Title: Entanglement, quantum randomness, and complexity beyond scrambling

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Abstract: <p>The entanglement properties of random quantum states or dynamics are important to the study of a broad spectrum of disciplines of physics, ranging from quantum information to high energy and many-body physics. This work investigates the interplay between the degrees of entanglement and randomness in pure states and unitary channels, by employing tools from random matrix theory, representation theory, combinatorics, Weingarten calculus etc. We reveal strong connections between designs (distributions of states or unitaries that match certain moments of the uniform Haar measure) and generalized entropies (entropic functions that depend on certain powers of the density operator), by showing that Renyi entanglement entropies averaged over designs of the same order are almost maximal. This strengthens the celebrated Page's theorem. Moreover, we find that designs of an order that is logarithmic in the dimension maximize all Renyi entanglement entropies, <span class="il" style="background-color:

transparent; color: rgb(0, 0, 0); font-family: arial,sans-serif; font-size: 12.8px; font-style: normal; font-variant: normal; font-weight: 400; letter-spacing: normal; orphans: 2; text-align: left; text-decoration: none; text-indent: 0px; text-transform: none; -webkit-text-stroke-width: 0px; white-space: normal; word-spacing: 0px;">and so are completely random in terms of the entanglement spectrum. Our results relate the behaviors of Renyi entanglement entropies to the complexity beyond scrambling/thermalization/<wbr style="background-color: transparent; color: rgb(0, 0, 0); display: inline-block; font-family: arial,sans-serif; font-size: 12.8px; font-style: normal; font-variant: normal; font-weight: 400; letter-spacing: normal; orphans: 2; text-align: left; text-decoration: none; text-indent: 0px; text-transform: none; -webkit-text-stroke-width: 0px; white-space: normal; word-spacing: 0px;" />chaos in terms of the degree of randomness, and suggest a generalization of the fast scrambling conjecture. (Refs: 1703.08104, 1709.04313)</p>

Outline

Intro: Physical background & mathematical problems

Mathematical/kinematic results: Generalized entanglement entropies of state/unitary designs

Physical implications: Randomness complexities beyond scrambling by Rényi entanglement entropies

Outlook

Introduction

Scrambling:

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- Closely related: quantum chaos, thermalization (no many-body localization)...
- Originated from study of black holes & quantum gravity: modeling Hawking radiation Page '93, Hayden-Preskill '07; fast scrambling conjecture Sekino-Susskind '08

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  $\diamond$ Complete (Haar) randomization $>$ scrambling
  Large complexity gap, rich physics after scrambling

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- A collective understanding: *effective* randomization
  ◇ Complete (Haar) randomization > scrambling
  Large complexity gap, rich physics after scrambling
- We consider: characterizing randomness & complexity behaviors beyond scrambling by entanglement (truly q)

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Math prob: entanglement properties of random/pseudorandom states/dynamics

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- Entanglement entropy (von Neumann) of Haar-random states—Page's theorem:
  The von Neumann entropy of small subsystems of a pure state, averaged over the Haar/uniform measure, is nearly maximal.
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  1. Haar has *high* complexity: requires exp gates/random bits to even approximate; maximizes higher order entropies.
  2. Max vN entropy has *low* complexity: can be achieved by less random distributions (designs), can be efficiently implemented; higher order entropies not necessarily max.

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This work: higher order entanglement entropies of pseudorandom states/unitary channels (designs)

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- Mathematical machinery from: random matrix theory, representation theory, combinatorics, free prob theory, Weingarten calculus...
- Tight (in terms of complexity) results on degree of entanglement vs randomness

Generalized quantum entropy

Generalized q. entropy of order α: entropic functions of $\mathrm{tr}\{\rho^\alpha\}$.
A unified definition:

$$S_s^{(\alpha)}(\rho) = \frac{1}{s(1-\alpha)}\left[(\mathrm{tr}\{\rho^\alpha\})^s - 1\right].$$

Parameter s: family.

