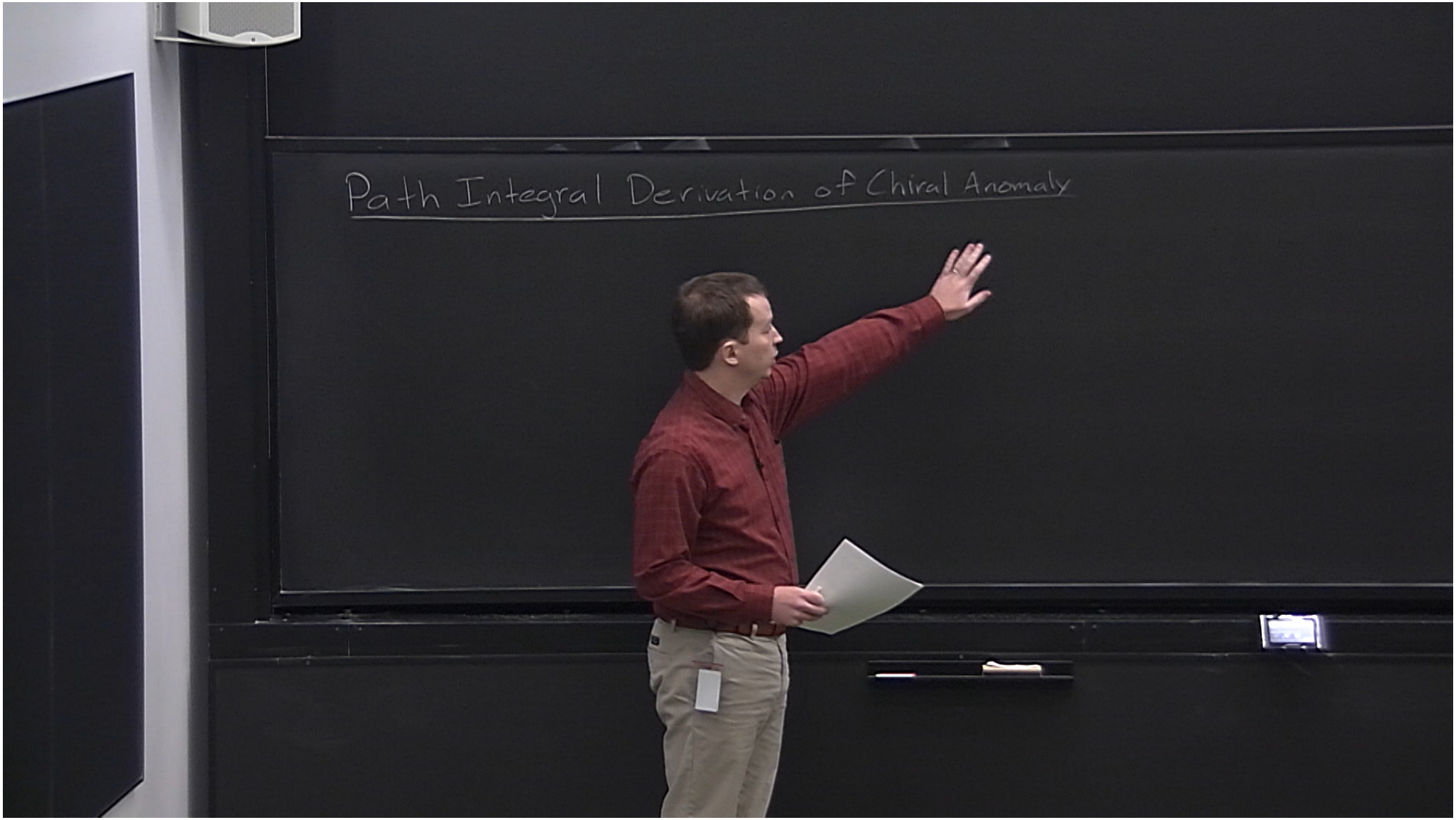
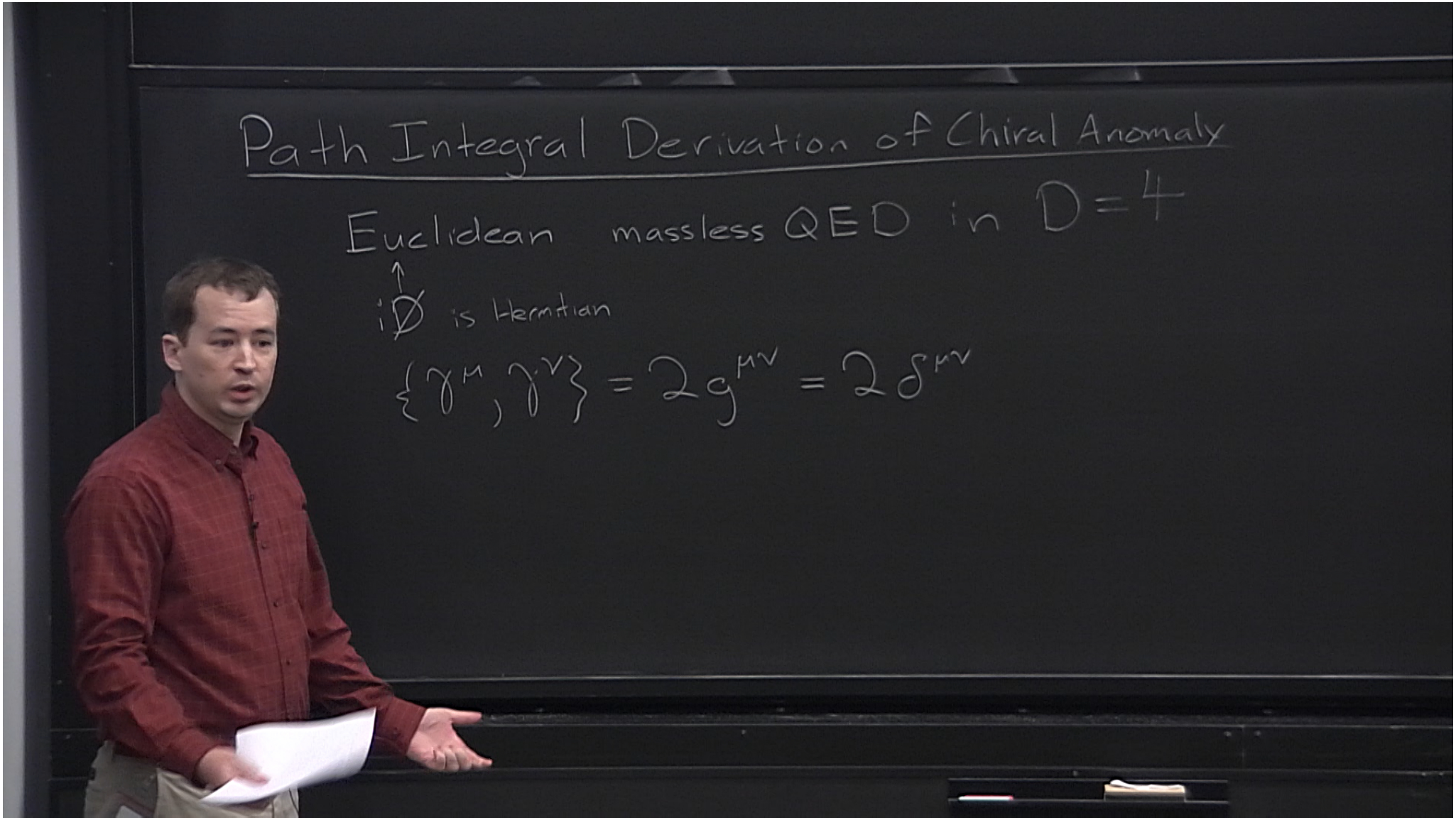
Title: PSI 17/18 - Quantum Field Theory III - Lecture 9

