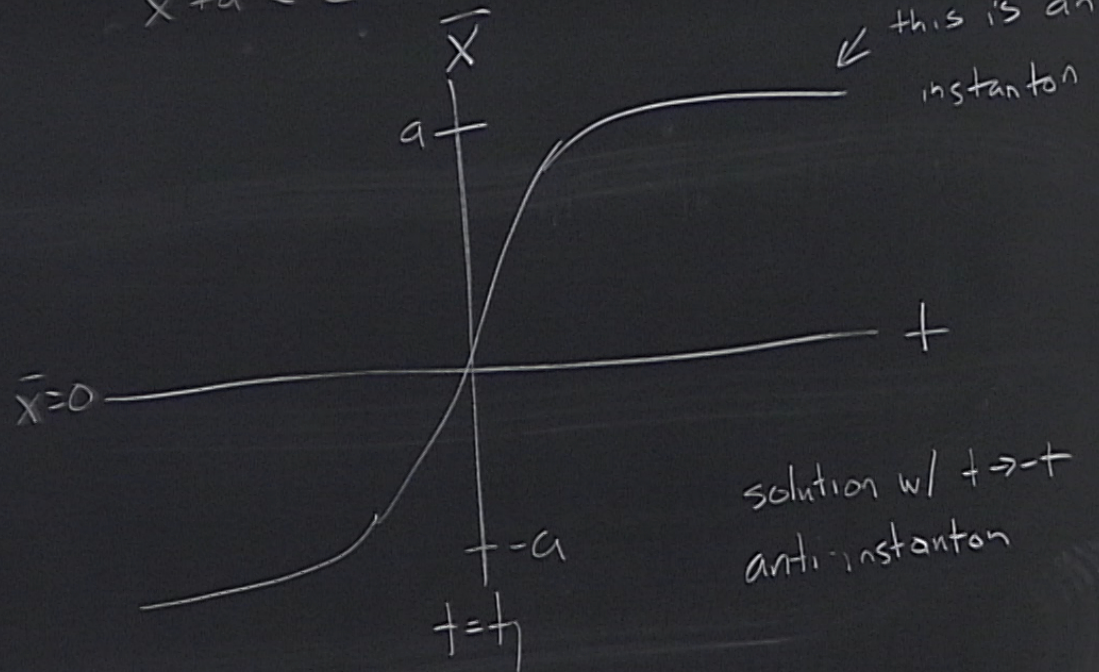
$$\bar{x} + a \approx e$$



$$V''(a) = \frac{\omega^2}{2} \left(\frac{1}{2} \right)^{1/2}$$

Similarly for $t < 0$ and $|t|$ large

$$\bar{x} + a \approx e^{-t}$$



Action of ^(anti-)instanton

$$S_E \equiv S_0 = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_E \left(\frac{1}{2} \left(\frac{d\bar{x}}{dt} \right)^2 + V(\bar{x}) \right)$$

Solutions to Euclidean EOM

$$= \int_{-a}^a d\bar{x} \frac{dt}{d\bar{x}} \left[\frac{1}{2} \dot{\bar{x}}^2 + V \right] \quad \left(\frac{d\bar{x}}{dt} = \sqrt{2} \right)$$