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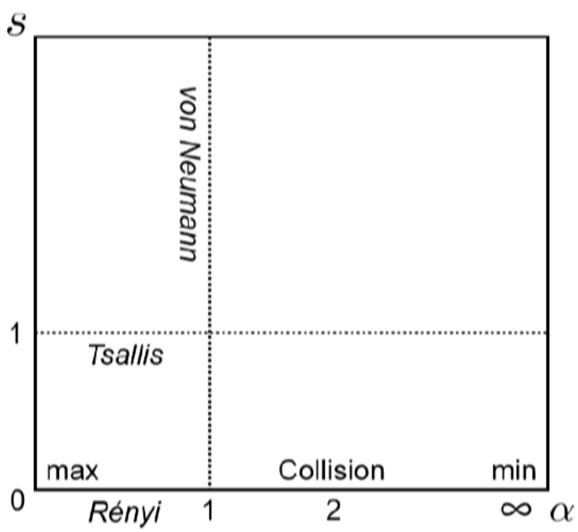

$s = 1$: Tsallis
$s \to 0$: Rényi ($\alpha \to \infty$: min)

$\alpha \to 1$: von Neumann

Rényi entropies

We focus on **Rényi entropies**:

$$S_R^{(\alpha)}(\rho) = \frac{1}{1-\alpha} \log \mathrm{tr}\{\rho^\alpha\}.$$

$\alpha \uparrow S_R^{(\alpha)} \downarrow$: more sensitive to nonuniformity in the spectrum.

Rényi entropies

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Can also analyze

- Other families (eg Tsallis, $s = 1$)
- Variant $S'_{\alpha} = \alpha^2 \partial_{\alpha}(\frac{\alpha-1}{\alpha} S_{\alpha})$: more relevant to thermo (free energy of modular H), holography (area law)

Rényi entropies

Example. Consider the spectrum with one large eigenvalue:

$$\boldsymbol{\lambda} = (\frac{1}{\sqrt{d}}, \underbrace{\frac{1-\frac{1}{\sqrt{d}}}{d-1}, \cdots, \frac{1-\frac{1}{\sqrt{d}}}{d-1}}_{d-1}).$$

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$$S_R^{(2)}(\boldsymbol{\lambda}) \geq \log d - 1.$$

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- Min entropy: far from max (gap $\Theta(\log d)$)

$$S_R^{(\infty)}(\boldsymbol{\lambda}) = -\log \lambda_{\max} = \frac{1}{2} \log d.$$

Designs

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- Finite-order approximation to the Haar measure (pseudorandom, info-theoretic).
- Degree-t polynomials, t-fold twirling, t-th frame potential; t-wise independence, t-independent hash functions...
- Approximate versions: by polynomials; by frame (TPE)...

Designs

- **State design**: An ensemble/dist. ν of pure state vectors in dimension d is a (complex projective) t-*design* if

$$\mathbb{E}_\nu\, p(\psi) = \int \mathrm{d}\psi\, p(\psi) \quad \forall p \in \mathrm{Hom}_{(t,t)}(\mathbb{C}^d).$$

- **Unitary design**: An ensemble/dist. μ of unitary operators in dimension d is a *unitary* t-*design* if

$$\mathbb{E}_\mu\, p(U) = \int \mathrm{d}U\, p(U) \quad \forall p \in \mathrm{Hom}_{(t,t)}(\mathrm{U}(d)).$$

Integrals taken over: uniform measure on the complex unit sphere in $\mathbb{C}^d$/Haar measure on $\mathrm{U}(d)$.
$\mathrm{Hom}_{(t,t)}$: polynomials homogeneous of degree t both in the elements and in their complex conjugates.

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  $\mathrm{Hom}_{(t,t)}$: polynomials homogeneous of degree t both in the elements and in their complex conjugates.

Efficient to implement. Applications in: signal processing, randomized benchmarking, quantum data hiding, decoupling...

Results for random states

Consider a bipartite pure state $|\psi\rangle_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ have dimensions d_A and d_B respectively.

Results for random states

Order correspondence

Theorem: equal partition, asymptotic

Let ν_α be a projective α-design. Consider equal partitions $d_A = d_B$. As $d_A \to \infty$,

$$\mathbb{E}_{\nu_\alpha} S_R^{(\alpha)}(\rho_A) \geq \log d_A - \frac{\log \mathrm{Cat}_\alpha}{\alpha - 1} + O(d_A^{-2}).$$

$\mathrm{Cat}_\alpha := \frac{1}{\alpha+1}\binom{2\alpha}{\alpha}$: Catalan number. So,

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- Rényi-α entanglement entropy averaged over an α-design (expectation) is almost maximal.
- A state sampled from an α-design is very likely to exhibit almost maximal Rényi-α entanglement entropy.

