Title: PSI 17/18 - Quantum Field Theory III - Lecture 14

Date: Feb 14, 2018 11:30 AM

URL: http://pirsa.org/18020034

Abstract:

Instantons

solutions to Euclidean EOM w/ finite, non-zero action

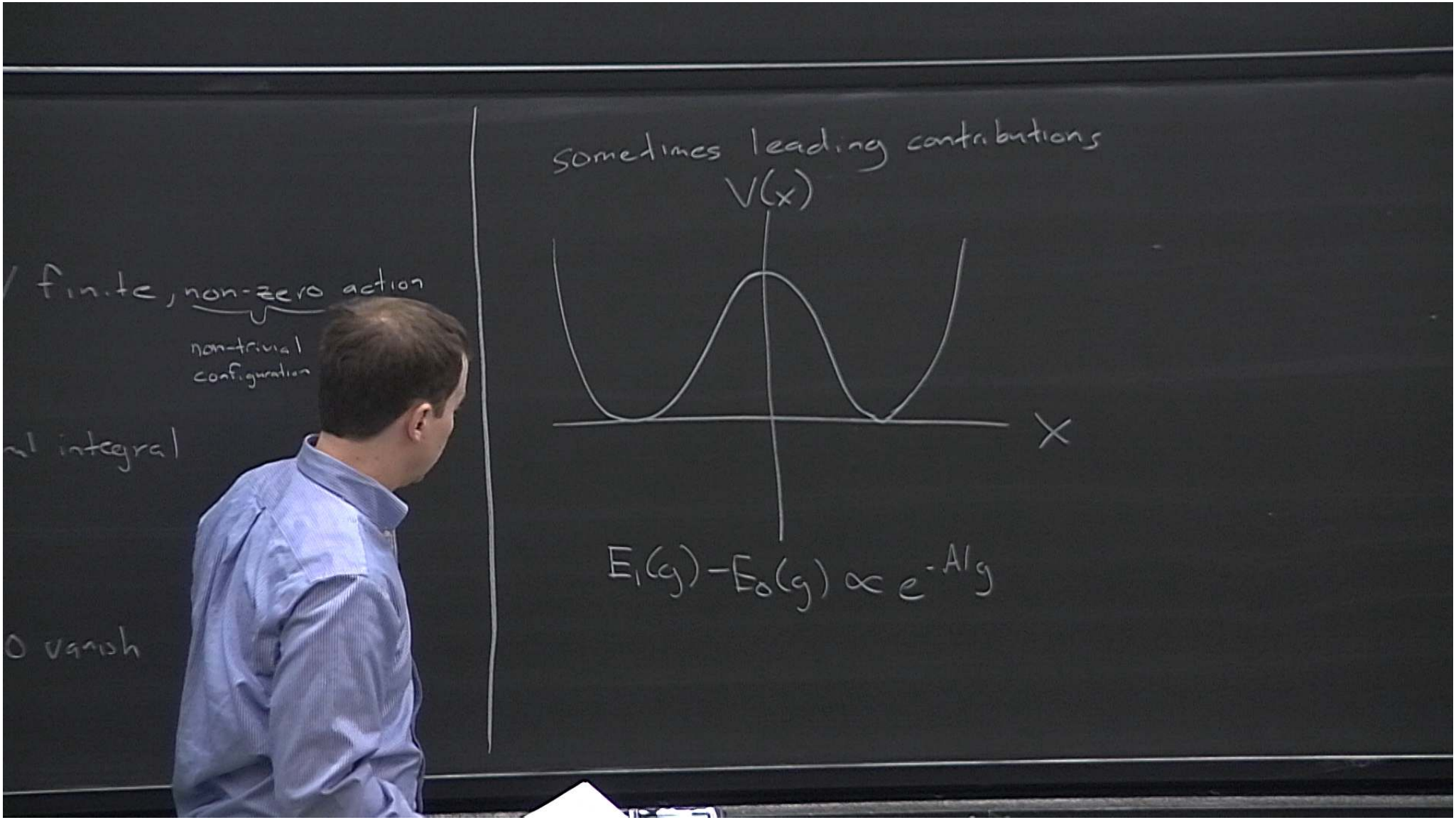
non-trivial configuration

non-trivial saddle points of functional integral

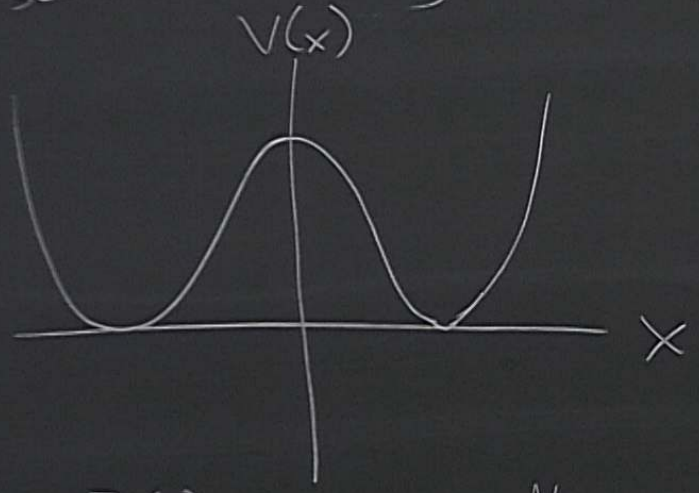
typically $\propto e^{-A/g}$

small coupling

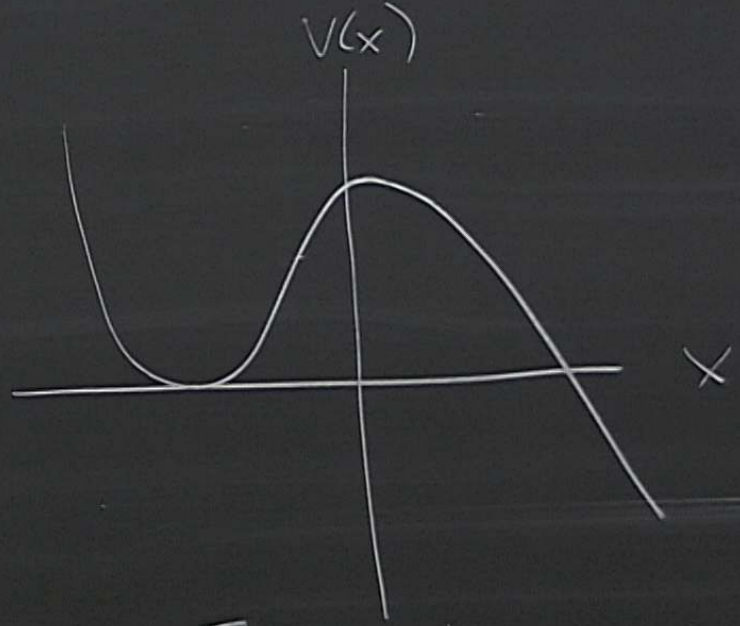
all terms in Taylor series about $g=0$ vanish


finite, non-zero action
non-trivial configuration
integral
vanish
sometimes leading contributions
V(x)
X
$E_1(g) - E_0(g) \propto e^{-A/g}$

