Title: Modules over factorization spaces, and moduli spaces of parabolic G-bundles.

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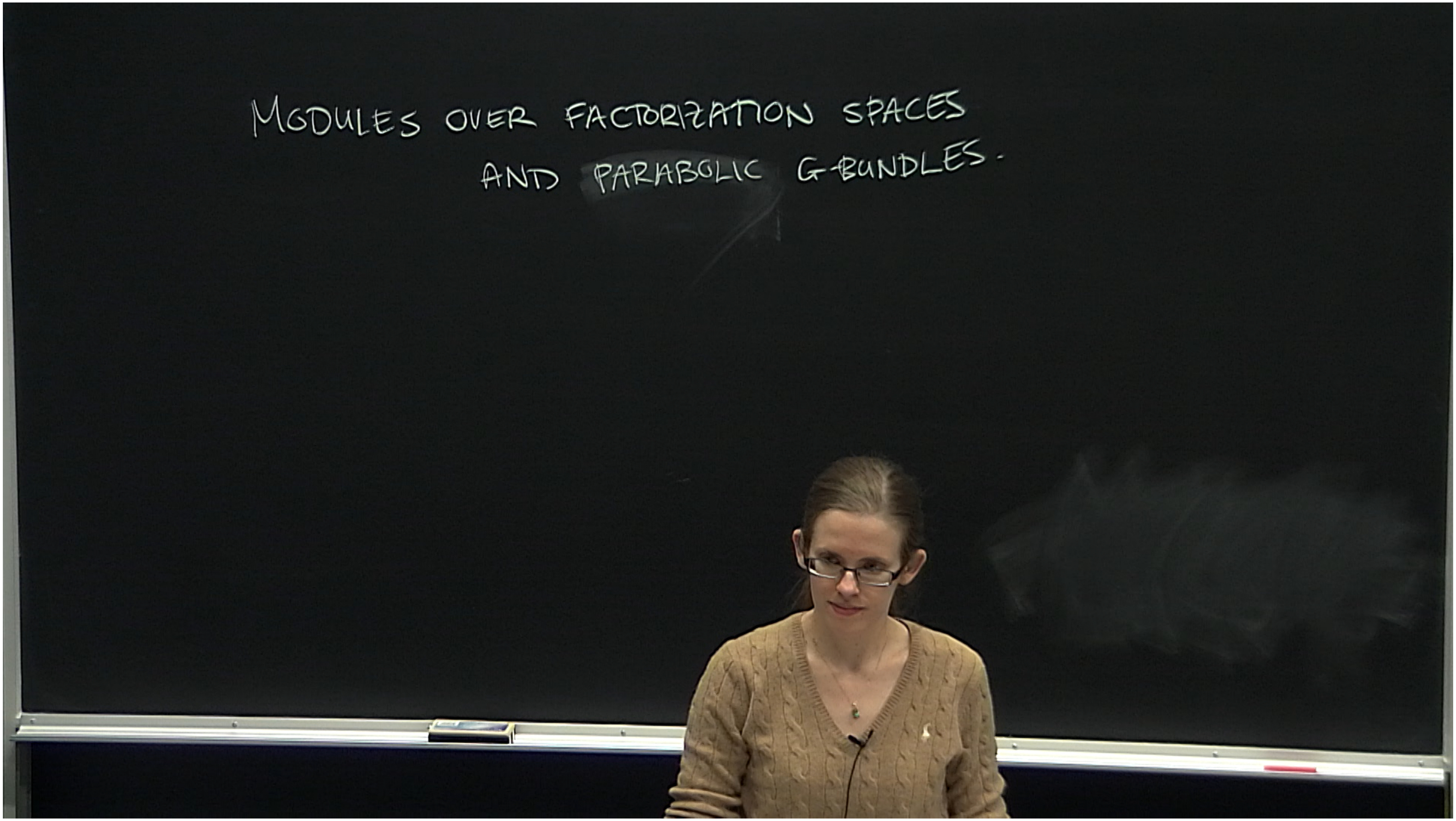
Abstract: <p>Factorization spaces (introduced by Beilinson and Drinfeld as "factorization monoids") are non-linear analogues of factorization algebras. They can be constructed using algebro-geometric methods, and can be linearised to produce examples of factorization algebras, whose properties can be studied using the geometry of the underlying spaces. In this talk, we will recall the definition of a factorization space, and introduce the notion of a module over a factorization space, which is a non-linear analogue of a module over a factorization algebra. As an example and an application, we will introduce a moduli space of principal G-bundles with parabolic structures, and discuss how it can be linearised to recover modules of the factorization algebra associated to the affine Lie algebra corresponding to a reductive algebraic group G.</p>


MODULES OVER FACTORIZATION SPACES
AND PARABOLIC G-BUNDLES.

