

Title: The SYK model and matter without quasiparticle

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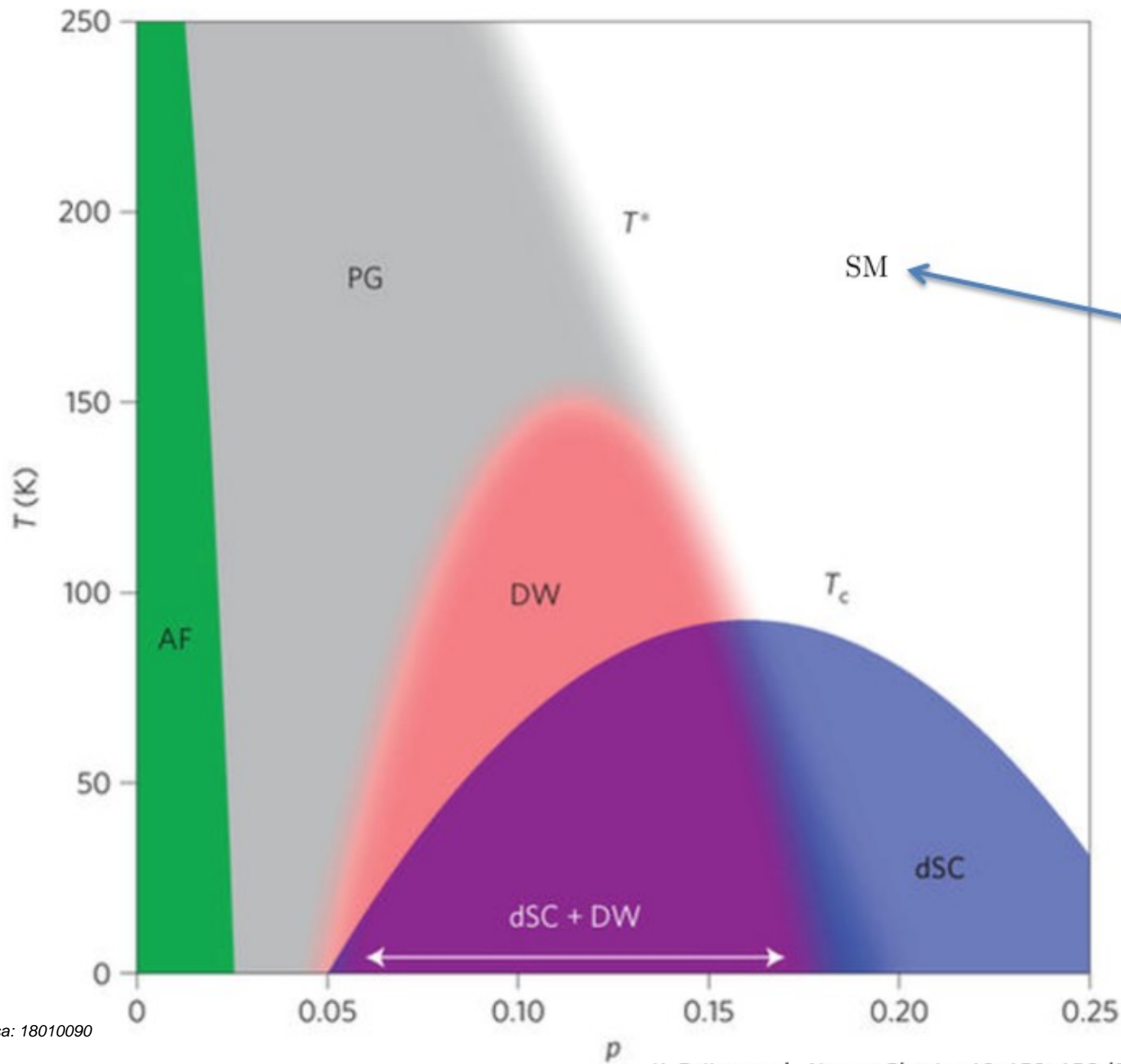
Abstract: <p>Many strongly-correlated systems like high T_c cuprates and heavy fermions have interesting features going beyond quasi-particle description. While Sachdev-Ye-Kitaev(SYK) models are exactly solvable models that can provide a platform to study these physics. In this talk, I will discuss interesting features about the SYK models, including extensive zero temperature entropy and maximally chaos. I will also show some generalization of the SYK models and discuss physical insights from them.</p>

The SYK models and matter without quasiparticle

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01/23/2018 PI



Strange Metal:
A metallic phase
which does not have
a quasiparticle
description

Strange Metal

- Linear T resistivity
- Linear H magnetoresistance
- $T \log(T)$ heat capacity
- Instability towards other order

Strange Metal

- **Linear T resistivity**

“A strongly correlated metal built from SYK models” X-Y Song et al 2017

- **Linear H magnetoresistance**

“Magnetotransport in a model of a disordered strange metal” A. A. Patel et al 2017

- **$T \log(T)$ heat capacity**

“Translationally invariant non-Fermi liquid metals with critical Fermi-surfaces: Solvable models” D. Chowdhury et al 2018

- **Instability towards other order**

“Instability of the non-fermi liquid state of the sachdev-Ye-Kitaev model” Z. Bi et al 2017

Motivation

- The SYK models are solvable non Fermi liquid models which provide a platform to study the strange metal physics
- NCFT1/NAdS2 holography, maximal chaos, relation to random matrix ensemble and black hole physics

My adventures with SYK

- Numerical study with exact diagonalization
Spectrum, thermal entropy, entanglement entropy
W. Fu, and S. Sachdev, PRB 94, 035135 (2016)
- Supersymmetric SYK models
Different origins of zero temperature entropy in N=1 and N=2 theories
W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, PRD 95, 026009(2017)
- Lattice construction of complex SYK model
Zero temperature entropy with different charge density, effective action, charge/energy diffusion
R. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen and S. Sachdev, PRB 95, 155131(2017)
- SYK boson/fermion model
Tunable Lyapunov exponent
W. Fu, C-M. Jian, Z Bi and C. Xu, in progress
- Quantum transitions of fractionalized SYK model

