Title: Non-abelian Hodge theory in dimension one, Fukaya categories and periodic monopoles.

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URL: http://pirsa.org/18010072

Abstract: <p>By the non-abelian Hodge theory of Carlos Simpson, harmonic bundles
interpolate between bundles with connections on a curve and
 Higgs bundes (precise formulations requires some additional data like parabolic structure and stability structure).

I will explain the framework for a generalization of the non-abelian Hodge theory
which unifies Simpson's story ("rational case") with those for q-difference
equations ("trigonometric case") and elliptic difference equations
("elliptic case").

This unification leads to a class of examples of the new notion of ``twistor families of categories".
In the rational,trigonometric and elliptic cases twistor families of categories involve partially wrapped Fukaya categories of certain complex symplectic surfaces, categories of
holonomic modules over quantizations of these surfaces and categories of coherent sheaves on the surfaces
with certain restrictions on the support.

In the trigonometric and elliptic cases doubly and triply periodic monopoles give an alternative description of harmonic objects, hence playing the same role
as harmonic bundles play in the case of Simpson theory.</p>

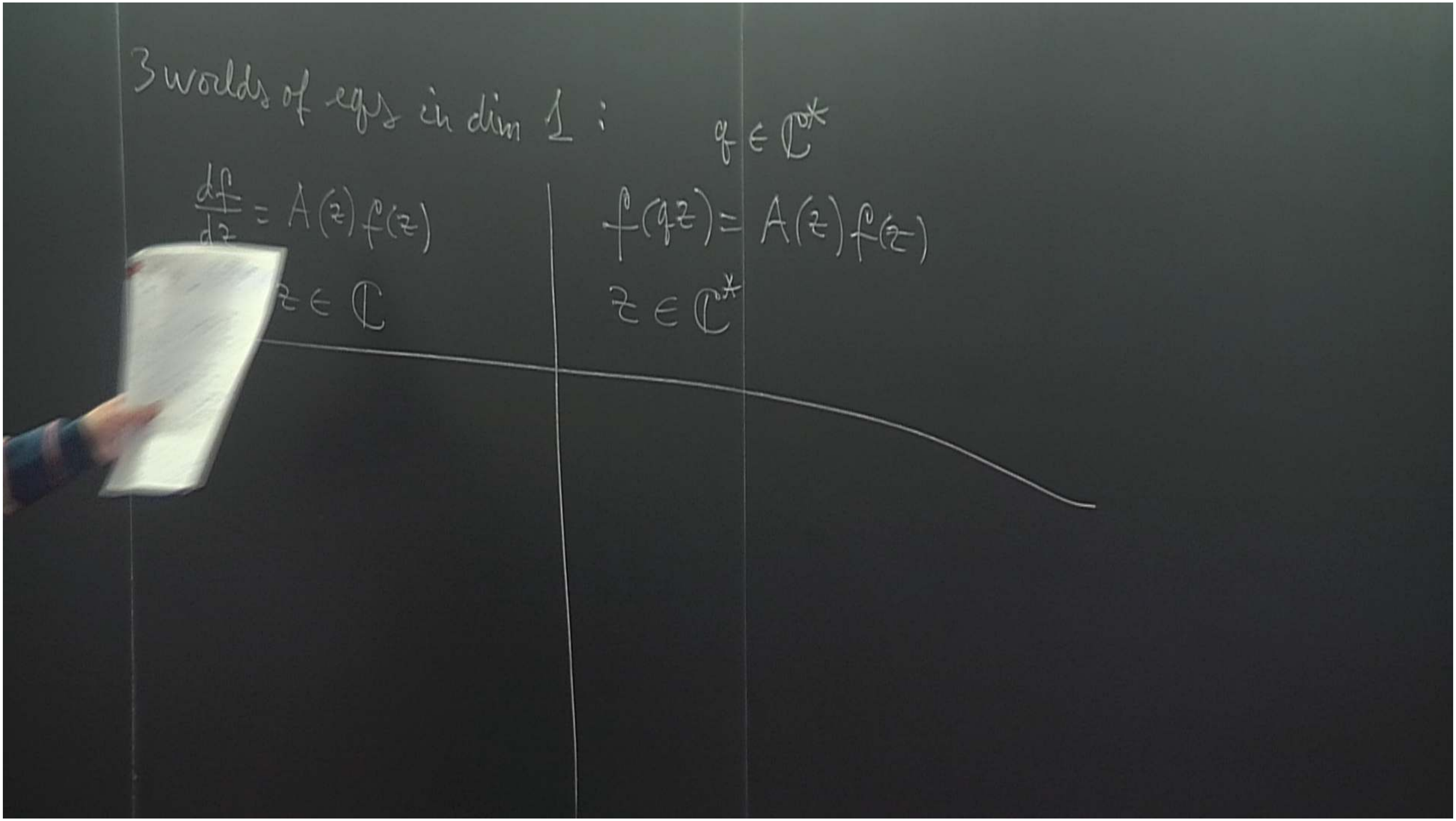
3 worlds of eqs in dim 1 :

