Title: Beilinson-Bernstein localization via the wonderful compactification

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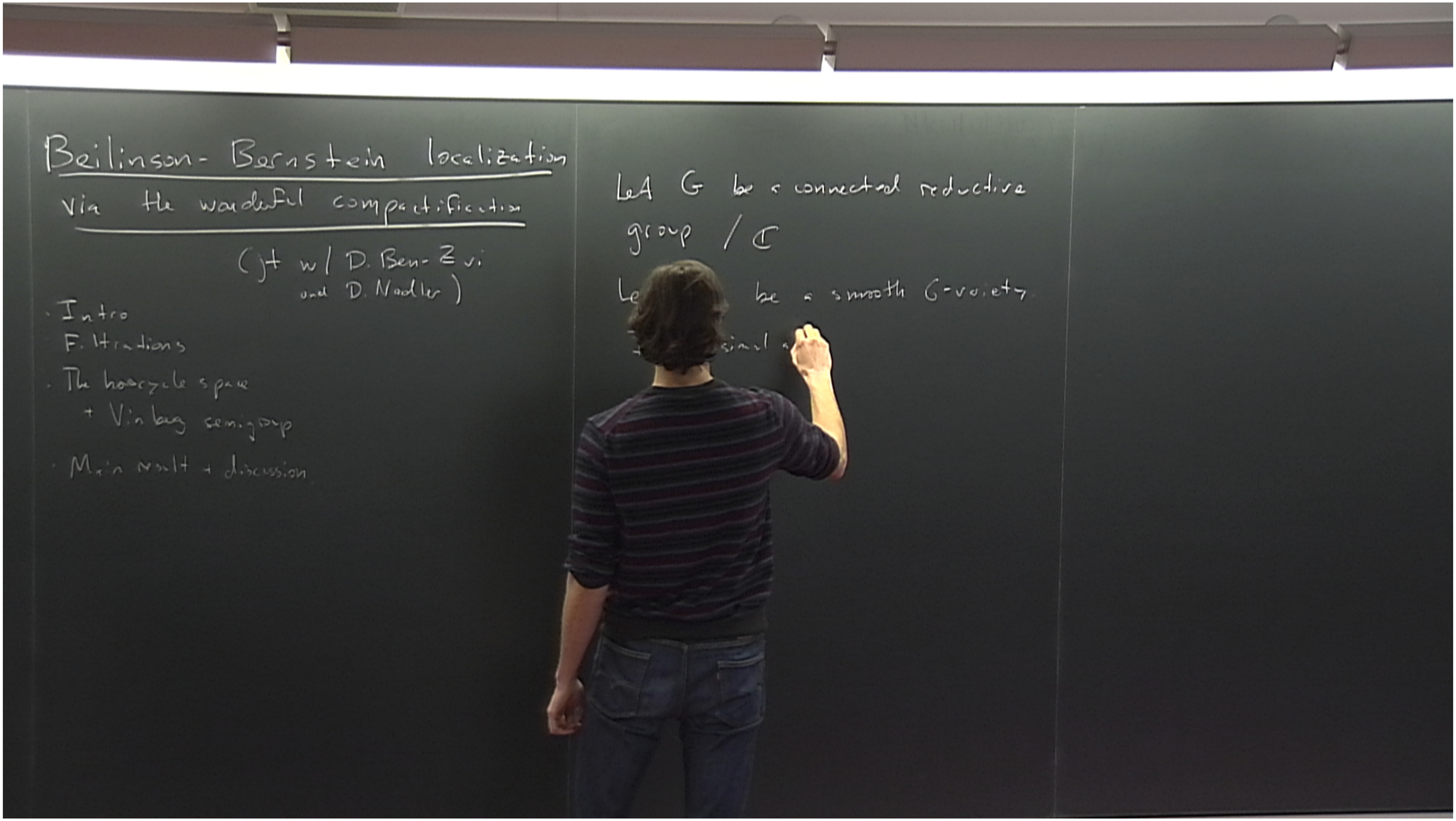
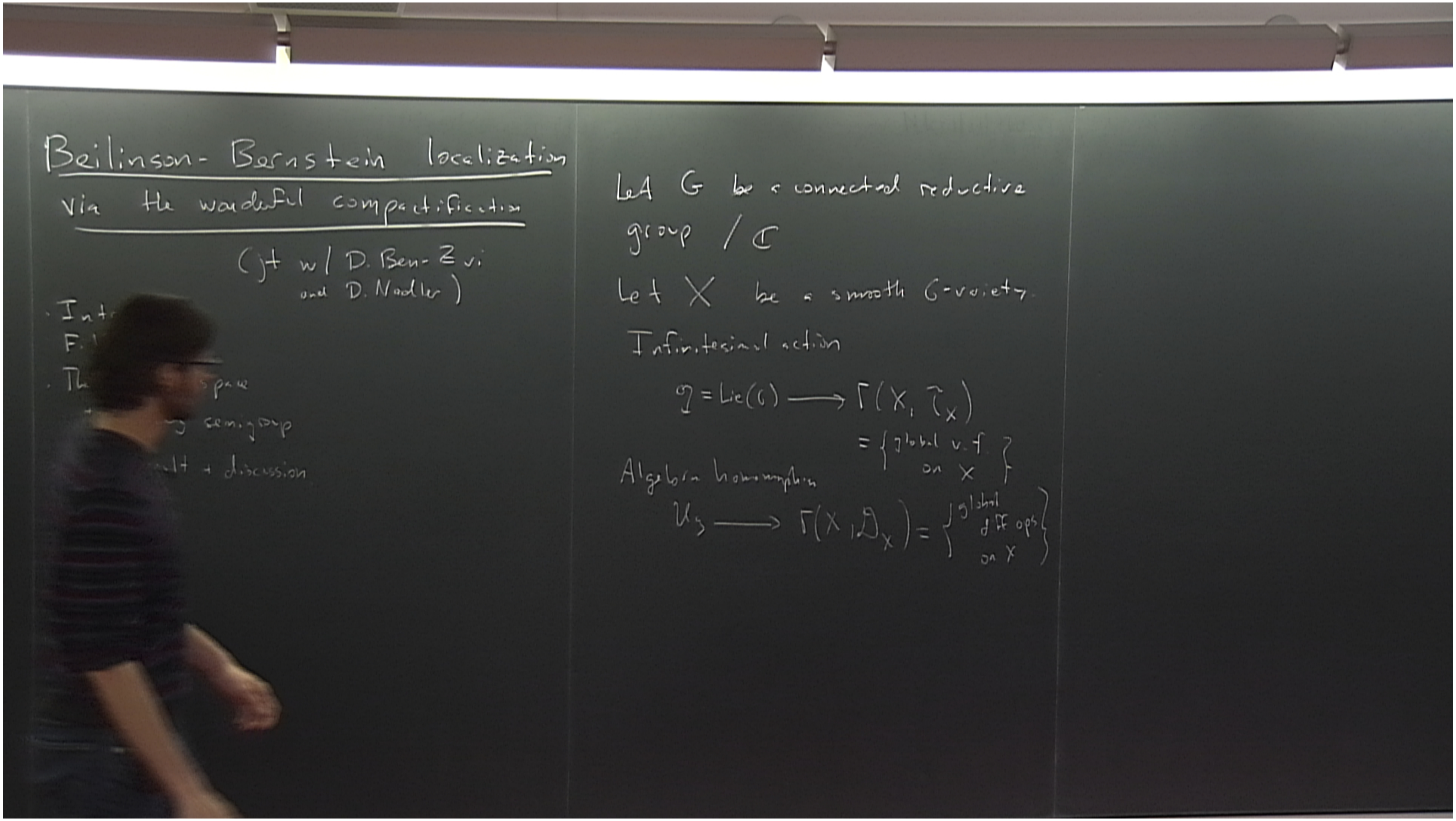
Abstract: <p>We explain how a doubled version of the Beilinson-Bernstein localization functor can be understood using the geometry of the wonderful compactification of a group. Specifically, bimodules for the Lie algebra give rise to monodromic D-modules on the horocycle space, and to filtered D-modules on the group that respect a certain matrix coefficients filtration. These two categories of D-modules are related via an associated graded construction in a way compatible with localization, Verdier specialization, and additional structures. This is joint work with David Ben-Zvi and David Nadler. </p>