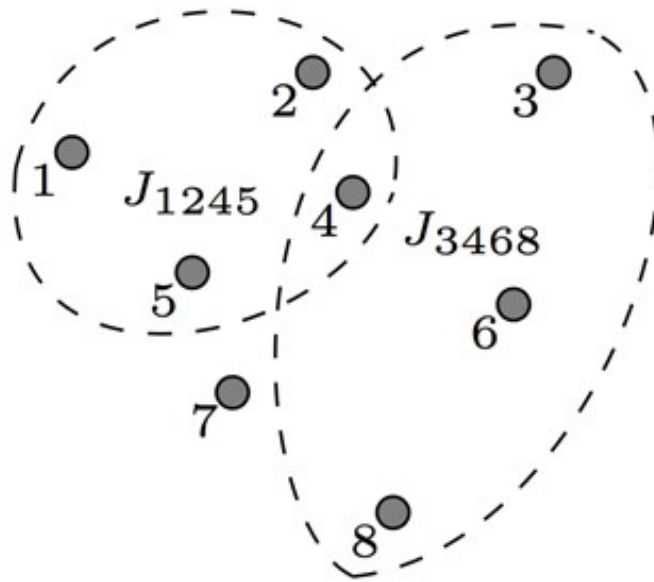
The SYK models

Complex fermion version

$$H = \frac{1}{(4/2)!} \sum_{j,k;l,m} J_{jk;lm} c_j^\dagger c_k^\dagger c_l c_m - \sum_i \mu c_i^\dagger c_i \quad \overline{J_{jk;lm}} = 0 \quad \overline{|J_{jk;lm}|^2} = \frac{2\mathcal{J}^2}{4N^3}$$

Majorana fermion version

$$H = \frac{1}{4!} \sum_{jklm} J_{jklm} \chi_j \chi_k \chi_l \chi_m \quad \overline{J_{jklm}} = 0 \quad \overline{J_{jklm}^2} = \frac{3!}{N^3} \mathcal{J}^2 \quad N: \text{site number}$$



$$N = 8$$

General q-body interaction

Complex fermion version

$$H = \frac{1}{(q/2)!} \sum_{j_1, \dots, j_{q/2}; k_1, \dots, k_{q/2}} J_{\star} c_{j_1}^{\dagger} \cdots c_{k_{q/2}}^{\dagger} c_{k_1} \cdots c_{j_{q/2}} - \sum_i^N \mu c_i^{\dagger} c_i \quad \overline{J_{\star}} = 0 \quad \overline{|J_{\star}|^2} = \frac{2\mathcal{J}^2}{qN^{q-1}}$$

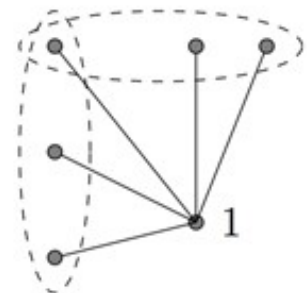
Majorana fermion version

$$H = \frac{1}{q!} \sum_{j_1, \dots, j_q} J_{\star} \chi_{j_1} \cdots \chi_{j_q} \quad \overline{J_{\star}} = 0 \quad \overline{J_{\star}^2} = \frac{q!}{N^{q-1}} \mathcal{J}^2$$

The model as fermion coupled to heat bath field:

$$c_1^{\dagger} \mathcal{B}_1 + h.c. \quad , \quad \mathcal{B}_1 = \frac{1}{(q/2 - 1)!} \sum J_{\star} c_{j_1}^{\dagger} \cdots c_{j_{q/2-1}}^{\dagger} c_{k_1} \cdots c_{k_{q/2}}$$

$$\langle \mathcal{B}_1 \mathcal{B}_1^{\dagger} \rangle \sim \mathcal{J}^2$$

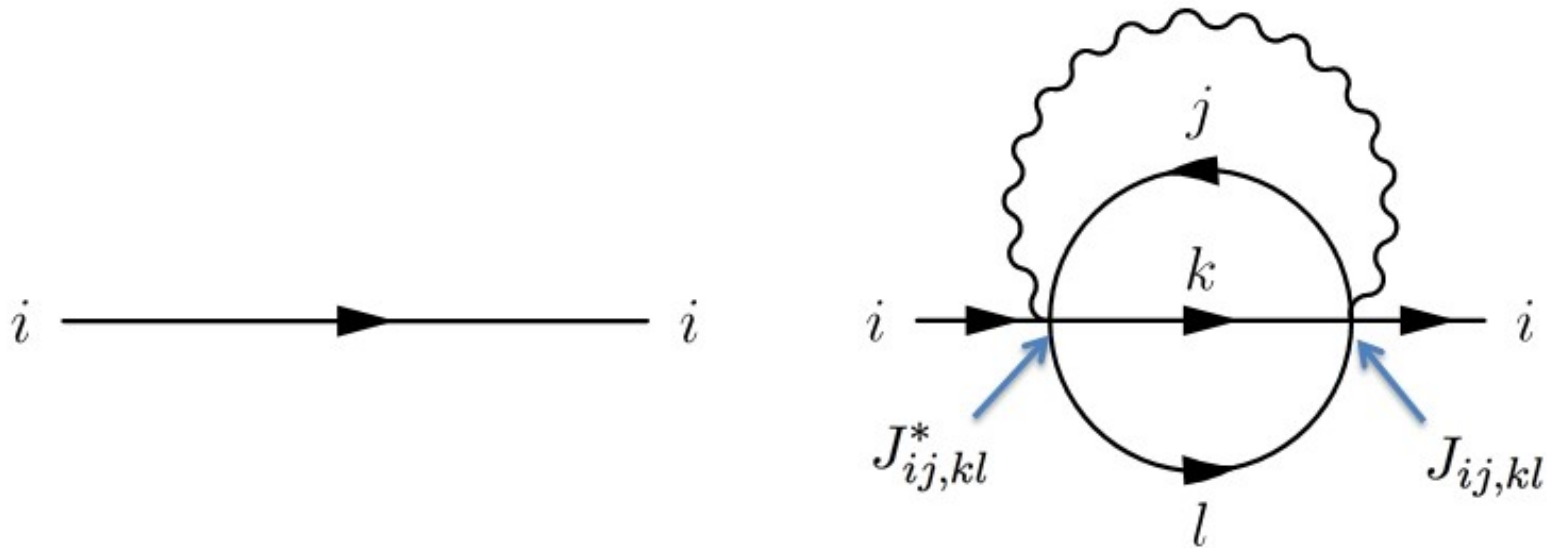


$$\mathcal{B}_1 \sim \sum_{k,l,m} J_{1k;lm} c_k^{\dagger} c_l c_m$$

Solve the model

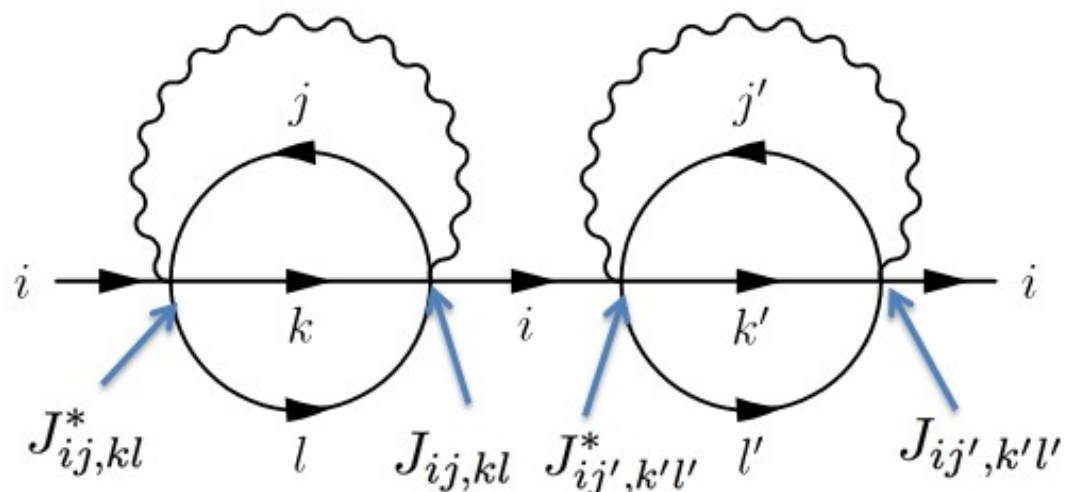
- Diagrammatics:

Two point function: $G(\tau_1, \tau_2) = -\langle T_\tau c(\tau_1) c^\dagger(\tau_2) \rangle$

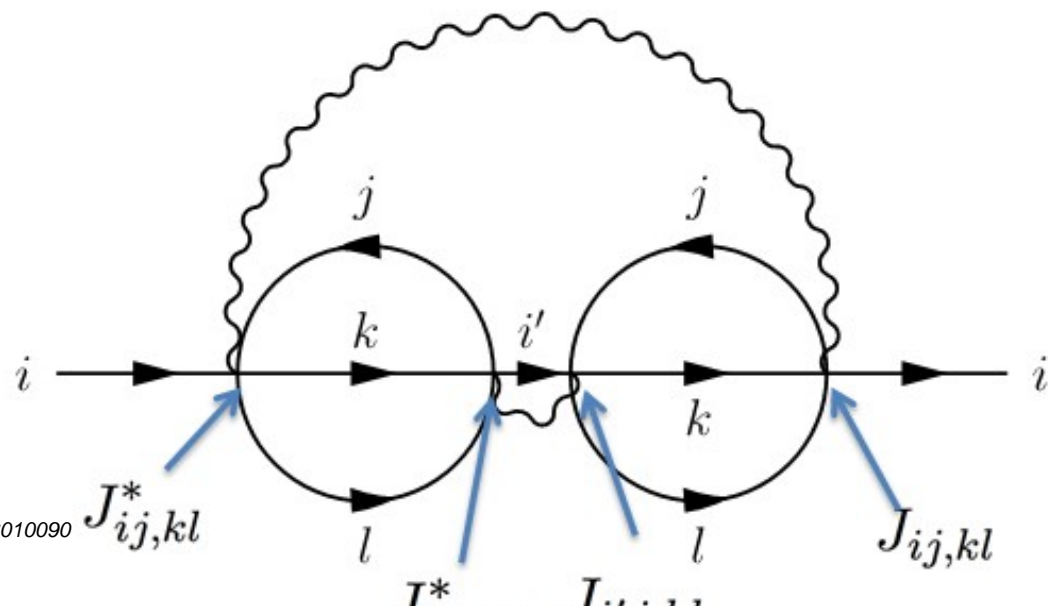


$$|J_{ij,kl}|^2 N^3 \sim \frac{\mathcal{J}^2}{N^3} N^3 \sim \mathcal{J}^2$$

Two point function: $G(\tau_1, \tau_2) = -\langle T_\tau c(\tau_1) c^\dagger(\tau_2) \rangle$

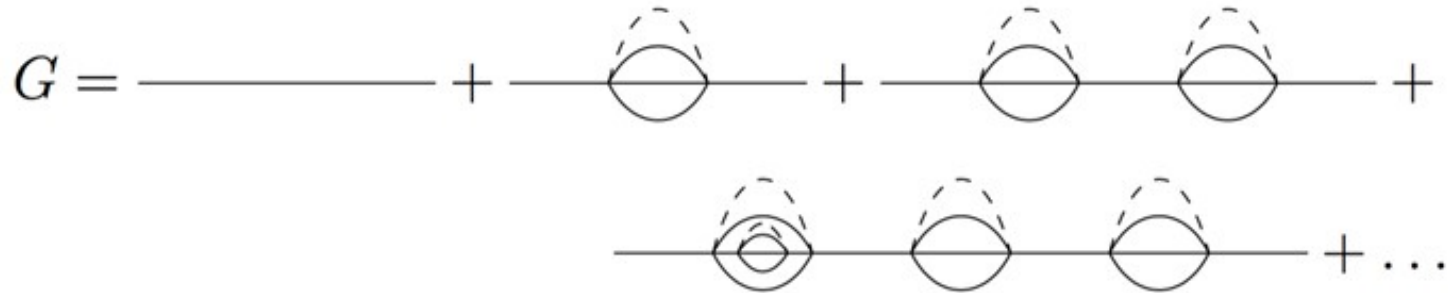


$$\left(\frac{\mathcal{J}^2}{N^3}\right)^2 N^6 \sim \mathcal{J}^4$$



$$\left(\frac{\mathcal{J}^2}{N^3}\right)^2 N^4 \sim \frac{\mathcal{J}^4}{N^2}$$

Large N limit:



Schwinger-Dyson equation:

$$G_0 = \text{---}$$

$$G = \text{---} = \text{---} + \text{---} \bigcirc \text{---}$$

$$G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)}$$

$$\Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$

$$\Sigma = \bigcirc$$

Solved numerically for any coupling;

analytically in the strong coupling limit $\beta J \gg 1$

Solve the model

- Effective action:

$$H = \frac{1}{(q/2)!} \sum_{j_1, \dots, j_{q/2}; k_1, \dots, k_{q/2}}^N J_{\star} c_{j_1}^{\dagger} \cdots c_{k_{q/2}}^{\dagger} c_{k_1} \cdots c_{j_{q/2}} - \sum_i^N \mu c_i^{\dagger} c_i \quad \overline{J_{\star}} = 0 \quad \overline{|J_{\star}|^2} = \frac{2\mathcal{J}^2}{qN^{q-1}}$$

Introduce replicas: $\log Z = \lim_{M \rightarrow 0} \frac{Z^M}{M}$

Integrate out disorder J field:

$$S_{eff} [c, c^{\dagger}] = \int d\tau \sum_{\alpha} c_{i\alpha}^{\dagger}(\tau) (\partial_{\tau} - \mu) c_{i\alpha}(\tau) - (-)^{q/2} \frac{\mathcal{J}^2}{qN^{q-1}} \int d\tau_1 d\tau_2 \sum_{\alpha, \beta} |c_{i\alpha}^{\dagger}(\tau_1) c_{i\beta}(\tau_2)|^q$$

Introduce Bi-local field G and Σ : Σ is the Lagrangian multiplier that enforces

$$G_{\alpha\beta}(\tau_1, \tau_2) = -\frac{1}{N} \sum_i c_{i\alpha}(\tau_1) c_{i\beta}^{\dagger}(\tau_2)$$

Then the effective action: $S_{eff}(c, G, \Sigma) \sim c^{\dagger}(\partial_{\tau} - \mu)c + \Sigma(G - c^{\dagger}c) + |G|^q$

Integrate out fermion field:

$$S_{eff}(G, \Sigma) = N \left\{ -\sum_n \log[-i\omega_n - \mu + \Sigma(i\omega_n)] - \int d\tau_1 d\tau_2 \left[\Sigma(\tau_1, \tau_2) G(\tau_2, \tau_1) + (-)^{q/2} \frac{N\mathcal{J}^2}{q} |G(\tau_2, \tau_1)|^q \right] \right\}$$

Solve the model

Saddle point approximation:

$$G(i\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(i\omega_n)}$$

$$\Sigma(\tau) = -(-)^{q/2} \mathcal{J}^2 G^{q/2}(\tau) G^{q/2-1}(-\tau)$$

In the low energy scaling limit: $\omega, T \ll J$ $i\omega_n + \mu - \Sigma(i\omega_n = 0)$ irrelevant

$$\int d\tau_2 G(\tau_1, \tau_2) \Sigma(\tau_2, \tau_3) = -\delta(\tau_1 - \tau_3)$$

$$\Sigma(\tau_1, \tau_2) = -(-)^{q/2} \mathcal{J}^2 G^{q/2}(\tau_1, \tau_2) G^{q/2-1}(\tau_2, \tau_1)$$

Reparameterization symmetry and U(1) symmetry:

$$\tau = f(\sigma)$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/q} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{G}(\sigma_1, \sigma_2)$$

$$\Sigma(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{1-1/q} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{\Sigma}(\sigma_1, \sigma_2)$$

If $G(\tau_1, \tau_2), \Sigma(\tau_1, \tau_2)$ is the solution, $\tilde{G}(\sigma_1, \sigma_2), \tilde{\Sigma}(\sigma_1, \sigma_2)$ is also the solution.

Solve the model

- Analytical solution:

In the low energy scaling limit: $\omega, T \ll J$ $i\omega_n + \mu - \Sigma(i\omega_n = 0)$ irrelevant

$$G(i\omega_n) = -\Sigma(i\omega_n)^{-1} \quad \longrightarrow \quad G(\tau) \sim \begin{cases} -|\tau|^{-2\Delta}, & \tau > 0, \\ e^{-2\pi\epsilon|\tau|^{-2\Delta}}, & \tau < 0, \end{cases} \quad \Delta = \frac{1}{q}$$

$$\Sigma(\tau) = -(-)^{q/2} \mathcal{J}^2 G^{q/2}(\tau) G^{q/2-1}(-\tau)$$

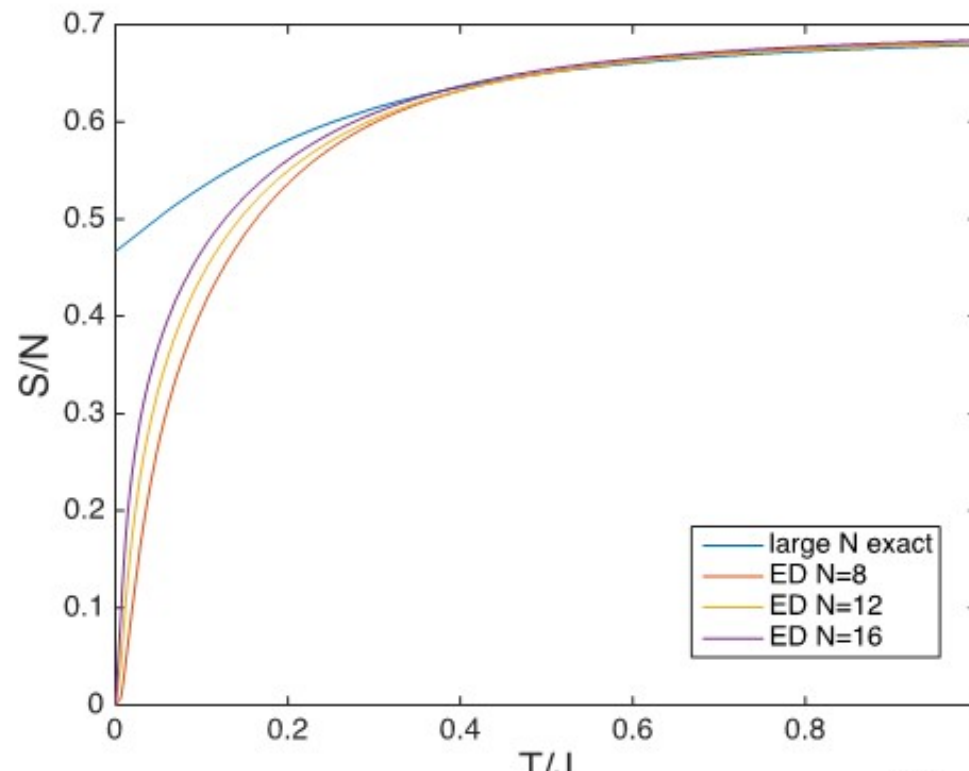
Finite temperature $\xrightarrow{\tau = \frac{1}{\pi T} \tan(\pi T \sigma)}$ $G(\sigma) \sim e^{-2\pi\epsilon\sigma/\beta} \left[\frac{\pi}{\beta \sin \frac{\pi\sigma}{\beta}} \right]^{2\Delta}$

0T entropy

- Numerical solution:

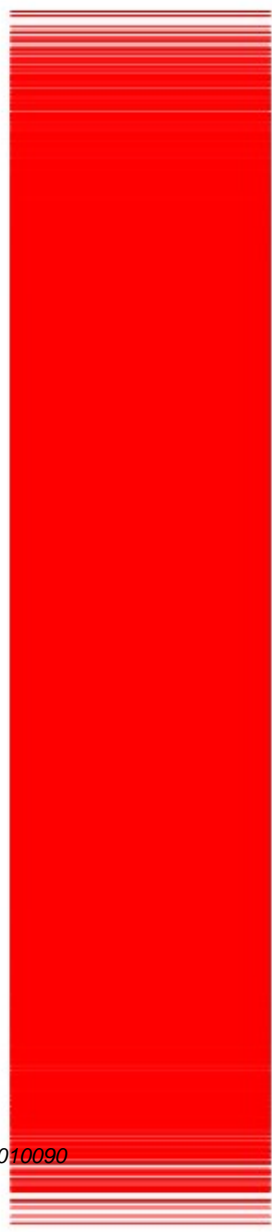
$$\frac{\mathcal{F}}{N} = T \sum_n \ln(-\beta G(i\omega_n)) - \int_0^\beta d\tau \frac{3}{4} \Sigma(\tau) G(-\tau)$$

$$\frac{S}{N} = -\frac{1}{N} \frac{\partial \mathcal{F}}{\partial T}$$



0T entropy

SYK:



Random hopping:



$\Delta E \sim e^{-N \ln 2}$

Two blue arrows originate from the equation $\Delta E \sim e^{-N \ln 2}$. One arrow points left towards the SYK energy spectrum, and the other points right towards the Random hopping energy spectrum, indicating that this equation describes the energy gap between the two models.