Sometimes leading contributions



$E_1(g) - E_0(g) \propto e^{-A/g}$



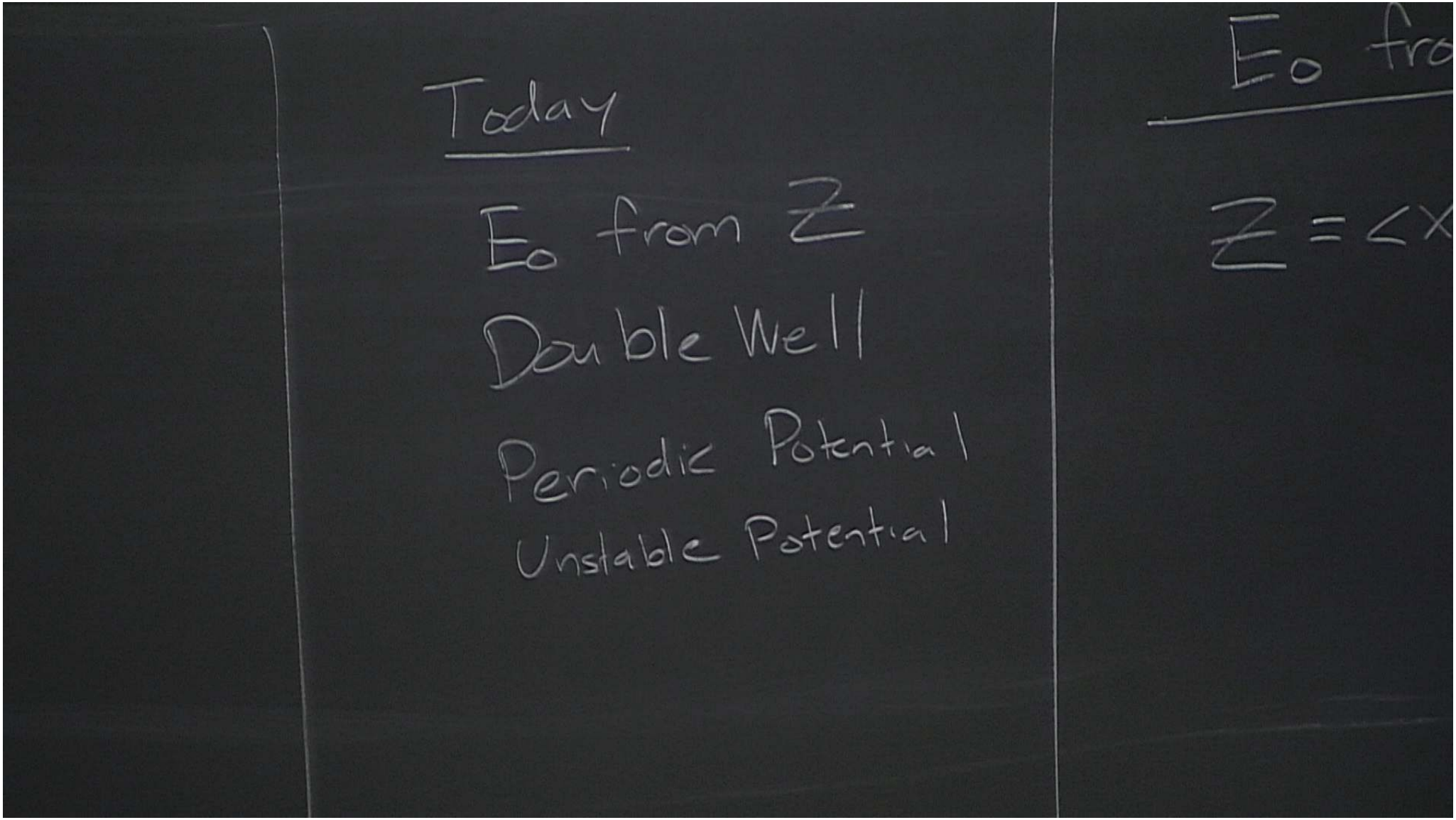
$\Gamma \propto e^{-A/g}$

$f(g) \sim \sum a_n g^n \overset{?}{+} e^{-A/g}$

↑ exact quantity

Outline

1. Instanton in QM
2. Decay of False Vacuum in QFT
   Resummation of Asymptotic Series + Instantons
3. Theta Vacua


Today
E₀ from Z
Double Well
Periodic Potential
Unstable Potential
E₀ fro
Z = <X

E_0 from Euclidean Path Integral