Results for random states

Order correspondence

Theorem: general partition, finite dimension

Let ν_α be a projective α-design. Let $q := \alpha^3/(32 d_B^2) < 1, h(q) := 1 + 2q/[3(1-q)]$. For all $d_A \le d_B, 0 \le \alpha \le \infty$,

$$\begin{aligned} \mathbb{E}_{\nu_\alpha} S_R^{(\alpha)}(\rho_A) &\ge \log d_A - \frac{2\alpha - \frac{3}{2}\log\alpha + \log h(q) - \frac{1}{2}\log\pi}{\alpha - 1} \\ &\ge \log d_A - 2. \end{aligned}$$

When $d_A < d_B$, the result can be improved as follows:

$$\mathbb{E}_{\nu_\alpha} S_R^{(\alpha)}(\rho_A) \ge \log d_A - 2\sqrt{\frac{d_A}{d_B}} - \log c,$$

where $c = 1$ if $\mathcal{H}$ is real and $c = 2$ if $\mathcal{H}$ is complex.

Results for random states

Order correspondence

Methods:

- Key observations: given an α-design ν_α, $\mathbb{E}_{\nu_\alpha} \mathrm{tr}\{\rho_A^\alpha\} = \int \mathrm{d}\psi \mathrm{tr}\{\rho_A^\alpha\}$, since $\mathrm{tr}\{\rho_A^\alpha\}$ only involves degree-α terms of entries of $|\psi\rangle$.

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  By the convexity of Rényi in $\operatorname{tr}\{\rho^\alpha\}$ and Jensen's ineq.,

$$\mathbb{E}_{\nu_\alpha} S_R^{(\alpha)}(\rho_A) \geq \frac{1}{1-\alpha} \log\left(\int \mathrm{d}\psi \operatorname{tr}\{\rho_A^\alpha\} \right).$$

Results for random states

Order correspondence

- Boils down to calculating the Haar integrals of $\mathrm{tr}\{\rho_A^\alpha\}$:

$$\int \mathrm{d}\psi \, \mathrm{tr}\{\rho_A^\alpha\} = \frac{1}{\alpha! D_{[\alpha]}} \sum_{\sigma \in S_\alpha} d_A^{\xi(\sigma\tau)} d_B^{\xi(\sigma)}.$$

$D_{[\alpha]} := \binom{d_A d_B + \alpha - 1}{\alpha}$: dim of the symmetric subspace of $\mathcal{H}^{\otimes\alpha}$
S_α: symmetric group of α symbols
$\xi(\sigma)$: number of disjoint cycles of permutation σ
τ: full cycle/1-shift (1 2 ... α)
Similar results Zyczkowsky-Sommers '01; Collins-Nechita '10, '11

Results for random states

Order correspondence

- Equal partitions, large d limit (basic result):
  Cycle Lemma. $\xi(\sigma\tau) + \xi(\sigma) \leq \alpha + 1$. Guarantees the $O(1)$ gap.
  After a little algebra:

$$\int d\psi \operatorname{tr}\{\rho_A^\alpha\} = \mathrm{Cat}_\alpha d_A^{-\alpha+1} + O\left(d_A^{-(\alpha+1)}\right).$$

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- General partitions, finite dimension (general result): more technical; various modern tools from Weingarten calculus, rep theory...Collins-Matsumoto '17; Goupil-Schaeffer '98...
  See full paper for details.