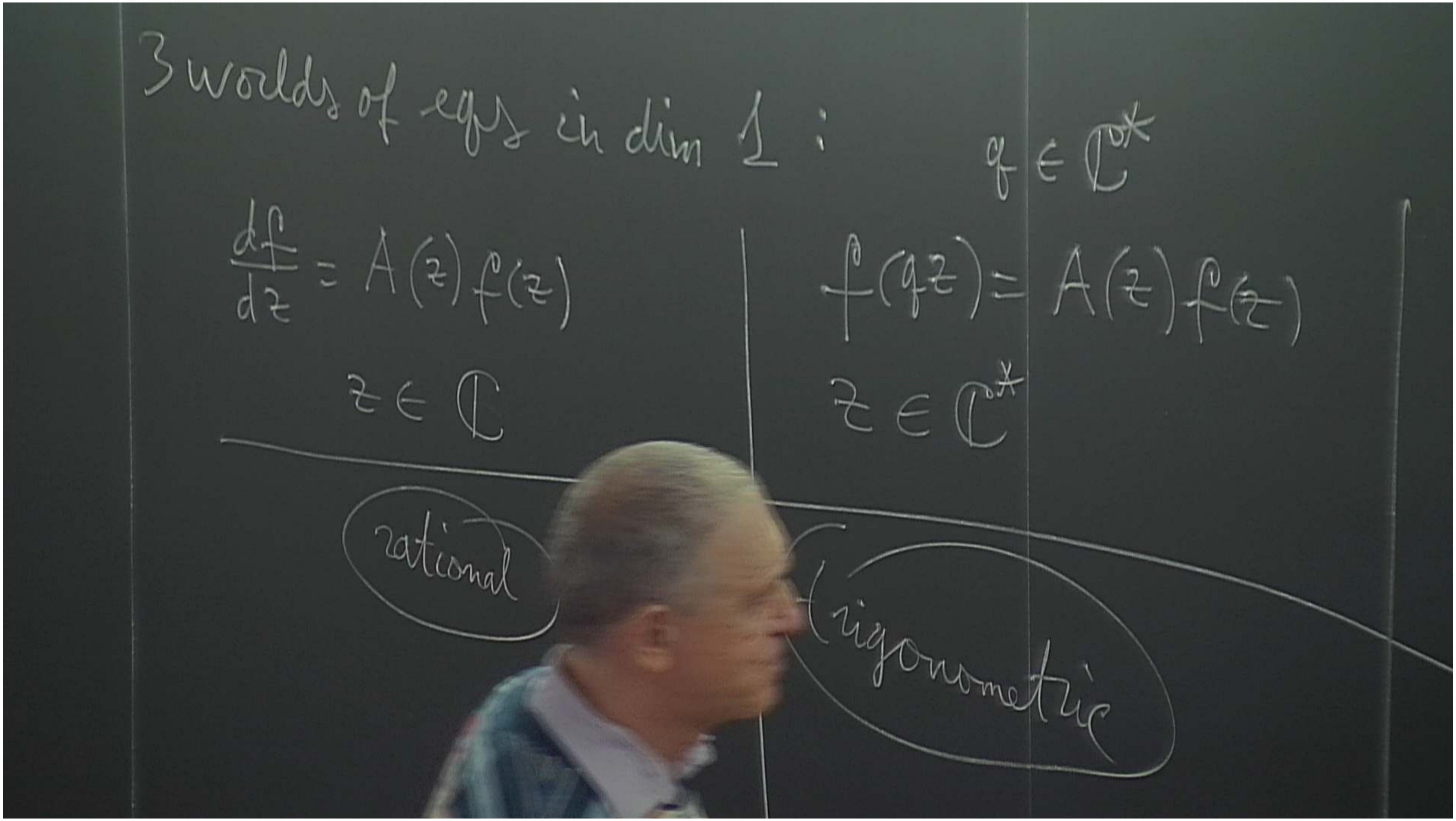
$q \in \mathbb{C}^*$

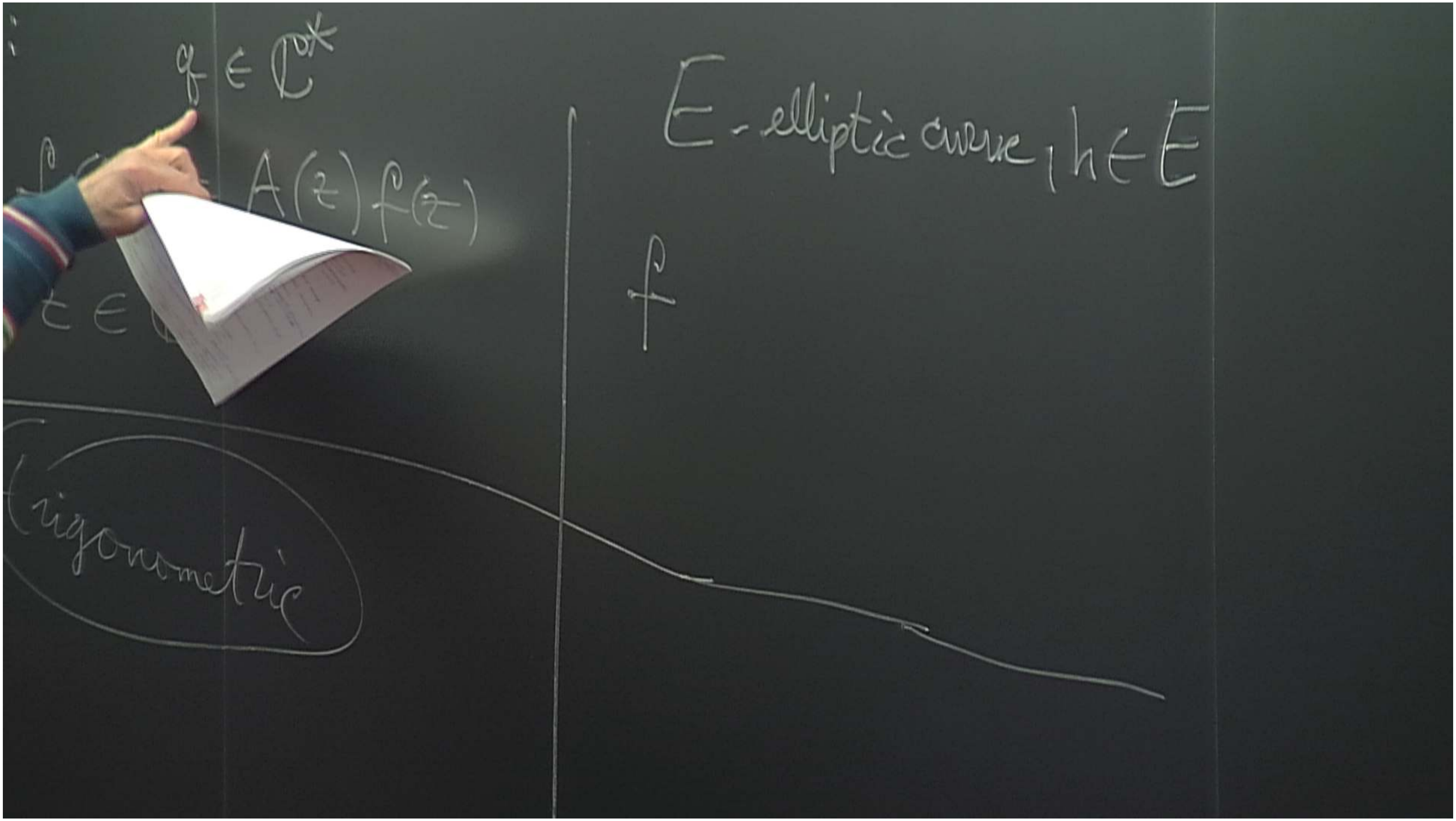
$\frac{df}{dz} = A(z) f(z)$

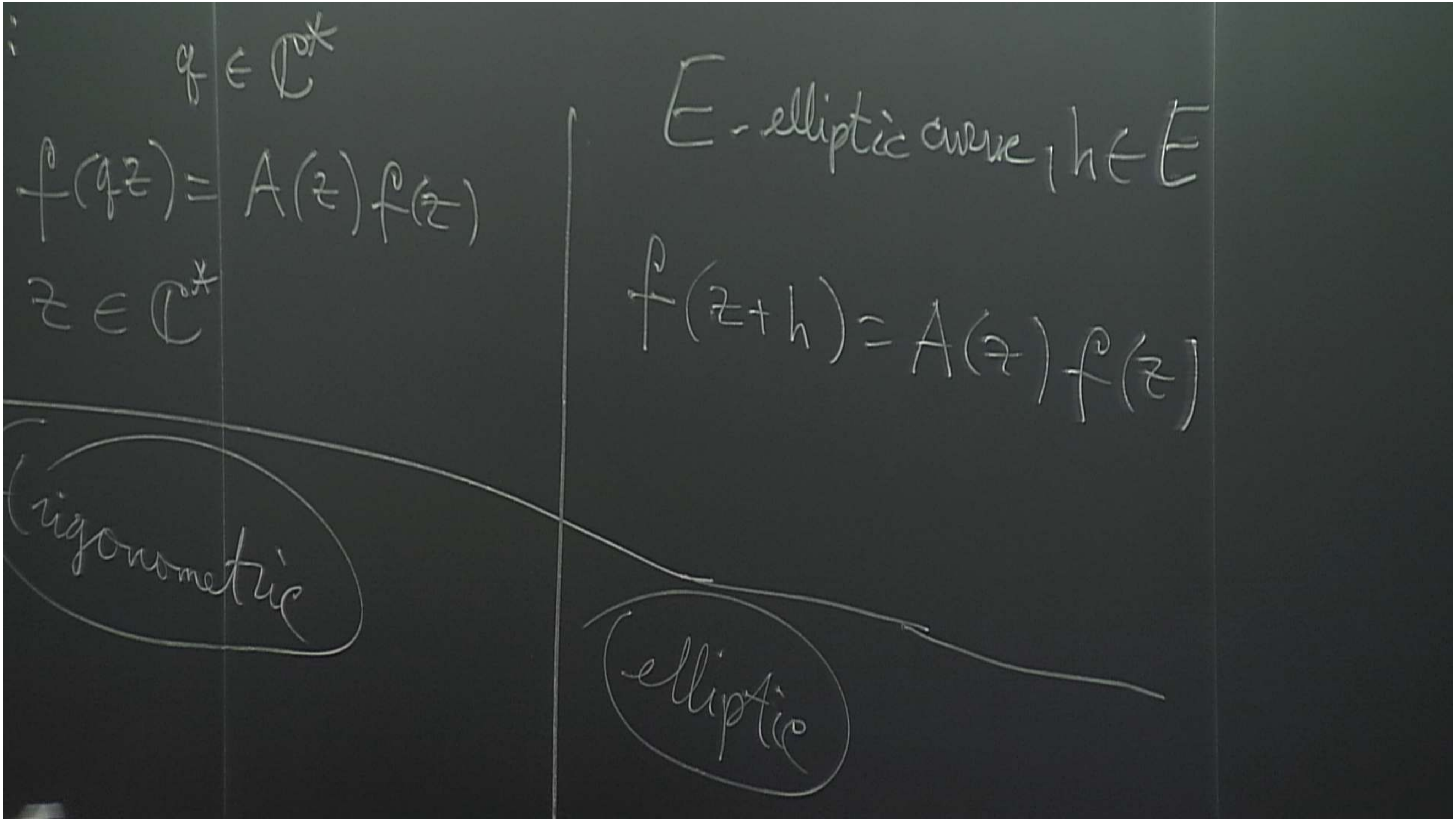
$z \in \mathbb{C}$

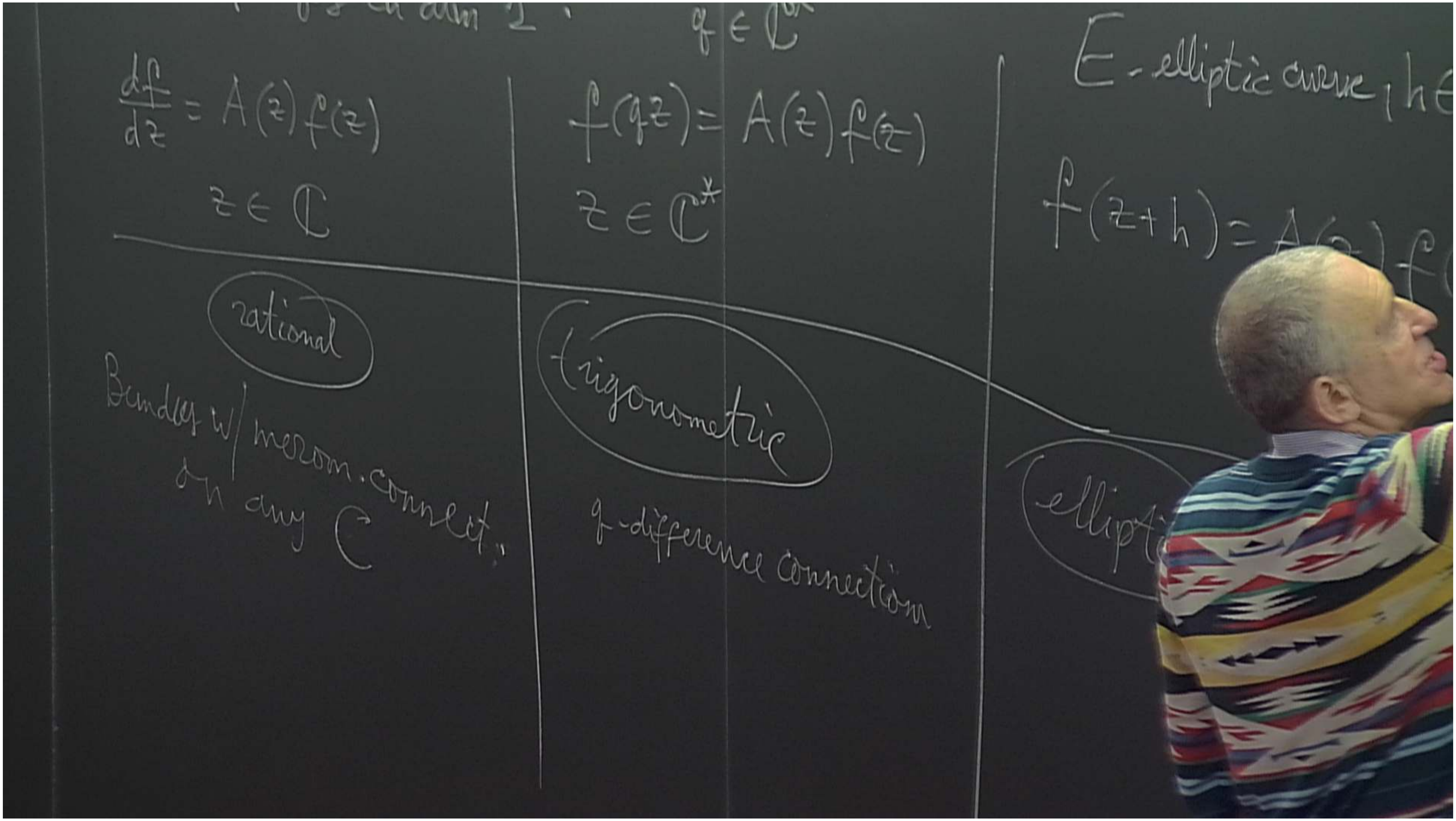
$f(qz) = A(z) f(z)$


3 worlds of eqs in dim 1 :
q ∈ ℂ*
df/dz = A(z) f(z)
z ∈ ℂ
f(qz) = A(z) f(z)
z ∈ ℂ*

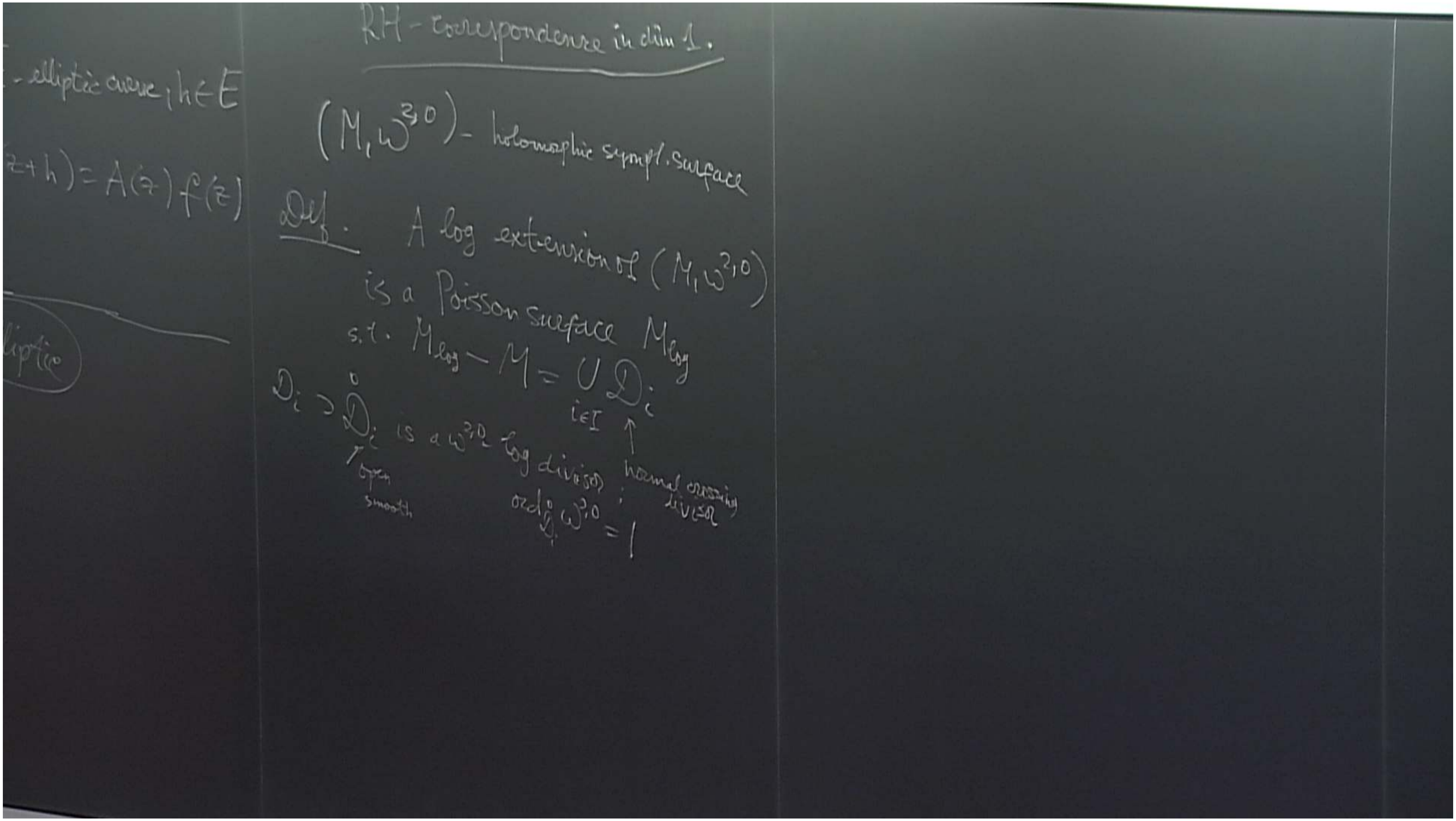

3 worlds of eqs in dim 1 :
q ∈ ℂ*
df/dz = A(z) f(z)
z ∈ ℂ
f(qz) = A(z) f(z)
z ∈ ℂ*
rational
trigonometric

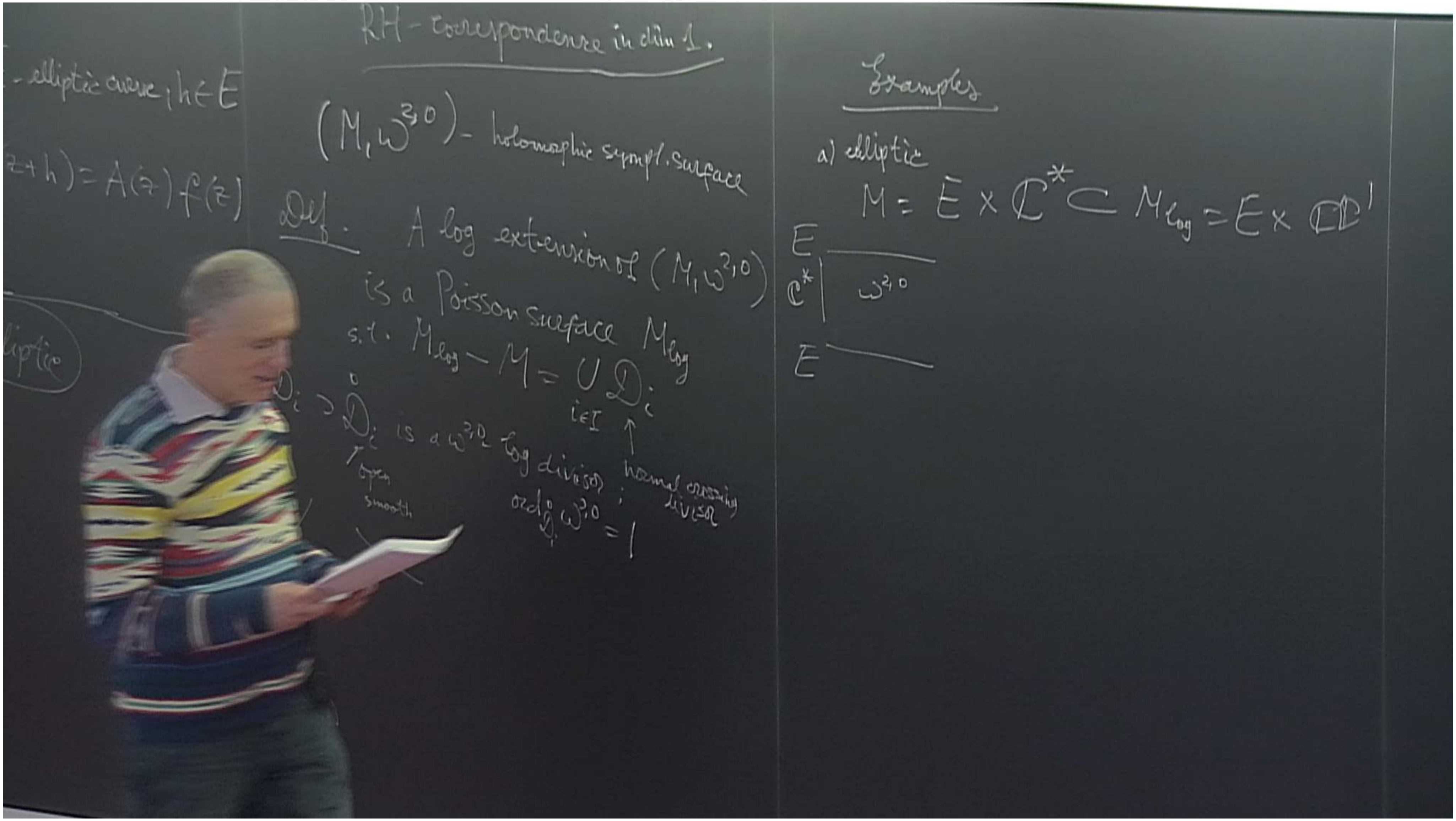

q ∈ ℂ*
A(z) f(z)
E - elliptic curve, h ∈ E
f
trigonometric


$q \in \mathbb{C}^*$
$f(qz) = A(z) f(z)$
$z \in \mathbb{C}^*$
trigonometric
E - elliptic curve, $h \in E$
$f(z+h) = A(z) f(z)$
elliptic



df/dz = A(z) f(z)
z ∈ ℂ
rational
Bundles w/ merom. connect. on any C
f(qz) = A(z) f(z)
z ∈ ℂ*
trigonometric
q-difference connection
E - elliptic curve, h ∈
f(z+h) = A
ellipt