$\Delta E \sim e^{-S_0}$

A blue arrow originates from the equation $\Delta E \sim e^{-S_0}$ and points left towards the SYK energy spectrum, indicating that this equation describes the energy gap for the SYK model.

$\Delta E \sim \frac{1}{N}$

A blue arrow originates from the equation $\Delta E \sim \frac{1}{N}$ and points right towards the Random hopping energy spectrum, indicating that this equation describes the energy gap for the Random hopping model.

0T entropy

There is a finite zero temperature entropy maximally at half-filling:

$$\frac{S_0}{N} = \frac{\text{Catalan}}{\pi} + \frac{\log 2}{4} \approx 0.4648$$

A. Georges, O. Parcollet, and S. Sachdev Phys. Rev. B 63, 134406 (2001)

The low energy sector cannot be described by polynomial in N number of parameters:

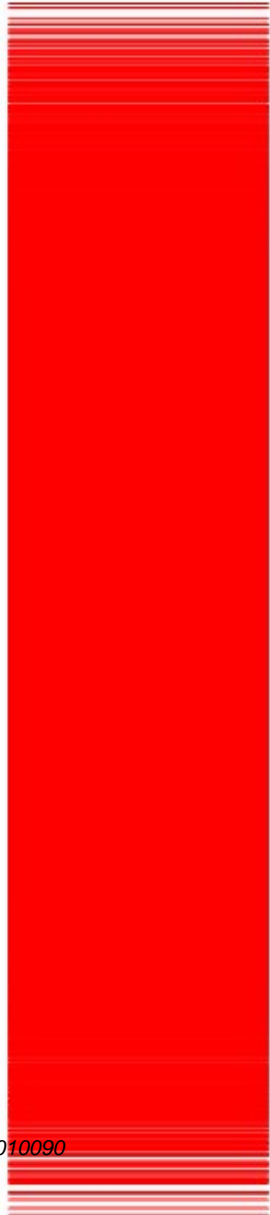
~~$$E[\{n_\alpha\}] = \sum n_\alpha \epsilon_\alpha + \sum F_{\alpha\beta} n_\alpha n_\beta + \dots$$~~

No quasiparticle description

$$\Delta E \sim e^{-N \ln 2}$$



$$\Delta E \sim e^{-S_0}$$

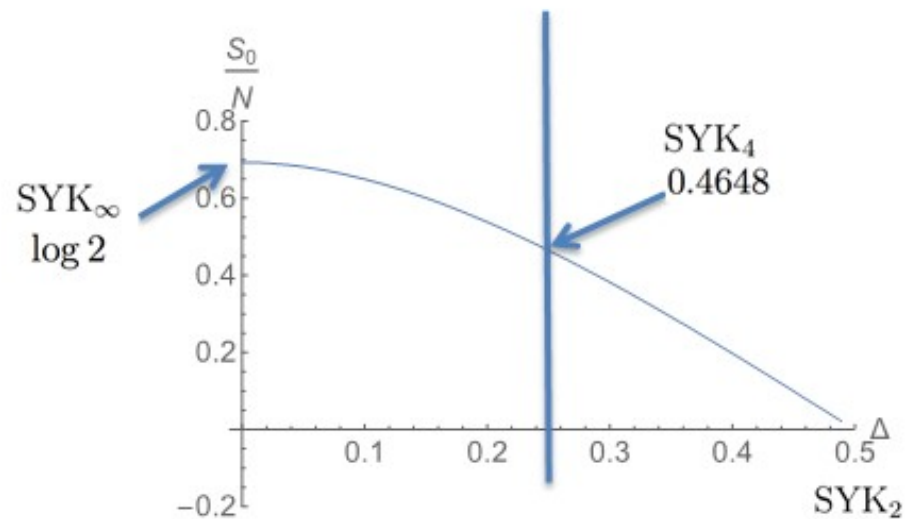


0T entropy

Plug in the conformal solution into free energy: $G(\sigma) \sim e^{-2\pi\epsilon\sigma/\beta} \left[\frac{\pi}{\beta \sin(\frac{\pi\sigma}{\beta})} \right]^{2\Delta}$

$$\partial_q \frac{\log Z}{N} = \beta \int d\sigma \mathcal{J}^2 G(\sigma)^{q-1} \log G(\sigma) = \beta(\text{constant}) + q\pi(q-2) \tan\left(\frac{\pi}{q}\right)$$

$$\frac{1}{N} \frac{\partial S_0}{\partial \Delta} = -\pi(1-2\Delta) \tan(\Delta\pi)$$



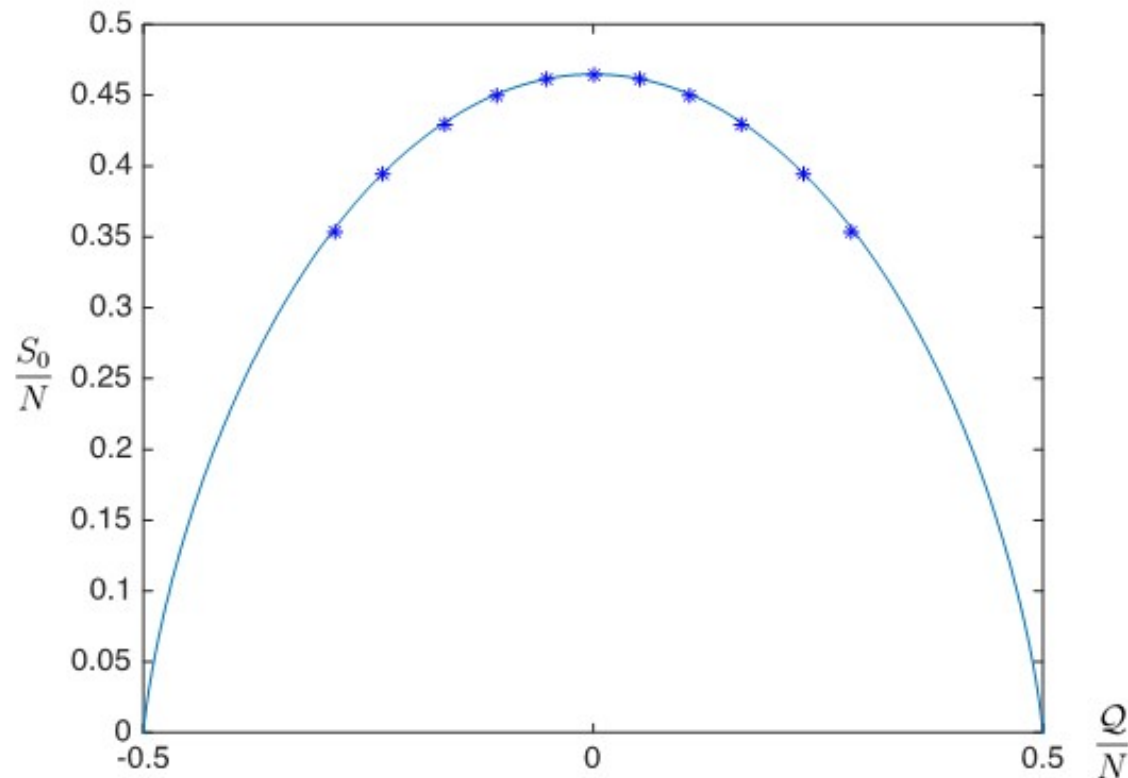
0T entropy

In general, particle-hole asymmetric case:

$$\left(\frac{S_0(\mathcal{E})}{N}, \frac{Q(\mathcal{E})}{N} \right)$$

A. Georges, O. Parcollet, and S. Sachdev PRB 63, 134406 (2001)

S. Sachdev, PRX 5, 041025 (2015)



Another aspect of the SYK models

Another aspect of the SYK models

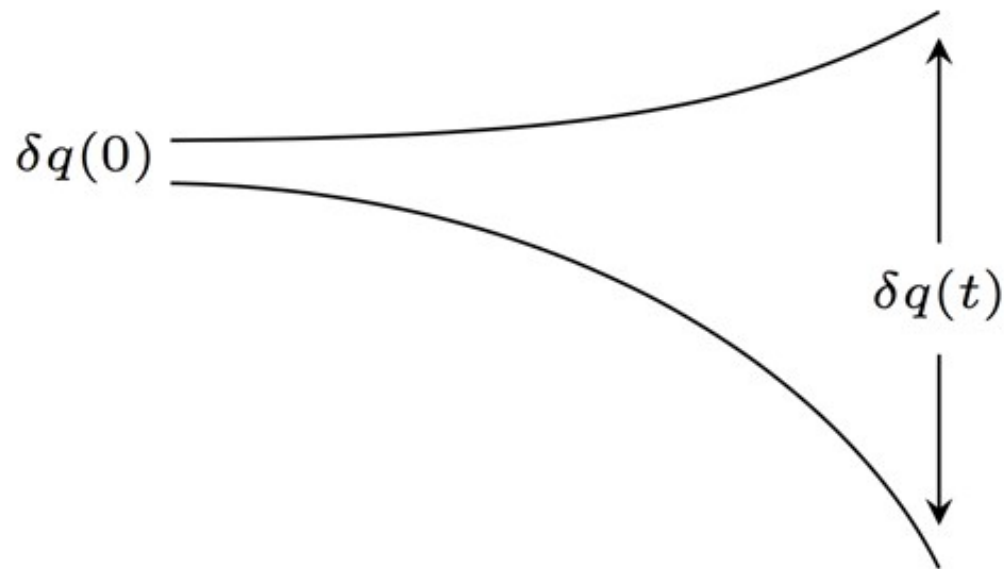
Chaos!

fast information scrambling

How to characterize chaos

Classical chaos $\{q(t), p(0)\} = \frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t}$

Lyapunov exponent: λ_L



Quantum analogy $\{q(t), p(0)\} \rightarrow \frac{i}{\hbar} [\hat{q}(t), \hat{p}(0)]$

Quantum chaos: growth of $C_{AB}(t) = \langle | [\hat{A}(t), \hat{B}(0)]|^2 \rangle_\beta$

Out of time ordered correlator

$$\begin{aligned}
 C_{AB}(t) &= \langle |[A(t), B(0)]|^2 \rangle_\beta \quad \begin{array}{c} \text{Normal order terms} \\ \downarrow \quad \downarrow \end{array} \\
 &= \langle A(t)B(0)B^\dagger(0)A^\dagger(t) \rangle_\beta + \langle B(0)A(t)A^\dagger(t)B^\dagger(0) \rangle_\beta \\
 &\quad - \langle A(t)B(0)A^\dagger(t)B^\dagger(0) \rangle_\beta - \langle B(0)A(t)B^\dagger(0)A^\dagger(t) \rangle_\beta \\
 &\quad \begin{array}{c} \uparrow \quad \uparrow \\ \text{Out of time ordered correlators(OTOC)} \end{array}
 \end{aligned}$$

Early time exponential deviation in OTOC causes exponential growth in C(t)

Normalized OTOC $\text{OTOC} \approx 1 - \#e^{\lambda_L t}$

A bound on chaos: $\lambda_L \leq \frac{2\pi}{\beta}$ J. Maldacena, S. H. Shenker and D. Stanford JHEP 08, 106 (2016)

SYK (all to all and q-local): C grows exponentially:

And saturates the bound at low temperature!

A quick recap

Symmetry of the action: Reparameterization and U(1)

$$\begin{aligned}\tau &= f(\sigma) \\ G(\tau_1, \tau_2) &= [f'(\sigma_1)f'(\sigma_2)]^{-1/q} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{G}(\sigma_1, \sigma_2) \\ \Sigma(\tau_1, \tau_2) &= [f'(\sigma_1)f'(\sigma_2)]^{1-1/q} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{\Sigma}(\sigma_1, \sigma_2)\end{aligned}$$

Conformal Green's function:

$$G(\sigma) \sim e^{-2\pi\mathcal{E}\sigma/\beta} \left[\frac{\pi}{\beta \sin(\frac{\pi\sigma}{\beta})} \right]^{2\Delta}$$

Residue symmetry of conformal solutions PSL(2,R) semiproduce U(1):

$$\begin{aligned}G(\tau_1, \tau_2) &= [f'(\tau_1)f'(\tau_2)]^\Delta G_c(f(\tau_1), f(\tau_2)) e^{i\phi(\tau_1) - i\phi(\tau_2)} \\ \frac{1}{\pi T} \tan(\pi T f(\tau)) &= \frac{a \tan(\pi T \tau) + b\pi T}{c \tan(\pi T \tau) + d\pi T} \quad -i\phi(\tau) = 2\pi\mathcal{E}T(\tau - f(\tau)) = -2\pi\mathcal{E}T\epsilon(\tau)\end{aligned}$$

Infinite many goldstone mode: including UV iw term, become soft mode

Analogy with ferromagnetic chain:



Out of time ordered correlator

Treat OTOC as two point function of bilocal fields:

$$F \sim \int DG D\Sigma G(t_1, t_2) G(t_3, t_4) e^{-S_{eff}}$$

After including UV terms, the Goldstone modes become pseudo Goldstone modes and produces Schwarzian action that gives maximal chaos:

$$F_{\text{big}}(t) \sim -e^{\frac{2\pi t}{\beta}}$$

Other sub-leading modes make much smaller contribution effectively reduce the Lyapunov exponent for a bit:

$$F(t) \sim -e^{\frac{2\pi t}{\beta}(1-\delta\lambda_L)} \quad , \quad \delta\lambda_L \sim \frac{1}{\beta\mathcal{J}}$$