Results for random states

Approximate designs

The above results are for exact designs. The following error bound shows that these results does not deviate much for approximate designs:

Theorem

Let $\tilde{\nu}_\alpha$ be an ϵ-approximate α-design ($\alpha \geq 2$), i.e., $\left\|\mathcal{F}_\alpha(\tilde{\nu}_\alpha) - P_{[\alpha]}\right\|_1 \leq \epsilon$, where $\mathcal{F}_\alpha(\tilde{\nu}_\alpha) := D_{[\alpha]}\, \mathbb{E}_{\tilde{\nu}_\alpha}(|\psi\rangle\langle\psi|)^{\otimes t}$ is the α-th frame operator of $\tilde{\nu}_\alpha$, and $P_{[\alpha]}$ is the projector onto the α-partite symmetric subspace of $(\mathbb{C}^d)^{\otimes\alpha}$. Then

$$\mathbb{E}_{\tilde{\nu}_\alpha} S_R^{(\alpha)}(\rho_A) \geq \frac{1}{1-\alpha} \log\left(\int \mathrm{d}\psi\, \mathrm{tr}\{\rho_A^\alpha\} + \frac{\epsilon}{D_{[\alpha]}} \right).$$

Key step: matrix Hölder

Results for random states

Log design maximizes min entropy

Is ∞-design necessary to maximize ∞-entropy (min entropy)?

Results for random states

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Is ∞-design necessary to maximize ∞-entropy (min entropy)?

Theorem

Let ν_α be a projective α-design, where $\alpha = \lceil (\log d_A)/a \rceil \leq (16 d_B^2)^{1/3}$ with $0 < a \leq 1$. Then

$$\mathbb{E}_{\nu_\alpha} S_{\min}(\rho_A) \geq \log d_A - 2 - a.$$

In particular, $\mathbb{E}_{\nu_\alpha} S_{\min}(\rho_A) \geq \log d_A - 3$ if $\alpha = \lceil \log d_A \rceil$.

Results for random states

Separation: existence of 2-design with non-maximal Rényi-3

Another natural question: Are Rényi entanglement entropies of different orders truly separated, in the sense that $\exists\ \alpha$-design s.t. Rényi entropy of order $> \alpha$ is bounded away from maximal?

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We construct such a separation for $\alpha = 2$:

Theorem

There exist a family of 2-designs such that, for all $\alpha > 2$, $\log d_A - S_R^{(\alpha)}(\rho_A) \in \Omega(\log d_A)$.

Based on the orbits of a special subgroup of the unitary group on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Representation theory.

Results for random states

Separation: existence of 2-design with non-maximal Rényi-3

Let $G = \mathrm{U}_A \otimes \mathrm{U}_B$ (NB: irreducible, not unitary 2-design). The orbit of $|\psi\rangle$ under the action of G forms a 2-design iff $\mathrm{tr}\{\rho_A^2\}$ is equal to the average over the uniform ensemble:

$$\mathrm{tr}\{\rho_A^2\} = \frac{d_A + d_B}{d_A d_B + 1}.$$

It holds if ρ_A has the following spectrum

$$\lambda_1 = \frac{d_A d_B + 1 + (d_A - 1)\sqrt{(d_A + 1)(d_A d_B + 1)}}{d_A(d_A d_B + 1)},$$
$$\lambda_2 = \cdots = \lambda_{d_A} = \frac{d_A d_B + 1 - \sqrt{(d_A + 1)(d_A d_B + 1)}}{d_A(d_A d_B + 1)}.$$

Suppose d_B/d_A is bounded by constant r. Then $\lambda_1 \geq (r d_A)^{-1/2}$. So

$$S_R^{(\alpha)}(\rho_A) \leq \frac{1}{1-\alpha} \log \lambda_1^\alpha \leq \frac{\alpha}{2(\alpha - 1)}(\log d_A + \log r).$$

Results for random unitaries

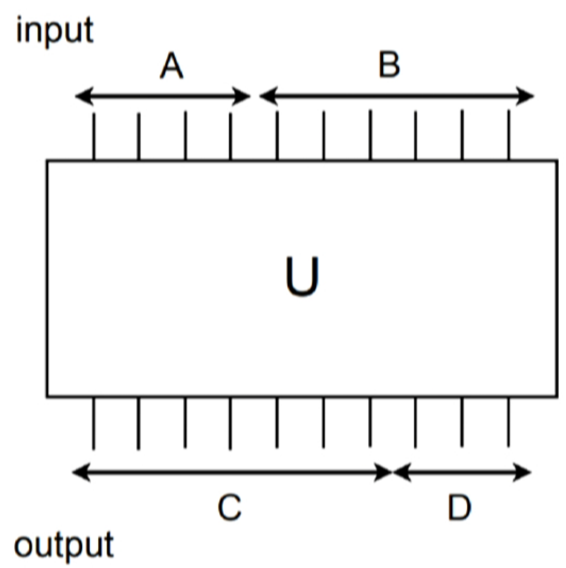
Model: entanglement/tripartite information of the Choi state

Intrinsic entanglement/scrambling properties of random unitary channels?