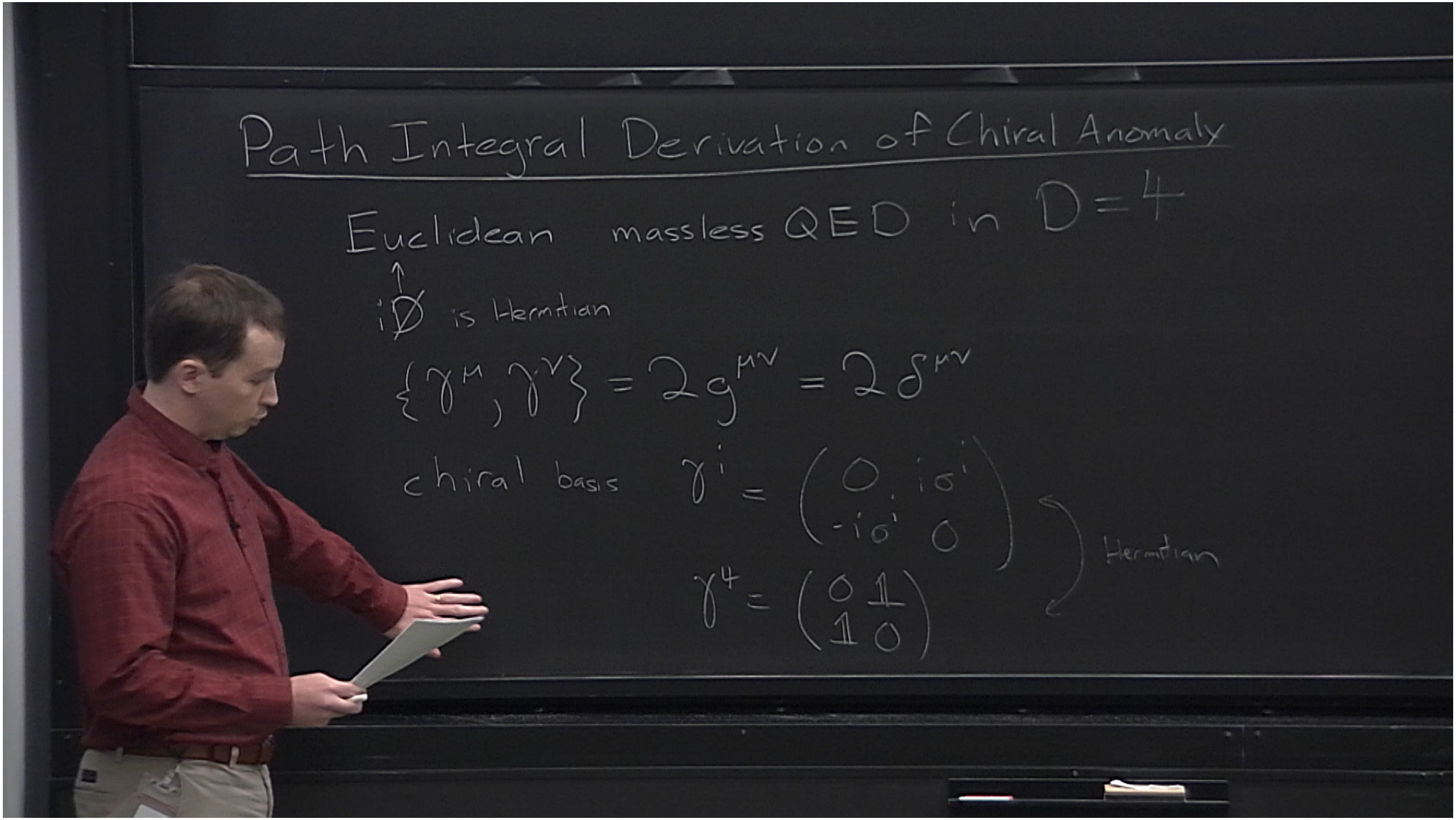
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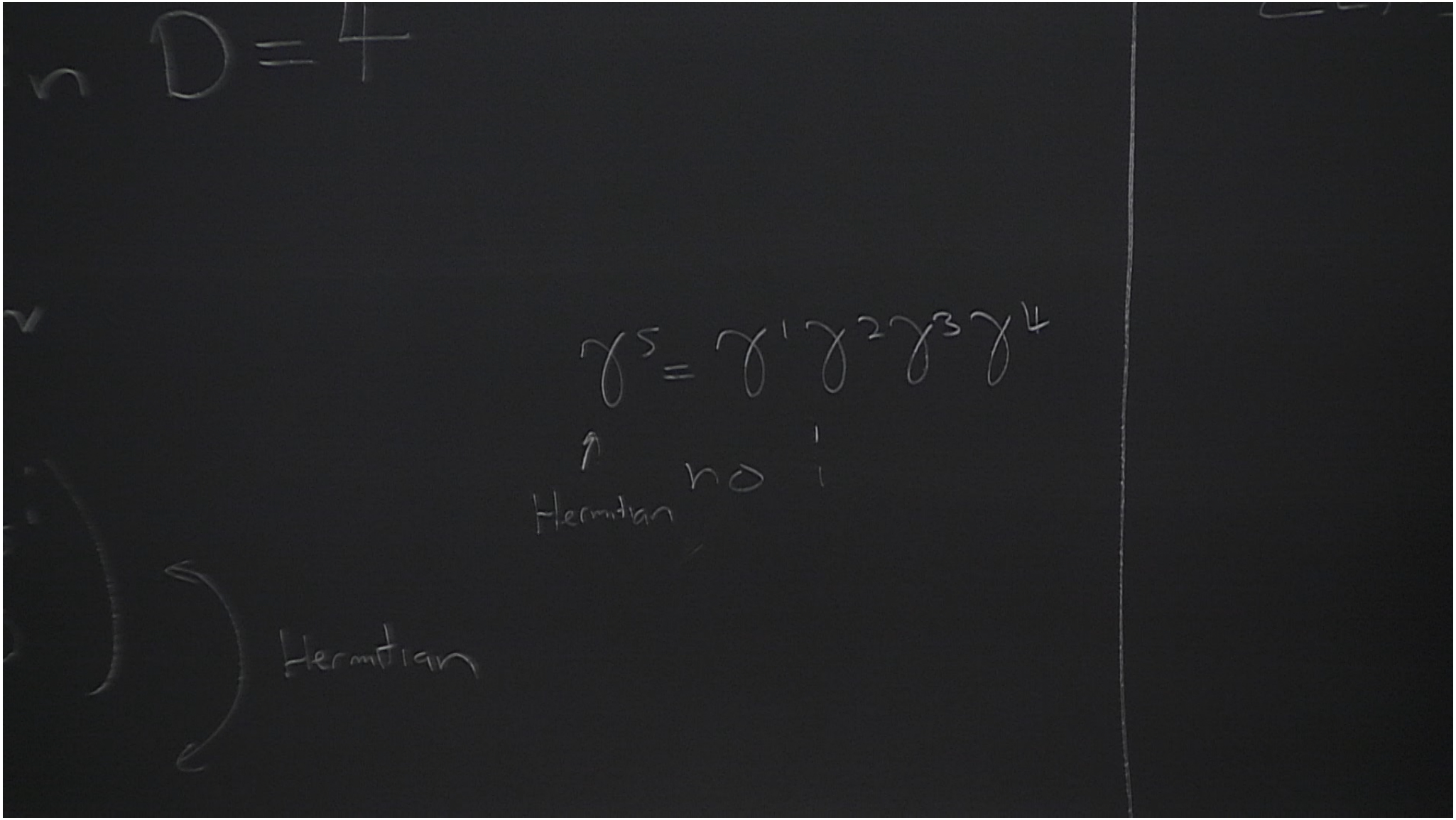
URL: http://pirsa.org/18020030

