Title: Quantum Field Theory for Cosmology (AMATH872/PHYS785) - Lecture 6

Date: Jan 23, 2018 04:00 PM

URL: http://pirsa.org/18010062

Abstract:

Note Title

QFT for Cosmology, Achim Kempf, Lecture 6

Recall:

- There are two basic mechanisms to increase the amplitudes of oscillators, i.e., also to excite a field's mode oscillators, i.e. to create particles:

  a.) A time-varying driving force $J(t)$

  b.) A time-varying spring "constant" $\omega(t)$

- We are presently considering case a):

$$\hat{H}(t) = \frac{1}{2}\hat{p}(t)^2 + \frac{\omega^2}{2}\hat{q}(t)^2 - J(t)\,\hat{q}(t)$$

with a temporary force: $J(t) = 0$ for all $t \notin [0,T]$

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Examples: 1. Temporary emission from antenna, 2. Brief interaction (scattering) of particles.

We defined a convenient variable $a(t)$,

$$a(t) := \sqrt{\frac{\omega}{2}}\,\hat{q}(t) + i\frac{1}{\sqrt{2\omega}}\,\hat{p}(t)$$

so that: $\hat{H}(t) = \omega\left(a^+(t)\,a(t) + \frac{1}{2}\right) - \frac{1}{\sqrt{2\omega}}\,J(t)\left(a^+(t) + a(t)\right)$

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- By using $a(t)$, the problem reduced to solving:

  * $i\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}}\, J(t)$ (EOM)

  * $[a(t), a^+(t)] = 1$ for all t (CCR)

We gave a convenient name to $a(t=0)$:

$a_{in} := a(t=0)$ (an operator on Hilbert space that we still have to choose.)

Then, as is easy to verify, the solution is:

$$a(t) = a_{in} e^{-i\omega t} + 1 \frac{i}{\sqrt{2\omega}} e^{-i\omega t} \int_0^t J(t') e^{i\omega t'} dt'$$

$$= \left(a_{in} + 1 \frac{i}{\sqrt{2\omega}} \int_0^t J(t') e^{i\omega t'} dt' \right) e^{-i\omega t}$$

And so, with the definition: $J_0 := 1 \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$

$$a(t) = \begin{cases} a_{in} e^{-i\omega t} & \text{for } t<0 \\ \text{see above} & \text{for } 0 \le t \le T \\ \underbrace{(a_{in} + J_0)}_{!!} e^{-i\omega t} & \text{for } T<t \end{cases}$$

Define: a_{out}

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- By using $a(t)$, the problem reduced to solving:

* $$i\dot{a}(t) = \omega\, a(t) - \frac{1}{\sqrt{2\omega}}\,J(t) \qquad \text{(EOM)}$$

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Define: a_{out}

Before and after the force action, we have an undriven harmonic oscillator, solved as always, by $a(t) = a_0\, e^{-i\omega t}$

Here:

$$a(t) = \begin{cases} a_{in}\, e^{-i\omega t} & \text{for } t<0 \\ a_{out}\, e^{-i\omega t} & \text{for } t>T \end{cases}$$

with $a_{out} = a_{in} + J_0$:

Conservation of the CCRs?

Notice that $[a_{in}, a_{in}^+] = 1$ implies $[a_{out}, a_{out}^+] = 1$.

In fact:

Proposition: If we can arrange for $[a_{in}, a^+_{in}] = 1$,

then $[a(t), a^+(t)] = 1$ follows for all $t \in \mathbb{R}$!

Proof:

Assume $[a_{in}, a^+_{in}] = 1$. Then:

$$[a(t), a^+(t)] = \Big[a_{in} e^{-i\omega t} + \underbrace{1 \frac{i}{\sqrt{2\omega}} \int_0^t \ldots dt'}_{\text{number}},\ a^+_{in} e^{+i\omega t} - \underbrace{1 \frac{i}{\sqrt{2\omega}} \int_0^t \ldots dt'}_{\text{number}}\Big]$$

$$= \underbrace{[a_{in}, a^+_{in}]}_{=1} e^{-i\omega t} e^{i\omega t}$$

$$= 1 \quad ✓$$

The initial period, $t<0$:

The dynamical variables:

We have $a(t) = a_{in}\, e^{-i\omega t}$ and therefore we also have the dynamics of all other variables, such as:

$$* \quad \hat{q}(t) = \frac{1}{\sqrt{2\omega}}\left(a^+_{in}\, e^{i\omega t} + a_{in}\, e^{-i\omega t}\right)$$

$$* \quad \hat{p}(t) = i\sqrt{\frac{\omega}{2}}\left(a^+_{in}\, e^{i\omega t} - a_{in}\, e^{-i\omega t}\right)$$

Exercise: verify

$$* \quad \hat{H}(t) = \omega\left(a^+(t)\, a(t) + \frac{1}{2}\right)$$

$$= \omega\left(a^+_{in} e^{i\omega t}\, a_{in} e^{-i\omega t} + \frac{1}{2}\right)$$

$$= \omega\left(a^+_{in}\, a_{in} + \frac{1}{2}\right)$$ is constant in time!

◻ The Hilbert space of states:

* As always, we can write arbitrary Hilbert space vectors as linear combinations of an arbitrary set of basis vectors.

* We could use, for example, the eigenbasis of $\hat{q}(t)$ (or the eigenbasis of $\hat{p}(t)$).

  But: In the Heisenberg picture, this would be inconvenient because $\hat{q}(t)$ has a different eigenbasis for each t.

* However, $\hat{H}$ is time independent (for $t<0$).

  $\rightsquigarrow$ Let us construct and use its eigenbasis:

The eigenbasis of $\hat{H}$ for $t<0$:

* We have

$$\hat{H}_{t<0} = \omega\left(a^+_{in} a_{in} + \frac{1}{2}\right)$$

with:
$$[a_{in}, a^+_{in}] = 1 \qquad \text{(CCR)}$$

* Assume there exists a vector, denoted say $|0_{in}\rangle$, which obeys:

$$a_{in}|0_{in}\rangle = 0$$

← the Hilbert space vector with zero length

* Then it is eigenvector of $H_{t<0}$:

Recall: the energy eigenvalues of any harmonic oscillator is $E_n = \hbar\omega(n+\frac{1}{2})$ i.e. we have here $E_0 = \hbar\omega\frac{1}{2}$ (with $\hbar = 1$).

$$\hat{H}_{t<0}|0_{in}\rangle = \omega\left(a^+_{in} a_{in} + \frac{1}{2}\right)|0_{in}\rangle = \frac{1}{2}\omega|0_{in}\rangle$$

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* Consider now the state $|1_{im}\rangle := a^+_{im}|0_{im}\rangle$:

$$\hat{H}_{(i)}|1_{im}\rangle = \hat{H}_{(i)} a^+_{im}|0_{im}\rangle = \omega\left(a^+_{im}a_{im} + \frac{1}{2}\right)a^+_{im}|0_{im}\rangle$$

$$= \left(\omega a^+_{im}\left(a^+_{im}\cancel{a_{im}} + 1\right) + \frac{\omega}{2}a^+_{im}\right)|0_{im}\rangle$$

$$= \omega \frac{3}{2} a^+|0_{im}\rangle$$

$$= \frac{3}{2}\omega\,|1_{im}\rangle$$

⇒ The state $|1_{im}\rangle$ is eigenstate of $\hat{H}$ with eigenvalue $\frac{3}{2}\omega$. So it must be the 1st excited state.

* Is the vector $|1_{im}\rangle$ normalized?

$$\langle 1_{im}|1_{im}\rangle = \langle 0_{im}|a_{im}a^+_{im}|0_{im}\rangle = \langle 0_{im}|a^+_{im}a_{im} + 1|0_{im}\rangle = \langle 0_{im}|0_{im}\rangle = 1$$

✓

* Is the vector $|1_{\text{in}}\rangle$ normalized?

$$\langle 1_{\text{in}}|1_{\text{in}}\rangle = \langle 0_{\text{in}}| a_{\text{in}}\, a^+_{\text{in}} |0_{\text{in}}\rangle = \langle 0_{\text{in}}| a^+_{\text{in}}\, a_{\text{in}} + 1 |0_{\text{in}}\rangle = \langle 0_{\text{in}}|0_{\text{in}}\rangle = 1$$