Results for random unitaries

Model: entanglement/tripartite information of the Choi state

Partition the input into A and B, and the output into C and D



We are interested in the entanglement between AC and BD (entropy of AC).

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Intrinsic entanglement/scrambling properties of random unitary channels?

Choi isomorphism:

$$\text{Unitary operator } U = \sum_{i,j=0}^{d-1} U_{ij} |i\rangle\langle j|$$

$$\Updownarrow$$

$$\text{Pure state } |U\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} U_{ji} |i\rangle_{in} \otimes |j\rangle_{out}$$

Results for random unitaries

Model: entanglement/tripartite information of the Choi state

(Negative) tripartite information Hosur-Qi-Roberts-Yoshida '15

$$-I_3(A:C:D) := I(A:CD) - I(A:C) - I(A:D).$$

Determined by $S(AC)$.

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- U generates global entanglement to "hide" local information of the input.
  Large $-I_3$ diagnoses 'scrambling' : local operators supported on A gets spread onto CD via global entanglement, and become invisible to local observers.
- Reduces to CMI by unitarity.

Results for random unitaries

Order correspondence

Theorem: equal partition, asymptotic

Let μ_α be a unitary α-design. Consider equal partitions of the input and output registers, $d_A = d_B = d_C = d_D$. As $d \to \infty$,

$$\mathbb{E}_{\mu_\alpha} S_R^{(\alpha)}(\rho_{AC}) \geq \log d - \frac{\log \mathrm{Cat}_\alpha}{\alpha - 1} + O(d^{-1}).$$

So,

$$\mathbb{E}_{\mu_\alpha} S_R^{(\alpha)}(\rho_{AC}) \geq \log d - O(1).$$

- Rényi-α entanglement entropy averaged over a unitary α-design (expectation) is almost maximal.
- A unitary sampled from a unitary α-design is very likely to exhibit almost maximal Rényi-α entanglement entropy.

Results for random unitaries

Order correspondence

Theorem: general partition, finite dimension

Let μ_α be a unitary α-design. Suppose $d > \sqrt{6}\alpha^{7/4}, d_A \leq d_B$. Then

$$\begin{aligned} & \mathbb{E}_{\mu_\alpha} S_R^{(\alpha)}(\rho_{AC}) \\ \geq \quad & \log d - \frac{\log \mathrm{Cat}_\alpha}{\alpha - 1} - \frac{\log\left[\frac{a_\alpha h(q)}{8}\left(7 + \cosh\frac{2\alpha(\alpha-1)}{d}\right)\right]}{\alpha - 1}, \end{aligned}$$

where $a_\alpha := \left(1 - \frac{6\alpha^{7/2}}{d^2}\right)^{-1}$.

Results for random unitaries

Order correspondence

<u>Methods</u>:

- By properties of designs and Jensen (similar as for states):

$$\mathbb{E}_{\nu_\alpha}\left[S_R^{(\alpha)}(\rho_{AC})\right] \geq \frac{1}{1-\alpha}\log\left(\int \mathrm{d}U \mathrm{tr}\left\{\rho_{AC}^{\alpha}\right\}\right).$$

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- Haar integrals of $\mathrm{tr}\{\rho_A^\alpha\}$:

$$\int \mathrm{d}U\mathrm{tr}\left\{\rho_{AC}^\alpha\right\} = \frac{1}{d^\alpha}\sum_{\sigma,\gamma\in S_\alpha} d_A^{\xi(\sigma\tau)} d_B^{\xi(\sigma)} d_C^{\xi(\gamma\tau)} d_D^{\xi(\gamma)}\mathrm{Wg}(d,\sigma\gamma^{-1}),$$

$\mathrm{Wg}(d,\sigma) := \frac{1}{(\alpha!)^2}\sum_{\lambda\vdash\alpha}\frac{\chi^\lambda(1)^2\chi^\lambda(\sigma)}{s_{\lambda,d}(1,\cdots,1)}$: Weingarten function.
$\lambda\vdash\alpha$ means λ is a partition of α
χ^λ, s_λ: corresponding character of S_α and Schur polynomial
Wg can be derived by various tools in representation theory, such as Schur-Weyl duality and Jucys-Murphy elements.