$$Z = \langle x_f | e^{-HT/\hbar} | x_i \rangle = \int_{x(-\frac{T}{2})=x_i}^{x(\frac{T}{2})=x_f} \mathcal{D}[x(t)]\, e^{-S_E[x]/\hbar}$$

$$H|n\rangle = E_n|n\rangle$$

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = \sum_n e^{-E_n T/\hbar} \langle x_f | n \rangle \langle n | x_i \rangle$$

$$E_0 = -\lim_{T\to\infty} \frac{\hbar}{T} \ln Z$$

E_0 from Euclidean Path Integral

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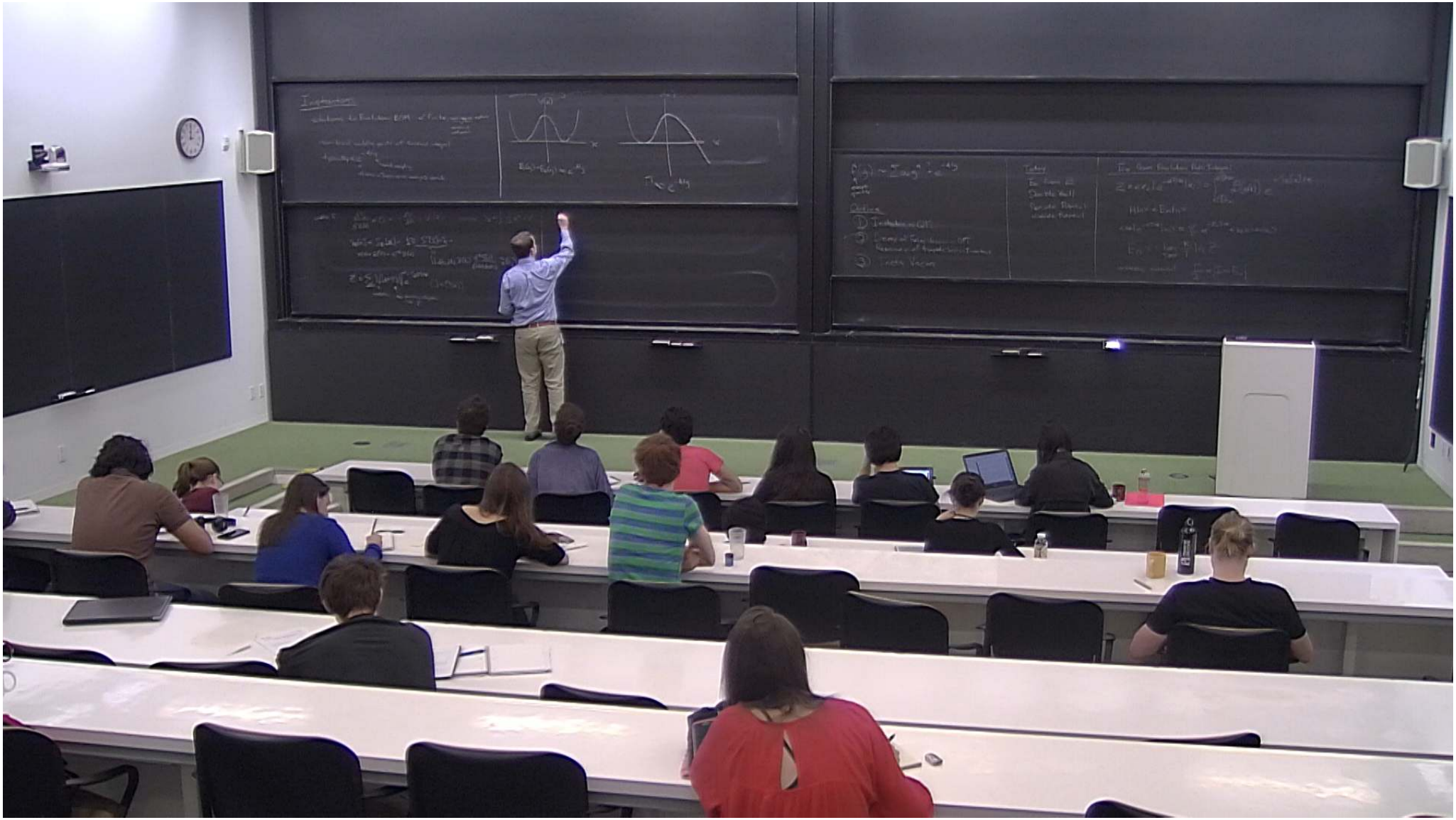
unstable potential $\quad \frac{\Gamma}{2} = |\mathrm{Im}\, E_0|$