Abstract:


Path Integral Derivation of Chiral Anomaly


Path Integral Derivation of Chiral Anomaly
Euclidean massless QED in D=4
$i\not{D}$ is Hermitian
$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\delta^{\mu\nu}$


Path Integral Derivation of Chiral Anomaly
Euclidean massless QED in D=4
$i\not{D}$ is Hermitian
$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\delta^{\mu\nu}$
chiral basis $\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}$
$\gamma^4 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$
Hermitian


n D = 4
$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$
Hermitian
no !
Hermitian

$$Z[A] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S_{QED}}$$

↑ no A integral

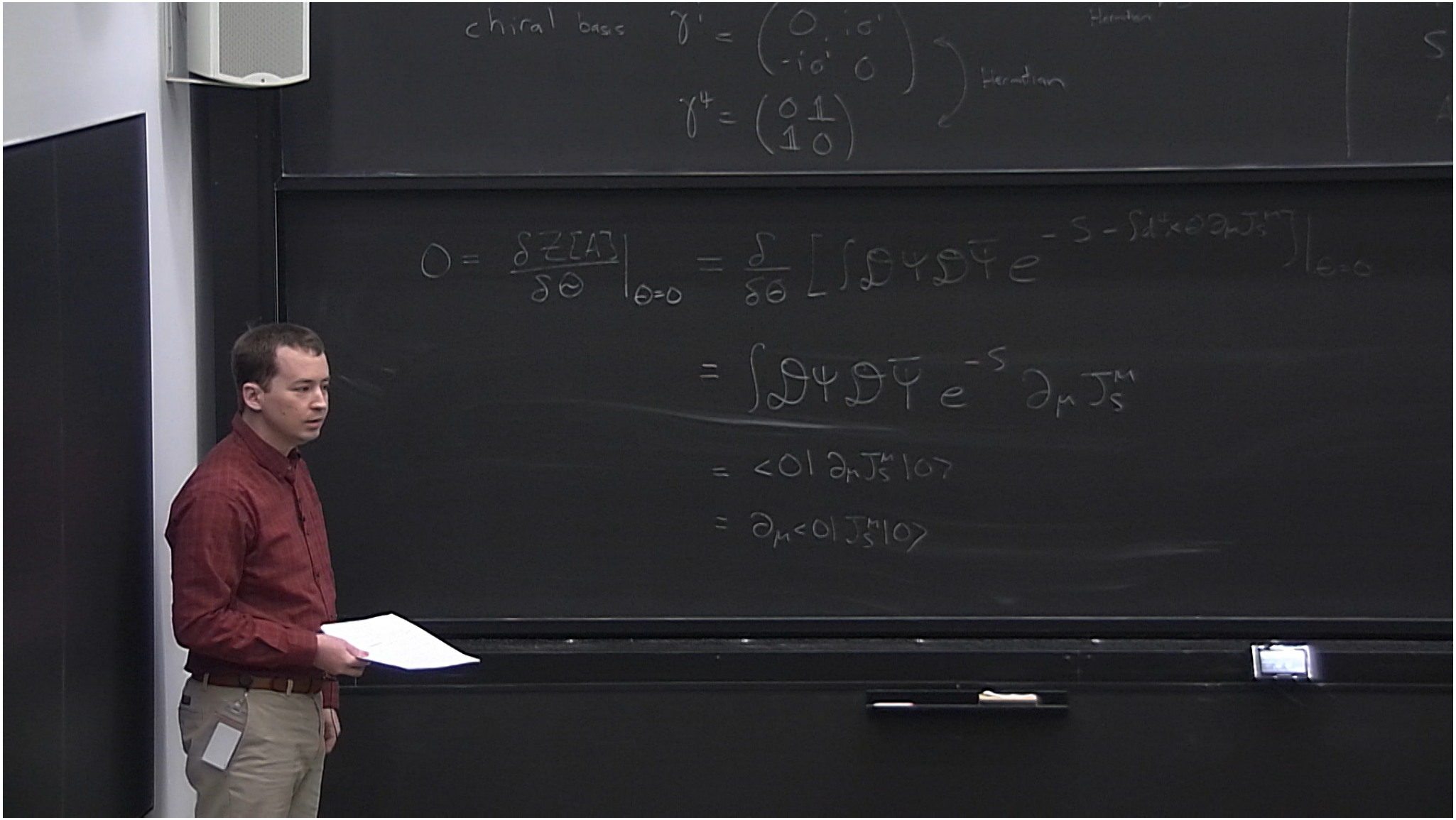
<u>local</u> chiral transformation

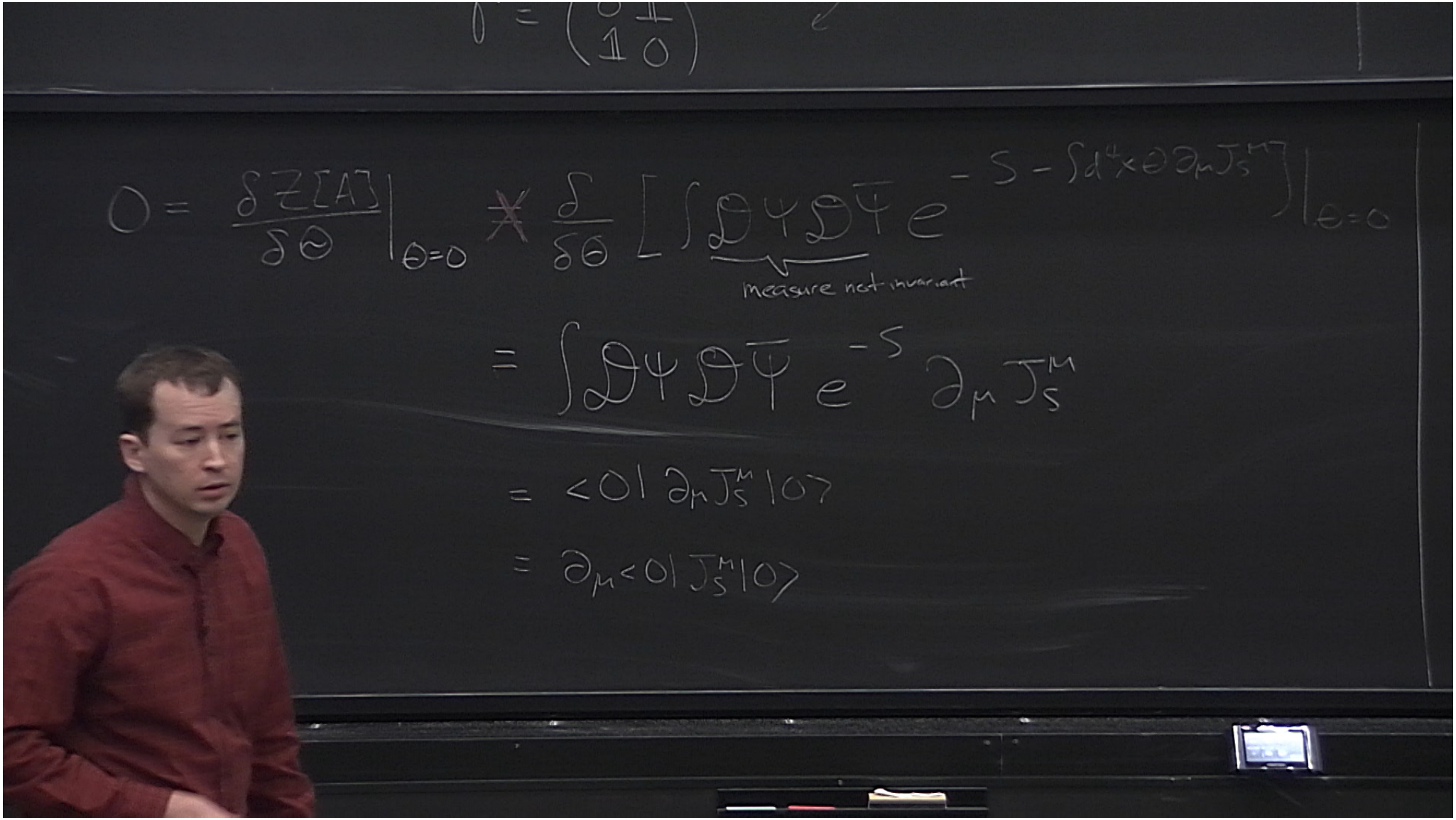
$$\psi \to \psi' = e^{i\theta(x)\gamma_5}\psi$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi} e^{i\theta(x)\gamma_5}$$