✓

* **Proposition:**

The set of vectors $\{|n_{\text{in}}\rangle\}_{n=0}^{\infty}$ defined through

$$|n_{\text{in}}\rangle := \frac{1}{\sqrt{n!}} (a^+_{\text{in}})^n \, |0_{\text{in}}\rangle$$

is orthonormal, i.e., $\langle n|n'\rangle = \delta_{n,n'}$. Exercise: verify

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* **Proposition:**

The vectors $|n_{in}\rangle$ are eigenvectors of $\hat{H}_{t<0}$:

$$\left.\begin{aligned} \hat{H}_{t<0} |n_{in}\rangle &= E_n |n_{in}\rangle \\ \text{with} \quad E_n &= \omega\left(n + \tfrac{1}{2}\right) \end{aligned}\right\} \text{Exercise: verify}$$

* **Proposition:** $\{|n_{in}\rangle\}$ is complete eigenbasis of $\hat{H}$.

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⇝ Summary re choice of basis for $t<0$:

- The Hamiltonian $\hat{H}(t)$ is constant for $t<0$.
- Thus it has one eigenbasis for all $t<0$, namely $\{|n_{in}\rangle\}$.
- We may expand every arbitrary vector $|\psi\rangle$ of the Hilbert space, $\mathcal{H}$, in this basis:

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n_{in}\rangle$$

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- E.g., the state of our quantum system could be:

$$|\psi\rangle = |5_{in}\rangle$$

- The system always stays in state $|\psi\rangle = |5_{in}\rangle$.

Recall:
- But $|\psi\rangle = |5_{in}\rangle$ generally ceases to be eigenvector of $\hat{H}(t)$ for $t>0$!

The period $t > T$: (after the force ceased to act)

- Once the driving force acts, $\hat{H}(t)$ starts to change.
- **But**: After the force finished, $t > T$, the Hamiltonian simply reads

$$\hat{H}(t) = \omega\left(a^{+}(t)\,a(t) + \frac{1}{2}\right) - \frac{a^{+}(t) + a(t)}{\sqrt{2\omega}}\,\underbrace{\cancel{J(t)}}_{=0} \quad \text{for } t > T$$

and from above, therefore:

$$\hat{H}(t) = \omega\left(a^{+}_{out}\,\cancel{e^{i\omega t}}\,a_{out}\,\cancel{e^{-i\omega t}} + \frac{1}{2}\right) \quad \text{with } a_{out} = a_{in} + J_0$$

$$\Rightarrow \quad \hat{H}_{t>T} = \omega\left(a^{+}_{out}\,a_{out} + \frac{1}{2}\right) \Rightarrow \hat{H} \text{ is then constant again!}$$

- Note: we can construct a basis from $a_{out}|0_{out}\rangle = 0$ etc.

Compare $t<0$ to $t>T$:

QFT:

A. Motion: $\bar{q}(t)$ (large $\bar{q}$ means large $\bar{\phi}_k$ means large waves)

B. Resonance: best $J(t)$? (consider e.g. antenna)

C. Energy expectation: $\bar{E}(t)$ (large $\bar{E}$ means large $\bar{E}_k$ means energy in mode k)

D. Energy eigenstates: $\{|E_n(t)\rangle\}$ (particle creation)

We will consider the example where the system starts out in the lowest energy state (the vacuum):

$$|\psi\rangle = |0_{in}\rangle$$

A. Motion $\bar{q}(t)$:

$$\bar{q}(t) = \langle \psi | \hat{q}(t) | \psi \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} (a^+(t) + a(t)) | 0_{in} \rangle$$

* For $t<0$ we obtain:

$$\bar{q}(t) = \frac{1}{\sqrt{2\omega}} \langle 0_{in} | a^+_{in} e^{i\omega t} + a_{in} e^{-i\omega t} | 0_{in} \rangle$$

$$= 0$$

This was expected since for $t<0$ the system's state $|0_{in}\rangle$ is the ground state of $\hat{H}(t)$.

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* For $t>T$ we obtain:

$$\bar{q}(t) = \langle \chi | \hat{q}(t) | \chi \rangle \qquad a_{out} = a_{in} + J_0$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+(t) + a(t) \right) | 0_{in} \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+_{out} e^{i\omega t} + a_{out} e^{-i\omega t} \right) | 0_{in} \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left((\cancel{a^+_{in}} + J_0^*) e^{i\omega t} + (\cancel{a_{in}} + J_0) e^{-i\omega t} \right) | 0_{in} \rangle$$

$$= \frac{1}{\sqrt{2\omega}} \left(J_0^* e^{i\omega t} + J_0 e^{-i\omega t} \right) \qquad (*)$$

Exercise: verify ↗

$$= \int_0^T \frac{\sin((t-t')\omega)}{\omega} J(t') \, dt'$$

(Remark: same as classical $q(t)$ due to Ehrenfest theorem)

⇒ $\bar{q}$ oscillates with frequency ω, as expected.