RH - correspondence in dim 1.
(M, ω^{2,0}) - holomorphic sympl. surface
Def. A log extension of (M, ω^{2,0}) is a Poisson surface M_log
s.t. M_log − M = ∪_{i∈I} D_i
D_i ⊃ D̊_i is a ω^{2,0} log divisor ; ord_{D̊_i} ω^{2,0} = 1
open smooth
normal crossing divisor
- elliptic curve ; h ∈ E
(z+h) = A(z) f(z)
elliptic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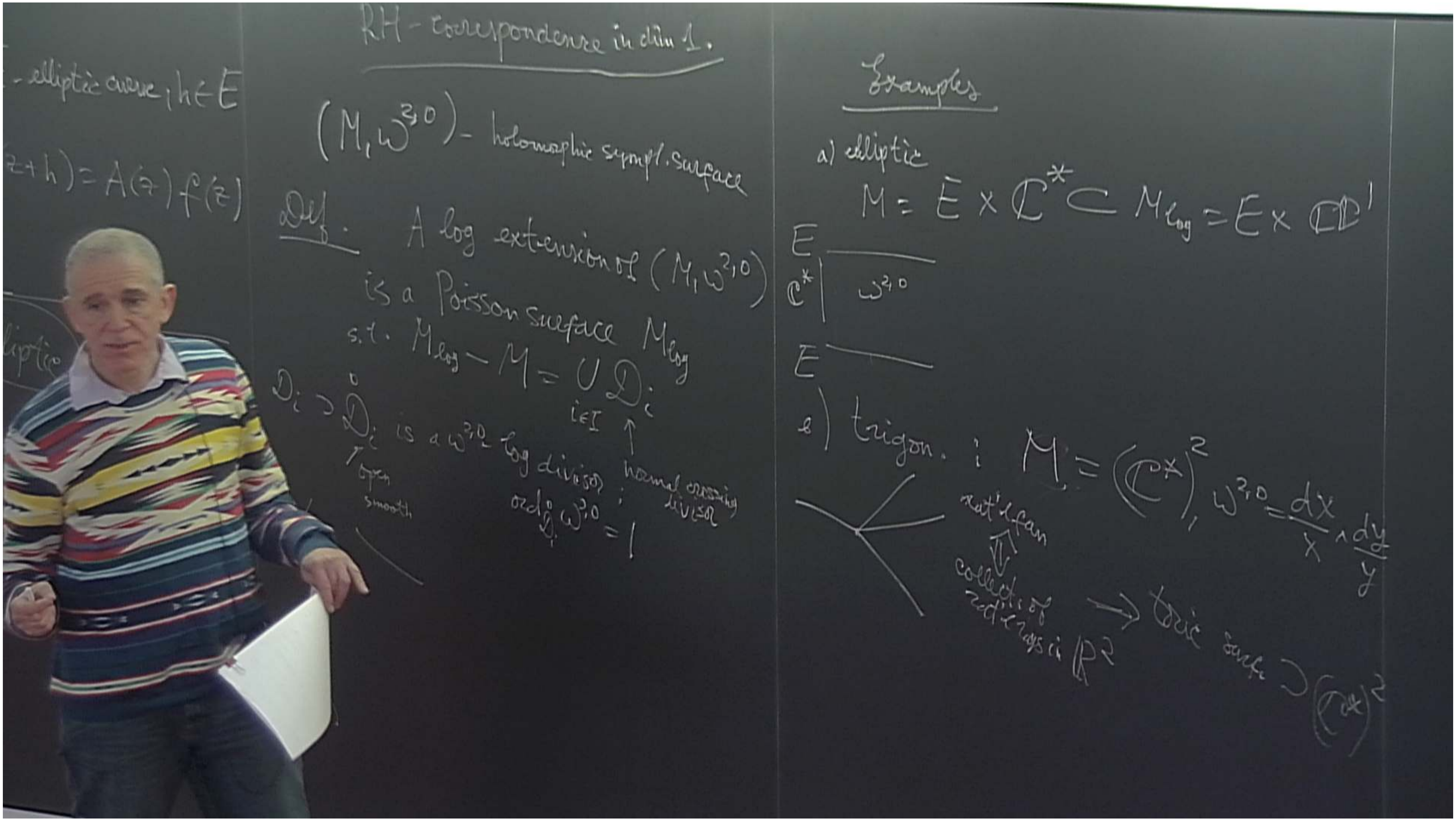
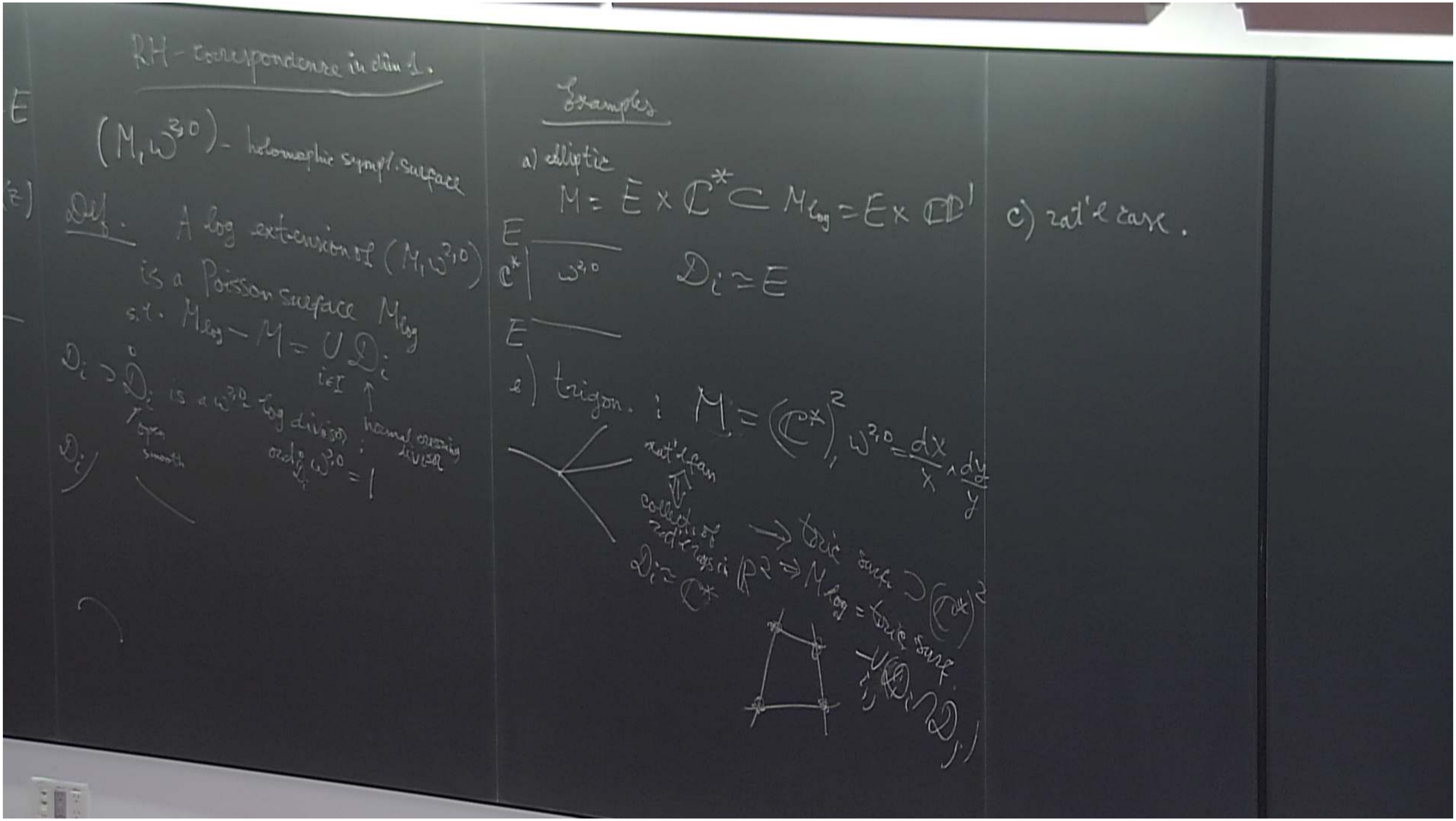
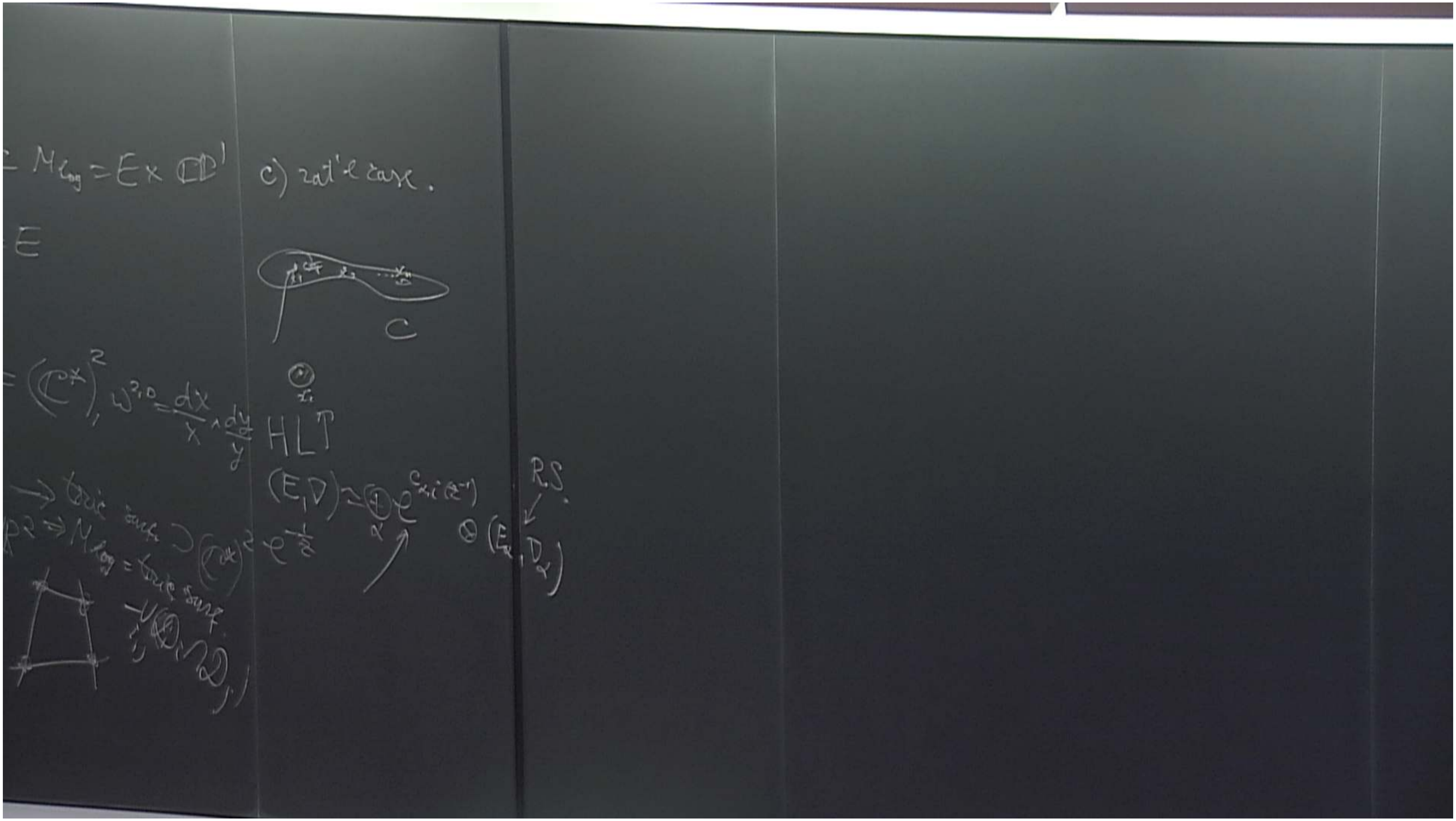
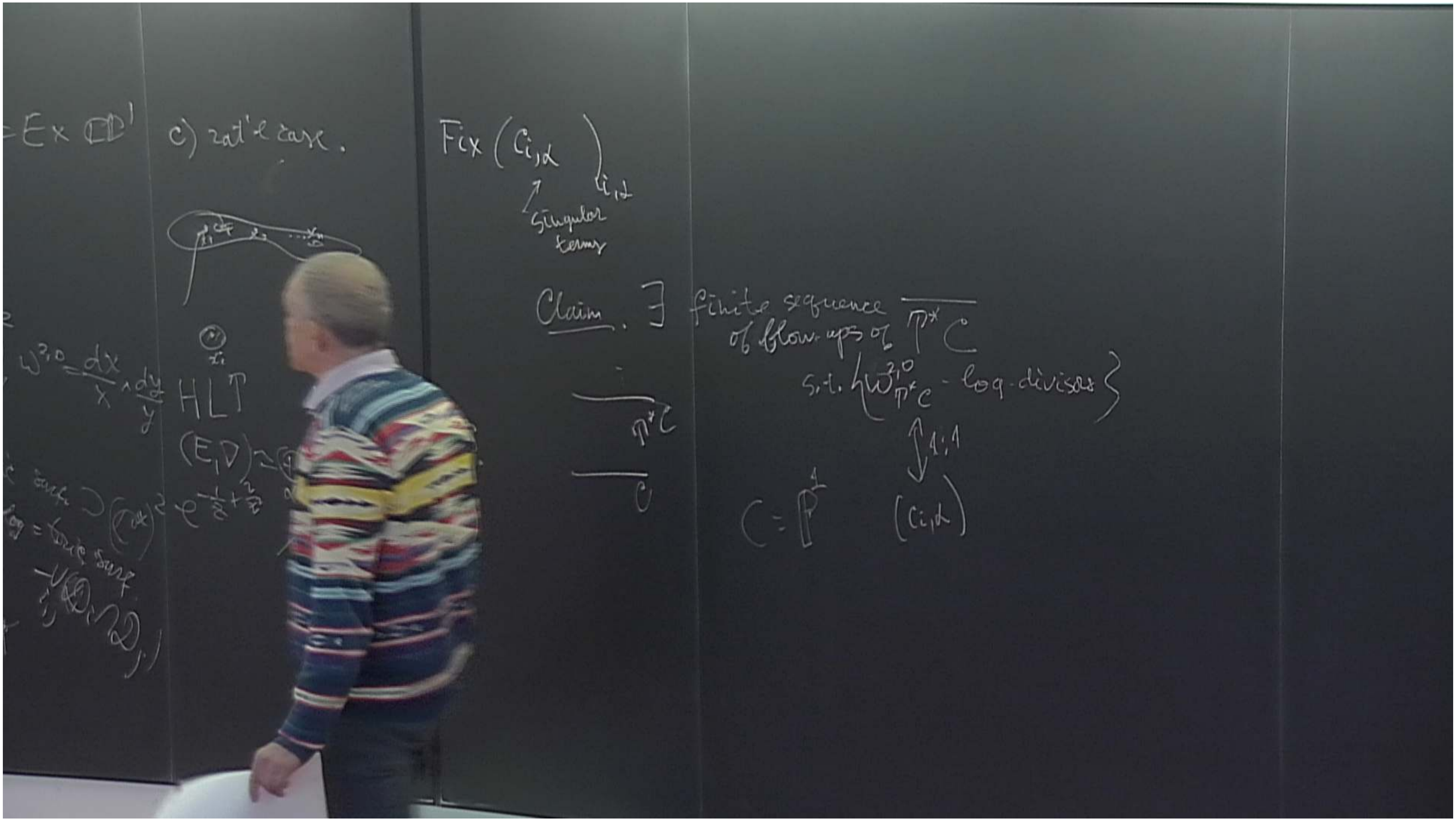
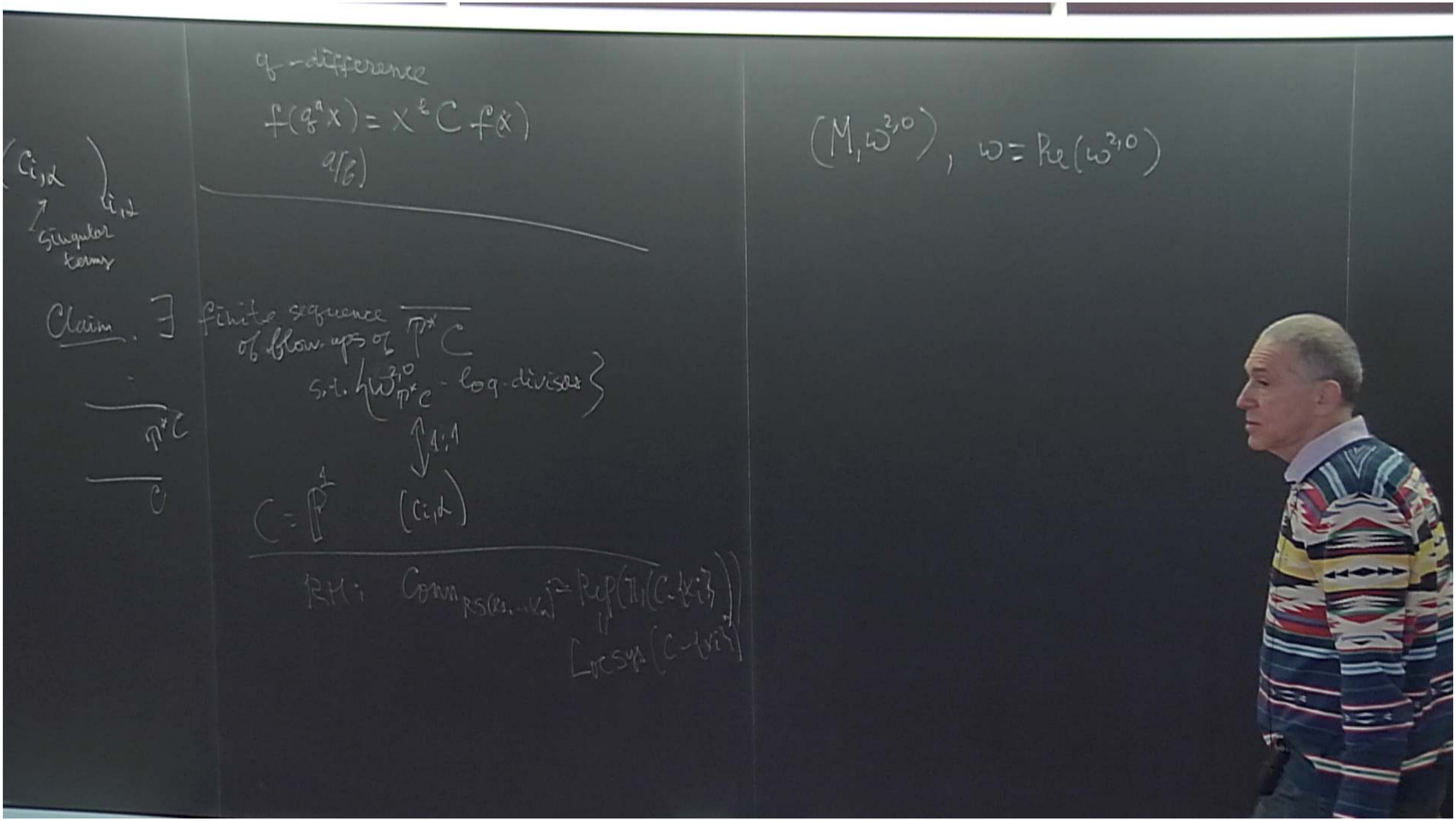
i∈I
normal crossing divisor
D_i ⊃ D̊_i is a ω^{2,0} log divisor ;
open smooth
ord ω^{2,0} = 1
Examples
a) elliptic
M = E × ℂ* ⊂ M_log = E × ℂℙ^1
E
ℂ*
ω^{2,0}
E
- elliptic curve ; h ∈ E
(z+h) = A(z) f(z)
elliptic