Beilinson-Bernstein localization
via the wonderful compactification
(jt w/ D. Ben-Zvi
and D. Nadler)
· Intro
· Filtrations
· The horocycle space
+ Vinberg semigroup
· Main result + discussion
Let G be a connected reductive
group / ℂ
be a smooth G-variety


Beilinson-Bernstein localization
via the wonderful compactification
(jt w/ D. Ben-Zvi
and D. Nadler)
semigroup
+ discussion
Let G be a connected reductive group / ℂ
Let X be a smooth G-variety.
Infinitesimal action
𝔤 = Lie(G) ⟶ Γ(X, T_X)
= {global v.f. on X}
Algebra homomorphism
U𝔤 ⟶ Γ(X, D_X) = {global diff ops on X}

...lization

...ation

Let G be a connected reductive group $/ \mathbb{C}$

Let X be a smooth G-variety

Infinitesimal action

$$\mathfrak{g} = \mathrm{Lie}(G) \longrightarrow \Gamma(X, T_X) = \left\{ \begin{array}{c} \text{global v.f.} \\ \text{on } X \end{array} \right\}$$

Algebra homomorphism

$$U\mathfrak{g} \longrightarrow \Gamma(X, \mathcal{D}_X) = \left\{ \begin{array}{c} \text{global} \\ \text{diff ops} \\ \text{on } X \end{array} \right\}$$

Localization functor

$$\mathrm{Loc.}\ U\mathfrak{g}\text{-mod} \longrightarrow D(X) = \begin{array}{c} \text{category of} \\ D\text{-modules} \\ \text{on } X \end{array}$$

$$M \longmapsto \mathcal{D}_X \otimes_{U\mathfrak{g}} M$$

Beilinson-Bernstein localization

$$X = G/B$$

$$\begin{tikzcd}
U\mathfrak{g}\text{-mod} \arrow[r] & D(G/B) \\
U\mathfrak{g}\text{-mod}_0 \arrow[u, hook] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Let G be a connected reductive group / $\mathbb{C}$

Let X be a smooth G-variety.

infinitesimal action

$$\mathfrak{g} = \mathrm{Lie}(G) \longrightarrow \Gamma(X, T_X) = \left\{ \begin{matrix} \text{global v.f.} \\ \text{on } X \end{matrix} \right\}$$

algebra homomorphism

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Localization functor

$$\text{Loc.} \quad U_{\mathfrak{g}}\text{-mod} \longrightarrow D(X) = \begin{matrix} \text{category of} \\ D\text{-modules} \\ \text{on } X \end{matrix}$$