A generalization: SUSY SYK model

$$Q = i \sum_{1 \leq i < j < k \leq N} C_{ijk} \psi^i \psi^j \psi^k \quad \overline{C_{ijk}} = 0 \quad , \quad \overline{C_{ijk}^2} = \frac{4J}{N^2}$$

$$\mathcal{H} = Q^2 = E_0 + \sum_{1 \leq i < j < k < l \leq N} J_{ijkl} \psi^i \psi^j \psi^k \psi^l \quad J_{ijkl} = -\frac{1}{8} \sum_a C_{a[ij} C_{kl]a}$$

Interaction linearized by introducing N bosons:

$$\mathcal{L} = \sum_i \left[\frac{1}{2} \psi^i \partial_\tau \psi^i - \frac{1}{2} b^i b^i + i \sum_{1 \leq j < k \leq N} C_{ijk} b^i \psi^j \psi^k \right]$$

$$G_f = \text{---} + \text{---} \text{---} \text{---}$$

$$\Delta_f = \frac{1}{6} \quad , \quad \Delta_b = \frac{2}{3}$$

$$G_b = \text{---} + \frac{1}{2} \text{---} \text{---} \text{---}$$

A further generalization:

$$H = (i)^2 \sum_{a=1}^M \sum_{i,j,k,l}^N C_{ij}^a C_{kl}^a \psi^i \psi^j \psi^k \psi^l$$

Interaction linearized by introducing M bosons:

$$\mathcal{L} = \sum_i^N \frac{1}{2} \psi^i \partial_\tau \psi^i - \sum_a^M \frac{1}{2} b^a b^a + i \sum C_{jk}^a b^a \psi^j \psi^k$$

$$G_f = \text{---} + \sqrt{\alpha} \text{---} \text{---} \text{---}$$

$$\alpha = \frac{M}{N}$$

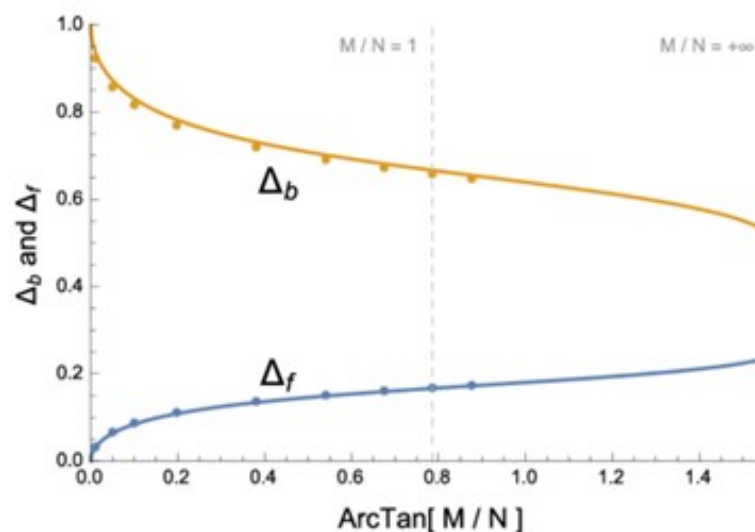
$$G_b = \text{---} + \frac{1}{2\sqrt{\alpha}} \text{---} \text{---} \text{---}$$

A further generalization:

$$H = (i)^2 \sum_{a=1}^M \sum_{i,j,k,l}^N C_{ij}^a C_{kl}^a \psi^i \psi^j \psi^k \psi^l$$

Interaction linearized by introducing bosons:

$$\mathcal{L} = \sum_i^N \frac{1}{2} \psi^i \partial_\tau \psi^i - \sum_a^M \frac{1}{2} b^a b^a + i \sum C_{jk}^a b^a \psi^j \psi^k$$



$$\alpha = \frac{M}{N}$$

At small α : $\Delta_f = \frac{\sqrt{\alpha}}{\pi}$

A further generalization:

$$H = (i)^2 \sum_{a=1}^M \sum_{i,j,k,l}^N C_{ij}^a C_{kl}^a \psi^i \psi^j \psi^k \psi^l$$

At small α , comparing with the free case, the anomalous dimension:

$$\Delta_f = \frac{\sqrt{\alpha}}{\pi} \rightarrow 0$$

What about the chaos behavior?

λ_L becomes smaller?

A further generalization:

Define OTOC:

$$\langle\langle\psi^i(t)\psi^j(0)\psi^i(t)\psi^j(0)\rangle\rangle_\beta = F(t)$$

Contribution from pseudo-Goldstone mode:

$$F_{\text{big}}(t) \sim -\frac{\Delta_f^2 \beta J}{4\pi(\alpha_f^S + \alpha_b^S)} e^{\frac{2\pi t}{\beta}}$$