$$= \int_{-a}^a d\bar{x} \sqrt{2V(\bar{x})}$$

Action of (anti-) instanton

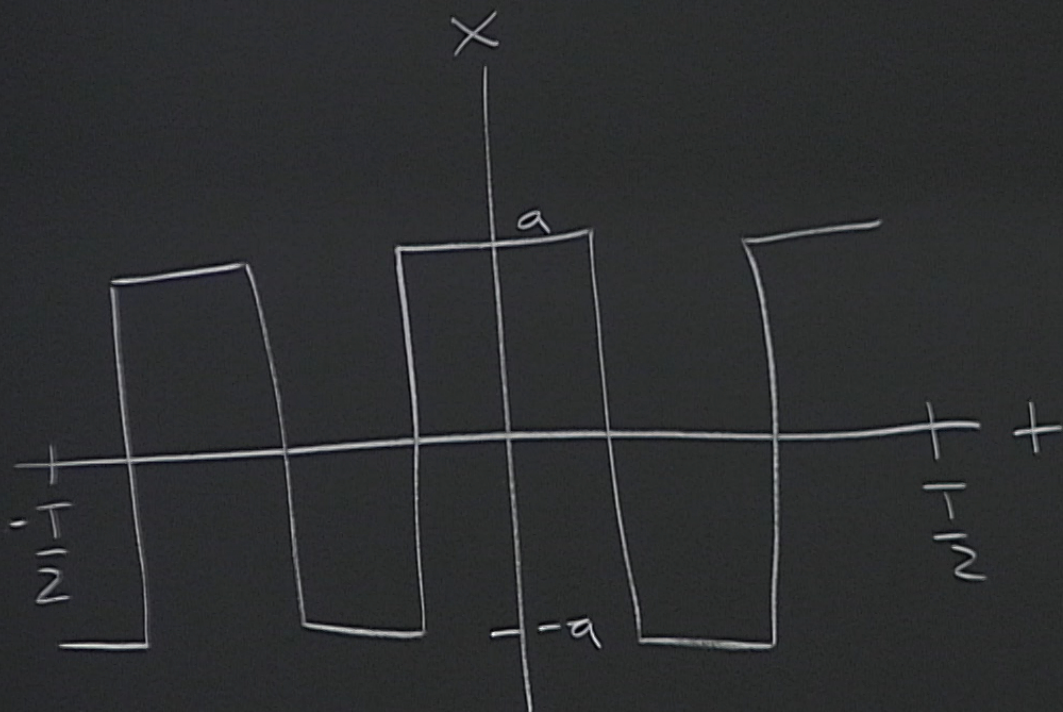
$$S_E \equiv S_0 = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_E \left(\frac{1}{2} \left(\frac{d\bar{x}}{dt} \right)^2 + V(\bar{x}) \right)$$

Solutions to Euclidean EOM $\Rightarrow V(\bar{x}) = 0$ (zero action)

$$= \int_{-a}^a d\bar{x} \frac{dt}{d\bar{x}} \left[\frac{1}{2} \dot{\bar{x}}^2 + V \right] \quad \left(\frac{d\bar{x}}{dt} = \sqrt{2V} \right)$$

$$= \int_{-a}^a d\bar{x} \sqrt{2V(\bar{x})}$$

$$\begin{aligned}
 & \langle a | e^{-HT/\hbar} | -a \rangle_{\text{one-instanton}} = K \int_{-\frac{T}{2}}^{\frac{T}{2}} dt e^{-S_E/\hbar} \\
 & \quad \uparrow \quad \quad \quad \uparrow \\
 & \quad \text{fluctuation about saddle} \quad \text{location of center is unspecified} \\
 & \quad (\det S'')^{-1/2}
 \end{aligned}$$



$$\text{action} \approx n S_0$$

↑
total number
of instantons
+ anti-instantons

become saddle points as $T \rightarrow \infty$

$$\langle a | e^{-HT/\hbar} | -a \rangle = C \sum_{n \text{ odd}} K^n \underbrace{\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{t_1} dt_2 \cdots \int_{-T/2}^{t_{n-1}} dt_n}_{\frac{T^n}{n!}} e^{-nS_0/\hbar}$$

$$= C \sum_{n \text{ odd}} K^n \frac{T^n}{n!} e^{-nS_0/\hbar}$$

$$= \frac{C}{2} \left(e^{kT} e^{-S_0/\hbar} - e^{-kT} e^{-S_0/\hbar} \right)$$

$$\equiv A - B$$

$$= \langle -a | e^{-HT/\hbar} | a \rangle$$

③ Theta Vacua

$$\langle a | e^{-HT/\hbar} | a \rangle = \langle -a | e^{-HT/\hbar} | -a \rangle = C \sum_{n \text{ even}} \frac{k^n T^n}{n!} e^{-nS_0/\hbar}$$

$$= A + B$$

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|a\rangle \mp |-a\rangle) \text{ are energy eigenstates}$$

$$\text{lowest energy levels are } \frac{1}{2}\hbar\omega \pm \hbar k e^{-S_0/\hbar}$$

① Instanton in QFT

② Decay of False Vacuum in QFT
Resummation of Asymptotic Series + Instantons

③ Theta Vacua

$$E_0 = -\lim_{T \rightarrow \infty} \frac{\hbar}{T} \ln Z$$

unstable potential $\frac{\Gamma}{2} = |I|$

$$\langle a | e^{-HT/\hbar} | a \rangle = \langle -a | e^{-HT/\hbar} | -a \rangle = \left(\sum_{n \text{ even}} \frac{k^n T^n}{n!} e^{-nS_0/\hbar} \right)$$
$$= A + B$$

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|a\rangle \mp |-a\rangle) \text{ are energy eigenstates}$$

$$\text{lowest energy levels are } \frac{1}{2} \hbar \omega \pm \hbar K e^{-S_0/\hbar}$$