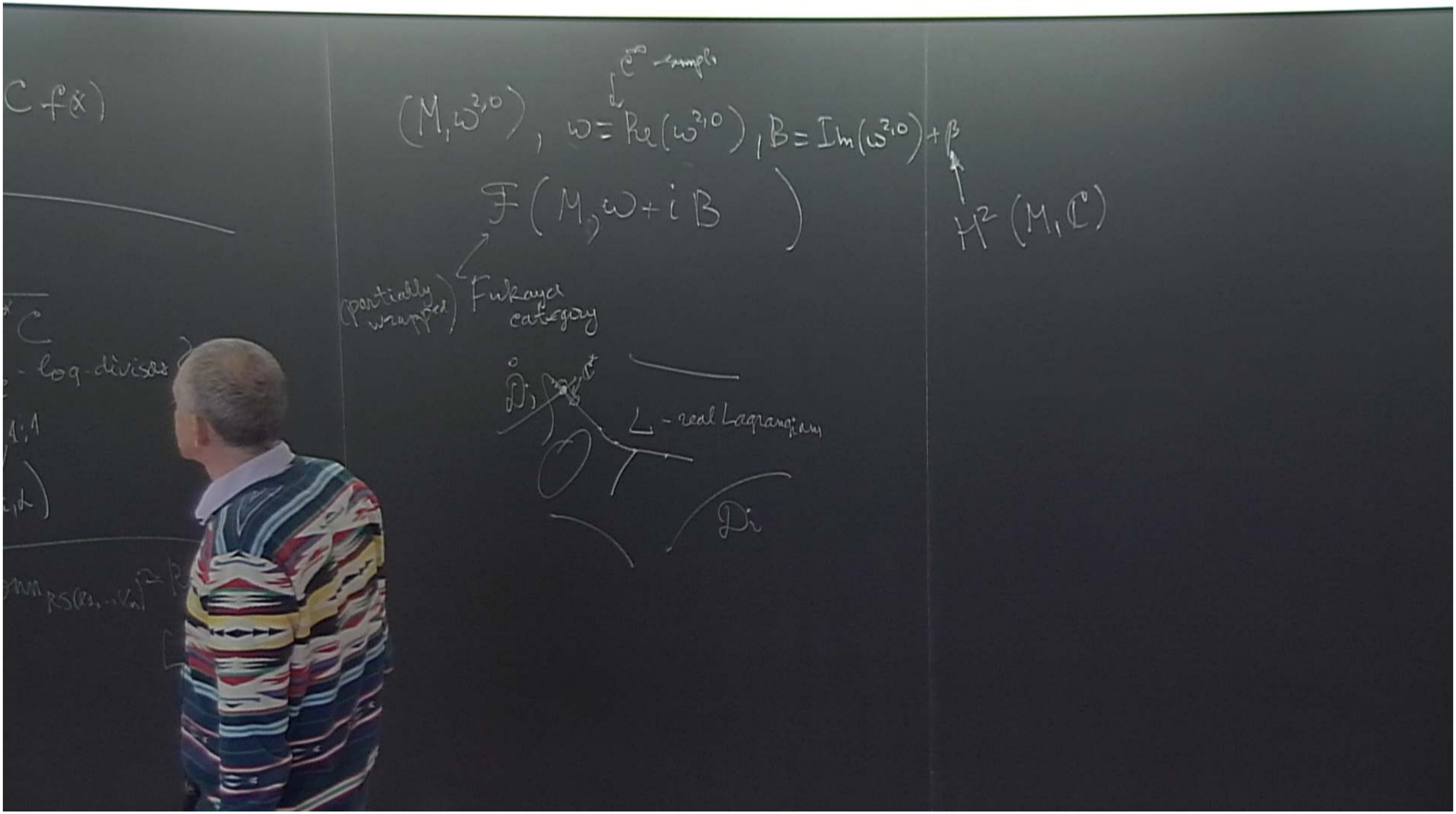

RH - correspondence in dim 1.
- elliptic curve ; h ∈ E
(M, ω^{2,0}) - holomorphic sympl. surface
Def. A log extension of (M, ω^{2,0}) is a Poisson surface M_log s.t. M_log − M = ∪ D_i
i∈I
normal crossing divisor
D_i ⊃ D̊_i is a ω^{2,0} log divisor ; ord ω^{2,0} = 1
open smooth
Examples
a) elliptic
M = E × ℂ* ⊂ M_log = E × ℙ^1
E
ℂ*
ω^{2,0}
E
b) trigon. : M = (ℂ*)^2, ω^{2,0} = dx/x ∧ dy/y
rat'l fan
collection of rat'l rays in ℝ^2
toric surf. ⊃ (ℂ*)^2

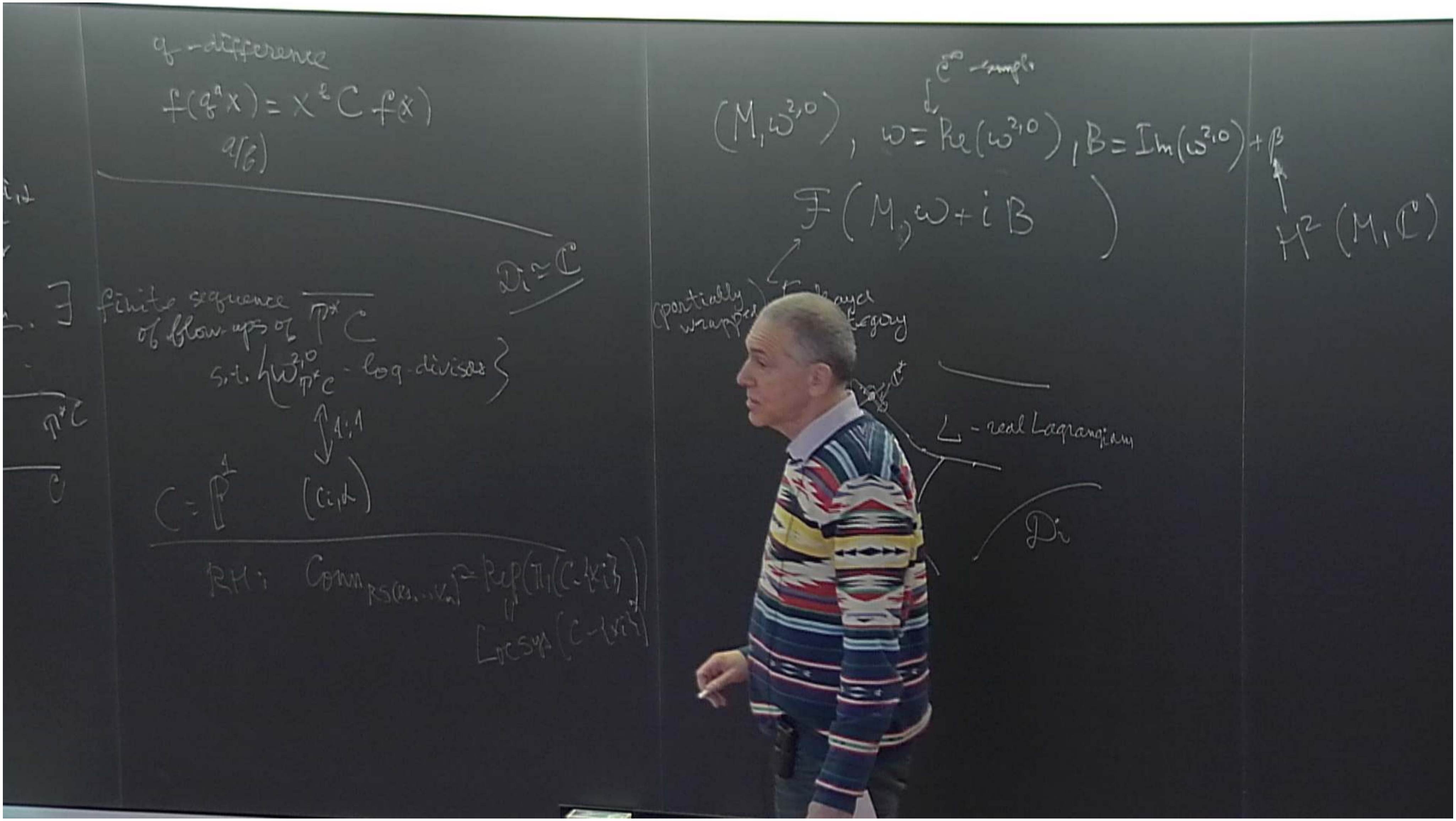

RH - correspondence in dim 1.
(M, ω^{2,0}) - holomorphic sympl. surface
Def. A log extension of (M, ω^{2,0}) is a Poisson surface M_log s.t. M_log − M = ∪ D_i
i∈I
normal crossing divisor
D_i is a ω^{2,0} - log divisor
open smooth
ord ω^{2,0} = 1
Examples
a) elliptic
M = E × ℂ* ⊂ M_log = E × ℂℙ¹
E
ℂ*
ω^{2,0}
D_i ≃ E
E
b) trigon. : M = (ℂ*)², ω^{2,0} = dx/x ∧ dy/y
rat'l fan
collection of rat'l rays in ℝ²
→ toric surf. ⊃ (ℂ*)²
⇒ M_log = toric surf.
D_i ≃ ℂ*
c) rat'l case.


c) rat'l case.
HLT
R.S.

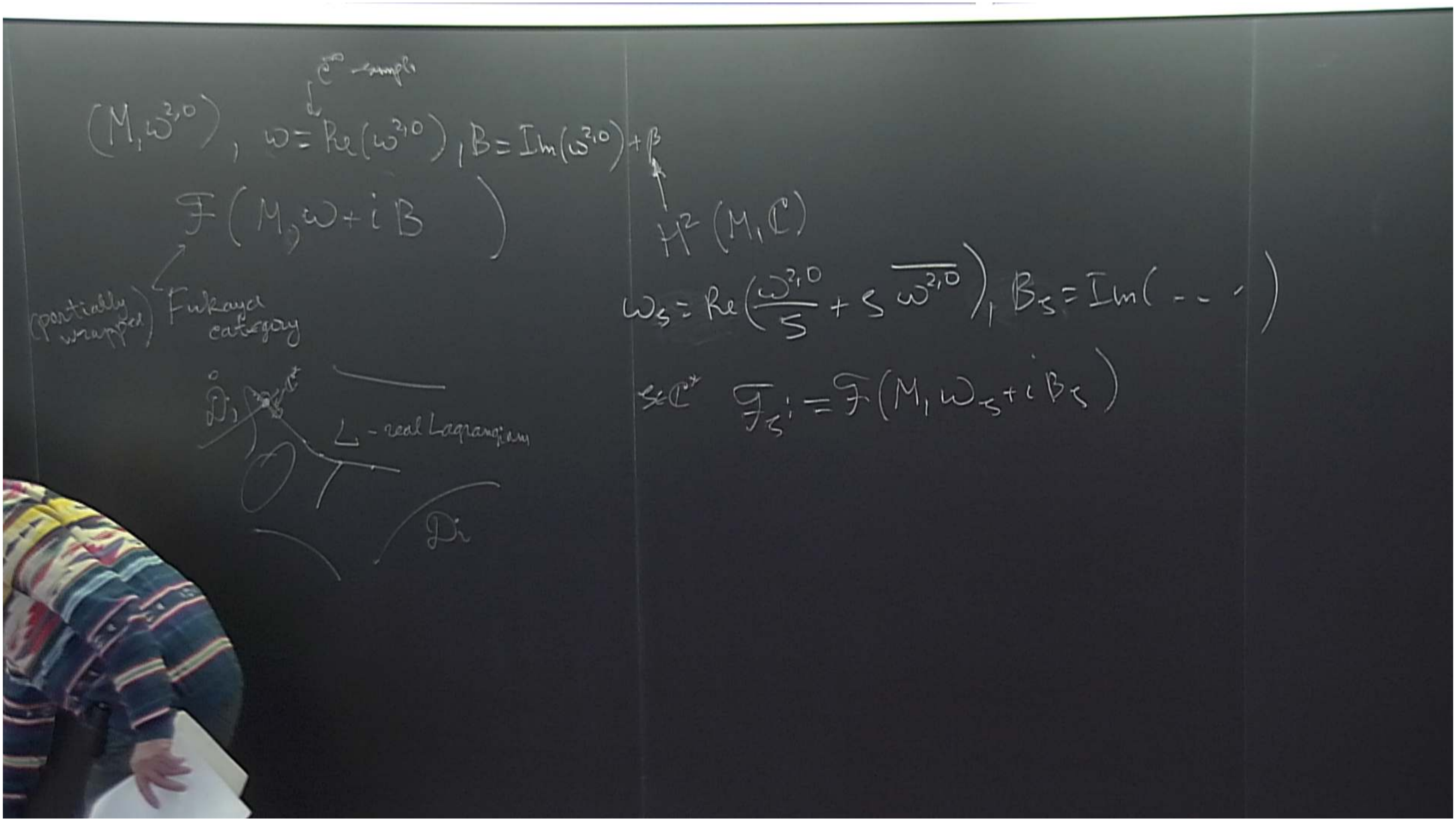

c) rat'l case.
Fix (Ci,α) i,α
Singular terms
HLT
Claim. ∃ finite sequence of blow-ups of π*C
s.t. {ω²,⁰ π*C - log-divisors}
1:1
C = P¹ (Ci,α)

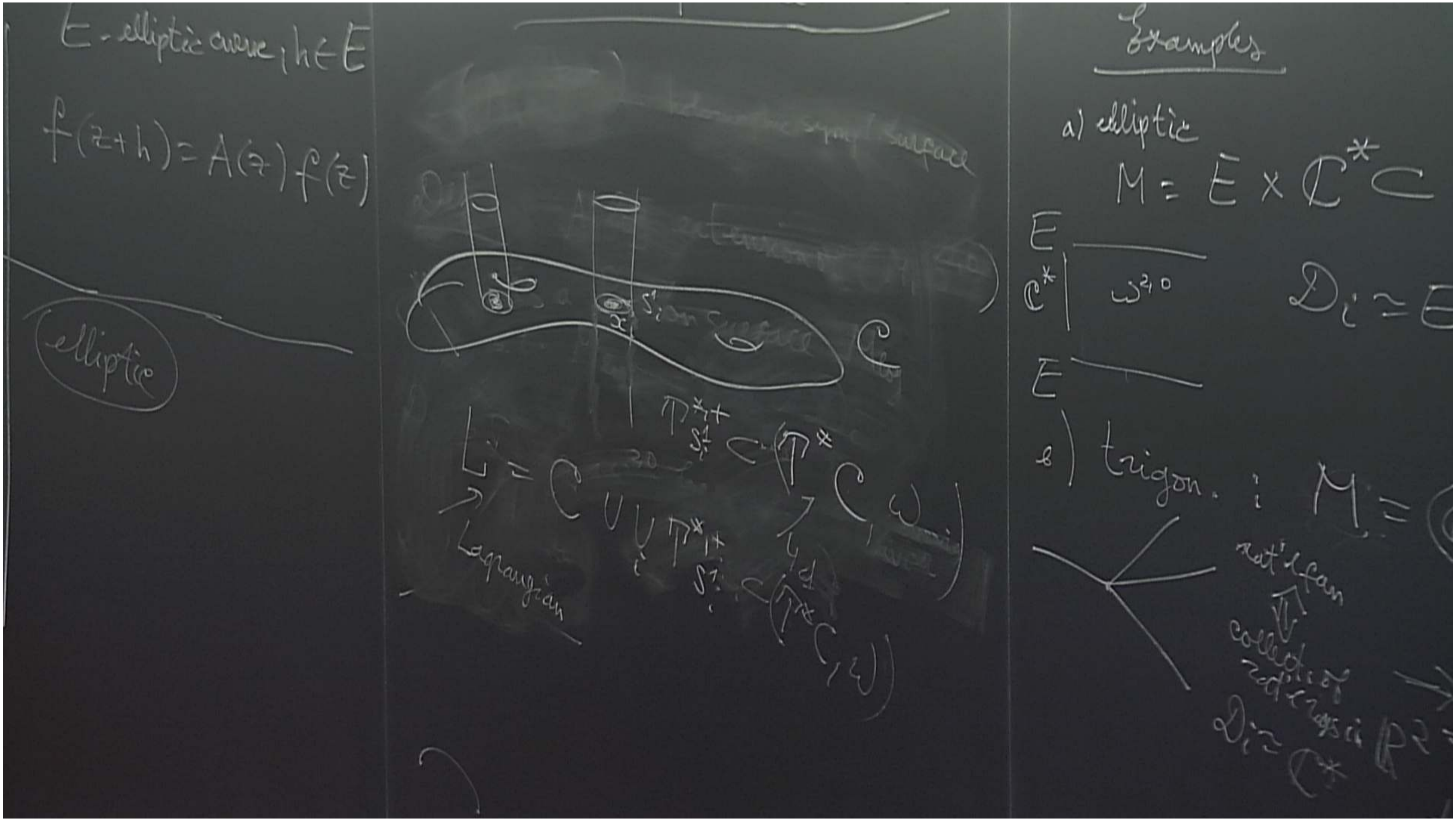

q-difference
f(q^a x) = x^k C f(x)
singular terms
Claim. ∃ finite sequence of blow ups of P*C
s.t. {ω^{2,0}_{P*C} - log-divisor}
1:1
C = P^1
RH: Conn
(M, ω^{2,0}), ω = Re(ω^{2,0})


(M, ω^{2,0}), ω = Re(ω^{2,0}), B = Im(ω^{2,0}) + β
F(M, ω + iB)
H^2(M, ℂ)
(partially wrapped) Fukaya category
L - real Lagrangian
log-divisor



q-difference
f(q^a x) = x^k C f(x)
Di ≃ ℂ
∃ finite sequence of blow-ups of ℙ*C
s.t. {ω^{2,0}_{ℙ*C} - log-divisor}
1:1
C = ℙ^1
RH: Conn
LocSys
(M, ω^{2,0}), ω = Re(ω^{2,0}), B = Im(ω^{2,0}) + β
F(M, ω + iB)
H^2(M, ℂ)
(partially wrapped) Fukaya category
L - real Lagrangian
Di


(M, ω^{2,0}), ω = Re(ω^{2,0}), B = Im(ω^{2,0}) + β
H^2(M, ℂ)
F(M, ω + iB)
(partially wrapped) Fukaya category
L - real Lagrangian
D_2
ω_s = Re(ω^{2,0}/s + s ω̄^{2,0}), B_s = Im(. . .)
s∈ℂ* F_s := F(M, ω_s + iB_s)


E - elliptic curve ; h ∈ E
f(z+h) = A(z) f(z)
elliptic
Examples
a) elliptic
M = E × C* ⊂
E
C*
ω ≠ 0
D_i ≅ E
E
b) trigon. : M =
Lagrangian
rat'l fan
collection of rat'l rays in R²
D_i ≅ C*
C

$\exists$ finite sequence of blow-ups of $\overline{T^* C}$

s.t. $\{\omega^{2,0}_{T^*C}$ - log-divisors$\}$

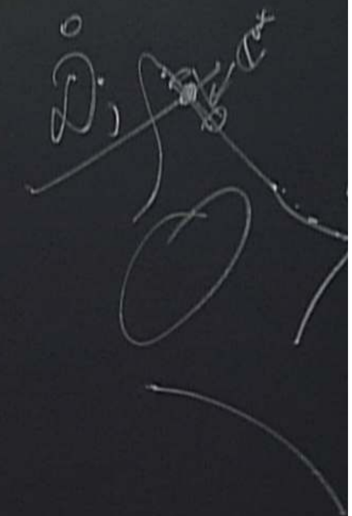
$\Big\updownarrow$ 1:1

(c_i, α)