Exercise: verify $\nearrow$ $$= \int_0^T \frac{\sin((t-t')\omega)}{\omega} J(t')\,dt'$$ (Remark: same as classical $q(t)$ due to Ehrenfest theorem)

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B. Resonance:

* The amplitude of the excited motion of the oscillator is determined by J_0, as equation (*) shows.

* We expect that the driving force $J(t)$ is most efficient at creating a large J_0 if it oscillates at roughly the oscillator's natural frequency ω.

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* We expect that the driving force $J(t)$ is most efficient at creating a large J_0 if it oscillates at roughly the oscillator's natural frequency ω.

* Indeed: J_0 is the Fourier component of $J(t)$ for the frequency ω on the interval $[0,T]$:

$$J_0 := \frac{i}{\sqrt{2\omega}} \int_0^T J(t') e^{i\omega t'} dt'$$

Thus, indeed, the more of the frequency ω is contained in $J(t)$, the larger is $|J_0|$.

* For $t>T$ we obtain:

$$a_{out} = a_{in} + J_0$$

$$\bar{q}(t) = \langle \psi | \hat{q}(t) | \psi \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+(t) + a(t)\right) | 0_{in} \rangle$$

$$= \langle 0_{in} | \frac{1}{\sqrt{2\omega}} \left(a^+_{out} e^{i\omega t} + a_{out} e^{-i\omega t}\right) | 0_{in} \rangle$$

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Thus, indeed, the more of the frequency ω is contained in $J(t)$, the larger is $|J_0|$.

C. Energy expectation

* For $t<0$ we have:

$$\bar{H}(t) = \langle \psi | \hat{H}(t) | \psi \rangle \quad \text{(always)}$$

$$= \langle 0_{in} | \omega (a^{+}_{in} a_{in} + \tfrac{1}{2}) | 0_{in} \rangle \quad (\text{for } t<0)$$

$$= \frac{\omega}{2}$$

i.e., the energy of the ground state of the Hamiltonian $\hat{H}_{t<0}$.

* For $t>T$ we have:

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$$= \omega \langle 0_{in} | (\cancel{a}^+_{in} + J_0^*)(\cancel{a}_{in} + J_0) + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \langle 0_{in} | J_0^* J_0 + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \left(\tfrac{1}{2} + |J_0|^2 \right) \qquad \text{which is elevated!}$$

Remark: We notice that the oscillator's energy

$$= \omega \langle 0_{in} | J_0^* J_0 + \tfrac{1}{2} | 0_{in} \rangle$$

$$= \omega \left(\tfrac{1}{2} + |J_0|^2\right)$$ which is elevated!

Remark: We notice that the oscillator's energy increases the more the larger $|J_0|$, i.e., from B, the closer the driving force is to the oscillator's natural frequency ω.

Remark: In QFT, say when electrical current drives electromagnetic field modes, the closer a mode's ω_k is to the frequency of the current, the more this mode gets excited.

<u>Implication:</u> $|0_{out}\rangle \neq |0_{in}\rangle = |\chi\rangle$

- Ground state $|0_{out}\rangle$ of

$$H_{t>T} = \omega\left(a^+(t)\,a(t) + \tfrac{1}{2}\right) = \omega\left(a^+_{out}\,a_{out} + \tfrac{1}{2}\right)$$

has eigenvalue $\omega/2$, i.e.:

$$a_{out}|0_{out}\rangle = 0.$$

- Therefore: $a_{out}\,|\chi\rangle = a_{out}\,|0_{in}\rangle$

$$= (a_{in} + J_0)\,|0_{in}\rangle$$

$$= J_0\,|\chi\rangle \neq 0$$

$\Rightarrow$ At late times: $|\chi\rangle \neq |0_{out}\rangle$

Q: So what kind of excited state is $|\chi\rangle$ at late times?