$$M \longmapsto \mathcal{D}_X \otimes_{U_{\mathfrak{g}}} M$$

Beilinson-Bernstein localization

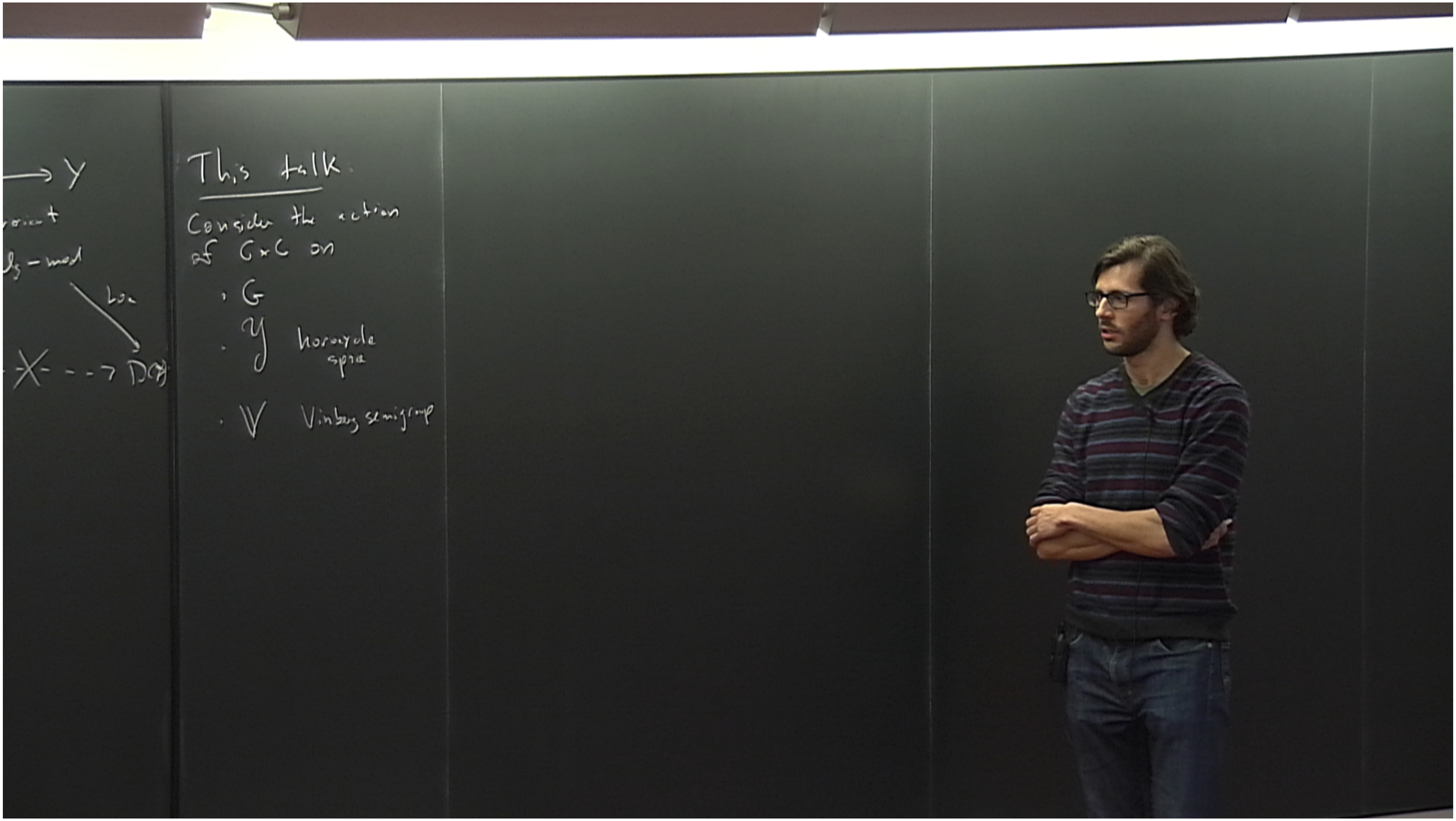
$$X = G/B$$

$$\begin{tikzcd} U_{\mathfrak{g}}\text{-mod} \arrow[r] & D(G/B) \\ U_{\mathfrak{g}}\text{-mod}_0 \arrow[u, hook] \arrow[ur, "\cong"'] & \end{tikzcd}$$

$$f: X \longrightarrow Y$$

G-equivariant

$$\begin{tikzcd} & U_{\mathfrak{g}}\text{-mod} \arrow[dl, "\text{Loc}"'] \arrow[dr, "\text{Loc}"] & \\ D(X) & \arrow[l, dashed] \arrow[r, dashed] \arrow[l, phantom, "\times"] & D(Y) \end{tikzcd}$$


This talk
Consider the action
of G×G on
G
Y horocycle space
V Vinberg semigroup

talk.

the action

C on

horocycle spra

Vinberg semigroup

Digression on hyperbolic geometry

$\mathbb{H} = PSL_2\mathbb{R}/SO_2$

$Y = \begin{pmatrix}\text{light}\\ \text{cone}\end{pmatrix} \setminus \{0\}$

$= PSL_2\mathbb{R} / \begin{bmatrix} 1 & * \\ 0 & \gamma \end{bmatrix}$

$= \{\text{horocycles in } \mathbb{H}\}$

$V = \begin{pmatrix}\text{filled-in}\\ \text{light cone}\end{pmatrix} \setminus \{0\}$

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d] & V \arrow[d] & \mathbb{H}\times\mathbb{R}_{>0} \arrow[l, hook'] \arrow[d] \\
0 \arrow[r, hook] & \mathbb{R}_{\geq 0} & \mathbb{R}_{>0} \arrow[l, hook']
\end{tikzcd}$$

$V/\mathbb{R}_{>0} = \overline{\mathbb{H}}$

$$\begin{tikzcd}
\mathrm{Fun}(\mathbb{H}) \arrow[d, "\substack{\text{wave}\\ \text{equation}}"'] & \mathrm{Fun}(Y) \\
\mathrm{Fun}(\mathbb{H}\times\mathbb{R}_{>0}) \arrow[r, "\text{“scattering theory”}"] & \mathrm{Fun}(V) \arrow[u, "\text{restrict}"']
\end{tikzcd}$$

This talk

$\mathbb{H} \rightsquigarrow G = G\times G/G_\Delta$

$Y \rightsquigarrow \mathcal{Y} = \frac{G/N \times N\backslash G}{T}$

$V \rightsquigarrow \mathbb{V}$

$\frac{\mathrm{Fin}}{\mathbb{H}} \rightsquigarrow$ D-modules

$\overline{\mathbb{H}} \rightsquigarrow \overline{G_{ad}}$ wonderful compactification.

Let G be a connected reductive group $/\mathbb{C}$

Let X be a smooth G-variety.

Infinitesimal action

$$\mathfrak{g} = \mathrm{Lie}(G) \longrightarrow \Gamma(X, \mathcal{T}_X) = \{\text{global v.f. on } X\}$$

Algebra homomorphism

$$U\mathfrak{g} \longrightarrow \Gamma(X, \mathcal{D}_X) = \{\text{global diff ops on } X\}$$

