Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 8

Date: Jan 30, 2018 04:00 PM

URL: http://pirsa.org/18010064

Abstract:

Note Title

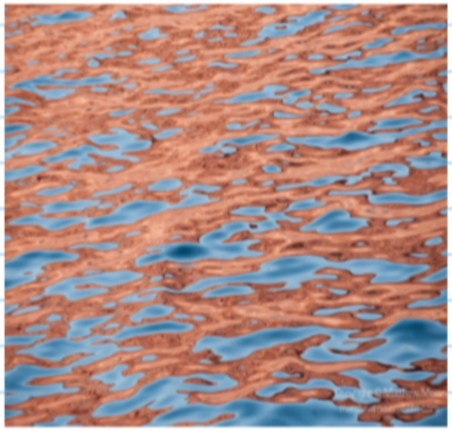
The Unruh effect (W. G. Unruh, 1976)

An accelerated ice cube will melt, even in vacuum.

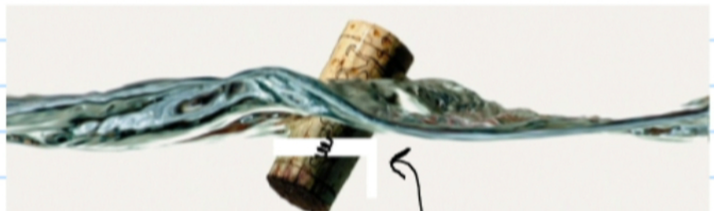
The Unruh effect is the observation, by accelerated observers,

of

Intuition:



A rigged cork can act as sender and detector.

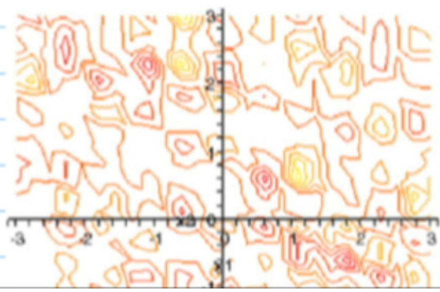


rigged

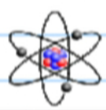
When accelerated, it gets excited - as if detecting.

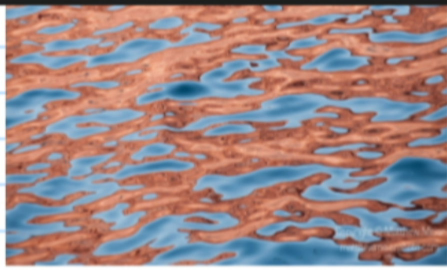
Similarly:

If simplified to a 2-level system, we say that we have an Unruh DeWitt detector system.



An atom or molecule can also both emit and detect particles. This can serve as the definition of particles.





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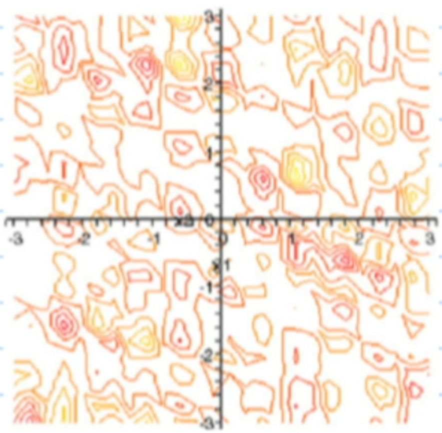

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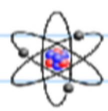

Unruh effect

When accelerated, expect particle emission and detection.

We'll consider detectors at rest and in motion:

* A detector at rest has: $x^\mu(\tau) = (\tau, 0, 0, 0)$

* Case of constant velocity:

$$x^\mu(\tau) = (a\tau, \vec{b}\,\tau)$$

with $a^2 - \vec{b}^{\,2} = 1$. Exercise: verify

* Case of constant acceleration in the x-direction:

$$x^0(\tau) = \alpha \sinh(\tau/\alpha)$$
$$x^1(\tau) = \alpha\,(1 + \sinh^2(\tau/\alpha))^{1/2}$$
$$x^2(\tau) = x^3(\tau) = 0$$

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Exercise: ☐ verify that $\ddot{x}^\mu \ddot{x}_\mu = \text{const}$

(i.e. for still small velocities)

☐ show that for $\tau \ll 1$: $x(\tau) \approx (\tau, a + b\tau^2)$

The quantum field

- We assume that, for an inertial observer, the field $\hat{\phi}$ is in the Minkowski vacuum. Recall:

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k}\vec{x}}\, \hat{\phi}_k(x^0)\, d^3k \quad \text{with} \quad \hat{\phi}_k(x^0) = \frac{1}{\sqrt{2}} \left(v_k^*(x^0)\, a_{\vec{k}} + v_{-k}(x^0)\, a^+_{-k} \right)$$

and $v_k(x^0) = \frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0}$ with $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

- Thus: $$\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{\sqrt{\omega_k}} e^{i\omega_k x^0 + i\vec{k}\vec{x}}\, a_k + \frac{1}{\sqrt{\omega_k}} e^{-i\omega_k x^0 + i\vec{k}\vec{x}}\, a^+_{-k} \right) d^3k$$

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- Note: $\hat\phi(x)$ acts on Hilbert space $\mathcal{H}^{field}$.