$C = \mathbb{P}^1$

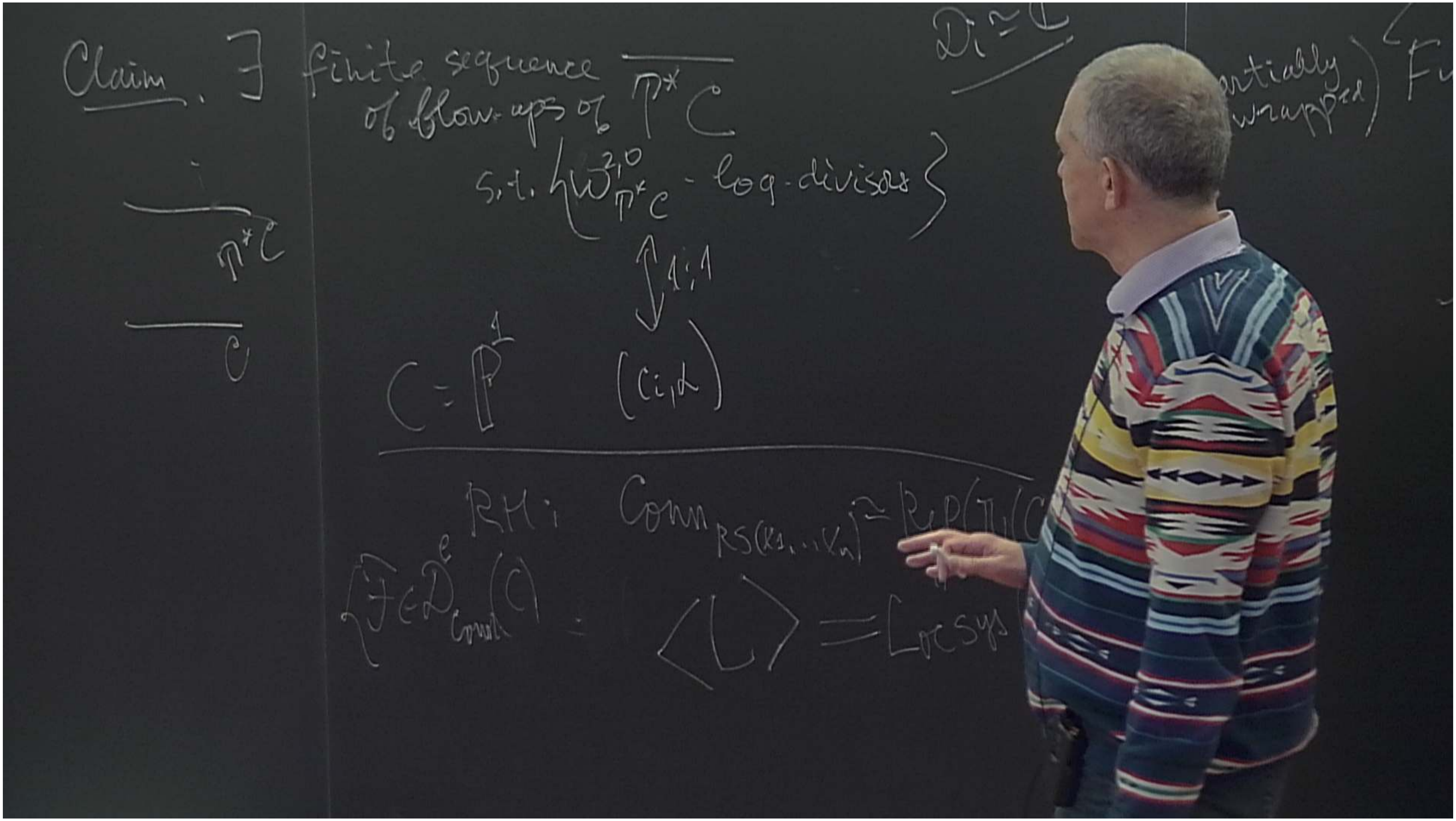
$D_i \simeq \mathbb{C}$

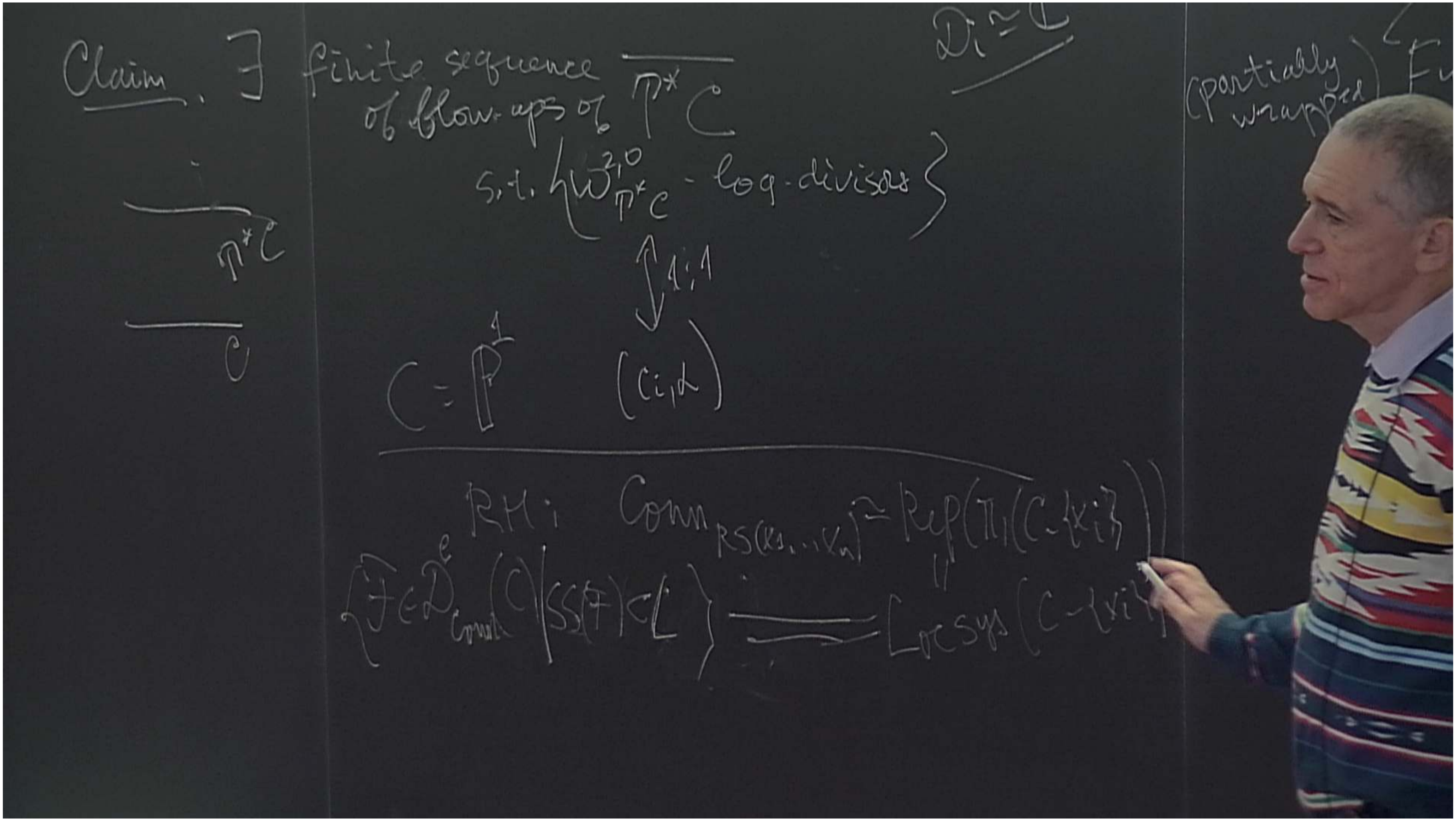
(partially wrapped) Fukaya category

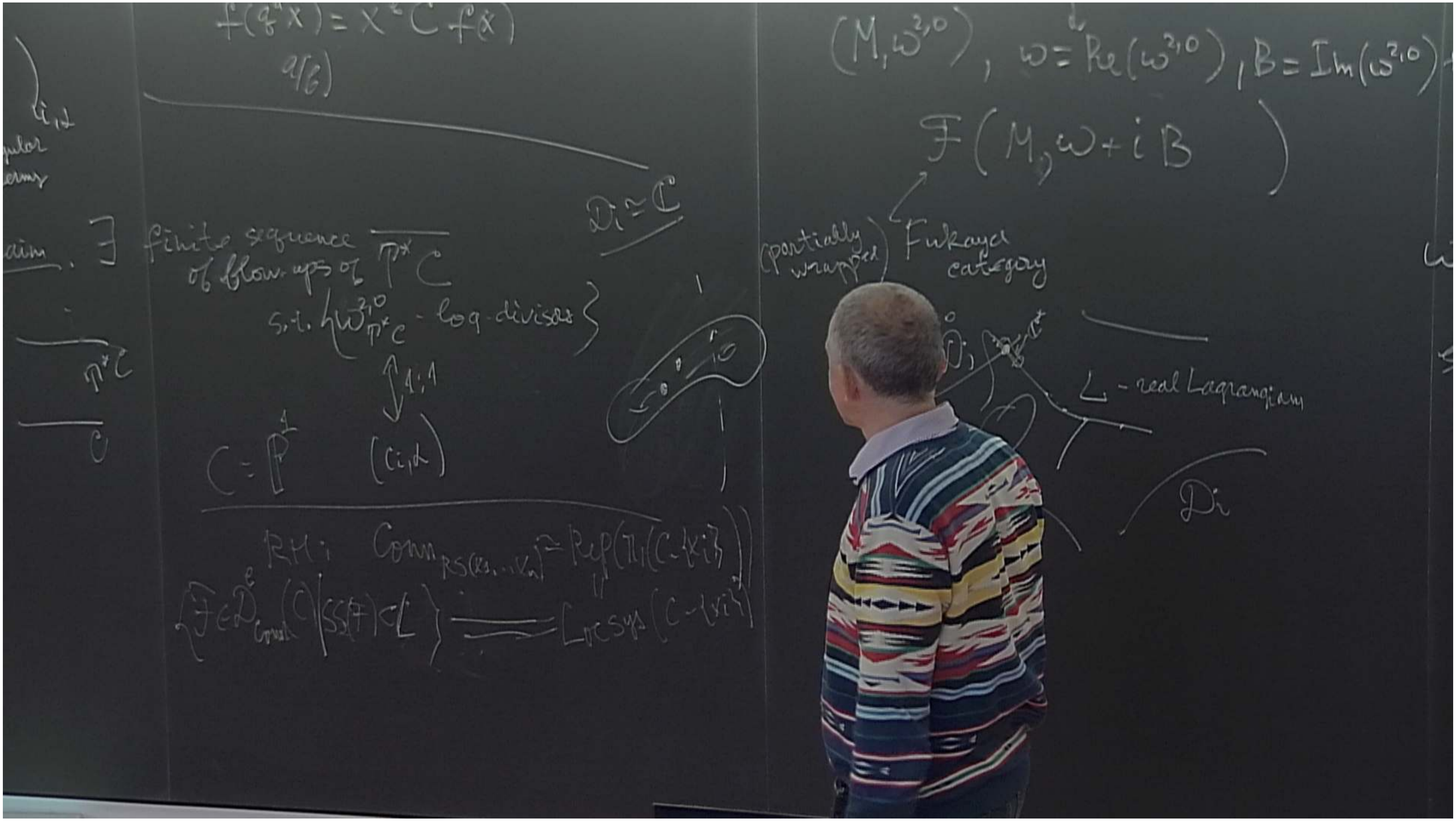


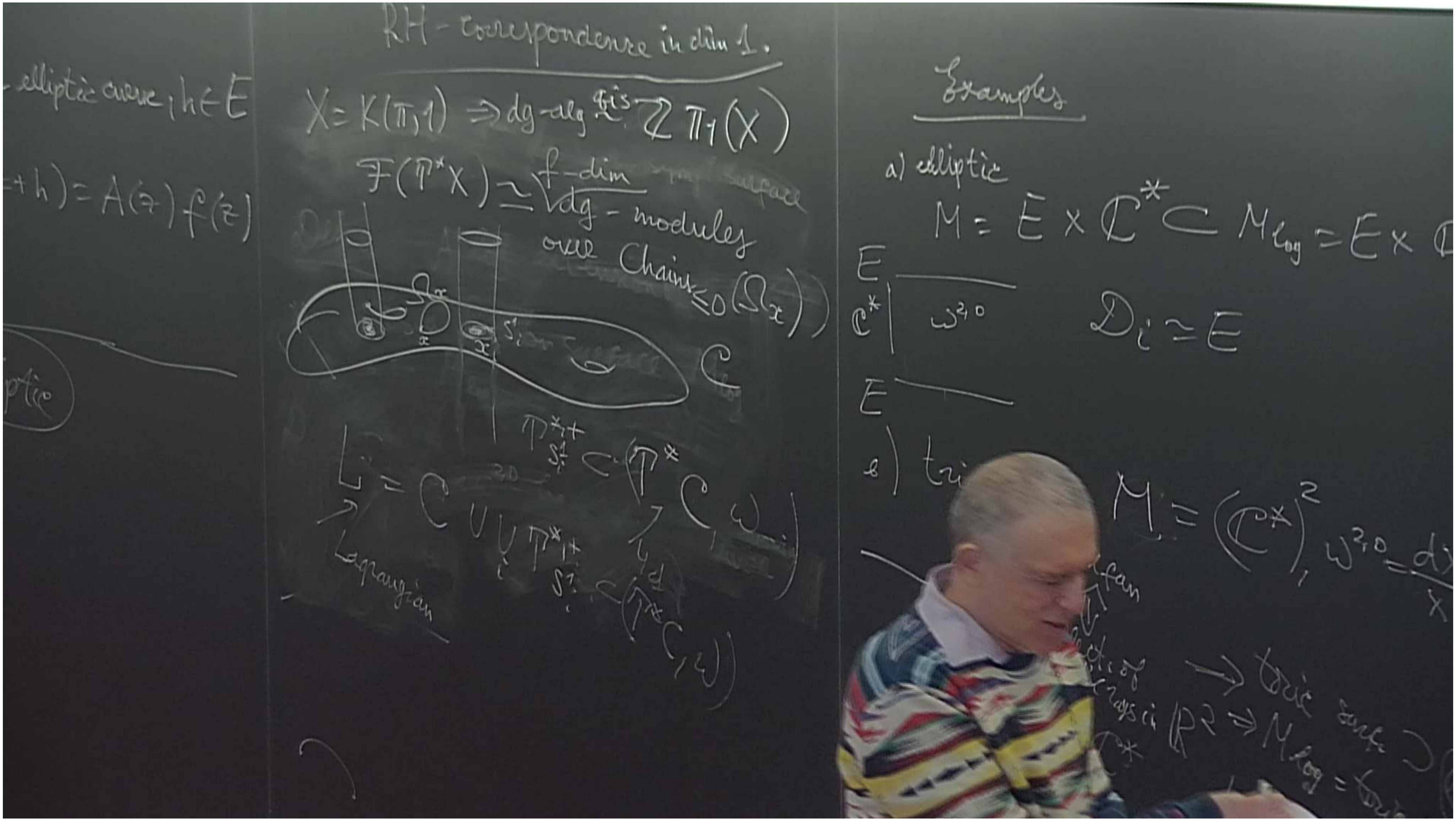
RH: $\mathrm{Conn}_{RS(x_1, \ldots, x_n)} \simeq \mathrm{Rep}(\pi_1(C - \{x_i\}))$

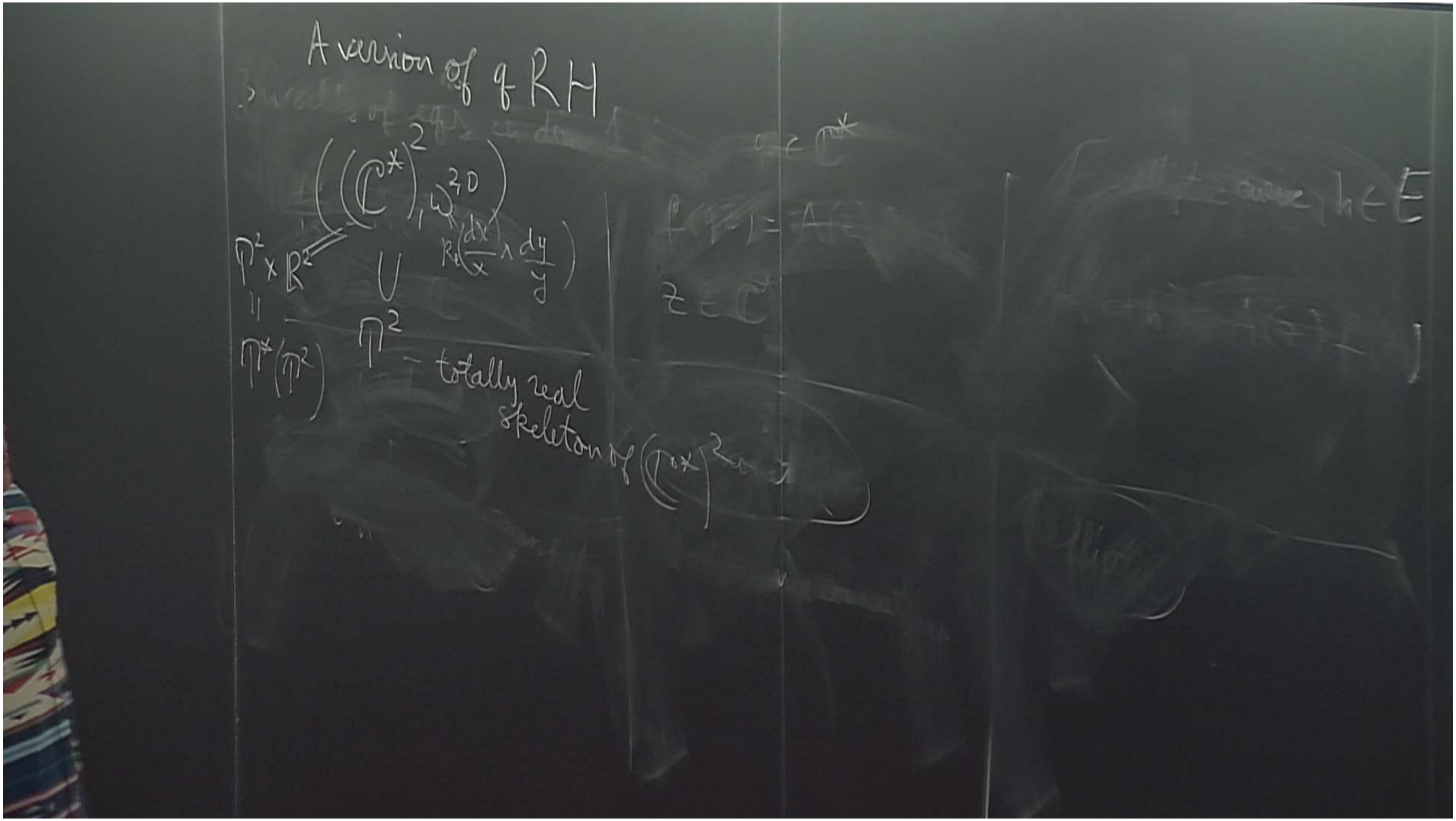
$\langle L \rangle = \mathrm{LocSys}(C - \{x_i\})$

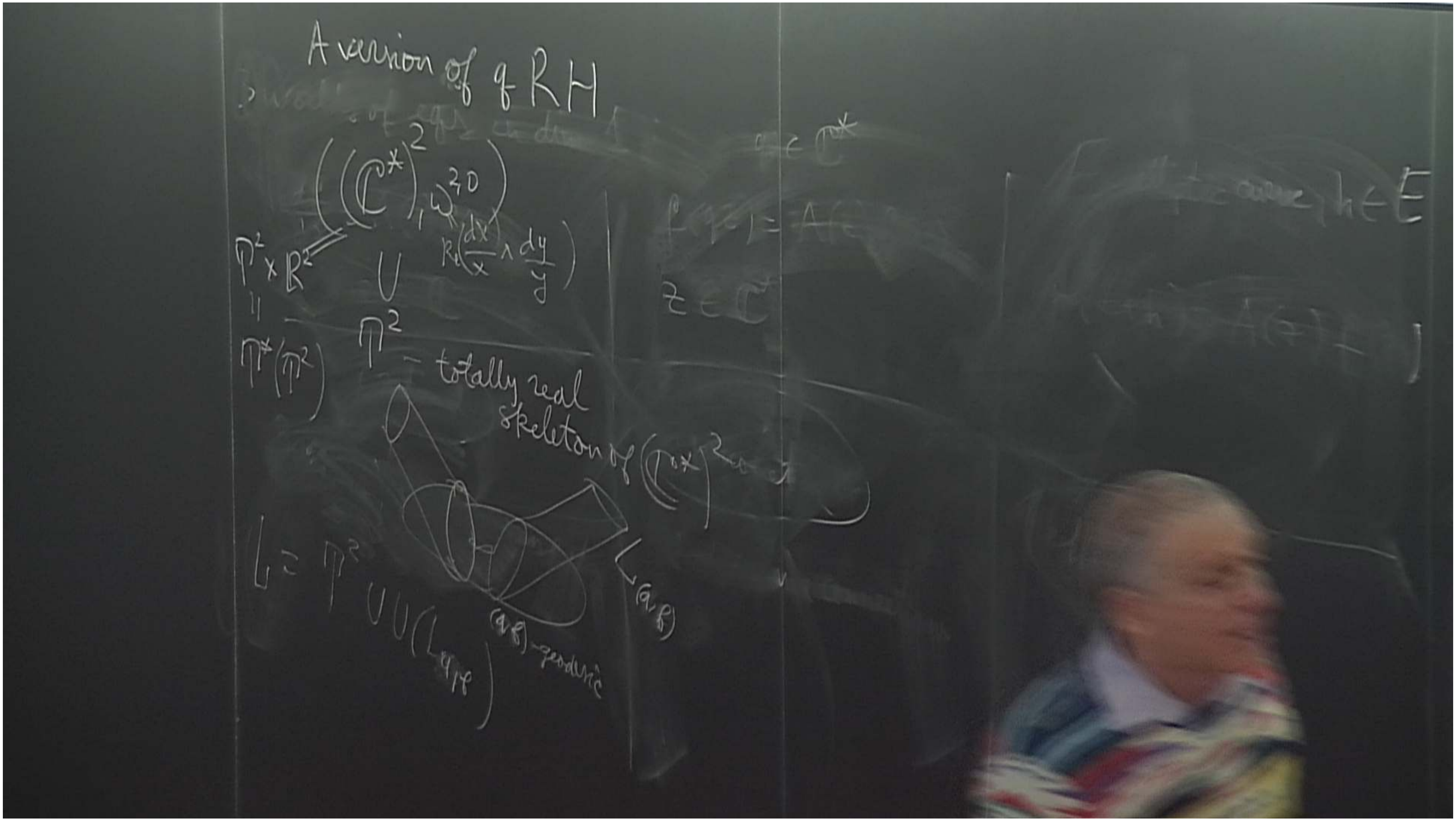

Claim. ∃ finite sequence
of blow-ups of
s.t.
log-divisors
1:1
RH:
Conn
LocSys
partially
wrapped

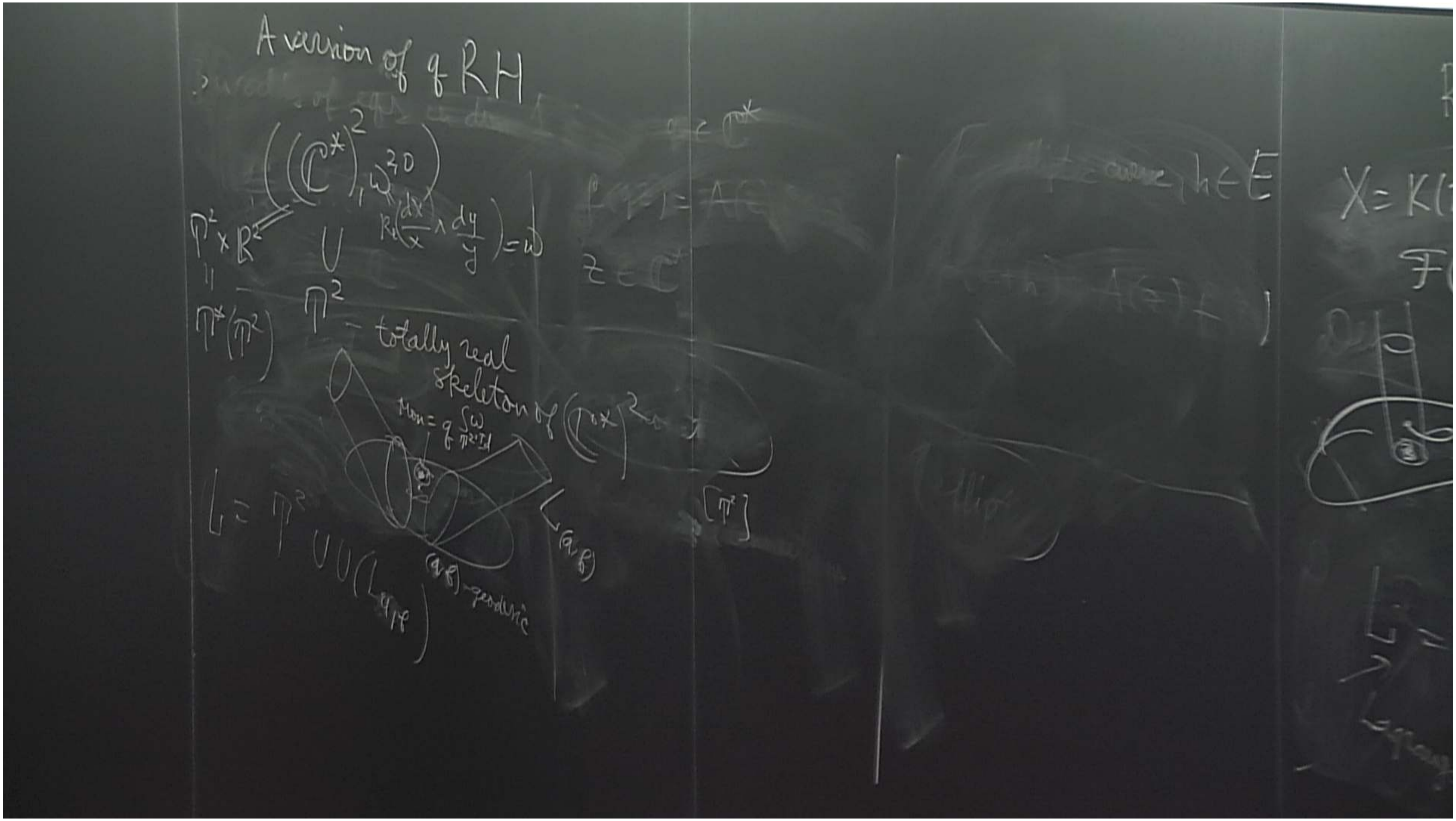

Claim. ∃ finite sequence of blow-ups of π*C
s.t. {ω^{2,0}_{π*C} - log-divisors}
1:1
C = P^1
RH: Conn_{RS(x_1,...,x_n)} ≅ Rep(π_1(C - {x_i}))
LocSys(C - {x_i})
(partially wrapped)