Localization

Loc: $U\mathfrak{g}$-mod

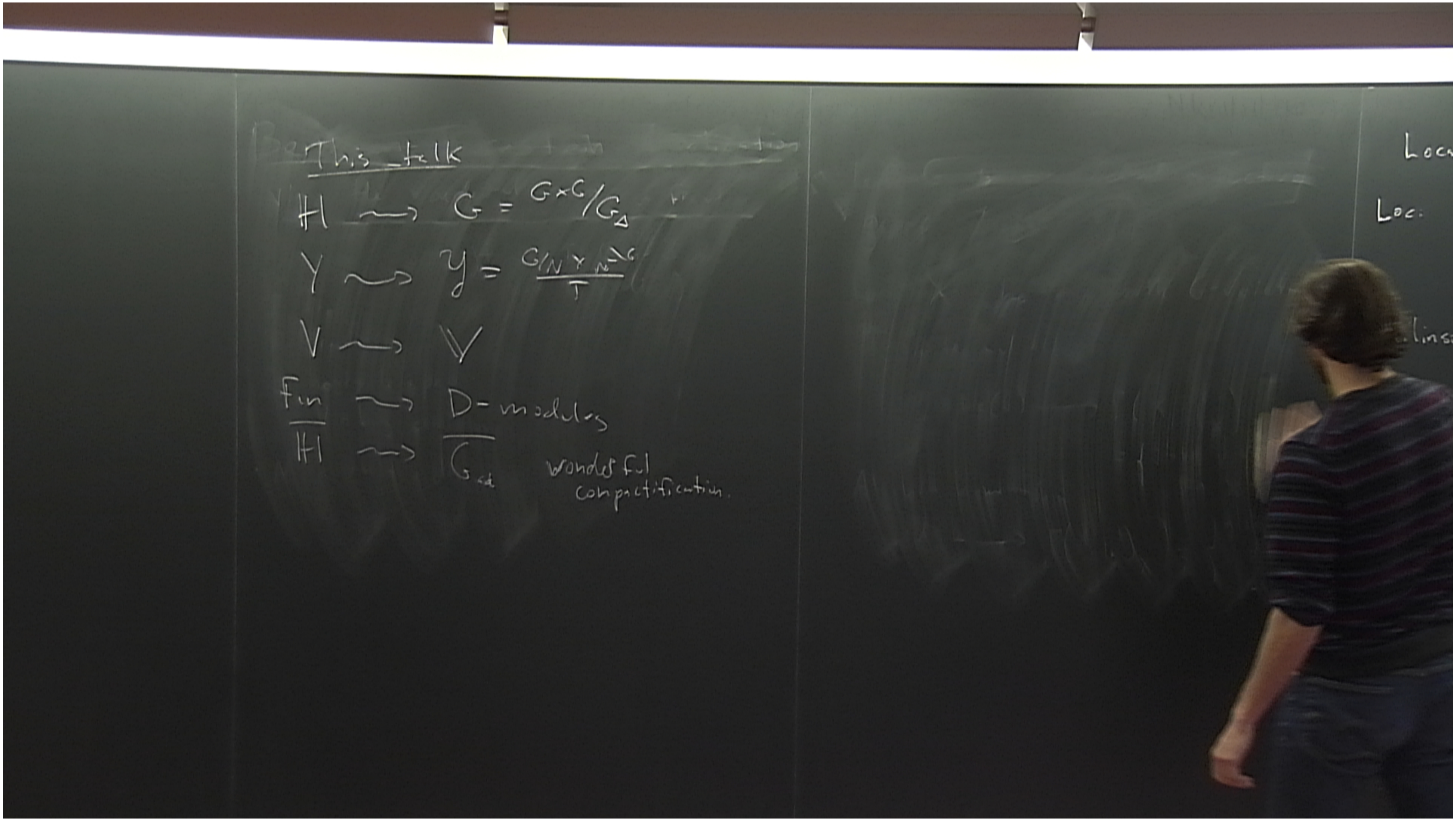
$M \mapsto$

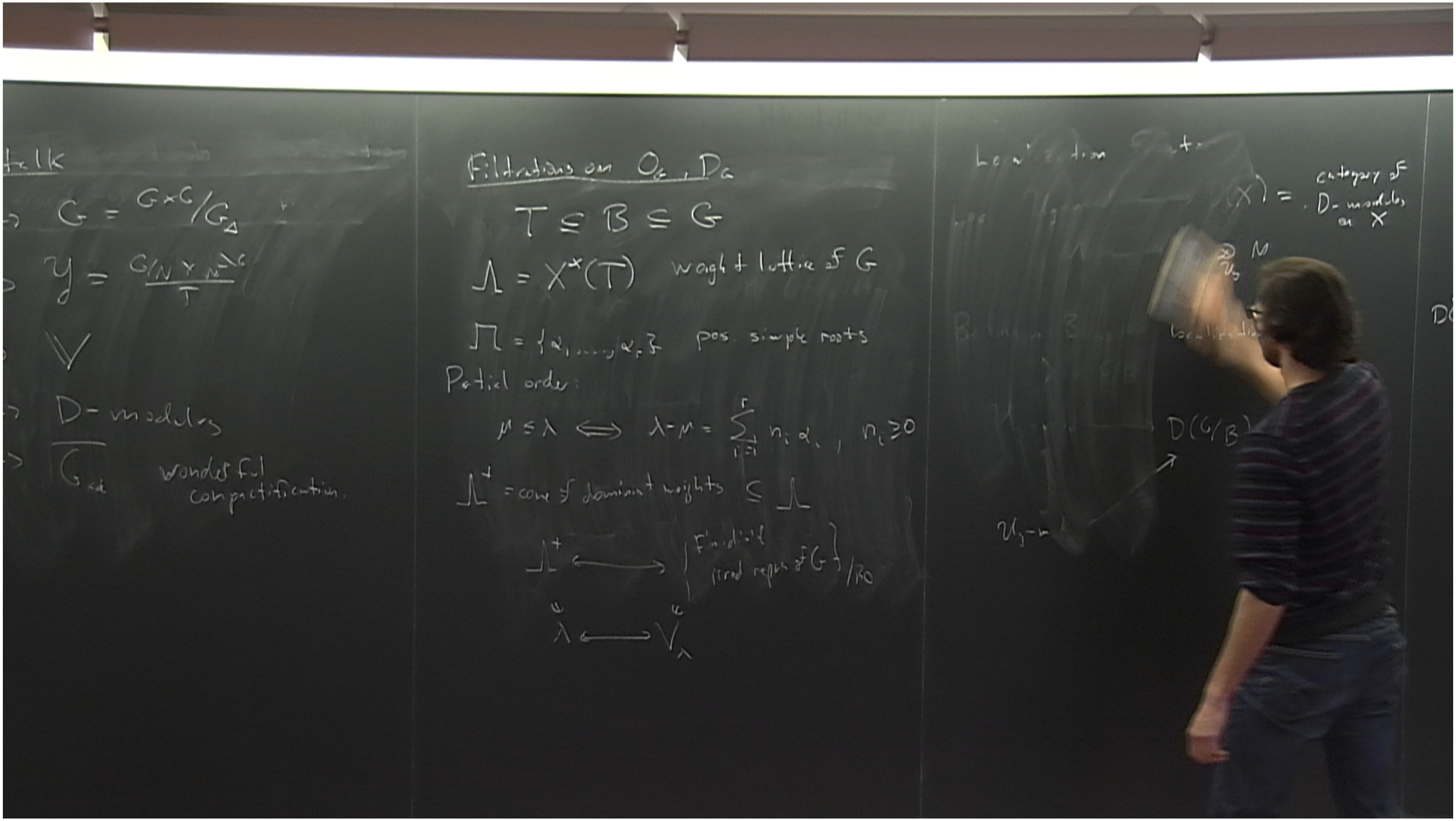
Beilinson - Berns

$X = G/$

$U\mathfrak{g}$-mod —

$U\mathfrak{g}$-mod$_0$


This talk
H ⇝ G = G×G/G_Δ
Y ⇝ y = (G/N × N⁻\G)/T
V ⇝ V
Fun/H ⇝ D-modules
H ⇝ G_ad
wonderful compactification.
Loc.


talk
$G = G \times G / G_\Delta$
$Y = \frac{G/N \times N^- \backslash G}{T}$
D-modules
$\overline{G}_{ad}$ wonderful compactification.
Filtrations on $\mathcal{O}_G$, D_G
$T \subseteq B \subseteq G$
$\Lambda = X^*(T)$ weight lattice of G
$\Pi = \{\alpha_1, \ldots, \alpha_r\}$ pos. simple roots
Partial order:
$\mu \leq \lambda \iff \lambda - \mu = \sum_{i=1}^{r} n_i \alpha_i, \ n_i \geq 0$
Λ^+ = cone of dominant weights $\subseteq \Lambda$
$\Lambda^+ \longleftrightarrow \{$ Finite-dim'l irred repns of G $\} / \text{iso}$
$\lambda \longleftrightarrow V_\lambda$
category of D-modules on X
$D(G/B)$

Filtrations on $\mathcal{O}_G$, $\mathcal{D}_G$

$$T \subseteq B \subseteq G$$

$\Lambda = X^*(T)$ weight lattice of G

$\Pi = \{\alpha_1, \dots, \alpha_r\}$ pos. simple roots

Partial order:

$$\mu \le \lambda \iff \lambda - \mu = \sum_{i=1}^{r} n_i \alpha_i, \quad n_i \ge 0$$

Λ^+ = cone of dominant weights $\subseteq \Lambda$

$$\Lambda^+ \longleftrightarrow \{\text{fin-dim'l irred repns of } G\}/\cong$$

$$\lambda \longleftrightarrow V_\lambda$$

If $V_\nu \subseteq V_\lambda \otimes V_\mu$, then $\nu \le \lambda + \mu$

Peter-Weyl theorem

There is an iso of $G \times G$ repns:

$$\bigoplus_{\lambda \in \Lambda^+} V_\lambda^* \otimes V_\lambda \xrightarrow{\sim} \mathcal{O}(G)$$

$$f \otimes v \longmapsto (g \longmapsto \langle f, g \cdot v \rangle)$$

There is a Λ-filtration on $\mathcal{O}(G)$:

$$\mathcal{O}(G)_{\le \lambda} = \bigoplus_{\mu \le \lambda} V_\mu^* \otimes V_\mu$$

This talk

Consider the ...
of $G \times G$ on

- G
- Y horosp...
- V Vinberg...