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A: Since $|\chi\rangle$ is eigenstate of a lowering operator,

$$a_{out}|\chi\rangle = J_0 |\chi\rangle$$

$|\chi\rangle$ is what is called a **Coherent State**.

Recall:

Coherent states saturate the uncertainty relation:

If $|\psi\rangle$ is a coherent state, then

$$\Delta q_{|\psi\rangle} \, \Delta p_{|\psi\rangle} = \frac{\hbar}{2}$$

→ These are the states which come closest to having

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$\rightsquigarrow$ These are the states which come closest to having definite values for both q and p, i.e., they are as close as possible to obeying:

$$\hat{q}|\psi\rangle = \langle \hat{q} \rangle |\psi\rangle \text{ and } \hat{p}|\psi\rangle = \langle \hat{p} \rangle |\psi\rangle$$

Exercise: Show that if $a|\alpha\rangle = \alpha|\alpha\rangle$, with $\alpha \in \mathbb{C}$

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and $\hat{q} = \frac{1}{\sqrt{2\omega}}(a^+ + a)$, $\hat{p} = i\sqrt{\frac{\omega}{2}}(a^+ - a)$ $\quad(*)$

Then, $\langle\alpha|\hat{q}|\alpha\rangle = \frac{1}{\sqrt{2\omega}}(\alpha^* + \alpha)$

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and: $\Delta q(t)\,\Delta p(t) = \frac{1}{2}$

Remarks:

- Notice that because $a|\alpha\rangle = \alpha|\alpha\rangle$, the operator a does **not** reduce the excitation (or particle) number of $|\alpha\rangle$.
- If the ω in (*) is chosen to be not the frequency of the harmonic oscillator of the Hamiltonian, then $|\alpha\rangle$ is called a Squeezed State.

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Q: Significance of driven harmonic oscillators always ending up in a coherent state for QFT?

A: Consider example of classical currents and charges driving the mode oscillators of the electromagnetic QFT:

Then, $\langle \alpha | \hat{q} | \alpha \rangle = \frac{1}{\sqrt{2\omega}} (\alpha^* + \alpha)$

$$\langle \alpha | \hat{p} | \alpha \rangle = i\sqrt{\frac{\omega}{2}} (\alpha^* - \alpha)$$

and: $\Delta q(t)\, \Delta p(t) = \frac{1}{2}$

Remarks:

- Notice that because $a|\alpha\rangle = \alpha|\alpha\rangle$, the operator a does **not** reduce the excitation (or particle) number of $|\alpha\rangle$.
- If the ω in (*) is chosen to be not the frequency of the harmonic oscillator of the Hamiltonian, then $|\alpha\rangle$ is called a Squeezed State.

Q: Significance of driven harmonic oscillators always

Q: Significance of driven harmonic oscillators always ending up in a coherent state for QFT?

A: Consider example of classical currents and charges driving the mode oscillators of the electromagnetic QFT:

- The charges and currents drive the EM oscillators into a coherent state. ← (In Heisenberg picture: State stays constant but its meaning relative to the then time-dependent operators changes)