MODULES OVER FACTORIZATION SPACES AND PARABOLIC G-BUNDLES.

I. Motivation

II. Factorization spaces

III Modules

I.

<u>Notation</u>. G simple simply connected alg. group $/\mathbb{C}$.

$\mathrm{Lie}(G) = \mathfrak{g}$. $h^\vee$ - dual Coxeter number.

Affine Lie alg. $\hat{\mathfrak{g}}$

Fix $\kappa \in \mathbb{N}$, $\tilde{\mathcal{O}}_\kappa$ - finite-length integrable $\hat{\mathfrak{g}}$-reps of level $\ell = \kappa - h^\vee$

<u>Fact</u>. $\tilde{\mathcal{O}}_\kappa$ semi-simple with simples $\mathbb{L}^\kappa_\lambda$, indexed by $\lambda \in P^+_\kappa$ $\leftarrow$ fundamental alcove

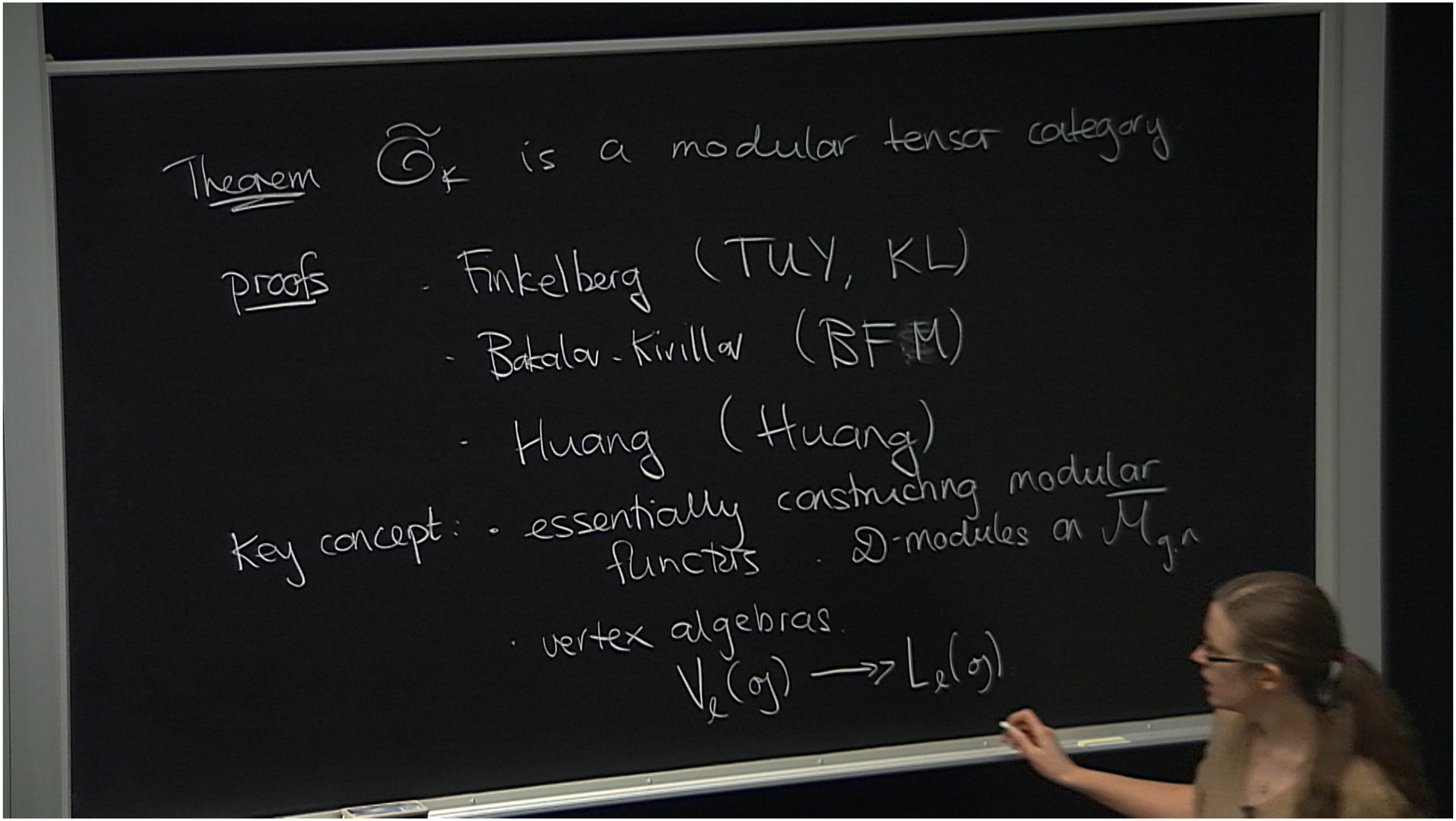
($\mathbb{L}^\kappa_\lambda$ = integrable quotient of the Weyl module $\mathbb{V}^\kappa_\lambda \twoheadrightarrow \mathbb{L}^\kappa_\lambda$