G

$n_i \geq 0$

If $V_\nu \subseteq V_\lambda \otimes V_\mu$, then $\nu \leq \lambda + \mu$

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$$m: \mathcal{U}_\mathfrak{g} \otimes \mathcal{U}_\mathfrak{g} \longrightarrow D_G$$

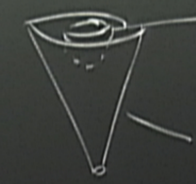
$\boxed{\text{Prop}}$

① $\mathrm{Image}(m) = \mathcal{U}_\mathfrak{g} \otimes_{Z(\mathcal{U}_\mathfrak{g})} \mathcal{U}_\mathfrak{g}$

② There is a Λ-filtration on D_G

$$(D_G)_{\leq \lambda} = \mathrm{Image}(m) \cdot \mathcal{O}(G)_{\leq \lambda}$$

Digression on hyp



$H =$

$Y =$

$=$

$=$

$V = \left(\begin{array}{c} \text{filled-in} \\ \text{light cone} \end{array} \right)$

$\mu : \mathcal{U}_\mathfrak{g} \otimes \mathcal{U}_\mathfrak{g} \longrightarrow D_G$

$\boxed{\text{Prop}}$

1. $\text{Image}(\mu) = \mathcal{U}_\mathfrak{g} \otimes_{Z(\mathcal{U}_\mathfrak{g})} \mathcal{U}_\mathfrak{g}$

2. There is a Λ-filtration on D_G

$$(D_G)_{\leq \lambda} = \text{Image}(\mu) \cdot \mathcal{O}(G)_{\leq \lambda}$$

$\boxed{\text{Ex}}$ $G = SL_2$, $\Lambda = \mathbb{Z} \supseteq \mathbb{Z}_{\geq 0} = \Lambda^+$

$\Pi = \{2\}$

$\cdots \leq -2 \leq 0 \leq 2 \leq \cdots$

$\cdots \leq -1 \leq 1 \leq 3 \leq \cdots$

$$\mathcal{O}(SL_2) = \mathbb{C}[a,b,c,d] \big/ ad - bc = 1$$

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & V \arrow[d] & \arrow[l] \\ 0 \arrow[r, hook] & \mathbb{R}_{\geq 0} & \arrow[l] \end{tikzcd}$$

$V / \mathbb{R}_{\geq 0} = \overline{\mathbb{H}}$

$\text{Fun}(\mathbb{H})$

$\downarrow$

$\text{Fun}(\mathbb{H} \times \mathbb{R}_{\geq 0}) \longrightarrow$

wave equation

on on D_G

$\mathrm{Image}(\mu) \cdot \mathcal{O}(G)_{\leq \lambda}$

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$$\mathcal{O}(SL_2) = \mathbb{C}[a,b,c,d] \big/ ad - bc = 1$$

$$U\mathfrak{sl}_2 \otimes U\mathfrak{sl}_2 \longrightarrow D_{SL_2} = \mathcal{O}(SL_2) \star U\mathfrak{sl}_2$$

$$X \otimes 1 \longmapsto X$$

$$1 \otimes E \longmapsto -a^2 E + c^2 F + ac H$$

$$1 \otimes F \longmapsto \cdots$$

$$1 \otimes H \longmapsto \cdots$$

Y

Fun

wave equation

Ex $G = SL_2$, $\Lambda = \mathbb{Z} \supseteq \mathbb{Z}_{\geq 0} = \Lambda^+$

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$$1 \otimes F \longmapsto \cdots$$

$$1 \otimes H \longmapsto \cdots$$

Horocycle + Vinberg

B^- opposite Borel, $B \cap B^- = T$

$N = R_{unip}(B)$, $N^- = R_{unip}(B^-)$

$Z(G)$ = center of G

$G_{ad} = G/Z(G)$, $T_{ad} = T/Z(G)$

Def The horocycle space for G is

$$\mathcal{Y} = \frac{G/N \times N^-\backslash G}{T} \circlearrowleft G \times G \times T$$

$$\Big\downarrow q \quad \text{principal } T\text{-bundle}$$

$$G/B \times B^-\backslash G$$

The affine closure of $\mathcal{Y}$ is

$\mathrm{Spec}(\mathrm{gr}\,\mathcal{O}(G))$

$\boxed{\text{Prop}}$ (Vinberg, ...) There is a smooth $G \times G \times T$ variety $\mathring{V}$ that fits into the following:

$$\begin{array}{ccccc} \mathcal{Y} & \hookrightarrow & \mathring{V} & \hookleftarrow & \frac{G \times T}{Z(G)} \\ \downarrow & & \downarrow \pi & & \downarrow \\ 0 & \hookrightarrow & \mathbb{A}^r & \hookleftarrow & T_{ad} \cong (\mathbb{C}^\times)^r \end{array}$$

$\lambda \longleftrightarrow V_\lambda$

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0 \arrow[r, hook] & \mathbb{A}^r & T_{ad} \cong (\mathbb{C}^\times)^r \arrow[l, hook']
\end{tikzcd}$$

where

- π is flat with smooth fibers
- All maps are $G \times G \times T$-equivariant ($G \times G \curvearrowright \mathbb{A}^r$ trivially)
- $T \curvearrowright \mathring{\mathbb{V}}$ is free,

$$\mathring{\mathbb{V}}/T = \overline{G_{ad}}$$ wonderful compactification

$\subseteq \ldots \otimes V_\lambda$

…th smooth fibers.