saddle $\bar{x}$ $\quad \frac{\delta S}{\delta \bar{x}(t)} = 0 = -\frac{d^2\bar{x}}{dt^2} + V'(\bar{x})$ $\quad$ assume $S_E = \int \frac{1}{2}\dot{x}^2 + V$

$$S_E[x] = S_E[\bar{x}] + \frac{1}{2}\tilde{x}\cdot S''[\bar{x}]\cdot\tilde{x} + \cdots$$

$$x(t) = \bar{x}(t) + \hbar^{1/2}\tilde{x}(t)$$

$$\iint dt_1 dt_2\, \tilde{x}(t_1)\frac{\delta^2 S[x]}{\delta x(t_1)\delta x(t_2)}\tilde{x}(t_2)$$

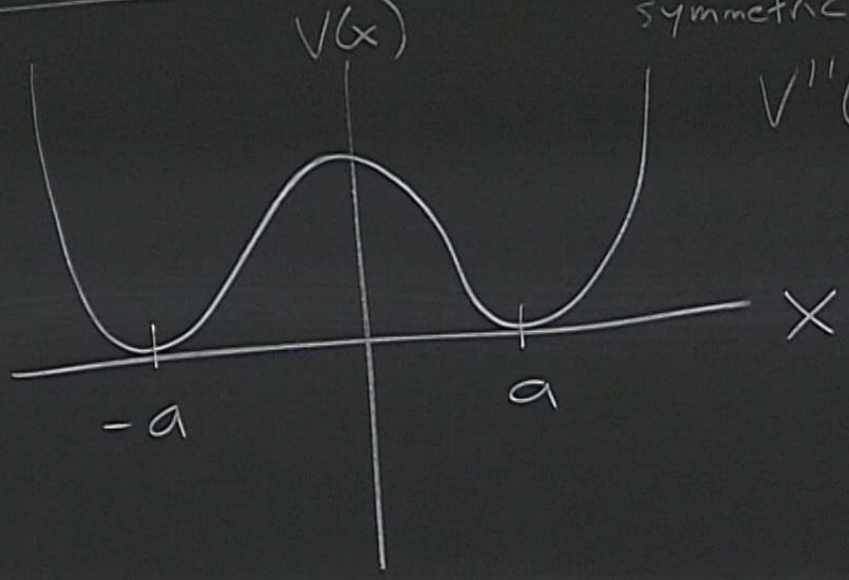
$$Z = \sum_{\text{saddles}} N\left|\det S''[\bar{x}]\right|^{-1/2} e^{-S_E[\bar{x}]/\hbar}\left(1+O(\hbar)\right)$$

↑ saddles — normalization — ↑ from Gaussian integration over $\tilde{x}$

$= \int \frac{1}{2}\dot{x}^2 + V$

$\tilde{x}(t_2)$

Double Well



symmetric $x \to -x$

$V''(\pm a) = \omega^2$

$E_1 - E_0 = ?$

compute $\langle \pm a | e^{-HT/\hbar} | \pm a \rangle$

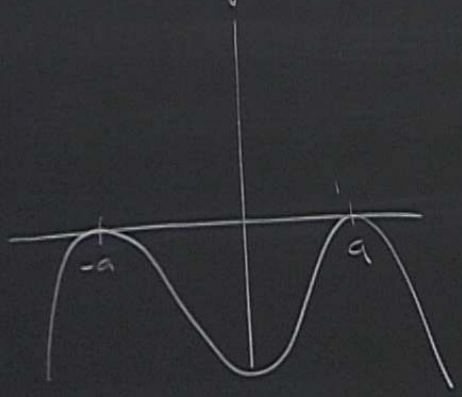
③ Theta Vacua

unstable potential

2

$\langle a | e^{-HT/\hbar} | -a \rangle$

$-\frac{d^2\bar{x}}{dt^2} + V'(\bar{x}) = 0$ ← EOM for a particle of unit mass in potential $-V$