Theorem $\widetilde{\mathcal{O}}_\kappa$ is a modular tensor category.

proofs

- Finkelberg (TUY, KL)
- Bakalov-Kirillov (BFM)
- Huang (Huang)

Key concept: essentially constructing modular functors · $\mathcal{D}$-modules on $\overline{\mathcal{M}}_{g,n}$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· Bakalov-Kirillov (BFM)
· Huang (Huang)
Key concept: · essentially constructing modular functors
· $\mathcal{D}$-modules on $\overline{\mathcal{M}}_{g,n}$
· vertex algebras
$V_\ell(\mathfrak{g}) \twoheadrightarrow L_\ell(\mathfrak{g})$

Goal: Rewrite everything in terms of factorization algebras. / spaces

$$\mathcal{U}_\ell(\mathfrak{g}) \longrightarrow \mathcal{L}_\ell(\mathfrak{g}).$$

① see the geometric origins of the proof.

② generalise.

Goal: Rewrite everything in terms of factorization algebras. / spaces

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1. see the geometric origins of the proof.
2. generalise.

First step - understand representations of factorization algebras geometrically.

II.

Notation $\quad X$ smooth variety$/\mathbb{C}$

- fSet : cat. of finite non-empty sets + surjections
- X^{fSet} :

$$\begin{aligned} \mathrm{fSet}^{op} &\longrightarrow \mathrm{Sch} \\ I &\longmapsto X^I \\ (\alpha : I \twoheadrightarrow J) &\longmapsto \left(X^J \xhookrightarrow[\Delta(\alpha)]{} X^I \right) \end{aligned}$$

- open embedding:

$$\begin{array}{c} U(\alpha) := \left\{ x^I \in X^I \;\middle|\; \begin{array}{l} x^{i_1} \neq x^{i_2} \\ \text{unless } \alpha(i_1) = \alpha(i_2) \end{array} \right\} \\ \Big\downarrow j(\alpha) \\ X^I \end{array}$$

Def A factorization space over X:

(0) $\forall I \in \mathrm{fSet}$. $\mathcal{Y}_{X^I} \xrightarrow[f_I]{} X^I$ w. connection.

(1) $\forall I \overset{\alpha}{\twoheadrightarrow} J$:

$$\begin{tikzcd} \mathcal{Y}_{X^J} \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y}_{X^I} \arrow[d] \\ X^J \arrow[r, hook] & X^I \end{tikzcd}$$

Ran's condition

(2)

$$\begin{tikzcd} U(\alpha) \times_{X^I} \mathcal{Y}_{X^I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] \arrow[rr, bend left] & \mathcal{Y}_{X^I} \arrow[d] & U(\alpha) \times_{X^I} \prod \mathcal{Y}_{X^{I_j}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod \mathcal{Y}_{X^{I_j}} \arrow[d] \\ U(\alpha) \arrow[r] & X^I & U(\alpha) \arrow[r] & X^I = \prod X^{I_j} \end{tikzcd}$$

+ compatibilities

Example: Beilinson-Drinfeld Grassmannian (X = curve)

$$\mathrm{Gr}_{G,X^I} : S \longmapsto \{ (x^I, \mathcal{P}, \sigma) \mid \cdot\ x^I : S \to X^I,\ \cdot\ \mathcal{P} \to S \times X \text{ G-bundle},\ \cdot\ \sigma : S \times X \setminus (\textstyle\bigcup_{i \in I} \Gamma_{x_i}) \to \mathcal{P} \}$$

$S \in \mathrm{Sch}^{op}$

$$\begin{array}{ccc} & \mathrm{Gr}_{G,X^I} & \\ \swarrow & \downarrow f_I & \\ \mathrm{Bun}_G & X^I & \end{array}$$

Theorem (BD) this space factorizations.

sketch of proof of (2). say $(x_1, x_2) \in X^2 \setminus \Delta$

Example: Beilinson-Drinfeld Grassmannian (X = curve)

$$\mathrm{Gr}_{G,X^I} : \underset{\mathrm{Sch}^{op}}{S \in} \longmapsto \left\{ (x^I, \mathcal{P}, \sigma) \;\middle|\; \begin{array}{l} \cdot\, x^I : S \to X^I \\ \cdot\, \mathcal{P} \to S \times X \quad \mathcal{P}: G\text{ bundle} \\ \cdot\, \sigma : S\times X \setminus \left(\bigcup_{i\in I} \Gamma_{x_i}\right) \to \mathcal{P} \end{array} \right\}$$

$$\begin{tikzcd} & \mathrm{Gr}_{G,X^I} \arrow[dl] \arrow[d, "f_I"] \\ \mathrm{Bun}_G & X^I \end{tikzcd}$$

Theorem (BD) this space factorizations.

sketch of proof of (2). say $(x_1, x_2) \in X^2 \setminus \Delta = \mathcal{U}$

Fibre of $(\mathrm{Gr}_{G,X})^2|_{\mathcal{U}}$ over (x_1, x_2):

$$\{((\mathcal{P}_1, \sigma_1), (\mathcal{P}_2, \sigma_2))\} = \{(\hat{\mathcal{P}}_1, \hat{\sigma}_1), (\hat{\mathcal{P}}, \hat{\sigma}_2))\} = \{(\mathcal{P}_{12}, \sigma$$

Next step: linearise $Gr_{G,\mathrm{Ran}X}$ to produce fact. algebras.

Note ① $Gr_{G,\mathrm{Ran}X}$ is unital

$$\begin{tikzcd} & Gr_{G,\mathrm{Ran}X} \arrow[dl, "\pi"'] \arrow[d, shift right] & \\ \mathrm{Bun}_G & \mathrm{Ran}X = \mathrm{colim}\, X^I \arrow[u, bend right, "u"'] & \end{tikzcd}$$

$$(x^I, \mathrm{Triv}, \tau) \longmapsfrom (x^I)$$

② Theorem (Laszlo. Sorger)
$\exists$ a line bundle $\mathcal{L}$ on Bun_G s.t. $\mathrm{Pic}(Gr_{G,\mathrm{Ran}X}) = \mathbb{Z}\pi^*\mathcal{L}$

(in types A/C, $\mathcal{L}$ = det.
B/D, $\mathcal{L}$ = Pfaffian).

Almost theorem. (BD, FBZ, BZ-N, G, ... Cliff-Kremnitzer)

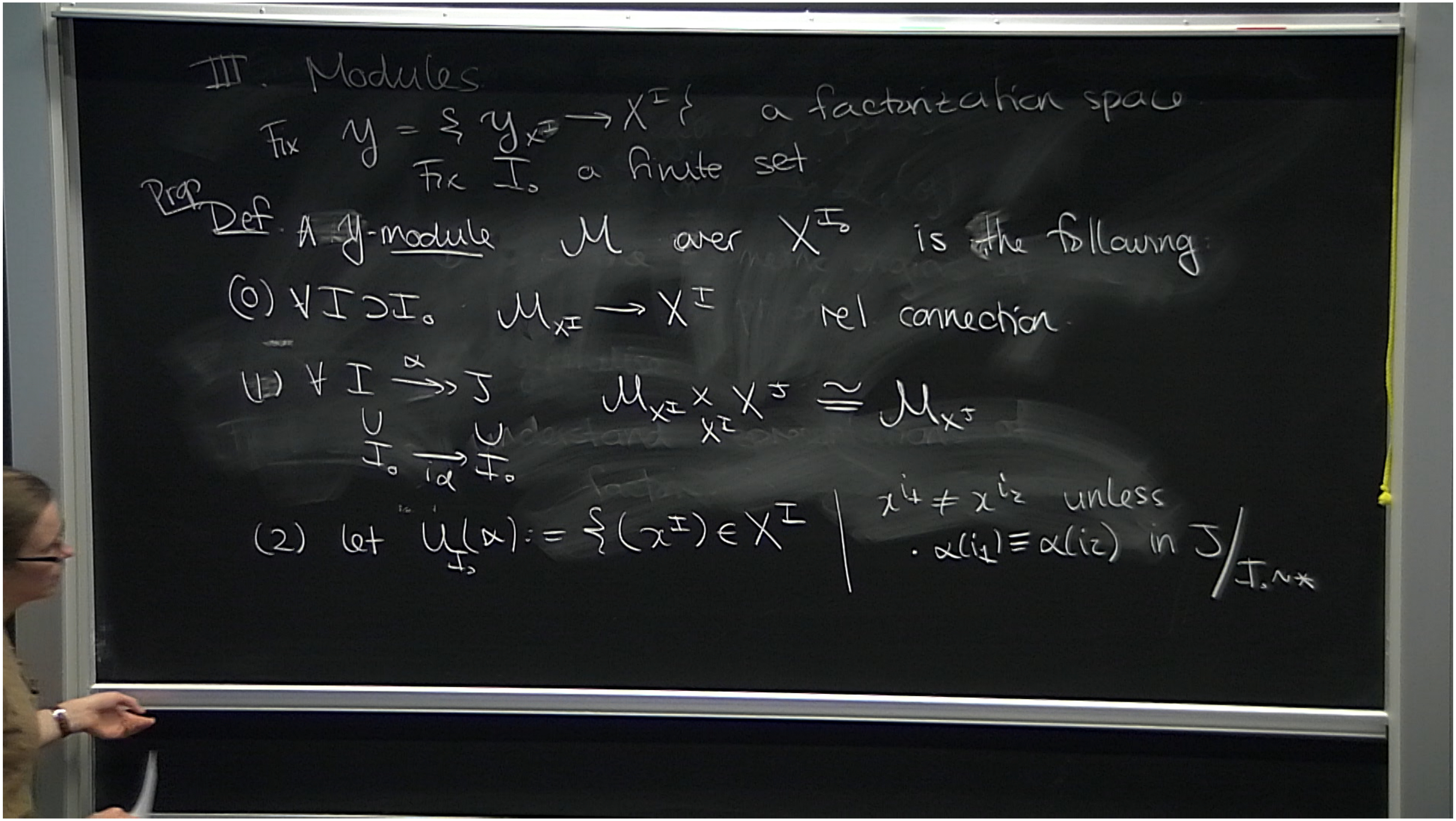
Fix a level ℓ, and consider $\Lambda^{\otimes \ell}$, $\Lambda^{\otimes \ell} \otimes u_!^{\otimes} \omega_{\mathrm{Ran} X}$

They give rise to factorization alg. on X.

$$\begin{array}{ccc} f_!^{G}(\Lambda^{\ell} \otimes u_!^{\otimes} \omega_{\mathrm{Ran} X}) & \longrightarrow & f_!^{G}(\Lambda^{\ell}) \\ \| \wr & & \| \wr \\ \mathcal{V}_{\mathfrak{g}}(\ell) & & \mathcal{L}_{\mathfrak{g}}(\ell) \end{array}$$

$\mathcal{L}\,\pi^* \mathcal{L}$

Λ



III. Modules
Fix $\mathcal{Y} = \{ \mathcal{Y}_{X^I} \to X^I \}$ a factorization space
Fix I_0 a finite set
Prop Def. A $\mathcal{Y}$-module $\mathcal{M}$ over X^{I_0} is the following:
(0) $\forall I \supset I_0$ $\mathcal{M}_{X^I} \to X^I$ rel connection.
(1) $\forall I \overset{\alpha}{\twoheadrightarrow} J$, $I \supset I_0 \xrightarrow{i_\alpha} I_0 \subset J$: $\mathcal{M}_{X^I} \times_{X^I} X^J \cong \mathcal{M}_{X^J}$
(2) let $\mathcal{U}_{I_0}(\alpha) := \{ (x^I) \in X^I \mid x^{i_1} \neq x^{i_2}$ unless $\cdot\ \alpha(i_1) \equiv \alpha(i_2)$ in $J/_{I_0 \sim *}$

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[rr, bend left, "\sim"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}_{X^I} \arrow[d] & * \arrow[r, hook] \arrow[d] & \left(\prod_{j\in J\setminus I_0} \mathcal{Y}_{J_j}\right) \times \mathcal{M}_{X^{I_*}} \arrow[d] \\ U_{I_0}(\alpha) \arrow[r] & X^I & U_{I_0}(X) \arrow[r, hook] & X^J = \left(\prod_{j\in J\setminus I_0} X^{J_j}\right) \times X^{I_*} \end{tikzcd}$$

$I_* = \alpha^{-1}(I_0)$

$\mathcal{Y} = \mathrm{Gr}_{G,\mathrm{Ran}X}$

<u>Main construction</u>

Fix $B \subset G$ Borel. $I_0 = \{1, \ldots, r\}$.

Define $\mathcal{M}^r_{G,X^I}$: $S \longmapsto \{(x^{J_1}, p^{I_0}, \mathcal{P}, \sigma, \tau^{I_0})\}$

$\cdot\ (p^{I_0}, x^{J_1}) : S \to X^I \quad (J_1 = I \setminus I_0)$

γ^{I_0} is a reduction of the structure group of $\mathcal{P}$ to $\mathcal{B}$ at $\{p^{I_0}\}$

Prop $\{\mathcal{M}^{\Gamma}_{G,X^I}\}$ is a module over $Gr_{G,\mathrm{Ran}X}$

Fact (LS) Given $\ell = \kappa - h^\vee$ and $\mu_1 \dots \mu_r \in P^+_\kappa$

$\exists$ a line bundle $\mathcal{L}(\ell,\underline{\mu})$ on $\mathcal{M}^{\Gamma}_{G,\mathrm{Ran}_I X}$

(Fibre of $\mathcal{M}^{\Gamma}_{G,X}$ at $p \in X$ is the affine flag variety $G(K)/I$

$\cdot\, \gamma^{I_0}$ is a reduction of the structure group of

$\mathcal{P}$ to B at $\{p^{I_0}\}$

Prop $\{\mathcal{M}^{\Gamma}_{G,X^I}\}$ is a module over $Gr_{G,\mathrm{Ran}X}$

Fact (LS) Given $\ell = \kappa - h^\vee$ and $\mu_1, \ldots, \mu_r \in P^+_\kappa$

$\exists$ a line bundle $\mathcal{N}(\ell, \underline{\mu})$ on $\mathcal{M}^{\Gamma}_{G,\mathrm{Ran}_I X}$

(Fibre of $\mathcal{M}^{\Gamma}_{G,X}$ at $p \in X$ is the affine flag variety $G(K)/I$

$G(K) \times_I \mathbb{C}(-\mu)$ on $G(K)/I$

Construction

$$\begin{tikzcd}
& \mathcal{M}^{\Gamma}_{G,\mathrm{Ran}_{I_0}X} \arrow[dl, "\pi^{\Gamma}"'] \arrow[dr, "f^{\Gamma}"] & \\
\mathrm{Bun}_G^{B,X^{I_0}} \arrow[dr, "p"'] & & \mathrm{Ran}_{I_0}X \arrow[dl, "q"] \\
& X^{I_0} &
\end{tikzcd}$$

Step 1 $\quad f^{\Gamma,0}_{!}(\Lambda(\ell\cdot\mu))$ is a factorization module for $\mathcal{L}_{\ell}(\mathfrak{g})$

Step 2 $\quad$ Claim the chiral homology of $f^{\Gamma,0}_{!}(\Lambda(\ell\cdot\mu))$ gives the conformal blocks of $\{\mathbb{L}^{k}_{\mu_1},\dots,\mathbb{L}^{k}_{\mu_r}\}$

Sketch. (LS) $f \in X^r$

$$\mathrm{Pic}(\mathrm{Bun}_G^{\mathcal{B},f}) \cong \mathbb{Z} \times \prod_{i=1}^{r} X(\mathcal{B})$$

$$\mathcal{L}(\ell,\underline{\mu})_f \longleftarrow \ell,\ \underline{\mu}$$

$$N(\ell,\underline{\mu}) = \pi^{r!}\mathcal{L}(\ell,\underline{\mu})$$

$$\int_X^{ch} f^{r,\otimes}_{!}(N(\ell,\underline{\mu})) = q^{\mathcal{D}}_{!}\left(f^{r,\otimes}_{!}(N(\ell,\underline{\mu}))\right)$$

$$\sim p^{\mathcal{D}}_{!}\left(\mathrm{Ind}^{\mathcal{D}}_{\mathcal{O}}\left(\underbrace{\pi^{r}_{!}\,\pi^{r!}}_{\cong\ \mathrm{id}}\mathcal{L}(\ell,\underline{\mu})\right)\right)$$

$$= p^{\mathcal{O}}_{!}(\mathcal{L}(\ell,\underline{\mu})) = \text{conformal blocks.}$$