$C \times C \times T$ - equivariant

$\mathbb{A}^r$ trivially)

is free,

$= \overline{G_{ad}}$ wonderful compactification

$\boxed{Ex}$ $G = SL_2$

$$\mathcal{Y} = \frac{\mathbb{C}^2\setminus 0 \times \mathbb{C}^2\setminus 0}{\mathbb{C}^\times} = \{rk\, 1\} \subseteq Mat_{2\times 2}$$

$$\downarrow q$$

$$\mathbb{P}^1 \times \mathbb{P}^1 \qquad q(M) = (\ker(M), \operatorname{Im}(M))$$

$$\begin{array}{ccccc} \mathcal{Y} & \hookrightarrow & \mathring{V} = Mat_{2\times 2}\setminus 0 & \hookleftarrow & \frac{SL_2 \times \mathbb{C}^\times}{\mathbb{Z}/2\mathbb{Z}} = GL_2 \\ \downarrow & & \downarrow \det & & \downarrow \\ 0 & \hookrightarrow & \mathbb{C} & \hookleftarrow & \mathbb{C}^\times \end{array}$$

$$\overline{PSL_2} = \mathbb{P}^3$$

$= U_g \otimes_{Z(U_g)} U_g$

- filtration

D_G

$(D)_{\leq\lambda} = \quad O(G)_{\leq\lambda}$

$\boxed{Ex}$

$O(S$

Ex $G = SL_2$

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$$\overline{PSL_2} = \mathbb{P}^3$$

Construction

$$Rees(\mathcal{O}(G)) = \bigoplus_{\lambda \in \Lambda} \mathcal{O}(G)_{\leq\lambda} z^\lambda$$

$$\subseteq \mathcal{O}(G) \otimes \mathbb{C}[\Lambda]$$

$$\Big\uparrow$$

$$\mathbb{C}[z^{\alpha_1}, \dots, z^{\alpha_r}]$$

polynomial algebra

Let $\mathbb{V} = Spec(Rees(\mathcal{O}(G)))$ Vinberg semigroup

$T \circlearrowright \mathbb{V}$

let $\mathring{\mathbb{V}} = \mathbb{V}^{\chi-ss}$ for $\chi \in (\Lambda^+)_{reg}$.

$$\bigoplus_{\lambda \in \Lambda} \mathcal{O}(G)_{\leq\lambda} z^\lambda \subseteq \mathcal{O}(G) \otimes \mathbb{C}[\Lambda]$$

Vinberg semigroup

$(\mathcal{O}(G))$

$\chi \in (\Lambda^+)_{reg}$.

$\boxed{\text{Ex}}$ Let $\mathcal{D}_\pi \subseteq \mathcal{D}_{\mathcal{Y}}$ be the sheaf of relative diff ops (wrt π)

Horocycle + Vinberg

B^- opposite Borel, $B \cap B^- = T$

$N = R_{unip}(B)$, $N^- = R_{unip}(B^-)$

$Z(G)$ = center of G

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$$\mathcal{Y} \xrightarrow{q} G/B \times B^-\backslash G \quad \text{principal } T\text{-bundle}$$

$\bigoplus_{\lambda \in \Lambda} \mathcal{O}(G)_{\leq\lambda} z^\lambda$

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Ex Let $\mathcal{D}_\pi \subseteq \mathcal{D}_{\mathbb{V}}$ be the sheaf of relative diff ops (wrt π)

- $Rees(D_G)$ defines a q-coh sheaf on $\mathbb{V}$ whose restriction to $\mathring{\mathbb{V}}$ is $\mathcal{D}_\pi$
- Functor

$$D(G)^{filt} \longrightarrow D_\pi(\mathring{\mathbb{V}}) = \mathcal{D}_\pi\text{-mod}$$

$$M \longmapsto \bigoplus_{\lambda \in \Lambda} M_{\leq\lambda} z^\lambda$$

$G/B \times B^{\vee c}$

$T/_{Z(G)}$

G is

$G \times G \times T$

T-bundle

Let $\mathcal{D}_\pi \subseteq \mathcal{D}_{\mathring{\mathbb{V}}}$ be the sheaf of relative diff ops (wrt π)

- $\mathrm{Rees}(D_G)$ defines a q-coh sheaf on $\mathbb{V}$ whose restriction to $\mathring{\mathbb{V}}$ is $\mathcal{D}_\pi$

- Functor
$$D(G)^{filt} \longrightarrow D_\pi(\mathring{\mathbb{V}}) = \mathcal{D}_\pi\text{-mod}$$
$$M \longmapsto \bigoplus_{\lambda \in \Lambda} M_{\leq \lambda} z^\lambda$$

$$\mathcal{U}(k) \longrightarrow \Gamma(X, \mathcal{D}_X)$$

If $K \curvearrowright X$, then $\mathcal{D}_X$ has a natural K-equivariant structure.

There is a notion of a weakly K-equivariant D-module on X

Category: $D(X)^{K}$

Main Result

$\boxed{\text{Thm}}$ [BZ - G - Nadler] There are well-defined functors that make the following diagram commute

$$\begin{tikzcd}
& \mathcal{U}_{\mathfrak{g}} \otimes_{Z(\mathcal{U}_{\mathfrak{g}})} \mathcal{U}_{\mathfrak{g}}\text{-mod} \arrow[ddl, bend right, "\mathrm{Loc}"'] \arrow[d, "\mathrm{Loc}"] \arrow[ddr, bend left, "\mathrm{Loc}"] & \\
& D_{\pi}(\mathring{\mathcal{Y}}/T) \arrow[dl, "\text{restriction to } \pi^{-1}(0)"'] & \\
D(\mathcal{Y}/T) & & D(G)^{\mathrm{filt}} \arrow[ul, "\mathrm{Rees}"'] \arrow[ll, "\text{assoc. gr}"]
\end{tikzcd}$$

where

...ilt

[...-G-Nadler] There are well-defined ...s that make the following commute

$$\begin{tikzcd}
 & \mathcal{U}_g \otimes_{Z(\mathcal{U}_g)} \mathcal{U}_g\text{-mod} \arrow[d, "\mathrm{Loc}"] \arrow[rd, bend left, "\mathrm{Loc}"] \arrow[ld, bend right] & \\
\ldots & D_\pi(\tilde{\mathcal{Y}}/T) \arrow[l, "\text{restriction to } \pi^{-1}(0)"'] & \mathcal{D}(G)^{filt} \arrow[l, "\mathrm{Rees}"'] \arrow[ll, bend left, "\text{assoc. } \mathrm{gr}"]
\end{tikzcd}$$

Computation in the proof.

$$\mathrm{gr}\left(\mathrm{Loc}_Z(M)\right) = \mathrm{gr}\left(\mathcal{D}_G \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M\right)$$

$M \in \mathcal{U}_g \otimes \mathcal{U}_g\text{-mod}$, over $Z(\mathcal{U}_g)$

$$= \mathrm{Rees}\left(\mathcal{D}_G \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M\right)\Big|_{\mathcal{Y}}$$

$$= \bigoplus_{\lambda \in \Lambda} \left(\mathcal{D}_G \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M\right)_{\leq \lambda} z^{\lambda} \Big|_{\mathcal{Y}}$$

$$= \left(\bigoplus_{\lambda \in \Lambda} (\mathcal{D}_G)_{\leq \lambda} z^{\lambda}\right) \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M \Big|_{\mathcal{Y}}$$

$$= \mathcal{D}_\pi\Big|_{\mathcal{Y}} \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M = \mathcal{D}_{\mathcal{Y}} \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M$$

$$= \mathrm{Loc}_{\mathcal{Y}}(M)$$

...putation in the proof:

$$(\ldots M)) = gr(\mathcal{D}_G \otimes_{U_g \otimes_Z U_g} M)$$

$U_g \otimes U_g$-mod

$$= \ldots \otimes_{U_g \otimes_Z U_g} M) \Big|_Y$$

$$= \bigoplus_{\lambda \in \Lambda} (\ldots M)_{\le \lambda} z^\lambda \Big|_Y$$

$$= \Big(\bigoplus_{\lambda \in \Lambda} \ldots \otimes_{U_g \otimes_Z U_g} M \Big|_Y$$