$E = \frac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2 - V(\bar{x})$ constant of motion

there is a solution with $E=0$

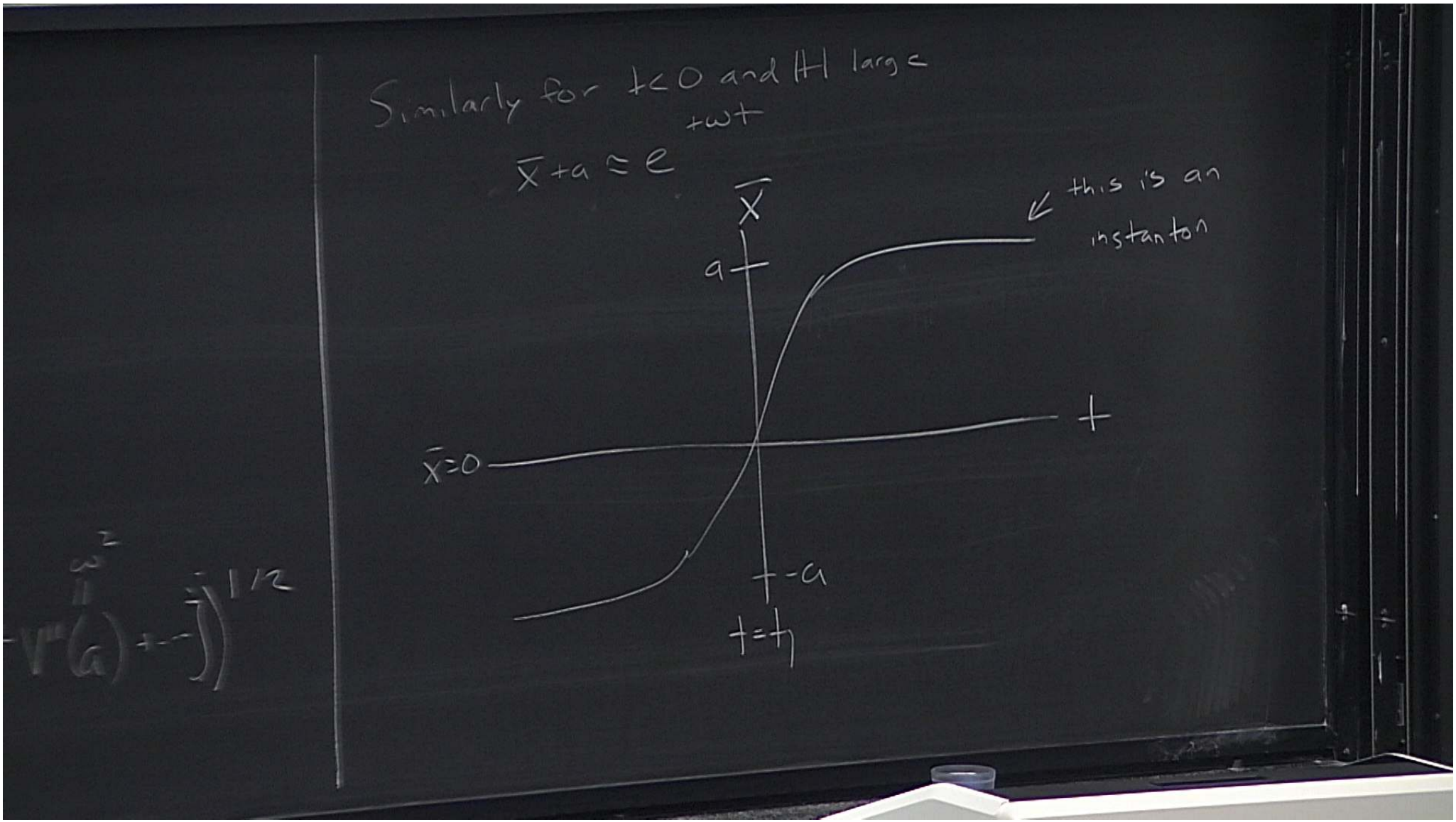
for large t $\bar{x}(t) \approx a$

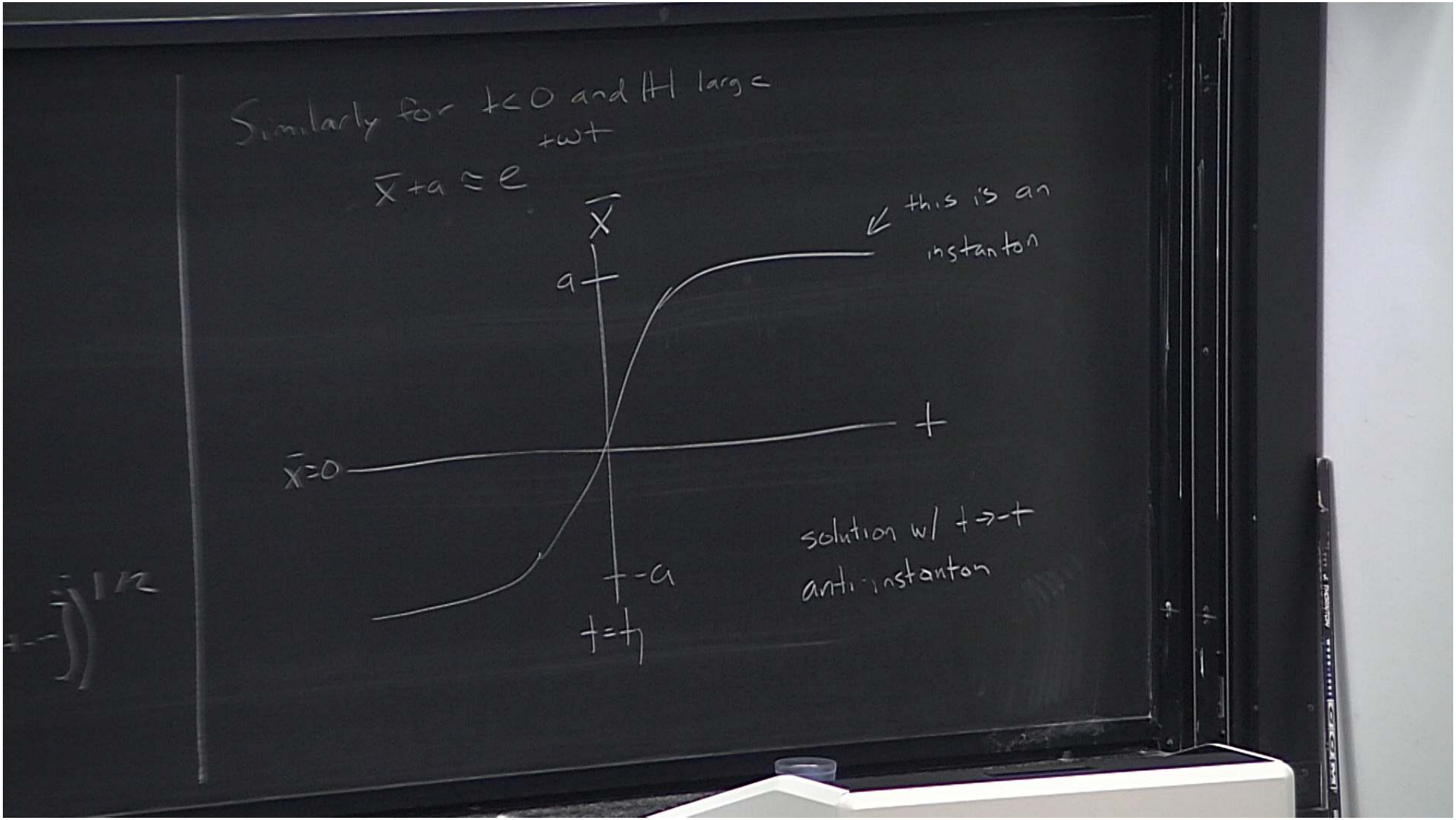
$\frac{d\bar{x}}{dt} = \sqrt{2V(\bar{x})} \approx \left(2\left[V(a) + (a-\bar{x})V'(a) + \frac{1}{2}(a-\bar{x})^2 V''(a) + \dots\right]\right)^{1/2}$

($V(a) \to 0$, $V'(a) \to 0$, $V''(a) = \omega^2$)

$\approx (a-\bar{x})\omega$

$\to \boxed{a-\bar{x} \approx e^{-\omega t}}$


Similarly for t<0 and |t| large
$\bar{x} + a \simeq e^{+\omega t}$
$\bar{x}$
a
-a
t
$\bar{x}=0$
t=t1
this is an instanton


Similarly for t<0 and |t| large
$\bar{x}+a \simeq e^{+\omega t}$
$\bar{x}$
a
this is an instanton
t
$\bar{x}=0$
solution w/ t→-t
anti-instanton
t=-a
t=t_f