$$S \to S' = S + \int d^4x\, \theta(x)\partial_\mu J_5^\mu$$

$$A_\mu \to A'_\mu = A_\mu$$


chiral basis
$\gamma^i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}$
$\gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Hermitian
Hermitian
$0 = \left.\frac{\delta Z[A]}{\delta \omega}\right|_{\theta=0} = \frac{\delta}{\delta\theta}\left[\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S - \int d^4x\, \theta\, \partial_\mu J_5^\mu}\right]\Big|_{\theta=0}$
$= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-S}\, \partial_\mu J_5^\mu$
$= \langle 0|\, \partial_\mu J_5^\mu\, |0\rangle$
$= \partial_\mu \langle 0|\, J_5^\mu\, |0\rangle$


$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$0 = \frac{\delta Z[A]}{\delta \omega}\Big|_{\theta=0} = \frac{\delta}{\delta \theta}\left[\int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S - \int d^4x \theta \partial_\mu J_5^\mu}\right]\Big|_{\theta=0}$
measure not invariant
$= \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S}\, \partial_\mu J_5^\mu$
$= \langle 0 | \partial_\mu J_5^\mu | 0 \rangle$
$= \partial_\mu \langle 0 | J_5^\mu | 0 \rangle$

$A_\mu \to A'_\mu = A_\mu$

$i\not{D} = i\not{\partial} - \not{A}$ is Hermitian

$i\not{D}\,\phi_n = \lambda_n \phi_n$ ← eigenfunctions

↑ real eigenvalues

$$\int d^4x\, \phi_n^\dagger(x)\, \phi_m(x) = \delta_{nm}$$

$$\sum_n \phi^\dagger_{n\alpha}(x)\, \phi_{n\beta}(y) = \delta(x-y)\,\delta_{\alpha\beta}$$

$$\psi(x) = \sum_n a_n \phi_n(x)$$

$$\bar{\psi}(x) = \sum_n \bar{b}_n \phi_n^+(x)$$

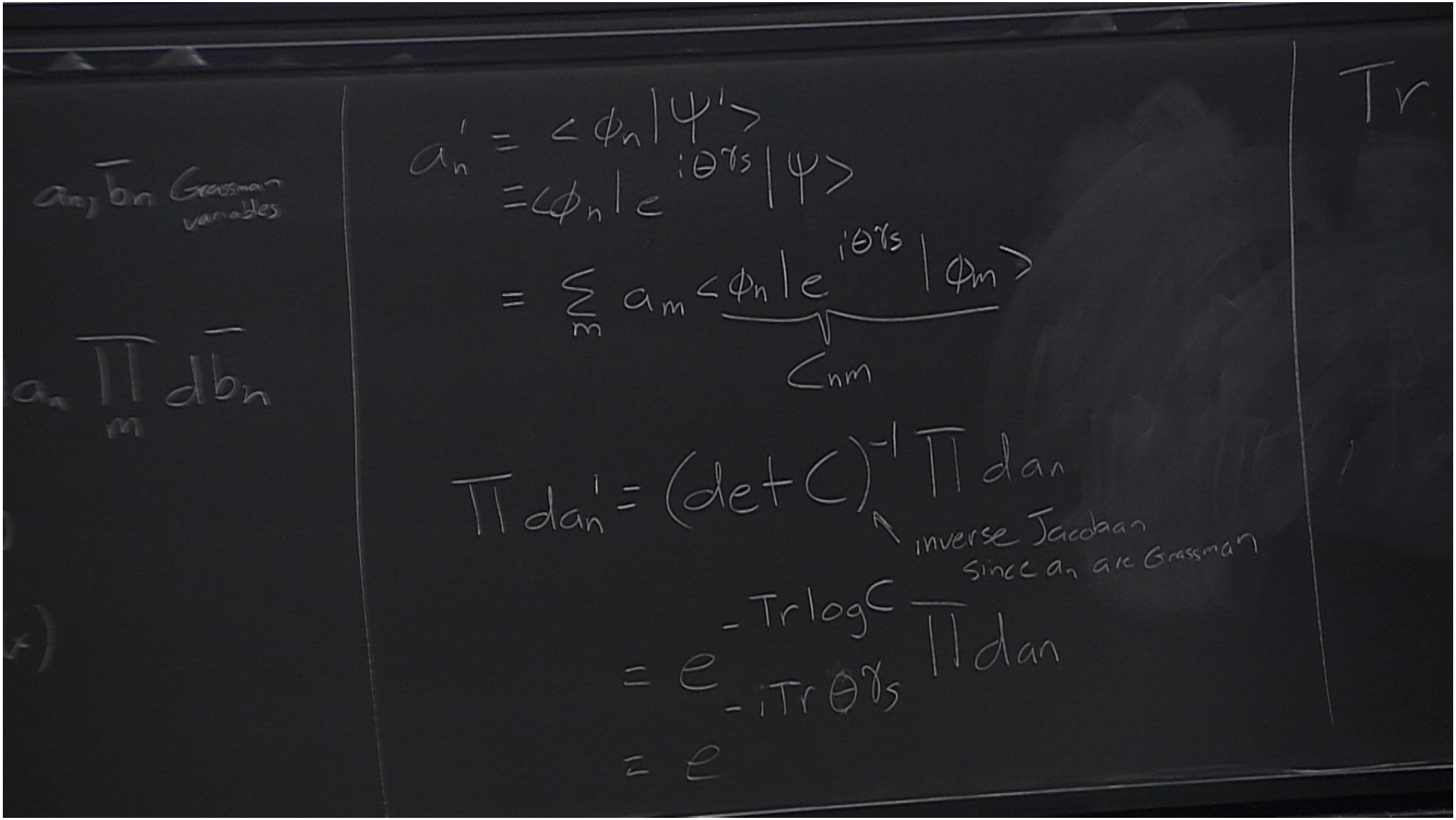
$a_n, \bar{b}_n$ Grassman variables

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_n da_n \prod_m d\bar{b}_n$$

$$\psi'(x) = \sum_n a_n' \phi_n(x)$$

$$\bar{\psi}'(x) = \sum_n \bar{b}_n' \phi_n^+(x)$$

a_n'


$a_n, \bar{b}_n$ Grassman variables
$\prod_n d\bar{b}_n$
$a_n' = \langle \phi_n | \psi' \rangle$
$= \langle \phi_n | e^{i\theta\gamma_5} | \psi \rangle$
$= \sum_m a_m \langle \phi_n | e^{i\theta\gamma_5} | \phi_m \rangle$
C_{nm}
$\prod da_n' = (\det C)^{-1} \prod da_n$
inverse Jacobian since a_n are Grassman
$= e^{-\mathrm{Tr}\log C} \prod da_n$
$= e^{-i\mathrm{Tr}\,\theta\gamma_5} \prod da_n$
$= e$
Tr

$$\mathrm{Tr}\,\theta\gamma_5 = \int d^4x \sum_n \phi_n^\dagger(x)\,\theta(x)\,\gamma^5_{\alpha\beta}\,\phi_n{}_\beta(x)$$

$$= \int d^4x\ \delta^{(4)}(x-x)\,\theta(x)\,\mathrm{tr}\,\gamma_5$$

$$= \infty \cdot 0$$

$$\delta^{(4)}(0) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\Big|_{x=y}$$

Need UV regulator

Insert a function $f(k^2)$ integral

Assume $f(0) = 1$

$$f(\infty) = f'(\infty) = f''(\infty) = \cdots = 0$$

e.g. $f(k^2) = e^{-k^2}$ or $\frac{1}{1+k^2}$

$$J^{(0)}(0) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}\Big|_{x=y}$$

Need UV regulator

$$\mathrm{Tr}\,\Theta\gamma_5 = \lim_{\Lambda\to\infty} \int d^4x \sum_n \Theta\, \phi_n^\dagger \gamma_5 f\left(-\frac{\lambda_n^2}{\Lambda^2}\right)\phi_n$$

$$= \lim_{\Lambda\to\infty} \int d^4x \sum_n \Theta\, \phi_n^\dagger \gamma_5 f\left(-\frac{(\not{D})^2}{\Lambda^2}\right)\phi_n$$

$$\overset{\text{Fourier transform}}{=} \lim_{\Lambda\to\infty} \int d^4x \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} \sum_n \Theta\, e^{-ik'x}\tilde{\phi}_n^\dagger(k')\gamma_5 \underbrace{f\left(-\frac{(\not{D})^2}{\Lambda^2}\right)\tilde{\phi}_n(k)e^{ikx}}$$

$$\not{D}\left(e^{ikx}g(x)\right) = e^{ikx}(i\not{k}+\not{D})g(x)$$

$$e^{ikx} f\left(-\frac{(i\not{D}-\not{k})^2}{\Lambda^2}\right)\tilde{\phi}_n(k)$$

$A_\mu \to A'_\mu = A_\mu$

$i\not{D} = i\not{\partial} - \not{A}$ is Hermitian

$i\not{D}\,\phi_n = \lambda_n \phi_n$ ← eigenfunctions

↑ real eigenvalues

$$\int d^4x\, \phi_n^\dagger(x)\, \phi_m(x) = \delta_{nm}$$

$$\sum_n \phi^\dagger_{n\alpha}(x)\, \phi_{n\beta}(y) = \delta(x-y)\,\delta_{\alpha\beta}$$

$$\sum \tilde{\phi}^\dagger_n(k')\, \tilde{\phi}_n(k) = (2\tilde{\pi})^4\, \delta^{(4)}(k-k')\, \underline{\mathbb{1}}$$

↑ spinor identity

$$\mathrm{Tr}\,\Theta\gamma_5 = \lim_{\Lambda\to\infty} \int d^4x \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\,\Theta\gamma_5\, f\!\left(\frac{-(i\not{D}-\not{k})^2}{\Lambda^2}\right)$$

$$\sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{-i\not{D}}{\Lambda}\right)^{2N} f^{(N)}\!\left(\frac{k^2}{\Lambda^2}\right)$$

$$\mathrm{tr}\,\gamma_5 = \gamma_5\gamma_\mu\gamma_\nu = 0$$

N=0 + N=1 vanish

N=2 term >> N=3 terms as $\Lambda\to\infty$

$$\mathrm{tr}\,\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4\,\epsilon_{\mu\nu\rho\sigma}$$

now in Euclidean

$$\mathrm{Tr}\,\Theta\gamma_5 = \lim_{\Lambda\to\infty} \frac{1}{32\pi^4} \int d^4x \int \frac{d^4k}{\Lambda^4}\, 4\,\epsilon_{\mu\nu\rho\sigma} D^\mu D^\nu D^\rho D^\sigma f''\!\left(\frac{k^2}{\Lambda^2}\right)$$

$$\int \frac{d^4k}{\Lambda^4} f''\left(\frac{k^2}{\Lambda^2}\right) \overset{q=\frac{k^2}{\Lambda^2}}{=} \int_0^\infty q\,dq\, f''(q) \int d\Omega_4$$

$$= \pi^2 \left[q f'(q)\Big|_0^\infty - \int_0^\infty dq f'(q) \right]$$

$$= \pi^2 \left(-f(q)\Big|_0^\infty \right)$$

$$= \pi^2$$

$\Lambda \to \infty$

$$\mathrm{Tr}\,\theta\gamma_5 = \lim_{\Lambda\to\infty} \int d^4x \frac{d^4k}{(2\pi)^4} \mathrm{tr}\,\theta\gamma_5 f\left(\frac{-(iD-k)^2}{\Lambda^2}\right)$$

$$\sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{-iD}{\Lambda}\right)^{2N} f^{(N)}\left(\frac{k^2}{\Lambda^2}\right)$$

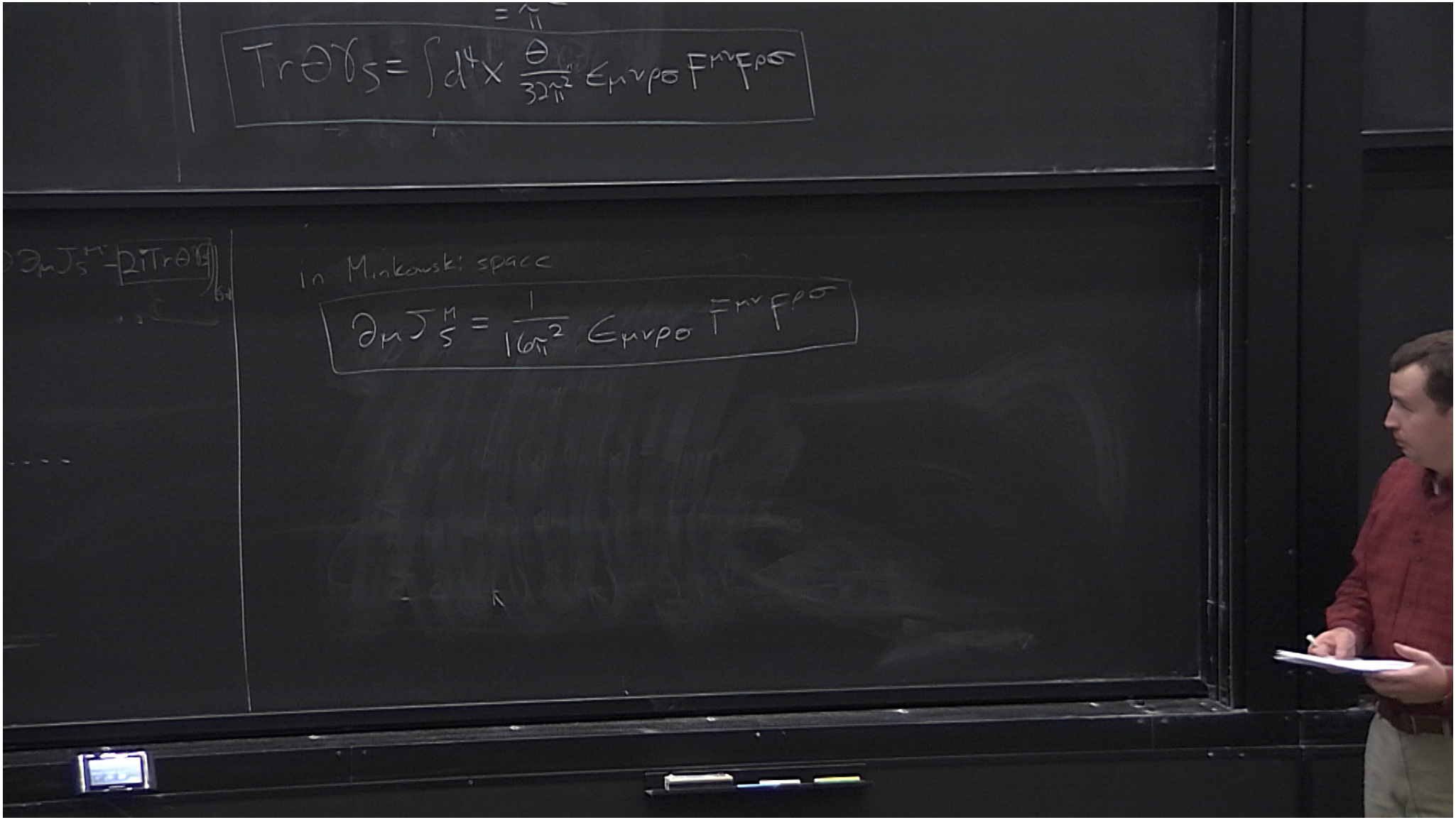
$\mathrm{tr}\,\gamma_5 = \gamma_5\gamma_\mu\gamma_\nu = 0$ $\quad N=0$ + $N=1$ vanish

$N=2$ term $\gg$ $N=3$ terms as $\Lambda\to\infty$

$\mathrm{tr}\,\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4\varepsilon_{\mu\nu\rho\sigma}$ $\quad$ now in Euclidean