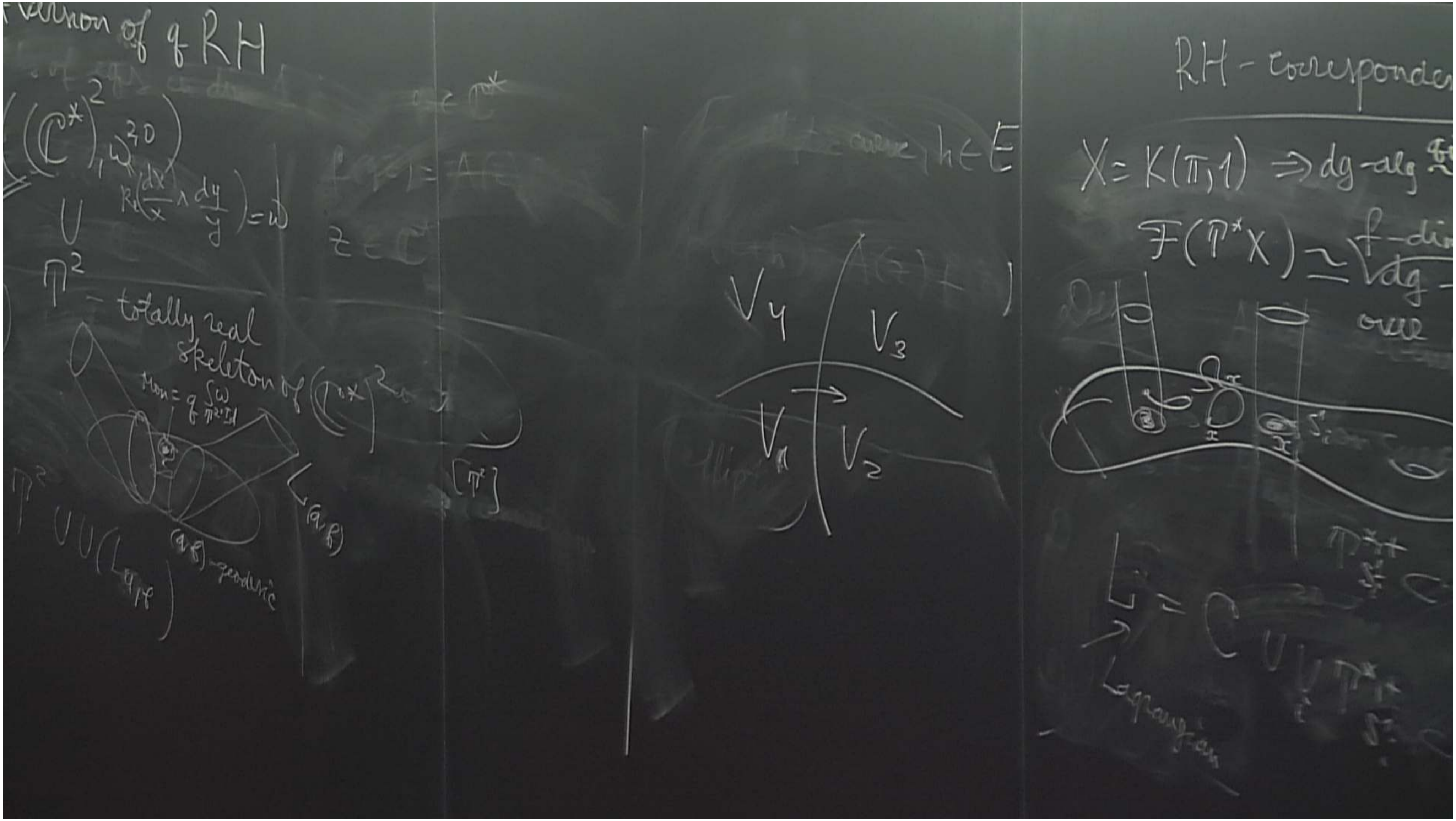

finite sequence of blow-ups of $\mathbb{P}^*C$
log-divisor
Fukaya category
(partially wrapped)
L - real Lagrangian
RH:
$\mathcal{F}(M, \omega + iB)$

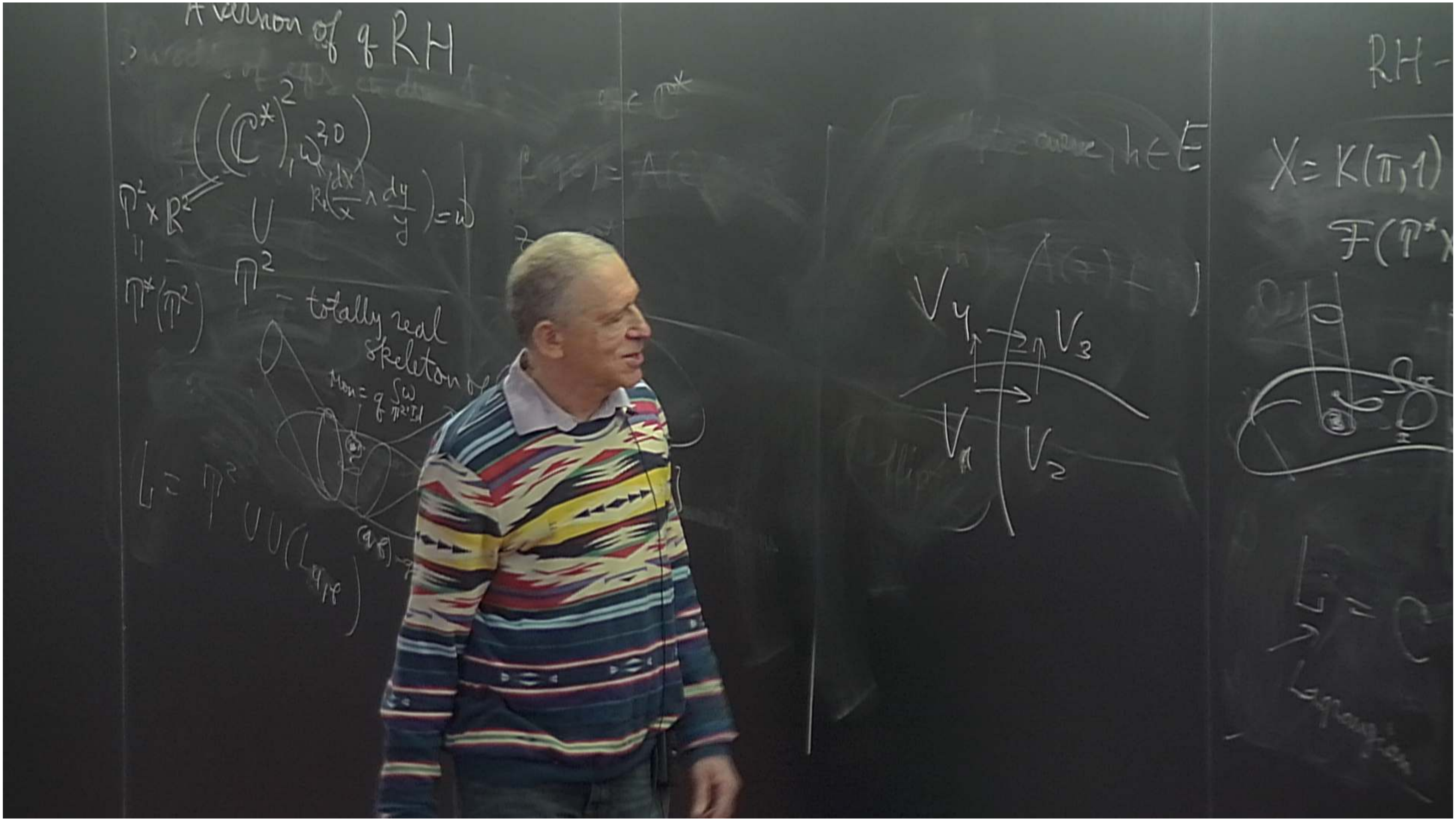

RH - correspondence in dim 1.
Examples
a) elliptic
dg - modules
over Chains
Lagrangian
fan
toric surf.

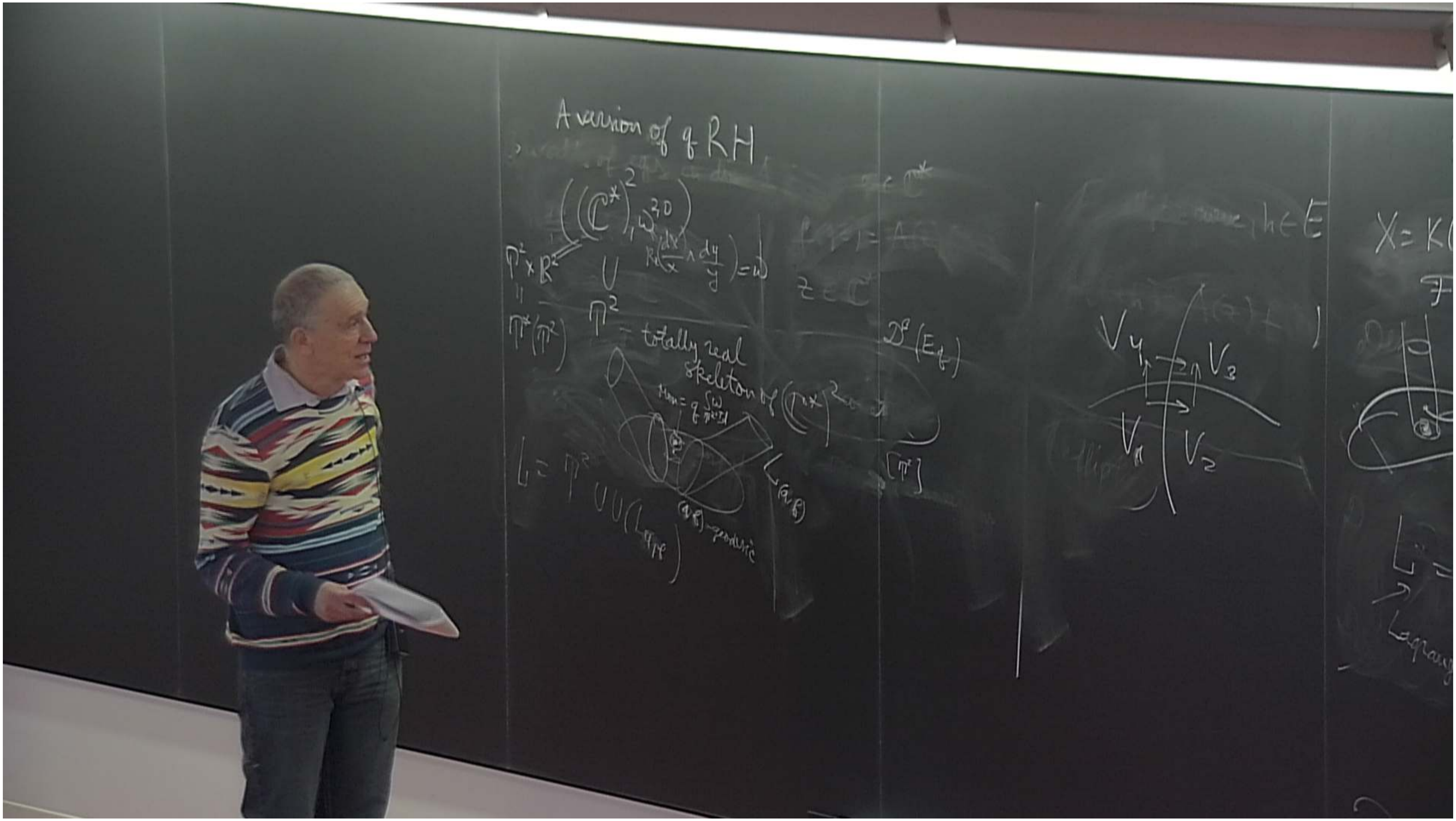

A version of q RH
((ℂ*)², ω)
T² × ℝ²
T*(T²)
T² – totally real
skeleton of (ℂ*)²
q ∈ ℂ*
z ∈ ℂ*