Results for random unitaries

Order correspondence

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- Results for unitaries analogous to those for states (analysis much more difficult).
- Gap constructions unknown.

Results for random unitaries

Approximate designs

Error bound for approximate unitary designs:

Theorem

Let $\tilde{\mu}_\alpha$ be an ϵ-approximate unitary α-design, i.e.,
$\left\| \mathcal{F}_\alpha(\tilde{\mu}_\alpha) - \int \mathrm{d}U U^{\otimes\alpha} \otimes U^{\dagger\otimes\alpha} \right\|_1 \leq \epsilon$, where
$\mathcal{F}_\alpha(\tilde{\mu}_\alpha) := \mathbb{E}_{\tilde{\mu}_\alpha}\left[U^{\otimes\alpha} \otimes U^{\dagger\otimes\alpha} \right]$ is the α-th frame operator of $\tilde{\mu}_\alpha$. Then

$$\mathbb{E}_{\tilde{\mu}_\alpha}[S_R^{(\alpha)}(\rho_{AC})] \geq \frac{1}{1-\alpha} \log\left(\int \mathrm{d}U \mathrm{tr}\left\{ \rho_{AC}^\alpha \right\} + \frac{1}{d^\alpha}\epsilon \right).$$

Key step: matrix Hölder

Results for random unitaries

Log design maximizes min entropy

Similar techniques and results for the min entropy:

Theorem

Let μ_α be a unitary α-design, where $1 \leq \alpha = \lceil \log d/a \rceil \leq \sqrt{d}/2$ and $a > 0$; then

$$\mathbb{E}_{\nu_\alpha} S_{\min}(\rho_{AC}) \geq \log d - 2 - a.$$

In particular, $\mathbb{E}_{\nu_\alpha} S_{\min}(\rho_{AC}) \geq \log d - 3$ if $\alpha \geq \lceil \log d \rceil$.

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- "Gap 2-design": non-flat entanglement spectrum/non-trivial order dependence for Renyi

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Scrambling complexities above info scrambling in terms of *degree of randomness/designs* by Rényi entanglement entropy:

- Hierarchy of complexities by design, between max entanglement entropy and max randomness;
- Rényi-α entanglement entropy as diagnostics of the randomness complexity of α-designs (higher OTOC, frame potential Roberts-Yoshida '17);

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- Fast α-design conjecture: $\approx O(\alpha \log n)$
  Nakata-Hirche-Koashi-Winter '17
- Log-designs achieve max-scrambling.

Some potential extensions

Direct open problems:

- Separations for higher orders and the unitary case?
- Negative tripartite Rényi information (no subaddivity)?

Some potential extensions

Strong entanglement is typical:

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Strong entanglement is typical:

- More precise probabilistic statements: measure concentration, Levy's lemma
- Strong typicality statements on eg 'almost' absolutely maximally entangled states/perfect tensors, locally maximally entangled states Raamsdonk et al. '17: exact cases highly non-trivial, not robust

Some potential extensions

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- Possible non-integer design constructions? Negative results so far.

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Random tensor network with non-flat entanglement spectrum:

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- Why $\approx 1/\sqrt{d}$ large eigenvalue? Gravity dual? More physical justifications? Helpful for studying some holographic statements?

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Further study on randomness complexities beyond scrambling/chaos/thermalization by the tool of Rényi in physical dynamics

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- Thermalization of Rényi entropies in chaotic models (especially fast scramblers)
  Do they thermalize? Thermalizing times?

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- Thermalization of Rényi entropies in chaotic models (especially fast scramblers)
  Do they thermalize? Thermalizing times?
  Calculation of Rényi (replica, twisted operator) in 1d SYK after a quench in the scrambling regime: Rényi not thermalized
  Gu-Lucas-Qi '17

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- Models for designs with conserved quantities?

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More connections to the study of quantum gravity, holography, many-body physics, quantum statistical mechanics and quantum information...

Thanks for your attention!