Action of (anti-) instanton

$$S_E \equiv S_0 = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_E \left(\frac{1}{2} \left(\frac{d\bar{x}}{dt} \right)^2 + V(\bar{x}) \right)$$

$$= \int_{-a}^{a} d\bar{x} \, \frac{dt}{d\bar{x}} \left[\frac{1}{2} \dot{\bar{x}}^2 + V \right] \qquad \left(\frac{d\bar{x}}{dt} = \sqrt{2} \right.$$

$$= \int_{-a}^{a} d\bar{x} \sqrt{2V(\bar{x})}$$

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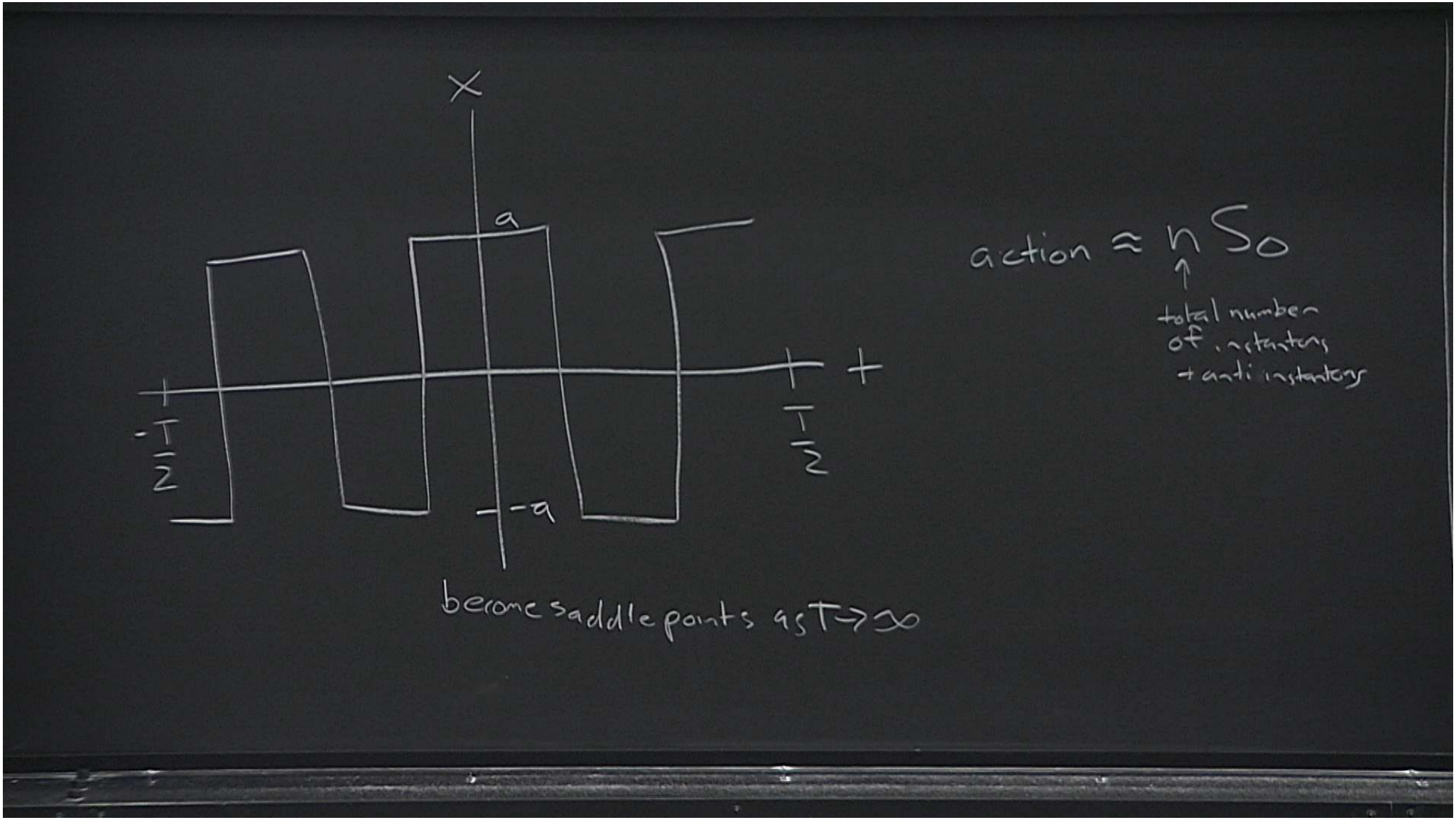
$$\langle a | e^{-HT/\hbar} | -a \rangle_{\substack{\text{one} \\ \text{-instanton}}} = K \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, e^{-S_E/\hbar}$$