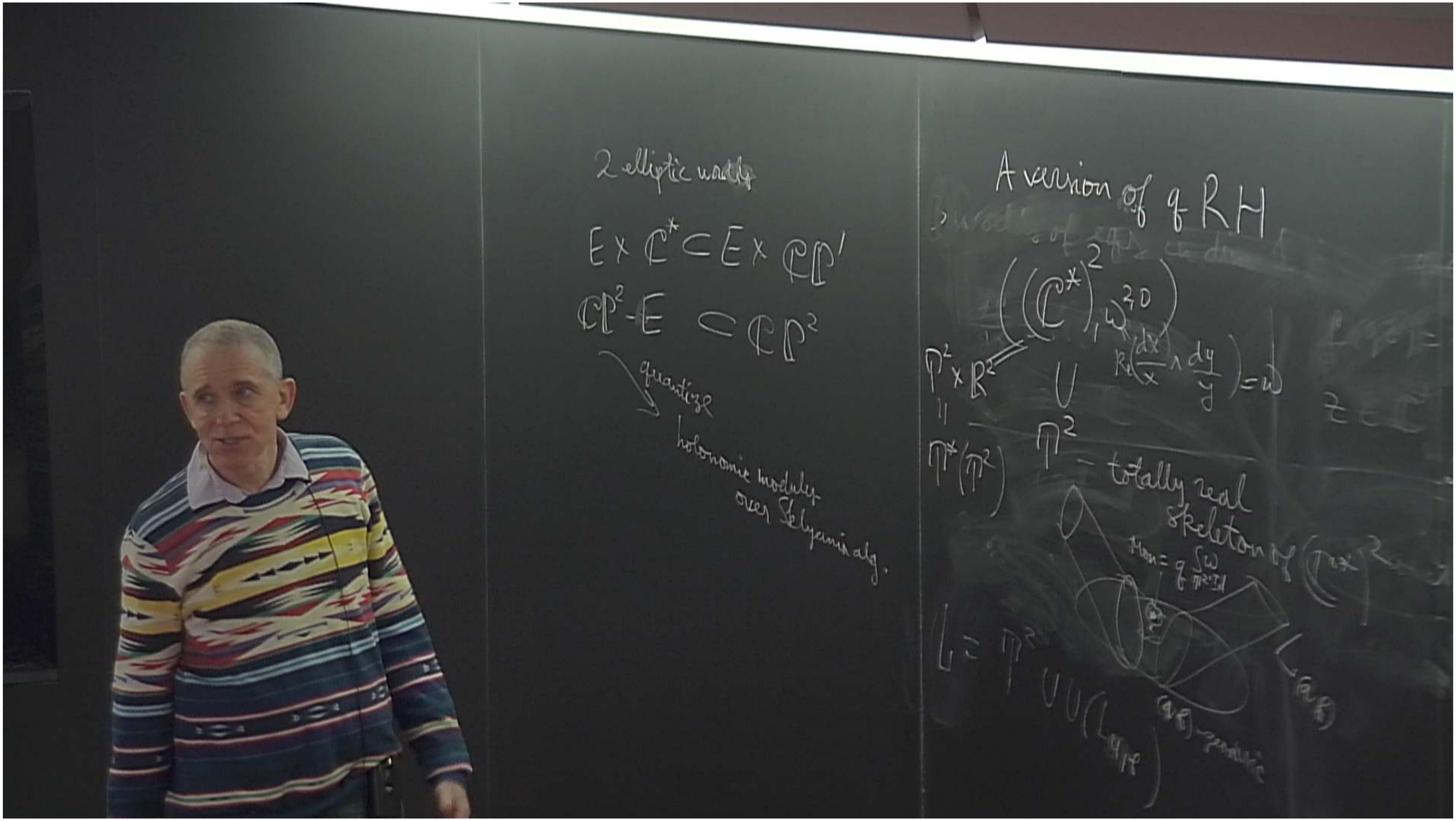

A version of q RH
totally real
skeleton of
(q,p)-geodesic

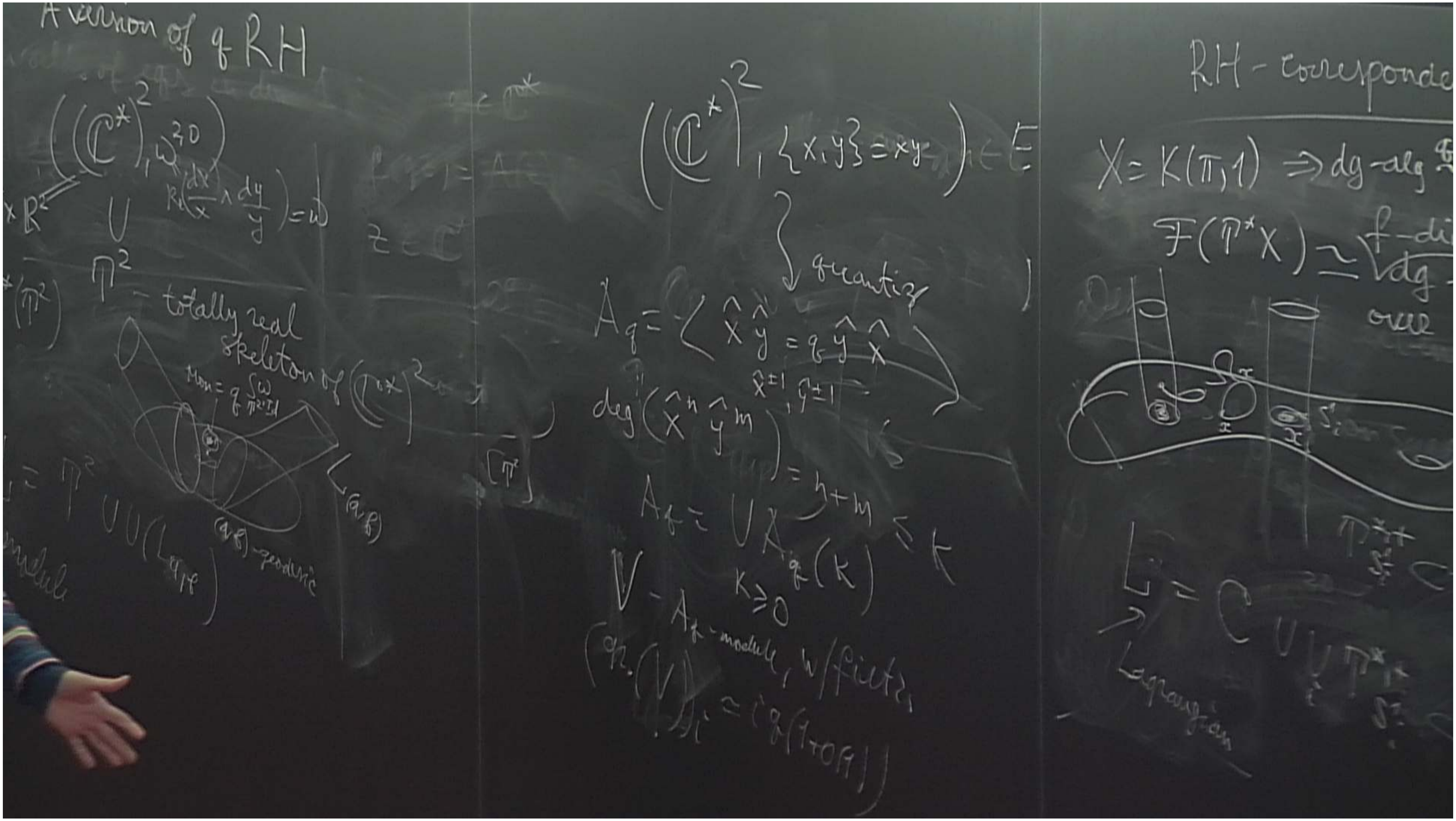

A version of q RH
totally real
skeleton of
(q,p)-geodesic

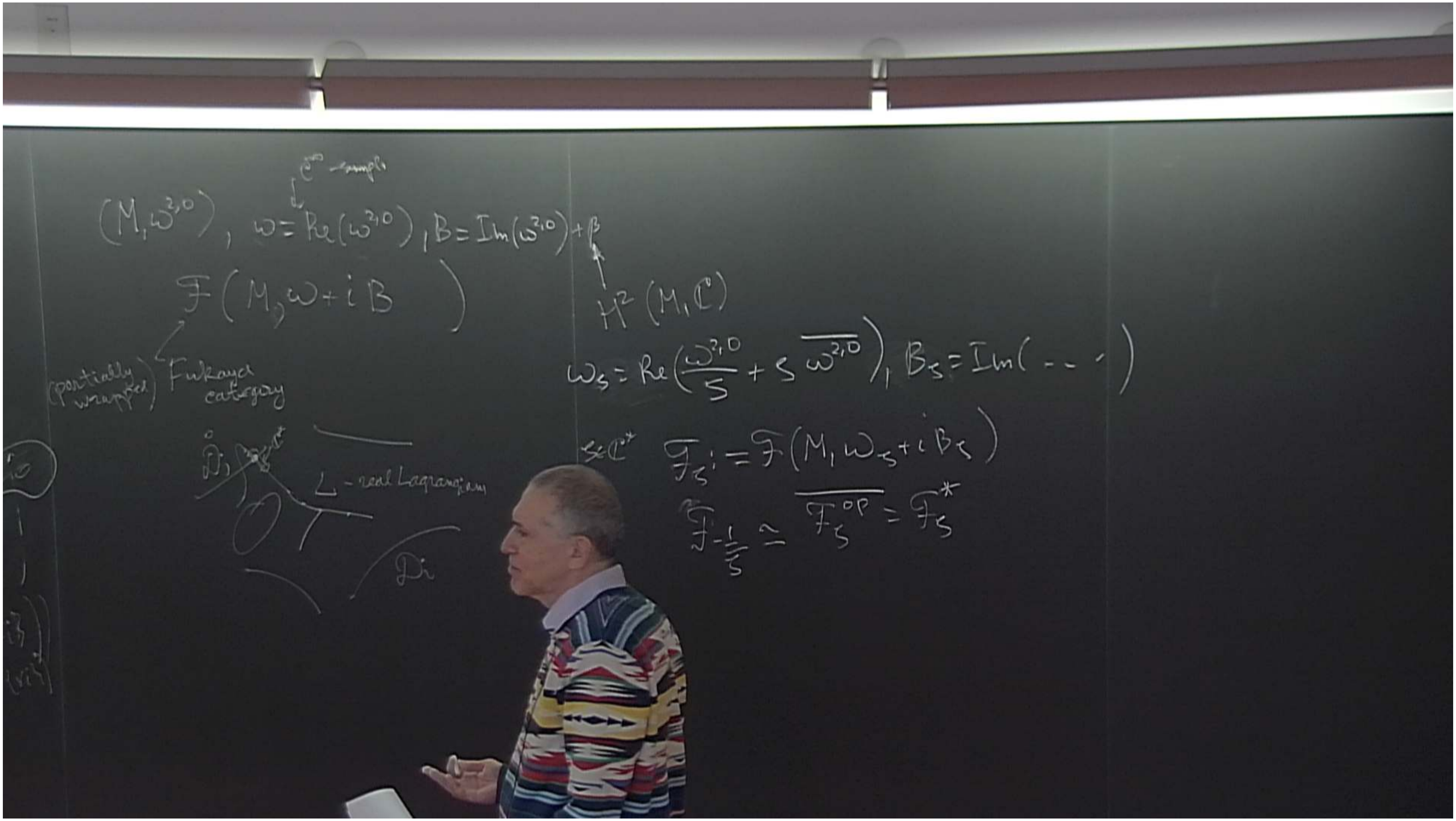

of q RH
totally real
skeleton of
(q,ρ)-geodesic
V_4
V_3
V_1
V_2
RH - correspondence
X = K(π,1) ⇒ dg-alg
over
Lagrangian


A version of q RH
totally real
skeleton
RH
X = K(π, 1)
Lagrangian


A version of q RH
totally real
skeleton of


E × C* ⊂ E × CP'
CP² - E ⊂ CP²
quantize
holonomic modules
over Sklyanin alg.
A version of q RH
totally real
skeleton of


A version of q RH
totally real skeleton of
(q,f)-geodesic
quantiz
RH - corresponde
Lagrangian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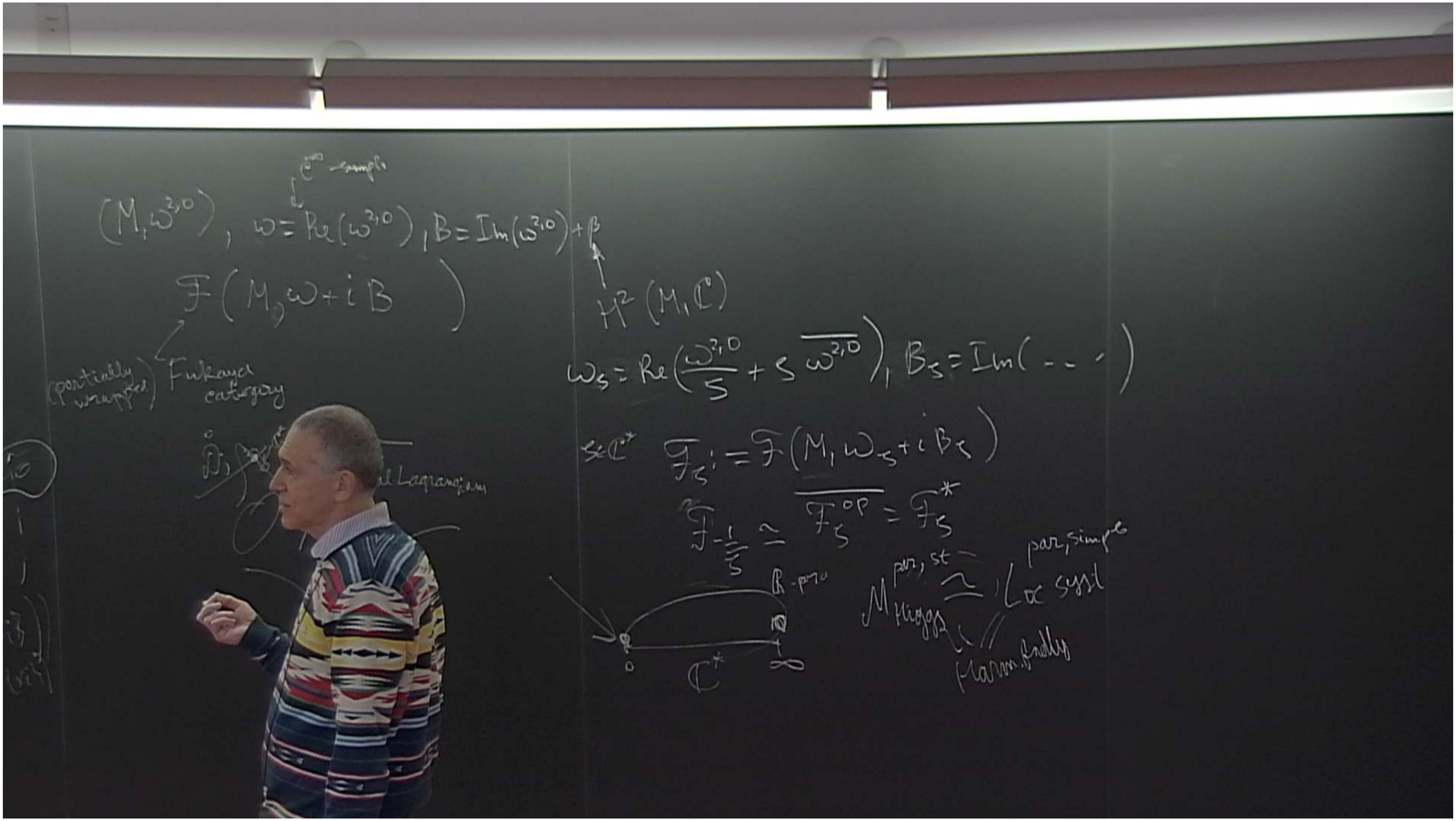
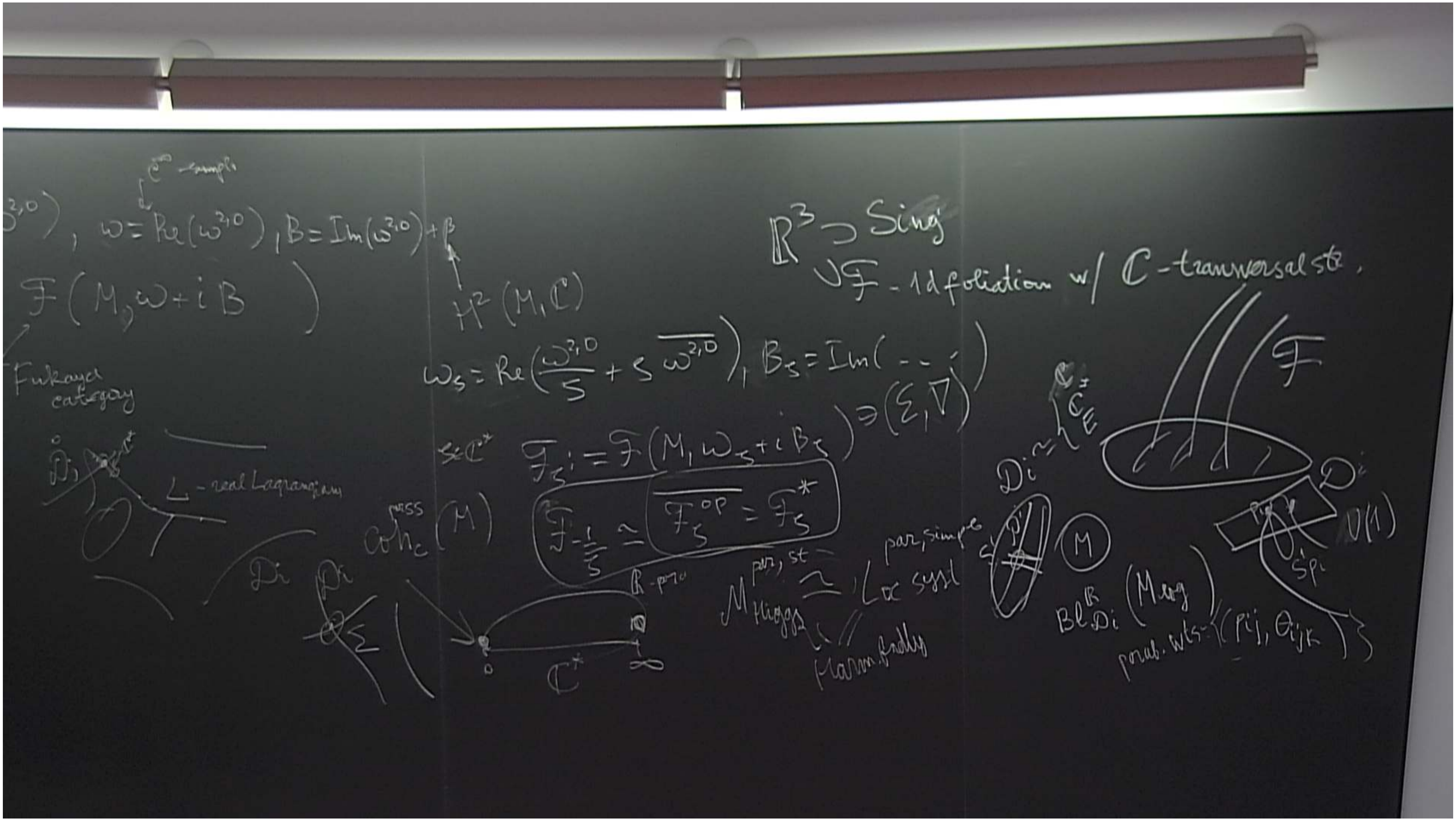
H^2(M, ℂ)
F(M, ω + iB)
(partially wrapped) Fukaya category
L - real Lagrangian
ω_s = Re(ω^{2,0}/s + s ω̄^{2,0}), B_s = Im(...)
s ∈ ℂ^*
F_s := F(M, ω_s + iB_s)
F_{-1/s̄} ≃ F_s^{op} = F_s^*


(M, ω^{2,0}), ω = Re(ω^{2,0}), B = Im(ω^{2,0}) + β
H^2(M, ℂ)
F(M, ω + iB)
(partially wrapped) Fukaya category
ω_ζ = Re(ω^{2,0}/ζ + ζ ω^{2,0}), B_ζ = Im(. . .)
ζ∈ℂ* F_ζ := F(M, ω_ζ + iB_ζ)
F_ζ^op = F_ζ^*
Lagrangian
ℝ-pro
ℂ*
0
∞
par, st
M_Higgs
par, simple
Loc syst
Harm. bndls


Fukaya category
L - real Lagrangian
1d foliation w/ C-transversal str.