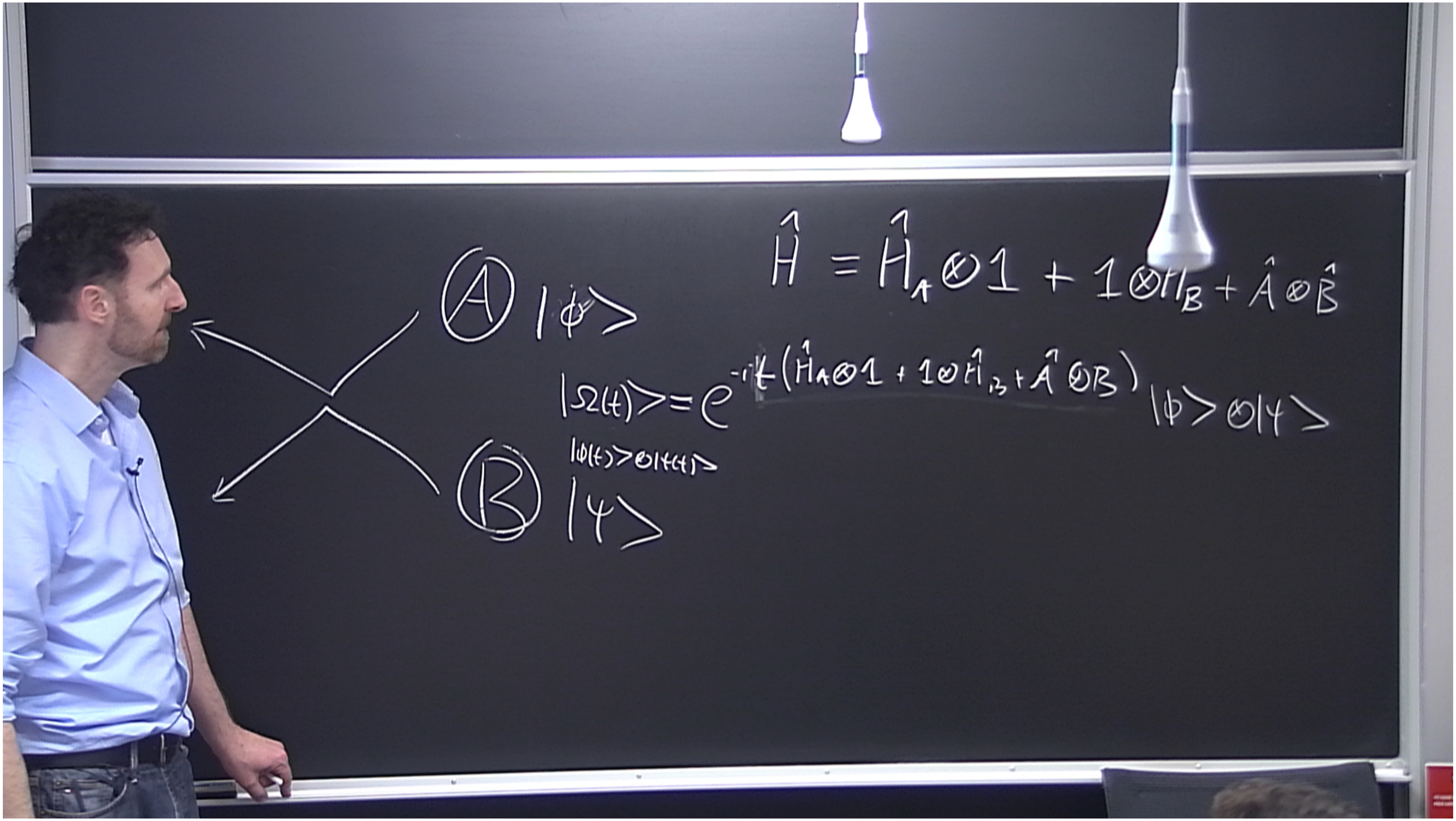
- ⇒ The $\hat{q}_k$, $\hat{p}_k$ of the EM field (essentially the $\hat{E}_i$ and $\hat{B}_i$ fields) will be as sharp as possible, i.e., the EM QFT's state is close to being eigenstate to $\hat{E}_i$ and $\hat{B}_i$.