$$= \ldots M = \mathcal{D}_Y \otimes_{U_g \otimes_Z U_g} M$$

$$= \mathrm{Loc}_Y(M)$$

Remarks

① Relation to Beilinson-Bernstein localization

$\lambda \in \mathfrak{t}^*$

$$U_g \otimes_Z U_g\text{-mod} \longrightarrow \mathcal{D}(Y/T)$$

$$\uparrow$$

$$U_g/\ker(\chi_\lambda)\text{-mod} \otimes U_g/\ker(\chi_\lambda)\text{-mod}$$

Construction

$$\mathrm{Rees}(\mathcal{O}(G)) = \bigoplus_{\lambda \in \Lambda} \mathcal{O}(G)_{\le \lambda} z^\lambda \subseteq \mathcal{O}(G) \otimes \mathbb{C}[\ldots$$

$$\uparrow$$

$$\mathbb{C}[z^{\alpha_1}, \ldots, z^{\alpha_r}]$$

polynomial algebra

Let $\mathbb{V} = \mathrm{Spec}(\mathrm{Rees}(\mathcal{O}(G)))$ Vinberg semigroup

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Let $\mathring{\mathbb{V}} = \mathbb{V}^{\chi\text{-ss}}$ for $\chi \in (\Lambda^+)_{\mathrm{reg}}$.

putation in the proof:

$$M)) = gr\left(\mathcal{D}_G \otimes_{U_g \otimes U_g} M\right)$$

$\cap$ $U_g \otimes U_g$-mod $Z(U_g)$

$$= Rees\left(\mathcal{D}_G \otimes_{U_g \otimes_Z U_g} M\right)\Big|_{\mathcal{Y}}$$

$$= \bigoplus_{\lambda \in \Lambda}\left(\mathcal{D}_G \otimes_{U_g \otimes U_g} M\right)_{\leq \lambda} z^\lambda \Big|_{\mathcal{Y}}$$

$$= \left(\bigoplus_{\lambda \in \Lambda}(\mathcal{D}_G)_{\leq \lambda} z^\lambda\right) \otimes_{U_g \otimes_Z U_g} M \Big|_{\mathcal{Y}}$$

$$= \mathcal{D}_\Pi\Big|_{\mathcal{Y}} \otimes_{U_g \otimes_Z U_g} M = \mathcal{D}_{\mathcal{Y}} \otimes_{U_g \otimes_Z U_g} M$$

$$= Loc_{\mathcal{Y}}(M)$$

Remarks

① Relation to Beilinson-Bernstein localization

$\lambda \in \mathfrak{t}^*$

$$U_g \otimes_Z U_g\text{-mod} \xrightarrow{Loc} D(\mathcal{Y}/T)$$

$$U_g/_{Ker(\chi_\lambda)}\text{-mod} \boxtimes U_g/_{Ker(\chi_\lambda)}\text{-mod} \xrightarrow{Loc_\lambda} D_\lambda(G/B) \boxtimes D_\lambda(B^{\vee})$$

[B-B] If λ is dominant+regular, then Loc_λ is an equivalence of abelian categories.

[BZ-N] On the level of dg categories, Loc gives an equivalence with $D(\mathcal{Y}/T)^{W}$ ← weyl-monod

Construction

$$Rees(\mathcal{O}(G)) = \bigoplus_{\lambda \in \Lambda} \mathcal{O}(G)_{\leq \lambda} z^\lambda \subseteq \mathcal{O}(G) \otimes \mathbb{C}[\ldots]$$

$\cup$

$\mathbb{C}[z^{\alpha_1}, \ldots, z^{\alpha_r}]$ polynomial algebra

Let $\mathbb{V} = Spec(Rees(\mathcal{O}(G)))$ Vinberg semigroup

$\circlearrowright T$

Let $\mathring{\mathbb{V}} = \mathbb{V}^{\chi\text{-ss}}$ for $\chi \in (\Lambda^+)_{reg}$.

Remarks

① Relation to Beilinson–Bernstein localization

$\lambda \in \mathfrak{t}^*$

$$\begin{array}{ccc} U\mathfrak{g} \otimes_Z U\mathfrak{g}\text{-mod} & \underset{\Gamma}{\overset{\mathrm{Loc}}{\rightleftarrows}} & D(Y/T) \\ \uparrow & & \uparrow \\ U\mathfrak{g}/\ker(\chi_\lambda)\text{-mod} \boxtimes U\mathfrak{g}/\ker(\chi_\lambda)\text{-mod} & \underset{\Gamma}{\overset{\mathrm{Loc}_\lambda}{\rightleftarrows}} & D_\lambda(G/B) \boxtimes D_\lambda(B^-\backslash G) \end{array}$$

[B-B] If λ is dominant + regular, then Loc_λ is an equivalence of abelian categories.

[BZ-N] On the level of dg categories, Loc gives an equivalence with $D(Y/T)^{W}$ (weyl monod)

② Relation to Verdier specialization

(Assume $G = G_{ad}$)

$G \xhookrightarrow[\text{open } G\times G \text{ orbit}]{i} \overline{G} \xhookleftarrow[\text{closed orbit}]{j} G/B \times B^-\backslash G$

$G/B \times B^-\backslash G \cong \bigcap_{i=1}^{r} Z_i$ (smooth irred. boundary divisors)

$\rightsquigarrow$ (mult.)-V-filtration on $\mathcal{D}_{\overline{G}}$ relative to $G/B \times B^-\backslash G$ by Λ

$\rightsquigarrow \Lambda$-filtration on $j_* \mathcal{D}_G$ and $\Gamma(\overline{G}, j_* \mathcal{D}_G) = \mathcal{D}_G$