K: fluction about saddle $(\det S'')^{-1/2}$

t: location of center is unspecified


x
a
-a
t
$-\frac{T}{2}$
$\frac{T}{2}$
action ≈ n S_0
total number of instantons + anti instantons
become saddle points as $T \to \infty$

$$\langle a | e^{-HT/\hbar} | -a \rangle = C \sum_{n \text{ odd}} k^n \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{t_1} dt_2 \cdots \int_{-T/2}^{t_{n-1}} dt_n \, e^{-nS_0/\hbar}$$

$$= C \sum_{n \text{ odd}} \frac{k^n T^n}{n!} e^{-nS_0/\hbar} \frac{I^n}{n!}$$

$$= \frac{C}{2} \left(e^{kTe^{-S_0/\hbar}} - e^{-kTe^{-S_0/\hbar}} \right)$$

$$\equiv A - B$$

$$= \langle -a | e^{-HT/\hbar} | a \rangle$$

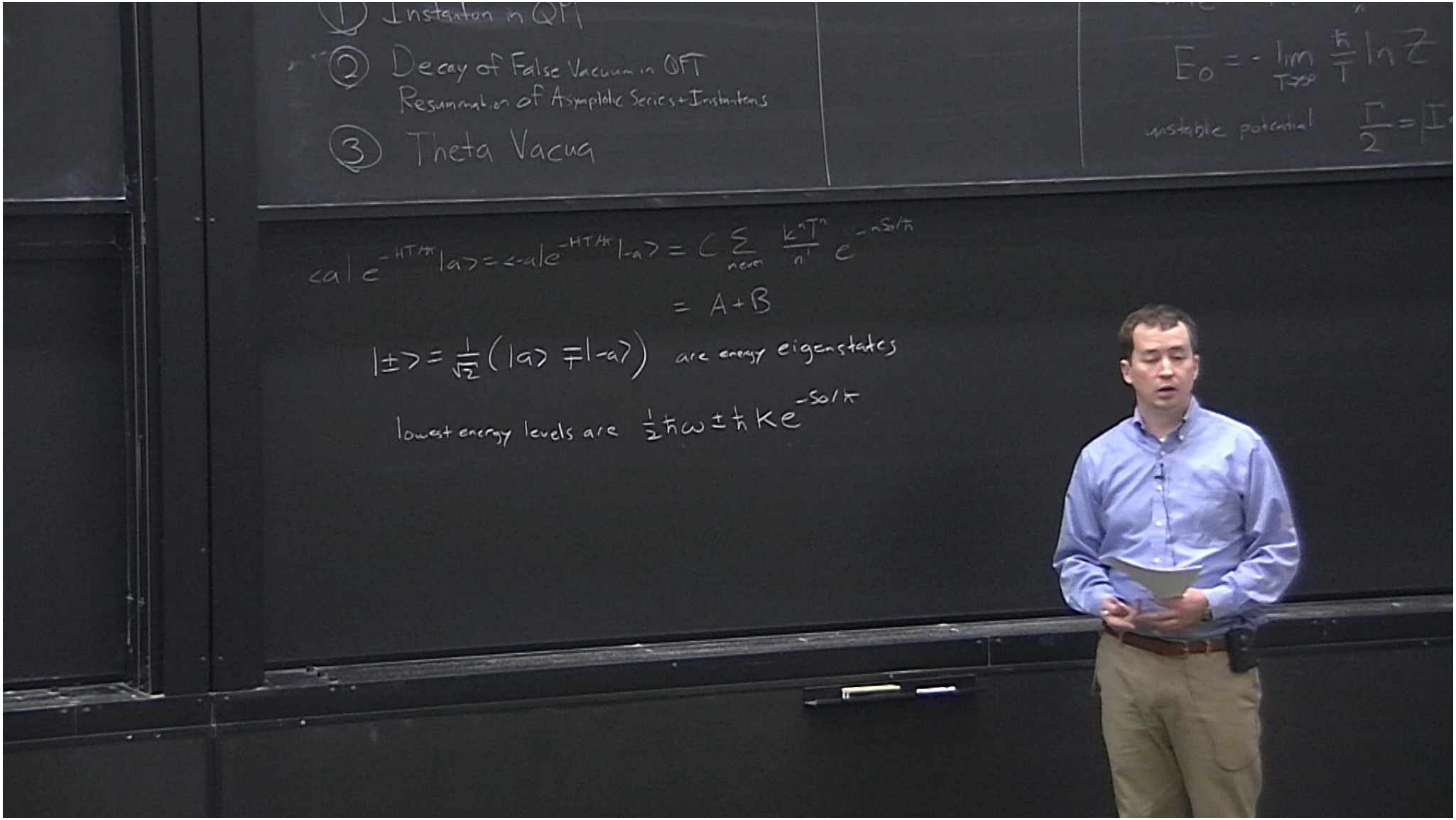
③ Theta Vacua

$$\langle a|e^{-HT/\hbar}|a\rangle = \langle -a|e^{-HT/\hbar}|-a\rangle = C\sum_{n\,\text{even}} \frac{k^n T^n}{n!} e^{-nS_0/\hbar}$$

$$= A + B$$

$|\pm\rangle = \frac{1}{\sqrt{2}}(|a\rangle \mp |-a\rangle)$ are energy eigenstates

lowest energy levels are $\frac{1}{2}\hbar\omega \pm \hbar K e^{-S_0/\hbar}$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