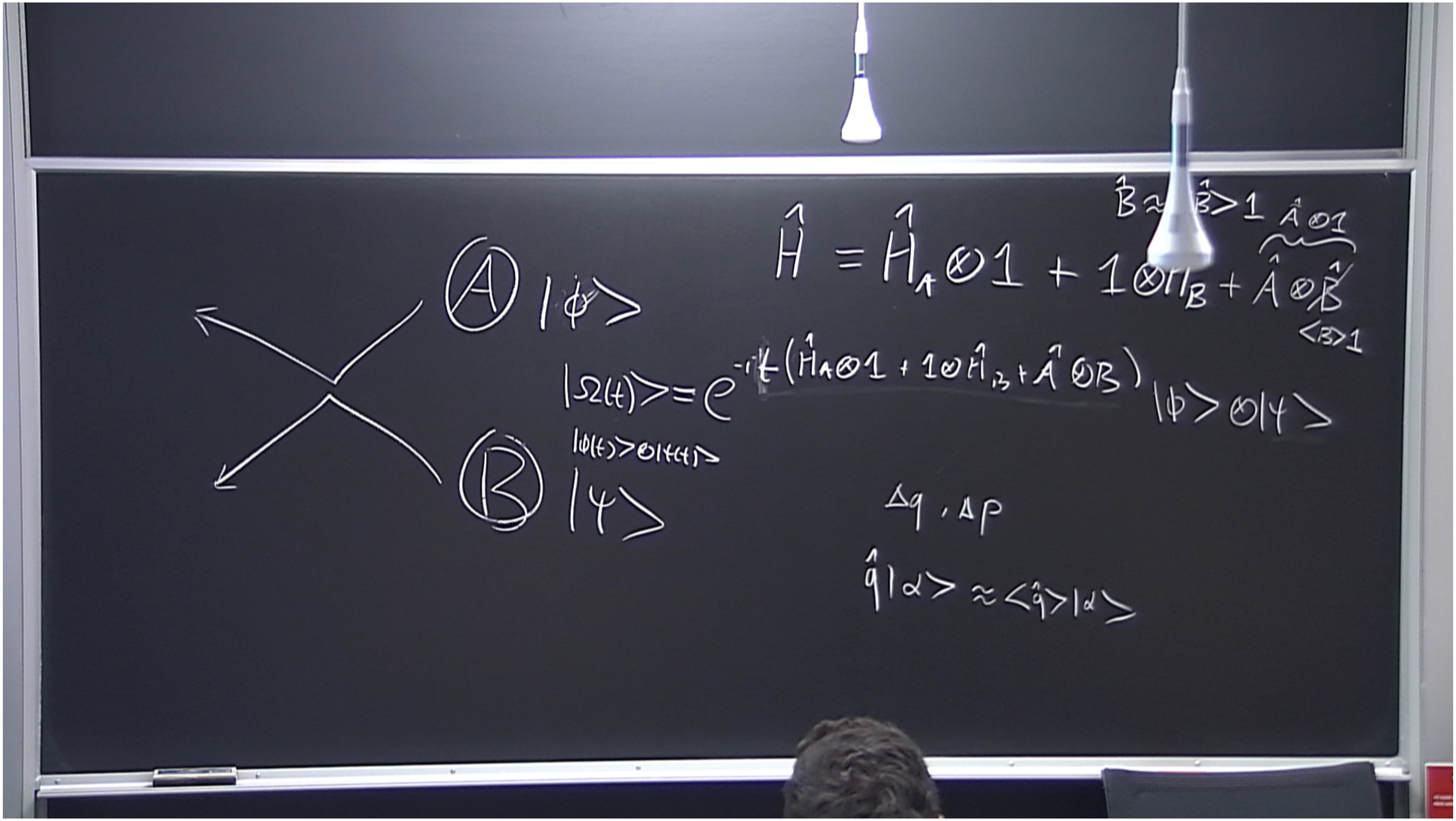
Q: Implications for how, e.g., an electron interacts with such an EM field?

A: It explains why it is nearly legal to do what one is taught in a first course in quantum mechanics:

"In order to describe the quantum mechanics of an electron interacting with a background EM field, simply put electric and magnetic fields as number-valued functions into the Schrödinger equation."

It is indeed often a good approximation to replace the operators $\hat{E}_i(x,t)$ and $\hat{B}_i(x,t)$ by their expectation values, if the QFT is in a coherent state.


A $|\phi\rangle$
B $|\psi\rangle$
$\hat{H} = \hat{H}_A \otimes 1 + 1 \otimes \hat{H}_B + \hat{A} \otimes \hat{B}$
$|\Omega(t)\rangle = e^{-it(\hat{H}_A \otimes 1 + 1 \otimes \hat{H}_B + \hat{A} \otimes B)} |\phi\rangle \otimes |\psi\rangle$
$|\phi(t)\rangle \otimes |\psi(t)\rangle$


$\hat{H} = \hat{H}_A \otimes 1 + 1 \otimes \hat{H}_B + \hat{A} \otimes \hat{B}$
$\hat{A} \otimes 1$
$\langle B \rangle \gg 1$
A $|\phi\rangle$
B $|\psi\rangle$
$|\Omega(t)\rangle = e^{-it(\hat{H}_A \otimes 1 + 1 \otimes \hat{H}_B + \hat{A} \otimes B)} |\phi\rangle \otimes |\psi\rangle$
$|\phi(t)\rangle \otimes |\psi(t)\rangle$
$\Delta q \cdot \Delta p$
$\hat{q}|\alpha\rangle \approx \langle \hat{q} \rangle |\alpha\rangle$