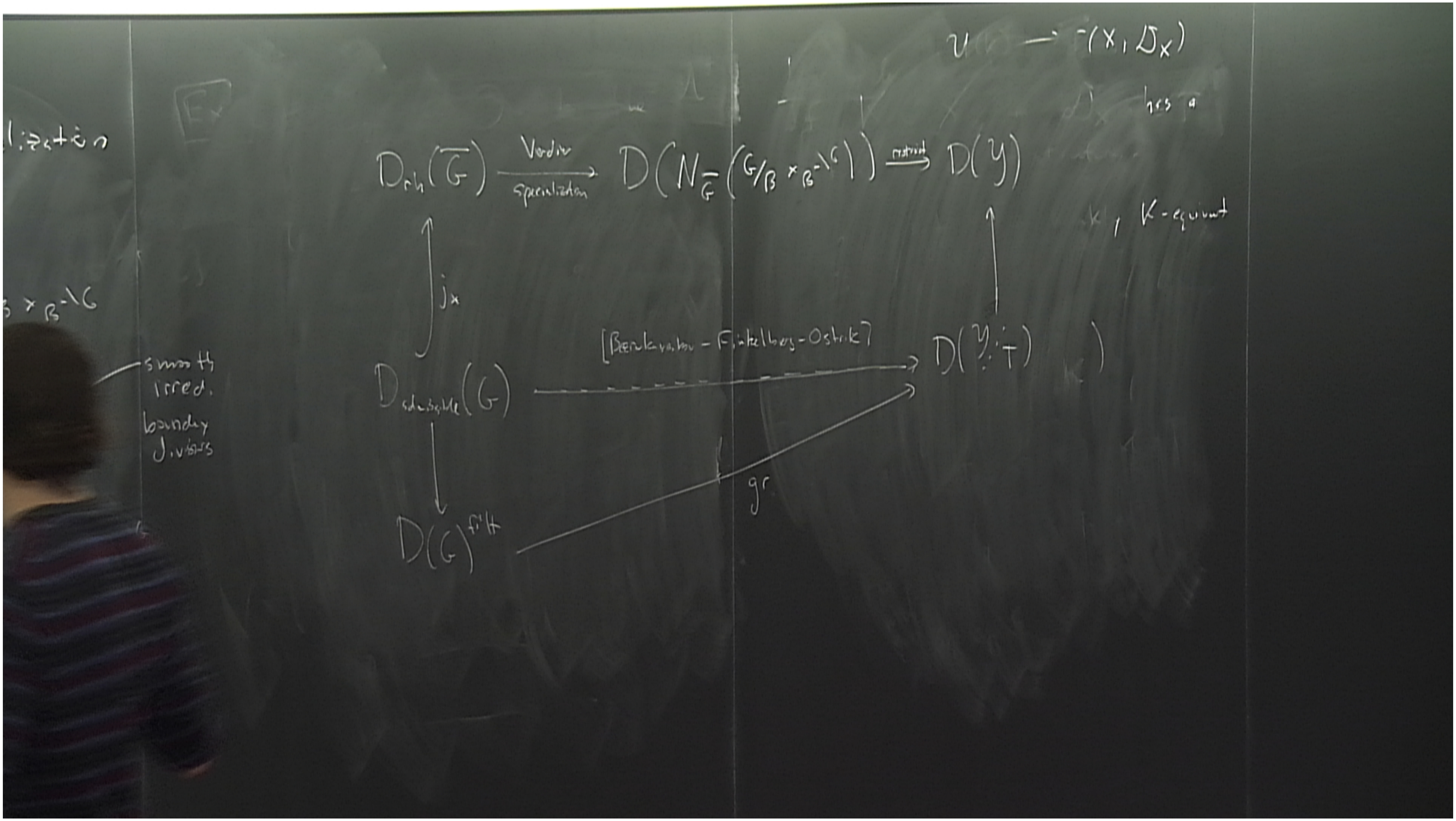
This matches with the earlier filtration

Ex Let $\mathcal{D}_\pi \subseteq \mathcal{D}$ … sheaf of relative … (w.r.t. π)

- $\mathrm{Rees}(\mathcal{D}_G)$ defines … whose restriction t…
- Functor

$D(G)^{\mathrm{filt}} \longrightarrow$

$M \longmapsto$


Verdier
specialization
restrict
j_*
[Bezrukavnikov - Finkelberg - Ostrik]
gr
smooth
irred.
boundary
divisors
K-equivariant

③ Generalization to symmetric spaces

$\theta : G \longrightarrow G$ involution

$K = G^{\theta}$

$P = MAN$

$\mathcal{G} = G \times \mathcal{B} / G_\Delta \rightsquigarrow G/K$

$\mathcal{Y} = \frac{G/N \times \pi^{-1}}{T} \rightsquigarrow G/MN$

Computation in the proof:

$gr(Loc_z(M)) = gr(\mathcal{D}_G \otimes_{\mathcal{U}_g \otimes \mathcal{U}_g} M)$

M: $\mathcal{U}_g \otimes \mathcal{U}_g$-mod, $Z(\mathcal{U}_g)$

$= Rees(\mathcal{D}_G \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M)$

$= \bigoplus_{\lambda \in \Lambda} (\mathcal{D}_G \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M)_{\leq \lambda} z^{\lambda}$

$= \left(\bigoplus_{\lambda \in \Lambda} (\mathcal{D}_G)_{\leq \lambda} z^{\lambda} \right) \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M \Big|_{\mathcal{Y}}$

$= \mathcal{D}_\pi \Big|_{\mathcal{Y}} \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} M = \mathcal{D}_{\mathcal{Y}} \otimes$

$= Loc_{\mathcal{Y}}($

③ Generalization to symmetric spaces

$\theta : G \longrightarrow G$ involution

$K = G^{\theta}$

$P = MAN$ maximally θ-split parabolic

$\mathcal{G} = {}^{G \times G}/_{G_\Delta} \rightsquigarrow G/K$

$\mathcal{Y} = \frac{G/N \times N^- \backslash G}{T} \rightsquigarrow G/_{MN}$

Computation in the pr

$gr(\mathrm{Loc}_\xi(M)) = gr($

M: $\mathcal{U}_g \otimes_{Z(\mathcal{U}_g)} \mathcal{U}_g$-mod

$= \mathrm{Rees}($

$= \bigoplus_{\lambda \in \Lambda} (\mathcal{D}_a \otimes_{\mathcal{U}_g \otimes \mathcal{U}_g}$

$= \left(\bigoplus_{\lambda \in \Lambda} (\mathcal{D}_c)_{\leq \lambda} \right.$

$= \Delta_\pi |_{\mathcal{Y}}$

(3) Generalization to symmetric spaces

$\theta : G \longrightarrow G$ involution

$K = G^{\theta}$

$P = MAN$ maximally θ-split parabolic

$\mathcal{G} = {}^{G \times \mathcal{U}}/_{G_\Delta} \rightsquigarrow G/K$ Chen - Yom Din

$\mathcal{Y} = \frac{G/N \times_{\mathfrak{n}} \mathfrak{n}^{-\perp}}{T} \rightsquigarrow G/MN$

Computation in the proof:

$gr\left(\mathrm{Loc}_\xi(M)\right) = gr\left(\mathcal{D}_G \ldots\right.$

$\mathcal{U}_g \otimes_{Z(\mathcal{U}_g)} \mathcal{U}_g$-mod

$= \mathrm{Rees}\left(\mathcal{D}_G \ldots\right.$

$= \left(\bigoplus_{\lambda \in \Lambda} \left(\mathcal{D}_G \otimes_{\mathcal{U}_g \otimes_{Z} \mathcal{U}_g} M\right)_{\leq} \ldots\right.$

$= \left(\bigoplus_{\lambda \in \Lambda} (\mathcal{D}_G)_{\leq \lambda} z^{\lambda}\right) \otimes_{\mathcal{U}_g} \ldots$

$= \mathcal{D}_{\Pi}\big|_{\mathcal{Y}} \otimes_{\mathcal{U}_g \otimes_Z \mathcal{U}_g} \ldots$