Contribution from sub-leading modes

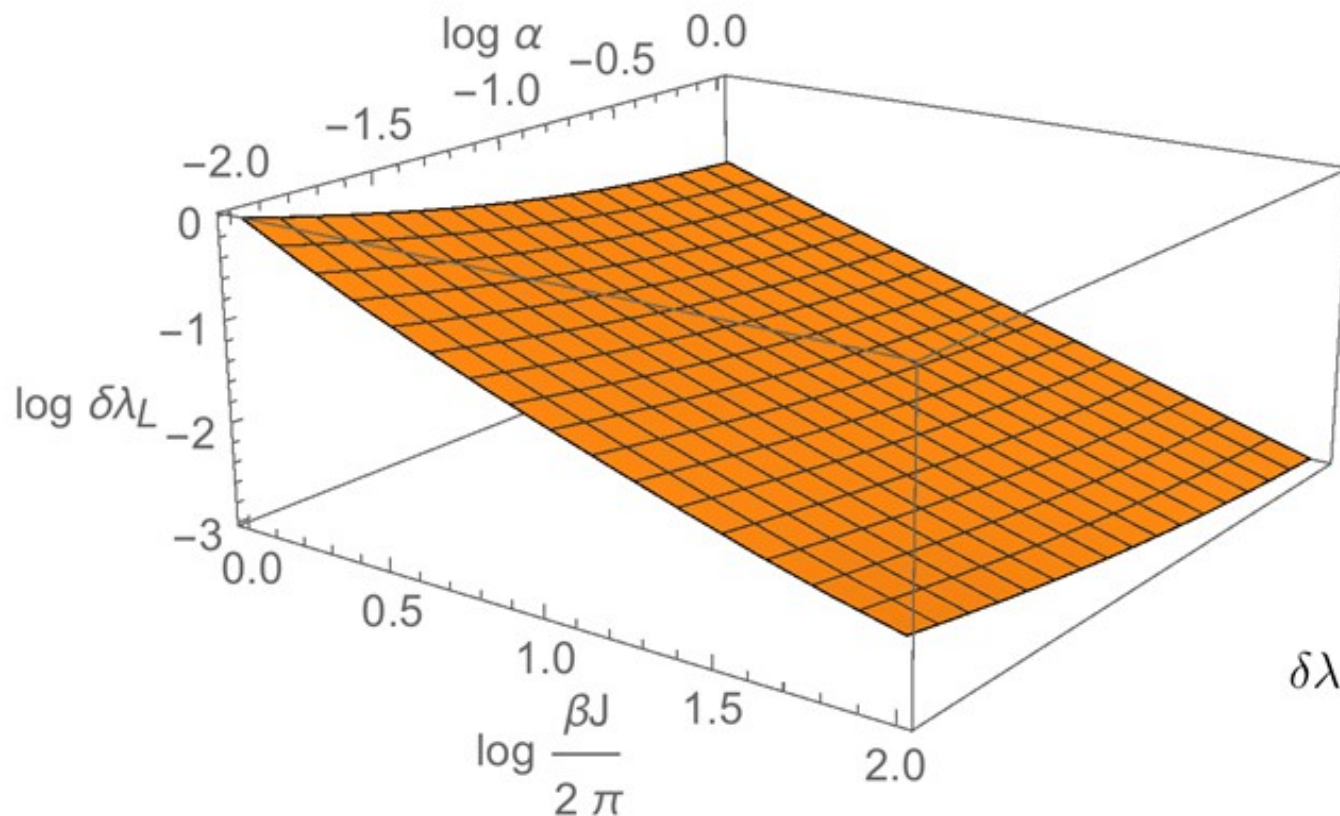
$$F_{\text{sub}}(t) \sim \frac{3}{2\pi k'_R(-1)} \frac{2\pi t}{\beta} e^{\frac{2\pi t}{\beta}}$$

Assume the full expression $F(t) \sim -\frac{\Delta_f^2 \beta J}{4\pi(\alpha_f^S + \alpha_b^S)} e^{\frac{2\pi t}{\beta}(1-\delta\lambda_L)}$

A further generalization:

Define OTOC:

$$\langle\langle\psi^i(t)\psi^j(0)\psi^i(t)\psi^j(0)\rangle\rangle_\beta = F(t)$$



Discussion

- Less chaotic towards free fermion limit
- How Lyapunov time is related to decoherence time?

$\delta\lambda_L$ \longleftrightarrow Decoherence factor \longleftrightarrow Branching time

A. Kitaev talk at IAS (2017)

- Does this have a thermal dynamical interpretation?

Quantum transition of SYK models

Fractionalize electrons:

$$c_{i\alpha} = f_{i\alpha} \sigma_i^z \quad \longrightarrow \quad \begin{aligned} c_{i\alpha p} &= f_{i\alpha} \phi_{ip} \\ \sum_{p=1}^M \phi_{ip}^2 &= M. \end{aligned}$$

Fractionalized particles carry Z2 charge

$$\begin{aligned} \mathcal{L} = & \frac{1}{2g} \sum_{i,p} (\partial_\tau \phi_{ip})^2 + \sum_{i,\alpha} f_{i\alpha}^\dagger \left(\frac{\partial}{\partial \tau} - \mu \right) f_{i\alpha} \\ & + \frac{1}{\sqrt{NM}} \sum_{i,j,p,\alpha} t_{ij} \phi_{ip} \phi_{jp} f_{i\alpha}^\dagger f_{j\alpha} + \frac{1}{\sqrt{NM}} \sum_{i>j,\alpha\beta} J_{ij} f_{i\alpha}^\dagger f_{i\beta} f_{j\beta}^\dagger f_{j\alpha}. \end{aligned}$$

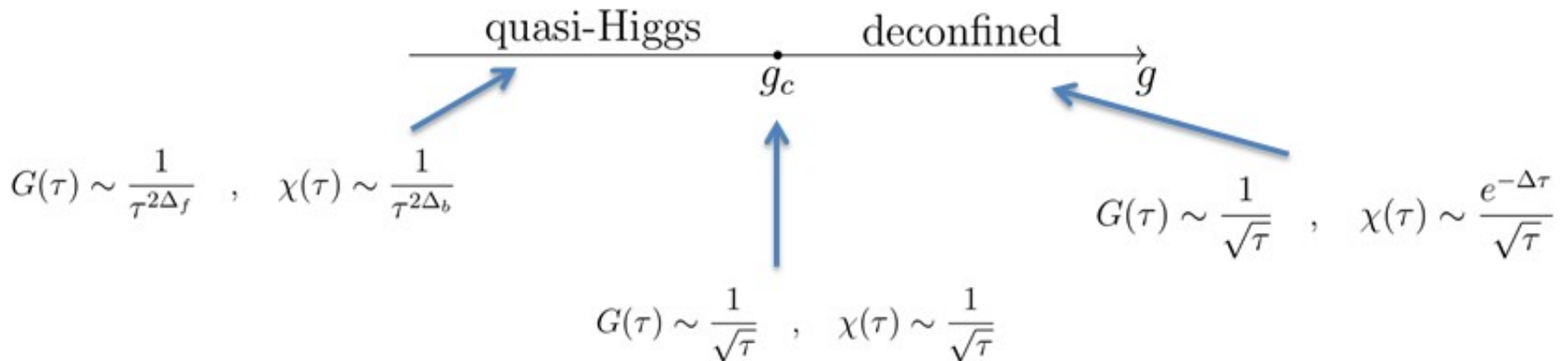
Quantum transition of SYK models

SYK type Schwinger-Dyson equations:

$$G(\omega_n) = \frac{1}{i\omega_n + \mu - \Sigma(\omega_n)} \quad , \quad \Sigma(\tau) = -J^2 G^2(\tau) G(-\tau) + \alpha \tilde{t}^2 G(\tau) \chi^2(\tau)$$

$$\chi(\omega_n) = \frac{1}{\omega_n^2 + \Delta_0^2 - P(\omega_n)} \quad , \quad P(\tau) = -2\tilde{t}^2 G(\tau) G(-\tau) \chi(\tau)$$

$$\frac{1}{\beta} \sum_{\omega_n} \chi(\omega_n) = \frac{1}{g}$$



Summary

- SYK models are matter without quasiparticle, it can reproduce strange metal properties and also has a close relation to gravity
 - Extensive 0T entropy
 - Chaotic behavior
 - Quantum transitions
- Other generalizations
 - Lattice model: transport properties
 - Witten-Gurau models

Outlook

- Relate chaos with decoherence
 - Entanglement and information dynamics
 - Time dependent SYK models
-
- My other interest: TQFT, duality...

Thank you!