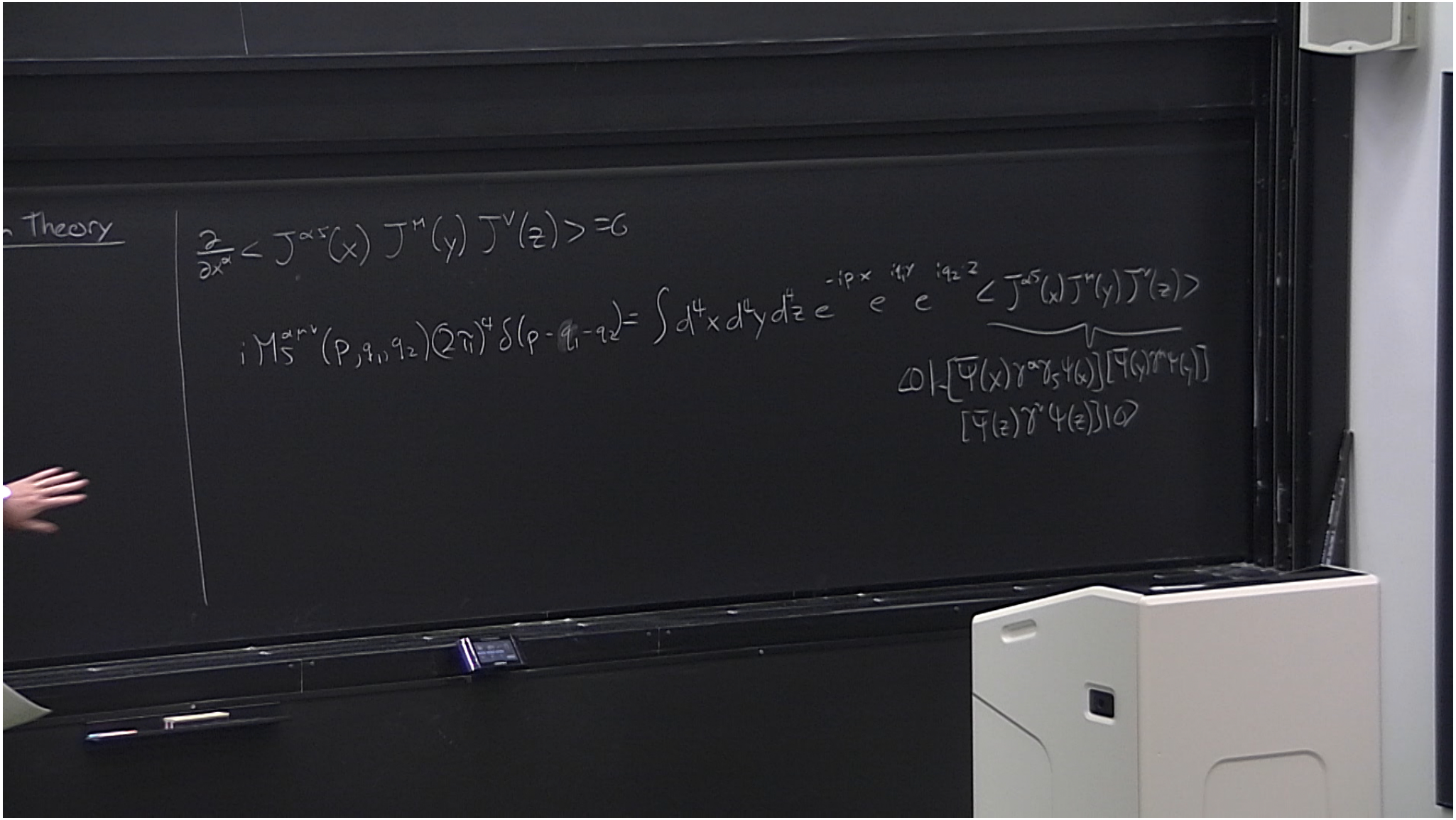
$$\mathrm{Tr}\,\theta\gamma_5 = \lim_{\Lambda\to\infty} \frac{1}{32\pi^4} \int d^4x \frac{d^4k}{\Lambda^4} 4\varepsilon_{\mu\nu\rho\sigma} D^\mu D^\nu D^\rho D^\sigma f''\left(\frac{k^2}{\Lambda^2}\right)$$

$\varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$

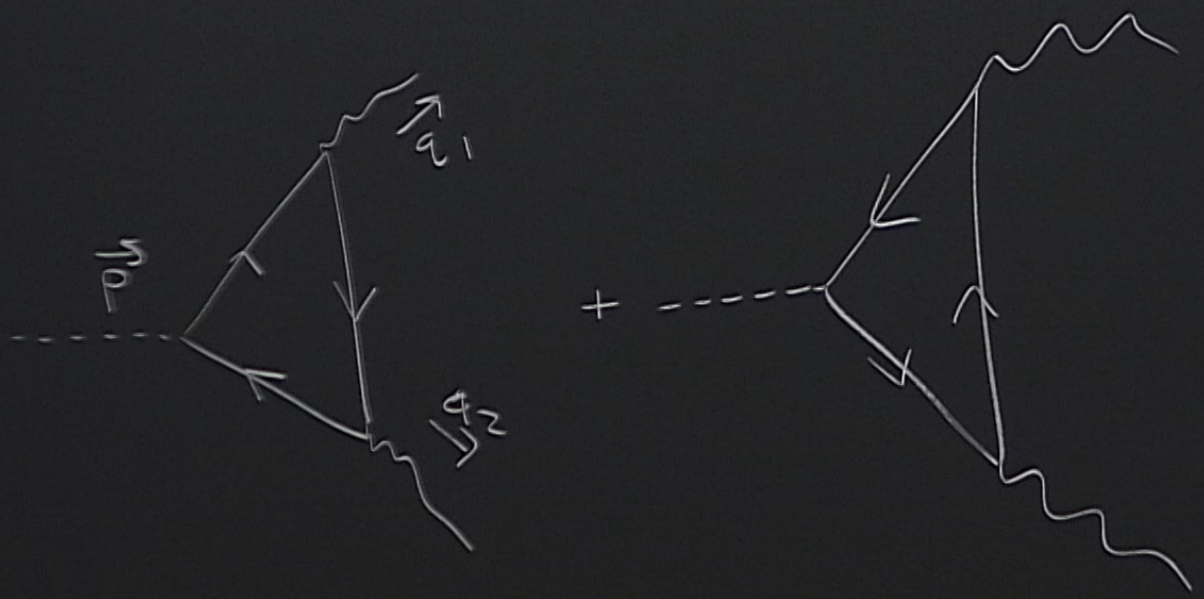

$\mathrm{Tr}\,\theta\gamma_5 = \int d^4x \, \frac{\theta}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$
in Minkowski space
$\partial_\mu J^\mu_5 = \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$

Chiral Anomaly from Perturbation Theory




Theory
$\frac{\partial}{\partial x^\alpha} \langle J^{\alpha 5}(x) J^\mu(y) J^\nu(z) \rangle = 0$
$i M_5^{\alpha\mu\nu}(p, q_1, q_2)(2\pi)^4 \delta(p - q_1 - q_2) = \int d^4x\, d^4y\, d^4z\, e^{-ipx} e^{iq_1 y} e^{iq_2 z} \langle J^{\alpha 5}(x) J^\mu(y) J^\nu(z) \rangle$
$\langle 0 | \{ [\bar{\psi}(x) \gamma^\alpha \gamma_5 \psi(x)] [\bar{\psi}(y) \gamma^\mu \psi(y)] [\bar{\psi}(z) \gamma^\nu \psi(z)] \} | 0 \rangle$

Chiral Anomaly from Perturbation Theory



$\frac{\partial}{\partial x^a} < J$

$i M_5^a$