Next: Consider a system (e.g. an atom) that can detect particles of the Klein Gordon field $\hat\phi$.

The detector system

- Small, localized system with path $x^\mu(\tau)$

  Inertial observer's cartesian coordinates.

  detector's eigentime

  E.g.: * An atom

  * An oscillator, such as the diatomic molecule H_2.

- First quantized.

- Hamiltonian $\hat{H}^{detector}$ acts on Hilbert space $\mathcal{H}^{detector}$.

- Assume $spec(H^{detector}) = \{E_0, E_1, E_2, \ldots\}$ is discrete.

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$$\boxed{\hat{H}^{total} = \hat{H}_0^{detector} \otimes 1 + 1 \otimes \hat{H}_0^{field} + \hat{H}^{interaction}}$$

- On the total Hilbert space:

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$$\hat{H}^{interaction}(\tau) = \varepsilon(\tau)\, \hat{Q}(\tau)\, \hat{\phi}(x^0(\tau), \vec{x}(\tau))$$

$\varepsilon(\tau)$: Detector efficiency (can also be used as on/off switch)

$\hat{Q}(\tau)$: An observable of the detector's quantum system

$\hat{\phi}(x^0(\tau), \vec{x}(\tau))$: The field $\hat{\phi}$ at the current detector location

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- Examples: $H^{int}(\tau) = \hat{S}_3(\tau) \otimes \hat{B}_3(x(\tau))$

detector is a spin. field is magnetic field.

or: $H^{int}(\tau) = -\frac{e}{mc}\, \hat{p}_i \otimes \hat{A}^i(x(\tau))$ (use Axial gauge: $\partial_i A^i = 0$)

Time evolution

- If we (realistically) assume that the detector efficiency $\varepsilon(\tau)$ is small, we can use perturbation theory.

- In this case, the Dirac picture of time evolution is convenient:

  * Operators evolve according to

$$\hat{H}^{free} = \hat{H}^{detector} \otimes 1 + 1 \otimes \hat{H}^{field} \qquad (*)$$

For example:

$$\hat{Q}(\tau) = e^{i\hat{H}^{free}\tau} (\hat{Q}_0 \otimes 1) e^{-i\hat{H}^{free}\tau}$$

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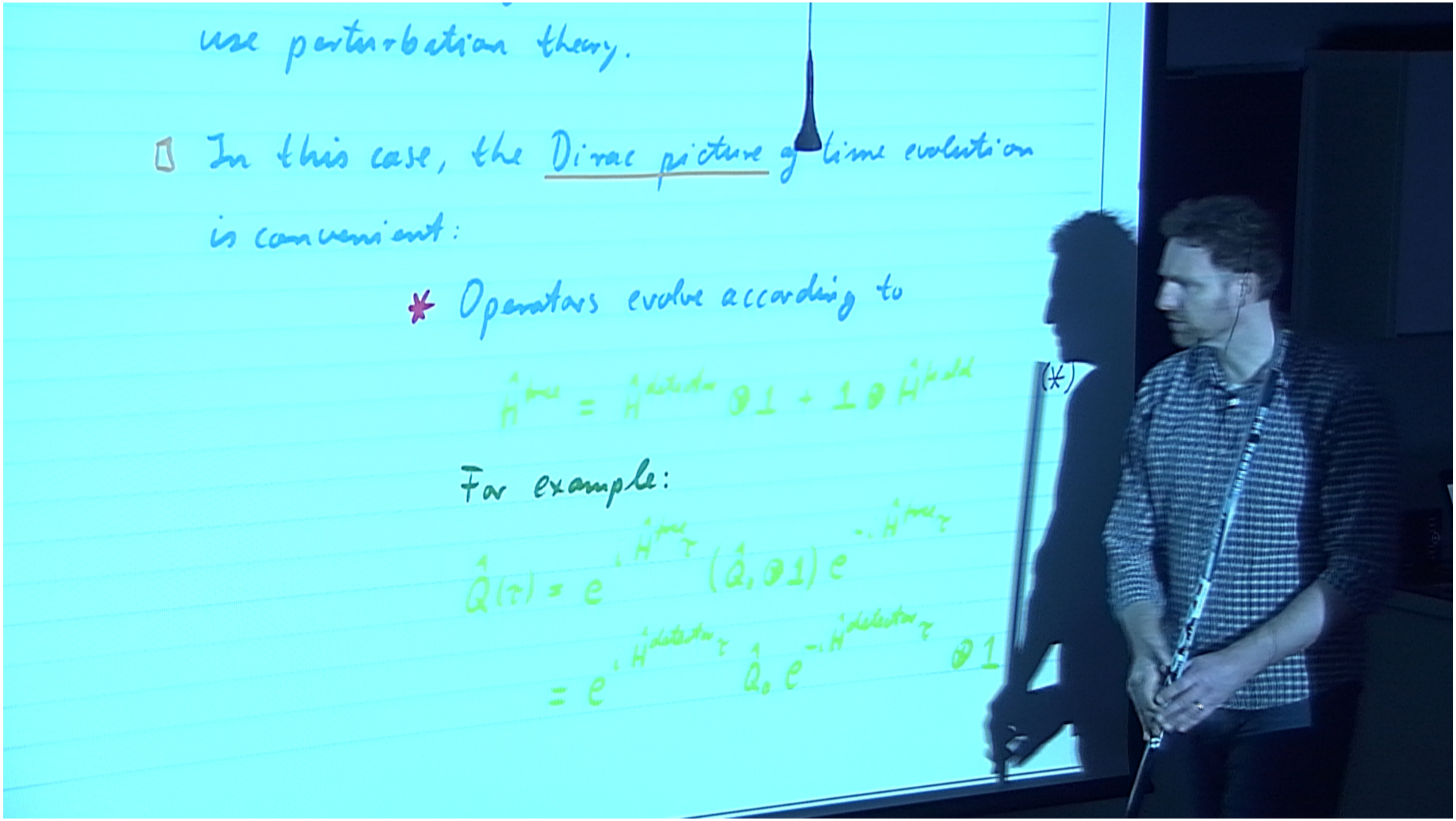
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* States evolve according to $\hat{H}^{int}(\tau)$, i.e., according to the time evolution operator:

$$\hat{U}(\tau) = T\exp\left(i\int_{\tau_i}^{\tau_f} \hat{H}^{interaction}(\tau')\,d\tau'\right)$$

time-ordering symbol

In $\hat{H}^{interaction}$ the operators are time dependent, evolving according to (*)

Perturbative ansatz

For small detector efficiency $\varepsilon(\tau)$ we can expand:

$$\hat{U}(\tau) = 1 + i\int_{-\infty}^{\tau} \varepsilon(\tau')\,\hat{Q}(\tau')\,\hat{\phi}(x^0(\tau'),\vec{x}(\tau'))\,d\tau' + \mathcal{O}(\varepsilon^2)$$

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- Note: We can set $\tau_i = -\infty$ since we can always switch $\varepsilon(\tau)$ on or off.

Initial conditions

- We assume that the quantum field $\hat{\phi}$ starts out in a state $|\alpha\rangle$ with $|\alpha\rangle$ = Minkowski vacuum, $|\alpha\rangle = |0\rangle$, or a 1-particle state $|\alpha\rangle = |1_k\rangle$.

- We assume that the detector starts out in its ground state $|E_0\rangle$.

- Thus, the combined system starts out in the state:

$$|\Psi_{in}\rangle = |E_0\rangle \otimes |\alpha\rangle$$

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$$|\Psi(\tau)\rangle = \hat{U}(\tau)|\Psi_{in}\rangle$$

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To this end, calculate:

$$p(\tau) := \Big(\langle E_n| \otimes \langle \Omega|\Big) |\psi(\tau)\rangle$$

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- **Note:** We will see that not all states $|\Omega\rangle$ yield a nonzero $p(\tau)$.

Total detection probability:

- The probability for detection eventually is:

(we may choose $\varepsilon(\tau)$ so as to make it finite)

$$p(\infty) \approx \langle E_n| \otimes \langle \Omega| \left(1 + i \int_{-\infty}^{+\infty} \varepsilon(\tau')\, \hat{Q}(\tau') \otimes \hat{\phi}(x(\tau'))\, d\tau' \right) |E_0\rangle \otimes |\alpha\rangle$$

Note: $\langle E_n | E_0 \rangle = 0 \Rightarrow 1^{st}$ term vanishes $\Rightarrow$

$$= i \int_{-\infty}^{+\infty} \varepsilon(\tau) \langle E_n| \hat{Q}(\tau) |E_0\rangle \langle \Omega| \hat{\phi}(x(\tau)) |\alpha\rangle \, d\tau$$

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Recall: $\hat{\phi}(x) = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{1}{\sqrt{2\omega_k}} e^{i\omega_k x^0 - i\vec{k}\vec{x}} a_{\vec{k}}^{+} + \frac{1}{\sqrt{2\omega_k}} e^{-i\omega_k x^0 + i\vec{k}\vec{x}} a_{\vec{k}} \right) d^3k$

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Note: We can now calculate all absorption and emission processes.

Here: Let's focus on particle detection in the vacuum, $|\alpha\rangle := |0\rangle$:

* In $\hat{\phi}(x)$, only the terms $\sim a^+_k$ can contribute, because $a_k |0\rangle = 0$

* Thus, in $|\Omega\rangle$ only the one-particle components contribute: