

Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 7

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Abstract:

QFT for Cosmology, Achim Kempf, Lecture 7

Note Title

The driven harmonic oscillator cont'd:

D. Energy eigenstates

* Recall $\hat{H}(t) = \omega (a^\dagger(t) a(t) + \frac{1}{2}) - \frac{i}{\sqrt{2\omega}} (a^\dagger(t) + a(t)) J(t)$

(Q: How come that $\hat{H}(t < 0) \neq \hat{H}(t > T)$?
A: We use the Heisenberg picture!)

$$= \begin{cases} \omega (a_{in}^\dagger a_{in} + \frac{1}{2}) & \text{for } t < 0 \\ \text{something} & \text{for } 0 \leq t \leq T \\ \omega (a_{out}^\dagger a_{out} + \frac{1}{2}) & \text{for } T < t \end{cases}$$

Here, $a_{in} := a(0)$, $a_{out} := a(T)$ and $a_{out} = a_{in} + J_0$

with: $J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$

* For $t < 0$, we diagonalized the Hamiltonian

$$\hat{H}(t) = \omega (a_{in}^\dagger a_{in} + \frac{1}{2}) = \hat{H}_{t=0} = \text{const.}$$

by using $[a_{in}, a_{in}^\dagger] = 1$ to construct its eigenbasis:

$$\hat{H}_{t=0} |n_{in}\rangle = E_n^{(in)} |n_{in}\rangle$$

Namely:

$$E_n^{(in)} = \omega (n + \frac{1}{2}) \quad , n = 0, 1, 2, 3, \dots$$

$$|n_{in}\rangle := \frac{1}{\sqrt{n!}} (a_{in}^\dagger)^n |0_{in}\rangle$$

Note: The set $\{|n_{in}\rangle\}$ is a Hilbert basis of the Hilbert space \mathcal{H} .

* By $t > T$, the Hamiltonian has become a different operator:

$$\hat{H}(t) = \omega \left(a_{\text{out}}^+ a_{\text{out}} + \frac{1}{2} \right) = \hat{H}_{t>T} = \text{const.}$$

What are its eigenvectors $|n_{\text{out}}\rangle$ and eigenvalues E_n^{out} ?

Observation:

We have:

$$[a_{\text{out}}, a_{\text{out}}^+] = 1$$

\Rightarrow we can construct the eigenbasis of $H_{t>T}$

with the same method as the eigenbasis of $H_{t<0}$:

* There is a unique vector $|0_{\text{out}}\rangle \in \mathcal{H}$ obeying:

$$a_{\text{out}} |0_{\text{out}}\rangle = 0$$

* We define the set of vectors $\{|n_{\text{out}}\rangle\}$:

$$|n_{\text{out}}\rangle := \frac{1}{\sqrt{n!}} (a_{\text{out}}^+)^n |0_{\text{out}}\rangle$$

* Proposition:

$$\hat{H}_{\text{EXT}} |n_{\text{out}}\rangle = E_n^{(\text{out})} |n_{\text{out}}\rangle \text{ with } E_n^{(\text{out})} = \omega(n + \frac{1}{2}) = E_n^{(\text{in})}$$

The operators \hat{H}_{EXT} and \hat{H}_{EXT} are different and have different eigenvectors: $|n_{\text{in}}\rangle$ and $|n_{\text{out}}\rangle$. Why are the eigenvalues the same? They both describe a free oscillator of frequency ω .

* Proposition:

The set $\{|n_{\text{out}}\rangle\}$ is a ON Hilbert basis of the Hilbert space \mathcal{H} .

How are the two bases related?

* Recall: Both, $\{|n_{in}\rangle\}$ and $\{|n_{out}\rangle\}$ are ON bases of \mathcal{H} .

\Rightarrow Each basis vector $|n_{in}\rangle$ is a linear combination of the basis vectors $\{|n_{out}\rangle\}$ and vice versa.

* Therefore, in particular:

There must exist coefficients $\Lambda_n \in \mathbb{C}$ so that:

$$|0_{in}\rangle = \sum_n \Lambda_n |n_{out}\rangle$$

\uparrow "Bogolubov Transformation"

* Meaning of the Λ_n ?

□ Recall: The system's state is frozen in state $|\psi\rangle = |0_{in}\rangle$.

□ Assume we measure at a time $t > T$ the energy,
i.e., we measure

$$H(t) = \omega (a_{out}^+ a_{out} + \frac{1}{2})$$

□ What is the probability amplitude for finding the energy eigenvalue E_n ?

□ Clearly:

$$\text{prob. amp. } (|n_{out}\rangle \text{ at } t > T) = \langle n_{out} | \psi \rangle$$

$$\text{i.e.: } \text{prob. } (|n_{out}\rangle \text{ at } t > T) = |\langle n_{out} | \psi \rangle|^2$$

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□ Calculate:

$$\begin{aligned}\langle n_{out} | \psi \rangle &= \langle n_{out} | 0_{in} \rangle \\ &= \langle n_{out} | \sum_m \Lambda_m | m_{out} \rangle \\ &= \Lambda_n\end{aligned}$$

⇒ If the oscillator started in its ground state, then at time $t > T$ the probability for finding the oscillator in its n 'th excited state is given by:

$$\text{prob.}(|n_{out}\rangle \text{ at } t > T) = |\Lambda_n|^2$$

Remark: In QFT, this will be the prob. for finding n particles after charges and currents $j(x,t)$ excited the vacuum.

Calculation of Λ_n :

Proposition: $\Lambda_n = e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n$

Proof: The claim is that $|0_{in}\rangle = \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle$.

We need to check that indeed: $a_{in} |0_{in}\rangle = 0$

Using $a_{out} = a_{in} + J_0$, we need to check: $(a_{out} - J_0) |0_{in}\rangle = 0$

Indeed:

$$\begin{aligned} & (a_{out} - J_0) \sum_n e^{-\frac{1}{2}|J_0|^2} \frac{1}{\sqrt{n!}} J_0^n |n_{out}\rangle \\ &= e^{-\frac{1}{2}|J_0|^2} (a_{out} - J_0) \sum_n J_0^n \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{n!}} (a_{out}^+)^n |0_{out}\rangle \end{aligned}$$

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 \end{aligned}$$

$$= e^{-\frac{1}{2}|j_0|^2} (a_{out} - j_0) e^{j_0 a_{out}^+} |0_{out}\rangle$$

$$= e^{-\frac{1}{2}|j_0|^2} (a_{out} e^{j_0 a_{out}^+} - j_0 e^{j_0 a_{out}^+}) |0_{out}\rangle$$

\parallel using $AB = [A, B] + BA$

$$-\frac{1}{2}|j_0|^2 \left(\underbrace{j_0 a_{out}^+ + 1}_{\text{using } [a, a^\dagger] = 1} - j_0 a_{out}^+ \right) |0_{out}\rangle$$

$$= e^{-\frac{i}{2}|J_0|^2} (a_{out} - J_0) e^{J_0 a_{out}^+} |0_{out}\rangle$$

$$= e^{-\frac{i}{2}|J_0|^2} \left(a_{out} e^{J_0 a_{out}^+} - J_0 e^{J_0 a_{out}^+} \right) |0_{out}\rangle$$

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$$= e^{-\frac{i}{2}|J_0|^2} \left([a_{out}, e^{J_0 a_{out}^+}] + e^{J_0 a_{out}^+} a_{out} - J_0 e^{J_0 a_{out}^+} \right) |0_{out}\rangle$$

$$\stackrel{(*)}{=} e^{-\frac{i}{2}|J_0|^2} \left((J_0 - J_0) e^{J_0 a_{out}^+} + e^{J_0 a_{out}^+} a_{out} \right) |0_{out}\rangle = 0 \quad \checkmark$$

Note: In the last step, $(*)$, we used that: $[a_{out}, e^{J_0 a_{out}^+}] = J_0 e^{J_0 a_{out}^+}$.

Exercise: Show that, more generally, $[a, a^+] = 1$ implies $[a, f(a^+)] = f'(a^+)$ by induction.

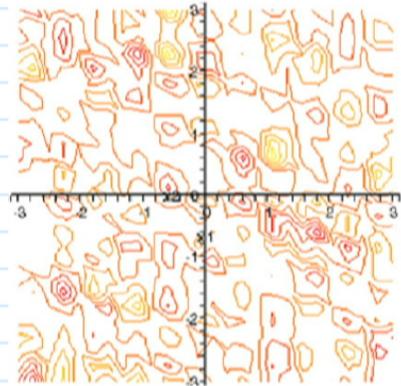
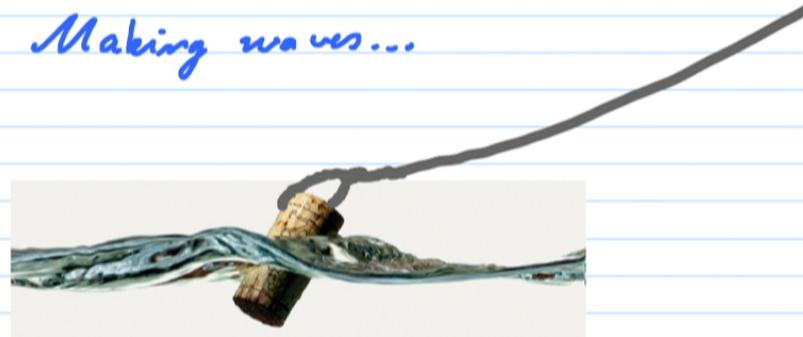
Hint: Show that: $[a, a^+] = 1$, $[a, a^{+2}] = 2a^+$, $[a, a^3] = 3a^2$, ..., $[a, a^{+n}] = n(a^+)^{n-1}$

Exercise: Verify that $|0_{in}\rangle = \sum_n e^{-\frac{i}{2}|J_0|^2} \frac{1}{n!} J_0^n |n_{in}\rangle$ obeys $\langle 0_{in} | 0_{in} \rangle = 1$.

Apply this strategy to the mode oscillators in QFT:



Making waves...



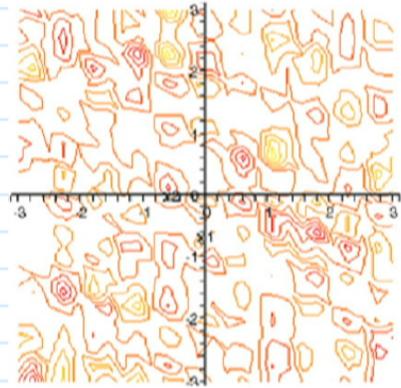
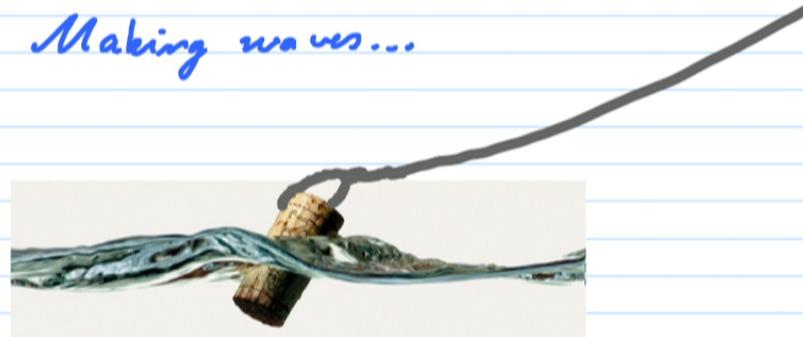
Making EM waves...

e^-

Apply this strategy to the mode oscillators in QFT:



Making waves...



Making EM waves...

e^-

Recall:

$$\hat{H}(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + j(x,t)\hat{\phi}(x,t) d^3x$$

Example interpretation:

- * $\hat{\phi}(x,t)$ may be viewed as a slightly simplified version of the quantum electromagnetic field.
- * $j(x,t)$ may be viewed as a simplified version of a given classical electric charge and current density functions.

Example:

A (Klein-Gordon) charge traveling a path $\tilde{x}^i(t)$:

Then: $j(x,t) = q \delta(x - \tilde{x}(t))$

In- and out periods

Let us consider the case where

$$J(x,t) = 0 \text{ for all } t \notin [0, T]$$

\Rightarrow It suffices to consider the periods $t < 0$ and $t > T$ in both of which $J(x,t) = 0$ (and then to relate the bases).

The free (i.e., undriven) QFT: ($t < 0$ or $t > T$)

$$\hat{H}(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t) (\Delta - m^2) \hat{\phi}(x,t) d^3x$$

* We need to solve:

$$\hat{\pi}(\mathbf{x}, t) - (\Delta - m^2) \hat{\phi}(\mathbf{x}, t) = 0$$

$$[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t)] = i \delta^3(\mathbf{x} - \mathbf{x}')$$

* Fourier transformed,

$$\hat{\phi}_k(t) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \hat{\phi}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3x$$

we need to solve:

$$\hat{\phi}_k''(t) + \underbrace{(k^2 + m^2)}_{\substack{= \sum_{i=1}^3 k_i^2 \\ \text{Definition: } =: \omega_k^2}} \hat{\phi}_k(t) = 0 \quad (\text{EOM})$$

$$[\hat{\phi}_k(t), \hat{\pi}_{k'}(t)] = i \delta^3(\mathbf{k} + \mathbf{k}') \quad (\text{CCRs})$$

* Recall: $\hat{\phi}^+(\mathbf{x}, t) = \hat{\phi}(\mathbf{x}, t)$ means $\hat{\phi}_k^+(t) = \hat{\phi}_k(t)$.

Solution strategy due to Fock:

* Proceed analogously to the driven oscillator, e.g., during $t < 0$:

□ Introduce new variables:

$$\text{QM:} \quad a(t) := \sqrt{\frac{\omega}{2}} \hat{q}(t) + i \frac{1}{\sqrt{2m\omega}} \hat{p}(t)$$

$$\text{QFT:} \quad a_k(t) := \sqrt{\frac{\omega_k}{2}} \hat{\phi}_k(t) + i \frac{1}{\sqrt{2m\omega_k}} \hat{\pi}_k(t)$$

□ Equation of motion and CCRs:

$$\text{QM:} \quad \dot{a}(t) = -i\omega a(t) \quad [a(t), a^\dagger(t)] = 1$$

$$\text{QFT:} \quad \dot{a}_k(t) = -i\omega_k a_k(t) \quad [a_k(t), a_k^\dagger(t)] = \delta(k-k')$$

Exercise: verify

△ Remark: Valid only while no force and while ω is constant.

▢ Solution, using an initial condition:

$$QM: \quad a(t) = e^{-i\omega t} a_{in}, \quad [a_{in}, a_{in}^+] = 1$$

$$QFT: \quad a_k(t) = e^{-i\omega_k t} a_{in_k}, \quad [a_{in_k}, a_{in_k}^+] = \delta^3(k-k')$$

▢ Explicitly \Rightarrow

$$QM: \quad \hat{q}(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^+ \right)$$

$$QFT: \quad \hat{\phi}_k(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega_k t}}{\sqrt{\omega_k}} a_{in_k} + \frac{e^{i\omega_k t}}{\sqrt{\omega_k}} a_{in_k}^+ \right) \quad (S)$$

Exercise:
verify \rightarrow

$$\left(\text{i.e.: } \hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in_k} e^{-i\omega_k t + ikx} + a_{in_k}^+ e^{i\omega_k t - ikx} \right) d^3k \right)$$

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$$QM: \quad \hat{q}(t) = \frac{1}{\sqrt{2}} \left(\frac{e^{-i\omega t}}{\sqrt{\omega}} a_{in} + \frac{e^{i\omega t}}{\sqrt{\omega}} a_{in}^+ \right)$$

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Exercise:
verify \rightarrow

$$\left(\text{i.e.: } \hat{\phi}(x,t) = \int \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(a_{in,k} e^{-i\omega_k t + ikx} + a_{in,k}^+ e^{i\omega_k t - ikx} \right) d^3k \right)$$

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The Hilbert space of states:

- * Analogous to the case of QM, there is a vector, $|0_{in}\rangle \in \mathcal{H}$, which obeys:

$$a_{in_k} |0_{in}\rangle = 0, \text{ now for all vectors } k.$$
$$(|0_{in}\rangle = \bigotimes_k |0_{in_k}\rangle)$$

- * The Hamiltonian reads (for $t < 0$):

Exercise: \rightarrow
verify \rightarrow

$$\hat{H} = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}_k^+ \hat{\pi}_k + \omega^2 \hat{\phi}_k^+ \hat{\phi}_k d^3k$$

\swarrow This ∞ is called a "Infrared divergence"

$$= \int_{\mathbb{R}^3} \omega_k (a_{in_k}^+ a_{in_k} + \frac{1}{2} \delta^3(0)) d^3k$$

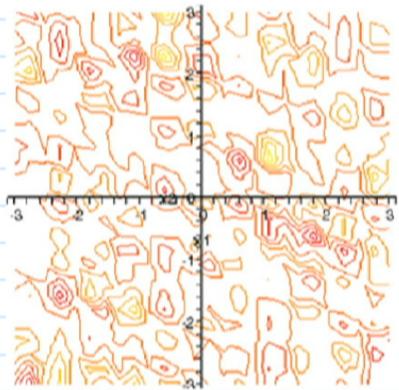
In a box: $\hat{H} = L^{-3/2} \sum_k \omega_k (a_{in_k}^+ a_{in_k} + \frac{1}{2})$ (because $\delta(k, k')$ is now $\delta_{k, k'}$)

Notice: The divergence $\sum_k L^{-3/2} \omega_k \frac{1}{2} = \infty$ is an "ultraviolet divergence".

Recall:



Making waves...



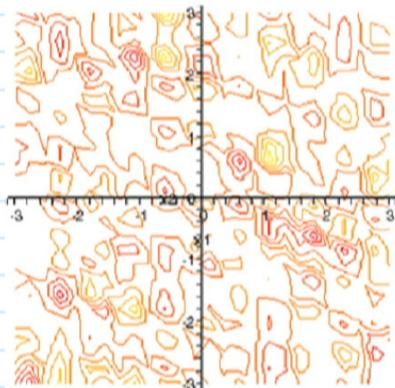
Making EM waves...

e^-
↳ Described by $J(x,t)$.

Upgrade: Give the charge $\rho(x,t)$
its own dynamics:



To cope with
a spring.

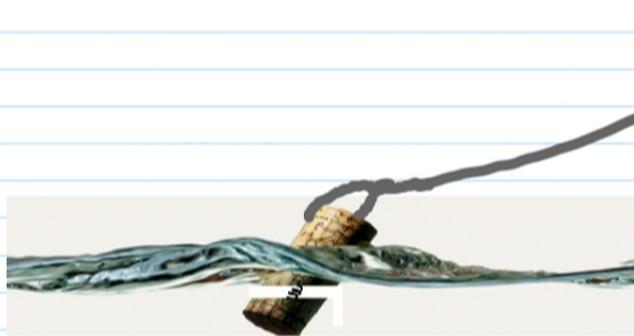


Often, only one atomic
transition is of interest.
Then, we can model the
atom as a 2-level system.

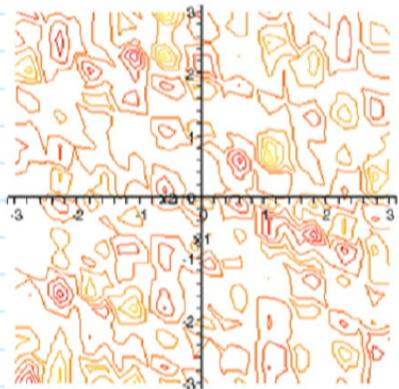
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Upgrade:

Give the charge $j(x,t)$
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To work with
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Described by QM, e.g. atom or qubit

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$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + \hat{j}(x,t)\hat{\phi}(x,t) d^3x$$

number-valued, classical
↓

and upgrade it to:

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \mathbb{1} \otimes \left(\hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) \right) + \hat{j}(x,t) \otimes \hat{\phi}(x,t) d^3x$$

↑
an operator acting on
the Hilbert space of the atom.

with $\hat{j}(x,t) = \hat{Q}(t) \delta(x - \tilde{x}(t))$

The Hilbert space: $\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{atom}} \otimes \mathcal{H}_{\text{field}}$

Simplified notation:

$$H(t) = \frac{1}{2} \int_{\mathbb{R}^3} \hat{\pi}^2(x,t) - \hat{\phi}(x,t)(\Delta - m^2)\hat{\phi}(x,t) + \hat{j}(x,t)\hat{\phi}(x,t) d^3x$$

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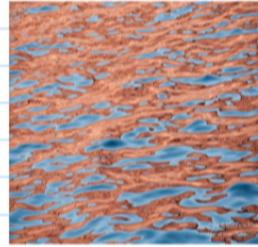
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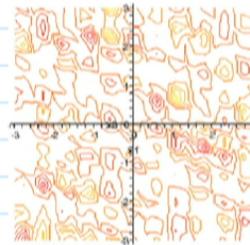
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The charged systems can act as emitters and as receivers of waves:

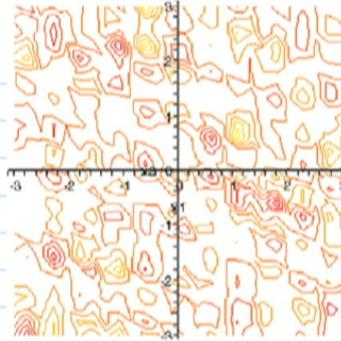


And, quantumly, they can act as emitters and receivers of particles!



Definition: (Unruh, deWitt):
A "particle", such as a photon
is what a "particle detector", such
as an atom, can detect, by
getting excited.

With acceleration:



An accelerated atom's charges can excite the field.

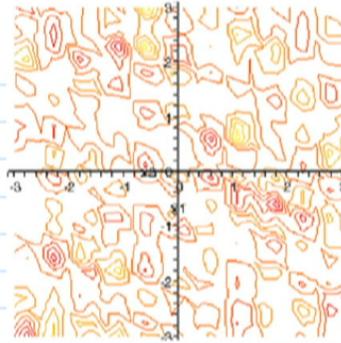
This, in turn can excite the atom: *the Unruh effect*

→ An accelerated atom may detect particles even if inertial observers only see the vacuum

→ Related to gravity via the equivalence principle

→ Related to Hawking radiation.

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